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On primal regularity estimates for set-valued mappings

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Abstract. We prove several generalizations of the results in [6] for set-valued mappings. In some cases, we improve also the statements for single-valued mappings. Linear openness of the set-valued mapping in question is deduced from the properties of its suitable approximation. This approach goes back to the classical Lyusternik-Graves theorem saying that a continuously differentiable single-valued mapping between Banach spaces is linearly open around an interior point of its domain provided that its derivative at this point is surjective. In this paper, we consider approximations given by a graphical derivative, a contingent variation, a strict pseudo H-derivative, and a bunch of linear mappings.

Key Words. metric regularity, open mapping theorem, linear openness, graphical derivative, one-sided directional derivative, strict pseudo H-differentiability, strict Fréchet prederivative, contingent variation, strong operator topology, weak operator topology

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1 Introduction

Metric regularity, linear openness and pseudo-Lipschitz property of the inverse are tree *equivalent* properties playing fundamental role in modern variational analysis and have been broadly covered in the recent monographs [3], [8], [20], and [24]. A survey on this topic together with a rich bibliography can be found in [17].

Banach's open mapping principle says that a continuous linear mapping between two Banach spaces is (linearly) open (at any point) if and only if it is surjective. In 1950, L.M. Graves generalized this statement proving that a single-valued mapping f acting between Banach spaces is (linearly) open at an interior point $\bar{x}$ of its domain provided that there exists a surjective continuous linear mapping A such that the difference $f - A$ is locally Lipschitz continuous at $\bar{x}$ with a sufficiently small Lipschitz modulus. This result remains true [5] when $\bar{x}$ is a boundary point of a closed convex subset K of a Banach space and the restriction of the approximating mapping A to K is open at $\bar{x}$. In variational analysis we work with mappings which may be non-smooth and also set-valued. Note that the mapping f in Graves' theorem is not necessarily differentiable at $\bar{x}$ but it is well approximated by one single-valued mapping around the point in question. K. Nachi and J.-P. Penot [21] extended this idea to a set-valued mapping F between Banach spaces by introducing a suitable derivative of F at the reference point which is supposed to

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be a (single-valued) continuous linear mapping. A different approach was used by B.H. Pourciau in [25] who proved that a Lipschitz-continuous single-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $m \leq n$, is linearly open at an interior point $\bar{x}$ of its domain provided that all the matrices in the Clarke's generalized Jacobian have full (row) rank. His result was extended to reflexive Banach spaces by D. Preiss and the second named author in [13] where f is approximated by a bounded convex set of continuous linear operators, and to general Banach spaces by Z. Páles in [23] under rather strong compactness assumption on the bunch of linear operators (with respect to the topology induced by the operator norm). Similarly as in the case of Graves' theorem, these results remain true [5, 6] when $\bar{x}$ is a boundary point of the domain of f provided that it is closed and convex. Of course, there are many other ways how to approximate a set-valued mapping (for more details, see bibliographical comments following the statements in Section 3). In this paper, we focus on approximations given by a graphical derivative, or by a contingent variation, or by a strict pseudo H-derivative, or by a bunch of continuous linear mappings. We obtain generalizations of the results mentioned above.

The paper is organized as follows. In the next section, we provide a background from regularity theory. The statements therein will be used in Section 3 which contains all our results together with relevant bibliographical comments.

Notations and terminology. When we write $f : X \to Y$ we mean that f is a (single-valued) mapping acting from X into Y while $F : X \rightrightarrows Y$ is a mapping from X into Y which may be set-valued. The set $\operatorname{dom} F := \{x : \ F(x) \neq \emptyset\}$ is the *domain* of F, the *graph* of F is the set $\operatorname{gph} F := \{(x, y) \in X \times Y : \ y \in F(x)\}$ and the *inverse* of F is the mapping $Y \ni y \longmapsto \{x \in X : y \in F(x)\} =: F^{-1}(y) \subset X$; thus $F^{-1} : Y \rightrightarrows X$. In any metric space $B(x, r)$ denotes the closed ball centered at x with a radius $r > 0$ and $\overset{\circ}{B}(x, r)$ is the corresponding open ball. B_X and S_X are respectively the *closed unit ball* and the *unit sphere* in a Banach space X. The *distance from a point* x to a subset C of a metric space (X, d) is $d(x, C) := \inf\{d(x, y) : \ y \in C\}$. We use the convention that $\inf \emptyset := +\infty$ and as we work with non-negative quantities we set $\sup \emptyset := 0$. If a set is singleton we identify it with its only element, that is, we write a instead of $\{a\}$. The symbol $\mathcal{L}(X, Y)$ denotes the space of all linear bounded operators from a Banach space X into a Banach space Y.

2 Background from regularity theory

Given two metric spaces X and Y, a set-valued mapping $F : X \rightrightarrows Y$ is called *open with a linear rate* near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there are positive numbers c and ε such that

$$(1) \quad B(y, ct) \subset F(B(x, t)), \quad \text{whenever} \quad (x, y) \in \big(B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon)\big) \cap \operatorname{gph} F \quad \text{and} \quad t \in (0, \varepsilon).$$

The supremum of $c > 0$ such that (1) holds for some $\varepsilon > 0$ is called *rate of openness* (*rate* or *modulus of surjection*) of F near $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur} F(\bar{x}, \bar{y})$. The mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is said to be *metrically regular* near $(\bar{x}, \bar{y})$ if there are $\kappa > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$(2) \qquad d\big(x, F^{-1}(y)\big) \leq \kappa d(y, F(x)) \quad \text{for all} \quad (x, y) \in U \times V.$$

The infimum of all $\kappa > 0$ such that (2) holds for some neighborhoods U and V is called the *rate* or *modulus of metric regularity* of F near $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$. It is well known that $\operatorname{reg} F(\bar{x}, \bar{y}) \cdot \operatorname{sur} F(\bar{x}, \bar{y}) = 1$ always holds under convention that $0 \cdot \infty = 1$ (see [17] for history of this equality). If $f : X \to Y$ is a single-valued mapping, then we write $\operatorname{sur} f(\bar{x})$ and $\operatorname{reg} f(\bar{x})$ instead of $\operatorname{sur} f\big(\bar{x}, f(\bar{x})\big)$ and $\operatorname{reg} f\big(\bar{x}, f(\bar{x})\big)$, respectively. An $A \in \mathcal{L}(X, Y)$ is metrically regular at any point if and only if it is surjective; therefore we write $\operatorname{sur} A$ and $\operatorname{reg} A$ only.

The theorems below will be proved via the following, nowadays called, A.D. Ioffe's criterion for regularity of mappings (accommodated a bit to our purposes); see [13, Corollary 1], and [16, Theorem 1b]. The proof of the first mentioned statement is based on a generalization of Caristi's principle while the latter theorem relies on Ekeland's variational principle. Elaborating the ideas from [13], we provide a direct iterative proof for the reader's convenience. In fact, Ekeland's variational principle can be proved via a quite similar iterative procedure [24, p. 62]. It turns out that a direct application of this criterion yields short proofs of well-known regularity statements [17].

Proposition 2.1. *Let (X, d) be a complete metric space and (Y, ϱ) be a metric space, let $\bar{x} \in X$ be given, and let $g : X \to Y$ be a continuous mapping, whose domain is all of X. Then* $\operatorname{sur} g(\bar{x})$ *equals to the supremum of all $c > 0$ for which there is an $r > 0$ such that for all $(x, y) \in B(\bar{x}, r) \times \big(B(g(\bar{x}), r) \setminus \{g(x)\}\big)$ there is an $x' \in X$ satisfying*

$$c\, d(x', x) < \varrho(g(x), y) - \varrho(g(x'), y). \tag{3}$$

Proof. Denote the above supremum by s. Fix any $c > 0$ for which there is $r > 0$ such that for every $y \in B(g(\bar{x}), r)$ and every $x \in B(\bar{x}, r)$, satisfying $g(x) \neq y$, there is an $x' \in X$ such that (3) holds. By the continuity of g, there is $\bar{t} \in (0, r)$ such that

$$B(g(u), c\bar{t}) \subset B(g(\bar{x}), r) \quad \text{and} \quad B(u, \bar{t}) \subset B(\bar{x}, r) \quad \text{whenever} \quad u \in B(\bar{x}, \bar{t}). \tag{4}$$

Fix any $t \in (0, \bar{t})$ and any $u \in B(\bar{x}, t)$. It suffices to show that $g\big(B(u, t)\big) \supset B(g(u), ct)$. Consider any fixed $y \in B(g(u), ct)$; we will find $x \in B(u, t)$ such that $y = g(x)$. If $y = g(u)$, take $x := u$ and we are done. Assume further that $y \neq g(u)$. We will construct a sequence $x_1, x_2, \ldots$ in $B(u, t)$ satisfying

$$c\, d(x_m, u) \leq \varrho(g(u), y) - \varrho(g(x_m), y), \quad m \in \mathbb{N}. \tag{5}$$

Clearly, $x_1 := u$ satisfies (5) with $m = 1$. Let $n \in \mathbb{N}$ and assume that $x_n \in B(u, t)$ was already found. If $g(x_n) = y$, then take $x := x_n$, and stop the construction. Assume further that $g(x_n) \neq y$. In view of (4), Axiom of choice, and the assumptions, there is an $x_{n+1} \in X$ such that

$$c\, d(x_{n+1}, x_n) < \varrho(g(x_n), y) - \varrho(g(x_{n+1}), y) \quad \text{and that} \quad d(x_{n+1}, x_n) \geq \tfrac{1}{2} s_n \tag{6}$$

where

$$s_n := \sup\big\{d(x', x_n) : \ x' \in X \ \text{ and } \ c\, d(x', x_n) < \varrho(g(x_n), y) - \varrho(g(x'), y)\big\}.$$

Note that $0 \leq s_n \leq \frac{1}{c}\varrho(g(x_n), y) < +\infty$. Adding the first inequality in (6) and (5), with $m := n$, we get (5) with $m := n + 1$. In particular, we have $cd(x_{n+1}, u) \leq \varrho(g(u), y) \leq ct$; thus $x_{n+1} \in B(u, t)$. If the process stops at some $n \in \mathbb{N}$, we are done.

Assume that this was not the case, that is, $g(x_n) \neq y$ for every $n \in \mathbb{N}$. From (6) we have, for all $1 \leq n < m$, that

$$\begin{aligned} 0 \leq c\, d(x_m, x_n) &\leq c\, d(x_m, x_{m-1}) + \cdots + c\, d(x_{n+1}, x_n) \\ &< \big(\varrho(g(x_{m-1}), y) - \varrho(g(x_m), y)\big) + \cdots + \big(\varrho(g(x_n), y) - \varrho(g(x_{n+1}), y)\big) \\ &= \varrho(g(x_n), y) - \varrho(g(x_m), y), \end{aligned}$$

and so, $\varrho(g(x_n), y) > \varrho(g(x_m), y)$. Thus $\lim_{n\to\infty} \varrho(g(x_n), y)$ exists and is finite, and consequently, (x_n) is a Cauchy sequence in the (complete) space X. Put $x := \lim_{n\to\infty} x_n$. Then clearly $x \in B(u, t)$. Suppose that $y \neq g(x)$. By the assumption, there is an $x' \in X$ such that $c\, d(x', x) < \varrho(g(x), y) - \varrho(g(x'), y)$. As $x_n \to x$ and $g(x_n) \to g(x)$, the continuity of d and $\varrho(g(\cdot), y)$ implies that, for all $n \in \mathbb{N}$ big enough, we have $c\, d(x', x_n) < \varrho(g(x_n), y) - \varrho(g(x'), y)$, and so $s_n \geq d(x', x_n)$. Thus $\limsup_{n\to\infty} s_n \geq d(x', x)$ (> 0). But, by (6), $s_n \leq 2d(x_{n+1}, x_n) \longrightarrow 0$ as $n \to \infty$, and so $x' = x$, a contradiction. Therefore $y = g(x)$. We proved that $g\big(B(u, t)\big) \supset B\big(g(u), ct\big)$. We thus showed that $c \leq \operatorname{sur} g(\bar{x})$, and therefore $s \leq \operatorname{sur} g(\bar{x})$.

Assume that $s < \operatorname{sur} g(\bar{x})$. Fix any $c \in \big(s, \operatorname{sur} g(\bar{x})\big)$. Find $\varepsilon > 0$ such that $B(g(x), ct) \subset g\big(B(x, t)\big)$ whenever $x \in B(\bar{x}, \varepsilon)$ and $t \in (0, \varepsilon)$. By the continuity of g, there is an $r \in (0, \varepsilon)$ such that $\varrho(g(x), y) < c\varepsilon$ for each $(x, y) \in B(\bar{x}, r) \times B(g(\bar{x}), r)$. Fix any $y \in B(g(\bar{x}), r)$ and any $x \in B(\bar{x}, r)$ with $g(x) \neq y$. Let $t := \varrho(g(x), y)/c$. Then $t \in (0, \varepsilon)$, and there is an $x' \in B(x, t)$ such that $y = g(x')$. Thus

$$0 < c\, d(x', x) \leq ct = \varrho(g(x), y) = \varrho(g(x), y) - \varrho(g(x'), y).$$

Hence $s \geq c'$ for any $c' \in (s, c)$, a contradiction. □

Although Proposition 2.1 is formulated for a single-valued function, it is well-known that the study of covering properties for a set-valued mapping $F : X \rightrightarrows Y$ can always be reduced to the study of the corresponding property for a simple single-valued mapping, namely, restriction of the canonical projection from $X \times Y$ onto Y, that is the asignment $\operatorname{gph} F \ni (x, y) \longmapsto y \in Y$ (e.g., see [16, Proposition 3]). Using this one gets the following statement for set-valued mappings.

Proposition 2.2 (general criterion for set-valued maps)**.** *Let (X, d), (Y, ϱ) be metric spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping whose graph is complete in a vicinity of $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then $\operatorname{sur} F(\bar{x}, \bar{y})$ equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $(x, v) \in \operatorname{gph} F \cap \big(B(\bar{x}, r) \times B(\bar{y}, r)\big)$ and any $y \in B(\bar{y}, r) \setminus \{v\}$ there is a pair $(x', v') \in \operatorname{gph} F$ such that*

$$c \max\{d(x, x'), \alpha\varrho(v, v')\} < \varrho(v, y) - \varrho(v', y). \tag{7}$$

Proof. Denote by s the supremum from the statement. First, we show that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq s$. If $s = 0$ we are done. Suppose that $s > 0$ and pick any $c \in (0, s)$. Find the corresponding $\alpha \in (0, 1/c)$ and $r > 0$ such that the property involving (7) holds. Define the (equivalent) metric $\tilde{\rho}$ on $X \times Y$ for each (u, w), $(u', w') \in X \times Y$ by $\tilde{\varrho}\big((u, w), (u', w')\big) := \max\{d(u, u'), \alpha\varrho(w, w')\}$. Find an $\tilde{r} \in (0, r)$ such that $\widetilde{X} := \big(B(\bar{x}, \tilde{r}) \times B(\bar{y}, \tilde{r}/\alpha)\big) \cap \operatorname{gph} F$ equipped with $\tilde{\varrho}$ is a complete metric space. Let $g := p_Y\restriction_{\widetilde{X}}$, where p_Y is the canonical projection from $X \times Y$ onto Y. Then g is a continuous mapping defined on the whole $\widetilde{X}$. Find an $r' \in (0, \alpha\tilde{r})$ such that $r'(2 + \alpha) < c\alpha\tilde{r}$.

Fix any $(x,v) \in B_{\widetilde{X}}((\bar{x},\bar{y}),r') = \operatorname{gph} F \cap \big(B(\bar{x},r') \times B(\bar{y},r'/\alpha)\big) \subset \operatorname{gph} F \cap \big(B(\bar{x},\tilde{r}) \times B(\bar{y},\tilde{r})\big)$ and any $y \in B(\bar{y},r') \setminus \{v\}$. Find a pair $(x',v') \in \operatorname{gph} F$ satisfying (7). Then

$$\begin{aligned} \tilde{\varrho}\big((x',v'),(\bar{x},\bar{y})\big) &\leq \tilde{\varrho}\big((x',v'),(x,v)\big) + r' \overset{(7)}{<} \frac{\varrho(v,y)}{c} + r' \leq \frac{\varrho(v,\bar{y}) + \varrho(\bar{y},y)}{c} + r' \\ &\leq \frac{r'/\alpha + r'}{c} + r' = \frac{r'}{c\alpha}(1+\alpha+c\alpha) < \frac{r'}{c\alpha}(2+\alpha) < \tilde{r}. \end{aligned}$$

Hence, $(x',v') \in \widetilde{X}$. Proposition 2.1 then implies that $\operatorname{sur} g(\bar{x},\bar{y}) \geq s$. Thus, for some $c' \in (0,s)$ (arbitrarily close to s), there is an $\varepsilon > 0$ such that for each $(x,y) \in \big(B(\bar{x},\varepsilon) \times (B(\bar{y},\varepsilon/\alpha) \cap B(\bar{y},\varepsilon))\big) \cap \operatorname{gph} F$ and each $t \in (0,\varepsilon)$ we have

$$B(y,c't) \subset g\Big(\operatorname{gph} F \cap \big(B(x,t) \times B(y,t/\alpha)\big)\Big).$$

Pick $\varepsilon' \in (0, \varepsilon \min\{1, 1/\alpha\})$. Fix any $(x,y) \in \big(B(\bar{x},\varepsilon') \times B(\bar{y},\varepsilon')\big) \cap \operatorname{gph} F$ and any $t \in (0,\varepsilon')$. Then for any $w \in B(y,c't)$ there is a $u \in B(x,t)$ such that $g(u,w) = w$, that is, $w \in F(u)$. Thus $B(y,c't) \subset F(B(x,t))$. As c' can be chosen arbitrarily close to s, we showed that $\operatorname{sur} F(\bar{x},\bar{y}) \geq s$.

Suppose that $\operatorname{sur} F(\bar{x},\bar{y}) > s$. Then there are $c' > s$ and $\varepsilon > 0$ such that

$$B(y,c't) \subset F(B(x,t)) \quad \text{whenever} \quad (x,y) \in \big(B(\bar{x},\varepsilon) \times B(\bar{y},\varepsilon)\big) \cap \operatorname{gph} F \quad \text{and} \quad t \in (0,\varepsilon).$$

Let $\alpha := 1/c'$. Define $\tilde{\varrho}$, $\tilde{r}$, $\widetilde{X}$, and g as before, but, in this case, we request that $\tilde{r} \in (0, \varepsilon \min\{1,\alpha\})$. Then

$$B(y,c't) \subset g\big(B_{\widetilde{X}}((x,y),t)\big) \quad \text{whenever} \quad (x,y) \in B_{\widetilde{X}}((\bar{x},\bar{y}),\tilde{r}) \quad \text{and} \quad t \in (0,\tilde{r}).$$

Pick any $c \in (s,c')$. Then $\operatorname{sur} g(\bar{x},\bar{y}) > c$. Proposition 2.1 implies that there is an $r' > 0$ such that for any $(x,v) \in \operatorname{gph} F \cap \big(B(\bar{x},r') \times B(\bar{y},c'r')\big)$ and any $y \in B(\bar{y},r') \setminus \{v\}$ there is a pair $(x',v') \in \widetilde{X} \subset \operatorname{gph} F$ such that (7) holds. Picking $r \in \big(0, \min\{r', c'r'\}\big)$ and noting that $\alpha = 1/c' < 1/c$, we get that $s \geq c$, a contradiction. $\square$

For subsets C and D of a metric space, the *excess* of C beyond D is defined by $e(C,D) := \sup_{x \in C} d(x,D)$. We now present a slightly improved version of [8, Theorem 5G.3] which concerns *perturbed metric regularity on a set* and can be of independent interest.

Theorem 2.3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $G : X \rightrightarrows Y$ be a set-valued mapping, and pick some $(\bar{x},\bar{y}) \in \operatorname{gph} G$. Assume that there are positive constants a and κ such that the set $\operatorname{gph} G \cap \big(B(\bar{x},a), \times B(\bar{y},a)\big)$ is closed and that*

$$e\big(G^{-1}(y) \cap B(\bar{x},a), G^{-1}(y')\big) \leq \kappa \|y - y'\| \quad \textit{for each} \quad y, y' \in B(\bar{y},a). \tag{8}$$

Let $\nu > 0$ be such that $\kappa\nu < 1$ and let $\kappa' > \kappa/(1-\kappa\nu)$. Then for every positive α and β such that

$$2\kappa'\beta + \alpha \leq a \quad \textit{and} \quad 2\beta + \nu(2\kappa'\beta + \alpha) \leq a \tag{9}$$

and for every mapping $g : X \to Y$ satisfying

$$\|g(\bar{x})\| \leq \beta \quad \textit{and} \quad \|g(x) - g(x')\| \leq \nu\|x - x'\| \quad \textit{for every} \quad x, x' \in B(\bar{x}, 2\kappa'\beta + \alpha), \tag{10}$$

the mapping $g+G$ has the following property: for every $y, y' \in B(\bar{y},\beta)$ and every $x \in (g+G)^{-1}(y) \cap B(\bar{x},\alpha)$ there exists an $x' \in (g+G)^{-1}(y')$ such that

$$\|x - x'\| \leq \kappa'\|y - y'\|.$$

Proof. Choose any α and β that satisfy (9) and any g as in the statement. Then

$$y - g(x) \in B(\bar{y}, a) \quad \text{for each} \quad (x, y) \in B(\bar{x}, 2\kappa'\beta + \alpha) \times B(\bar{y}, \beta). \tag{11}$$

Indeed, fix any such a pair (x, y). Then (10) and (9) imply that

$$\begin{aligned} \|y - g(x) - \bar{y}\| &\leq \|g(\bar{x})\| + \|g(\bar{x}) - g(x)\| + \|y - \bar{y}\| \leq \beta + \nu\|x - \bar{x}\| + \beta \\ &\leq 2\beta + \nu(2\kappa'\beta + \alpha) \leq a. \end{aligned}$$

Fix any two distinct y, $y' \in B(\bar{y}, \beta)$ and any $x \in (g + G)^{-1}(y) \cap B(\bar{x}, \alpha)$. Let $r := \kappa'\|y - y'\|$. As $r \leq 2\kappa'\beta$, the first inequality in (9) implies that

$$B(x, r) \subset B(\bar{x}, 2\kappa'\beta + \alpha) \subset B(\bar{x}, a).$$

Consider the mapping

$$X \ni u \longmapsto G^{-1}\big(y' - g(u)\big) =: \Phi_{y'}(u) \subset X.$$

It suffices to show that there is a fixed point x' of $\Phi_{y'}$ in $B(x, r)$, because then $x' \in (g + G)^{-1}(y')$ and the desired distance estimate holds.

To obtain such a point x' we are going to apply [8, Theorem 5E.2]. The set $\Omega := \operatorname{gph} \Phi_{y'} \cap (B(x, r) \times B(x, r))$ is closed. Indeed, pick any sequence (x_n, z_n) in Ω converging to a point $(\tilde{x}, \tilde{z}) \in X \times X$. Clearly, $(\tilde{x}, \tilde{z}) \in B(x, r) \times B(x, r)$. The definition of $\Phi_{y'}$ and (11) imply that

$$\big(z_n, y' - g(x_n)\big) \in \operatorname{gph} G \cap (B(x, r) \times B(\bar{y}, a)) \subset \operatorname{gph} G \cap (B(\bar{x}, a) \times B(\bar{y}, a)) \quad \text{for each} \quad n \in \mathbb{N}.$$

Passing to the limit we get that $\big(\tilde{z}, y' - g(\tilde{x})\big) \in \operatorname{gph} G$, that is, $(\tilde{x}, \tilde{z}) \in \operatorname{gph} \Phi_{y'}$. Thus $(\tilde{x}, \tilde{z}) \in \Omega$. Since $y \in g(x) + G(x)$, we have $x \in G^{-1}(y - g(x)) \cap B(\bar{x}, a)$. Then (11) and (8) imply that

$$\begin{aligned} d(x, \Phi_{y'}(x)) &= d\big(x, G^{-1}(y' - g(x))\big) \leq e\big(G^{-1}(y - g(x)) \cap B(\bar{x}, a), G^{-1}(y' - g(x))\big) \\ &\leq \kappa\|y - y'\| < \kappa'\|y - y'\|(1 - \kappa\nu) = r(1 - \kappa\nu). \end{aligned}$$

Let $u, v \in B(x, r)$ be arbitrary. Using (11) and (8), we get that

$$\begin{aligned} e\big(\Phi_{y'}(u) \cap B(x, r), \Phi_{y'}(v)\big) &= e\big(G^{-1}(y' - g(u)) \cap B(\bar{x}, r), G^{-1}(y' - g(v))\big) \\ &\leq e\big(G^{-1}(y' - g(u)) \cap B(\bar{x}, a), G^{-1}(y' - g(v))\big) \leq \kappa\|g(u) - g(v)\| \\ &\leq \kappa\nu\,\|u - v\|. \end{aligned}$$

The assumptions of [8, Theorem 5E.2] are verified. Therefore there is an $x' \in B(x, r) \cap \Phi_{y'}(x')$. This proves our theorem. □

In [8, Theorem 5G.3], it is supposed, instead of (8), that

$$d\big(x, G^{-1}(y)\big) \leq \kappa d\big(y, G(x)\big) \quad \text{for each} \quad (x, y) \in B(\bar{x}, a) \times B(\bar{y}, a).$$

Inequality (8) means that G has the Aubin property on the set $B(\bar{x}, a) \times B(\bar{y}, a)$, which, by [8, Theorem 5H.3], is equivalent to the metric regularity of G on the same set, that is,

$$d\big(x, G^{-1}(y)\big) \leq \kappa d\big(y, G(x) \cap B(\bar{y}, a)\big) \quad \text{for each} \quad (x, y) \in B(\bar{x}, a) \times B(\bar{y}, a).$$

Therefore our assumption is slightly weaker. Clearly, such an $a > 0$ exists provided that $\operatorname{sur} G(\bar{x}, \bar{y}) > 0$.

Of some importance for us will also be the following perturbation statement, which is a corollary to [8, Theorem 5E.5].

Theorem 2.4. *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, *and* $(P, \|\cdot\|)$ *be three Banach spaces, let* $(\bar{p}, \bar{x}) \in P \times X$ *be given, and let* $g : P \times X \to Y$ *and* $G : X \rightrightarrows Y$ *be two mappings defined in a vicinity of* $(\bar{p}, \bar{x})$, *and satisfying* $g(\bar{p}, \bar{x}) = 0$ *and* $G(\bar{x}) \ni 0$. *Suppose that there are neighborhoods* Q *of* $\bar{p}$ *and* V *of* $\bar{x}$ *along with positive constants* ν *and* γ *such that for each* p, $p' \in Q$ *and each* x, $x' \in V$ *we have*

$$\|g(p,x) - g(p,x')\| \leq \nu\|x - x'\| \quad \text{and} \quad \|g(p,x) - g(p',x)\| \leq \gamma\|p - p'\|. \tag{12}$$

Further, assume that $\operatorname{sur} G(\bar{x}, 0) > \tau > \nu$ *and consider the "solution mapping"* $S : P \rightrightarrows X$ *defined by*

$$S(p) := \{x \in X : \; g(p,x) + G(x) \ni 0\}, \quad p \in P.$$

Then there are neighborhoods Q' *of* $\bar{p}$ *and* V' *of* $\bar{x}$ *such that*

$$S(p) \cap V' \subset S(p') + \frac{\gamma}{\tau - \nu}\|p - p'\|B_X \quad \text{for every} \quad p, p' \in Q'.$$

3 Regularity statements

First, we use Proposition 2.1 to derive an open mapping theorem for directionally differentiable single-valued mappings. A mapping $g : X \to Y$ acting between Banach spaces X and Y is *one-sided directionally differentiable* at $x \in \operatorname{int} \operatorname{dom} g$ provided that for each $h \in X$ the limit

$$\lim_{t \downarrow 0} \frac{g(x + th) - g(x)}{t} =: d^+g(x)(h)$$

in the norm-topology of Y exists.

Theorem 3.1. *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *be Banach spaces, and let* $g : X \to Y$ *be given. Assume that there are* $\bar{x} \in X$ *and positive constants* ϱ *and* r *such that* g *is continuous and one-sided directionally differentiable on* $B(\bar{x}, r) \subset \operatorname{int} \operatorname{dom} g$ *and*

$$d^+g(x)(B_X) \supset \varrho B_Y \quad \text{for each} \quad x \in B(\bar{x}, r). \tag{13}$$

Then $\operatorname{sur} g(\bar{x}) \geq \varrho$.

Proof. Pick any $c \in (0, \varrho)$. Fix any $(x, y) \in B(\bar{x}, r/2) \times Y$ with $g(x) \neq y$. Let

$$z := \varrho \frac{y - g(x)}{\|y - g(x)\|} \neq 0.$$

By (13), there is an $h \in B_X$ such that $d^+g(x)(h) = z$; clearly $h \neq 0$. Use the definition of the one-sided directional derivative to find a $t \in \big(0, \min\{\|g(x) - y\|/\varrho, r/2\}\big)$ such that

$$\|g(x + th) - g(x) - t\, d^+g(x)(h)\| < (\varrho - c)t.$$

Let $x' := x + th$. Then $\|x' - \bar{x}\| \leq \|x - \bar{x}\| + \|x' - x\| \leq r/2 + t\|h\| < r$. Hence $x' \in \operatorname{dom} g$. Since $td^+g(x)(h) = tz$ and $\|x' - x\| \leq t$, we may estimate

$$\begin{aligned}
\|y - g(x')\| &= \|y - g(x + th)\| \leq \|y - g(x) - tz\| + \|g(x) - g(x + th) + tz\| \\
&= \|y - g(x)\| - t\varrho + \|g(x + th) - g(x) - td^+g(x)(h)\| \\
&< \|y - g(x)\| - t\varrho + (\varrho - c)t = \|y - g(x)\| - ct \leq \|y - g(x)\| - c\|x - x'\|.
\end{aligned}$$

Proposition 2.1 now says that $\operatorname{sur} g(\bar{x}) \geq c$. And letting $c \uparrow \varrho$ we conlude the proof. □

The above statement extends [8, Theorem 5K.1] where it is assumed that there exists $\kappa > 0$ such that for any $x \in B(\bar{x}, r)$ there exists a selection $\sigma(x; \cdot)$ for $[d^+ g(x)]^{-1}$ such that

$$\|\sigma(x; y)\| \le \kappa \|y\| \quad \text{for all} \quad y \in Y,$$

and its conclusion is that g^{-1} *has a local selection around* $g(\bar{x})$ *for* $\bar{x}$ *which is calm at* $g(\bar{x})$ *with modulus* κ*, that is, for any* $\kappa' > \kappa$ *there is a* $\delta > 0$ *and a mapping* $s : Y \to X$ *such that*

$$s(y) \in g^{-1}(y) \cap B\big(\bar{x}, \kappa' \|y - g(\bar{x})\|\big) \quad \textit{whenever} \quad y \in B(g(\bar{x}), \delta).$$

Indeed, we simply put $\varrho := 1/\kappa$ to conclude that $\operatorname{reg} g(\bar{x}) \le \kappa$ and then we take $x := \bar{x}$ in (2). Such a result was originally proved by I. Ekeland in [11] where the role of the local selection at a point x is played by the right-inverse of the Gateaux derivative of g at x. A finite-dimensional version of Theorem 3.1 for locally Lipschitz functions was proved in [26].

Now, we proceed to set-valued mappings. Given a set $S \subset X$ and an $x \in S$, the *contingent tangent cone* $T(S, x)$ of S at x is the collection of all $h \in X$ with the following property: there are sequences (t_k) in $(0, \infty)$ converging to 0 and (h_k) in X converging to h such that $x + t_k h_k \in S$ for all $k \in \mathbb{N}$. For a set-valued mapping $F : X \rightrightarrows Y$ the *contingent* or *graphical derivative* of F at $(x, y) \in \operatorname{gph} F$ is defined as the following set-valued assignment

$$X \ni h \longmapsto DF(x, y)(h) := \{v \in Y : \ (h, v) \in T(\operatorname{gph} F, (x, y))\}.$$

Theorem 3.2. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces, let* $F : X \rightrightarrows Y$ *be a mapping with closed graph, and let* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$*. Assume that there are positive constants* β*,* ϱ*, and* r *such that*

$$DF(x, v)(B_X) + \beta B_Y \supset (\beta + \varrho) B_Y \quad \textit{for every} \quad (x, v) \in \big(B(\bar{x}, r) \times B(\bar{y}, r)\big) \cap \operatorname{gph} F.$$

Then $\operatorname{sur} F(\bar{x}, \bar{y}) \ge \varrho$*.*

Proof. Fix any $c \in (0, \varrho)$. Pick $\gamma > 0$ such that $c(1 + \gamma) < \varrho$. Fix any $(x, v) \in \operatorname{gph} F \cap \big(B(\bar{x}, r) \times B(\bar{y}, r)\big)$ and any $y \in Y$ distinct from v. Let

$$z := (\beta + \varrho) \frac{y - v}{\|y - v\|} \neq 0.$$

By the assumption, there is a pair $(h, d) \in B_X \times Y$ such that $d \in DF(x, v)(h)$ and $\|z - d\| \le \beta$. Hence $\|d\| \le 2\beta + \varrho$. The definition of the graphical derivative yields a triple $(t, h', d') \in (0, \infty) \times X \times Y$ such that $v + td' \in F(x + th')$ with

$$(14) \quad (\beta + \varrho)t < \|y - v\|, \quad \|d - d'\| < \varrho - (1 + \gamma)c, \quad \|d'\| < (1 + \gamma)(2\beta + \varrho), \quad \text{and} \quad \|h'\| < 1 + \gamma.$$

Let $x' := x + th'$ and $v' := v + td'$. Then $(x', v') \in \operatorname{gph} F$. The first inequality in (14) implies that $\|y - v - tz\| = \|y - v\| - t(\beta + \varrho)$. Taking into account the second one, we conclude that

$$(15) \qquad \begin{aligned} \|y - v'\| &= \|y - v - td'\| \le \|y - v - tz\| + t(\|z - d\| + \|d - d'\|) \\ &< \|y - v\| - t(\beta + \varrho) + t(\beta + \varrho - (1 + \gamma)c) = \|y - v\| - t(1 + \gamma)c. \end{aligned}$$

The last two inequalities in (14) reveal that $\|v'-v\| = t\|d'\| < t(1+\gamma)(2\beta+\varrho)$ and $\|x'-x\| = t\|h'\| < t(1+\gamma)$. Thus, using (15), we have

$$\|y-v'\| < \|y-v\| - c{\cdot}\max\big\{\|x'-x\|, \|v'-v\|/(2\beta+\varrho)\big\}.$$

Now, Proposition 2.2 with $\alpha := 1/(2\beta+\varrho)$ says that $\operatorname{sur} F(\bar{x},\bar{y}) \geq c$. And letting $c \uparrow \varrho$ we conclude the proof. $\square$

The above statement, proved originally by J.P. Aubin in [1] in 1981 (see also [2, Theorem 5.4.3]), implies both Theorem 5.13 and Theorem 5.15 in [17] where an infinitesimal version of Proposition 2.2 is used. Here we present a bit simpler proof. Moreover, we think that there is a tiny gap in the proofs of both statements there since it is claimed that $DF(x,v)(B_X) \supset \beta B_Y$ means that for any $z \in Y$ with $\|z\| = \beta$ there is an $h \in X$ *with norm one* such that $z \in DF(x,v)(h)$. In general, this h can be zero unless we assume that $DF(x,v)(0) = 0$. Note that Theorem 3.1 can be derived from the last statement. However, we preferred to have a separate proof to keep it as simple as possible.

From Theorem 3.2, we can derive a constrained version of Theorem 3.1 which generalizes [2, Theorem 3.4.3] where the (single-valued) mapping in question is assumed to be Fréchet differentiable in a vicinity of the reference point.

Theorem 3.3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let K be a closed subset of X which contains a point $\bar{x} \in X$, and let $g : X \to Y$. Assume that there are positive constants β, ϱ, and r such that g is defined, Lipschitz continuous, and one-sided directionally differentiable on $B(\bar{x},r)$; and that*

$$d^+g(x)\big(T(x,K)\cap B_X\big) + \beta B_Y \supset (\beta+\varrho)B_Y \quad \textit{for every} \quad x \in K \cap B(\bar{x},r).$$

Then $\operatorname{sur} g\restriction_K(\bar{x}) \geq \varrho$.

Proof. Let $F : X \rightrightarrows Y$ be defined by $F(x) := g(x)$ if $x \in K \cap B(\bar{x},r)$ and $F(x) := \emptyset$ otherwise; thus, $\operatorname{gph} F = \operatorname{gph} g\restriction_K$. The conclusion will follow from Theorem 3.2 once we show that

$$d^+g(x)\big(T(x,K)\cap B_X\big) \subset DF(x,g(x))(B_X) \quad \text{for each} \quad x \in K \cap B(\bar{x},r).$$

To see this, fix any such x and then pick any $w \in d^+g(x)\big(T(x,K)\cap B_X\big)$. Find an $h \in T(x,K)\cap B_X$ such that $w = d^+g(x)(h)$. The definition of the tangent cone implies that there are sequences (t_k) in $(0,\infty)$ converging to 0 and (h_k) in X converging to h such that $x + t_k h_k \in K$ for all $k \in \mathbb{N}$. The Lipschitz continuity of g guarantees that

$$w_k := \frac{g(x+t_k h_k) - g(x)}{t_k} \longrightarrow d^+g(x)(h) = w \quad \text{as} \quad k \to \infty.$$

Moreover, $(x+t_k h_k, g(x)+t_k w_k) \in \operatorname{gph} g|_K = \operatorname{gph} F$ for all $k \in \mathbb{N}$. Therefore, $w \in DF(x,g(x))(h)$. Since $h \in B_X$, we get the desired inclusion. $\square$

A mapping $H : X \rightrightarrows Y$ whose graph is a cone in $X \times Y$ is called *positively homogeneous*. For such a mapping H, the *outer* and the *inner norm* are defined, respectively, by

$$\|H\|^+ := \sup_{\|x\|\leq 1}\ \sup_{y\in H(x)} \|y\| \quad \text{and} \quad \|H\|^- := \sup_{\|x\|\leq 1}\ \inf_{y\in H(x)} \|y\|.$$

If H is single-valued and defined on the whole X, the two numbers above coincide. For $H \in \mathcal{L}(X,Y)$ they reduce to the operator norm $\|H\|$. Under the notation just introduced, we see that Theorem 3.2 (with $\beta := 0$) immediately implies [17, Theorem 5.13]:

Corollary 3.4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $F : X \rightrightarrows Y$ be a mapping with closed graph, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$\operatorname{sur} F(\bar{x}, \bar{y}) \geq \liminf_{\operatorname{gph} F \ni (x,y) \to (\bar{x},\bar{y})} \frac{1}{\left\|[DF(x,y)]^{-1}\right\|^{-}}.$$

Note that in finite dimensions both the quantities above are equal (see [17, Theorem 7.3] for a five-line-proof). Originally, this result was proved in [9] (see also [8, Theorem 4B.1]). This formula for surjectivity modulus can be deduced from a more general statement proved by H. Frankowska in [14, Theorem 6.1 and Corollary 6.2] which is presented below and contains both Theorem 3.2 and Theorem 3.1.

For a set-valued mapping $F : X \rightrightarrows Y$ and $(x, y) \in \operatorname{gph} F$, the *contingent variation* of F at (x, y) is the set $F^{(1)}(x, y)$ of all vectors $d \in Y$ such that there are sequences (t_k) in $(0, \infty)$ converging to 0 and (d_k) in Y converging to d such that $y + t_k d_k \in F(B(x, t_k))$ for each $k \in \mathbb{N}$. It is easy to show that $DF(x, y)(B_X) \subset F^{(1)}(x, y)$ always and $DF(x, y)(B_X) = F^{(1)}(x, y)$ provided that X is finite-dimensional.

Theorem 3.5. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $F : X \rightrightarrows Y$ be a mapping with closed graph, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then $\operatorname{sur} F(\bar{x}, \bar{y})$ is equal to the supremum of all $\varrho > 0$ for which there is an $r > 0$ such that*

$$F^{(1)}(x, y) \supset \varrho B_Y \quad \textit{for every} \quad (x, y) \in \big(B(\bar{x}, r) \times B(\bar{y}, r)\big) \cap \operatorname{gph} F. \tag{16}$$

Proof. Denote by s the supremum from the statement. First, we show that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq s$. If $s = 0$ we are done, so suppose that $s > 0$. Fix any $\varrho \in (0, s)$ and then find an $r > 0$ such that (16) holds. Pick any $c \in (0, \varrho)$ and then a $\gamma > 0$ such that $c(1 + \gamma) < \varrho$. Fix any $(x, v) \in \big(B(\bar{x}, r) \times B(\bar{y}, r)\big) \cap \operatorname{gph} F$ and any $y \in Y$ distinct from v. Let

$$z := \varrho \frac{y - v}{\|y - v\|} \neq 0.$$

By (16), $z \in F^{(1)}(x, v)$. The definition of the contingent variation yields a triple $(t, x', z') \in (0, \infty) \times X \times Y$ such that $v + tz' \in F(x')$ with

$$\varrho t < \|y - v\|, \quad \|z - z'\| < \varrho - (1 + \gamma)c, \quad \|z'\| < (1 + \gamma)\varrho, \quad \text{and} \quad \|x' - x\| \leq t. \tag{17}$$

Let $v' := v + tz'$. Then $(x', v') \in \operatorname{gph} F$. The first inequality in (17) means that $\|y - v - tz\| = \|y - v\| - t\varrho$. Then, taking into account the second one here, we conclude that

$$\begin{aligned} \|y - v'\| &= \|y - v - tz'\| \leq \|y - v - tz\| + t\|z - z'\| < \|y - v\| - t\varrho + t(\varrho - (1 + \gamma)c) \\ &= \|y - v\| - t(1 + \gamma)c. \end{aligned}$$

The last two inequalities in (17) reveal that $\|x' - x\| < t(1 + \gamma)$ and $\|v' - v\| = t\|z'\| < t(1 + \gamma)\varrho$. Thus

$$\|y - v'\| < \|y - v\| - c \max\big\{\|x' - x\|, \|v' - v\|/\varrho\big\}.$$

Now, Proposition 2.2 with $\alpha := 1/\varrho$ says that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq c$. Letting $c \uparrow \varrho$, we conclude that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq \varrho$. Since $\varrho \in (0, s)$ was arbitrary, we obtain that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq s$ as claimed.

Assume that $\operatorname{sur} F(\bar{x}, \bar{y}) > s$. Pick a $\varrho \in (s, \operatorname{sur} F(\bar{x}, \bar{y}))$. Then there is an $r > 0$ such that

$$B(y, \varrho t) \subset F(B(x,t)) \quad \text{whenever} \quad (x,y) \in \big(B(\bar{x}, r) \times B(\bar{y}, r)\big) \cap \operatorname{gph} F \quad \text{and} \quad t \in (0, r).$$

In other words, for any x, y, t as above we have $\varrho B_Y \subset \big(F(B(x,t)) - y\big)/t$. This implies that $\varrho B_Y \subset F^{(1)}(x,y)$ for each $(x,y) \in \big(B(\bar{x}, r) \times B(\bar{y}, r)\big) \cap \operatorname{gph} F$. Thus $\varrho \leq s$, a contradiction. □

Next, we present a generalization of [6, Theorem 3.9] which was proved in [4] by using an iterative procedure. An application of the regularity criterion yields a substantially shorter proof. The space $\mathcal{L}(X,Y)$ will be considered with the topology induced by the operator norm.

Theorem 3.6. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $(\bar{x}, \bar{y}) \in X \times Y$, let positive constants r, β, and ϱ be given, and let $\mathcal{A} \subset \mathcal{L}(X,Y)$ be a compact convex set. Let $f : X \to Y$ and $F : X \rightrightarrows Y$ be such that F has closed graph, that $\bar{y} \in f(\bar{x}) + F(\bar{x})$ and that $\operatorname{dom} f \supset \operatorname{dom} F$. Further assume that:*

(A) for each x, $x' \in B(\bar{x}, r) \cap \operatorname{dom} f$ there is an $A \in \mathcal{A}$ such that $\|f(x) - f(x') - A(x - x')\| \leq \beta\|x - x'\|$,

(B) $\operatorname{sur}\big(f(\bar{x}) + A(\cdot - \bar{x}) + F(\cdot)\big)(\bar{x}, \bar{y}) \geq \beta + \varrho$ for each $A \in \mathcal{A}$, and

(C) for each $y \in Y$ and each $A \in \mathcal{A}$ the fiber $(f(\bar{x}) + A(\cdot - \bar{x}) + F(\cdot))^{-1}(y) \cap B(\bar{x}, r)$ is convex (possibly empty).

Then $\operatorname{sur}(f + F)(\bar{x}, \bar{y}) \geq \varrho$.

Proof. Without any loss of generality we assume that $\bar{x} = 0$, $\bar{y} = 0$, $f(0) = 0$, and $0 \in F(0)$. The condition (A) and the boundedness of $\mathcal{A}$ guarantee that f is Lipschitz continuous on $rB_X \cap \operatorname{dom} f$. Fix any $c \in (0, \varrho)$. Then pick a $\varrho' \in (c, \varrho)$.

Lemma 0. *There is an $r' \in (0, r/2)$ such that for each $A \in \mathcal{A}$, each $t \in (0, r')$, and each $(x, w) \in \operatorname{gph}(A + F) \cap \big(2r'B_X \times 2r'B_Y\big)$ we have*

$$(A + F)\big(\mathring{B}(x,t)\big) \supset B(w, (\beta + \varrho')t). \tag{18}$$

Proof. Pick a $\nu > 0$ such that $\varrho' + 3\nu < \varrho$. The compactness of $\mathcal{A}$ yields an $n \in \mathbb{N}$ and a subset $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$ such that $\mathcal{A} \subset \mathcal{B} + \nu B_{\mathcal{L}(X,Y)}$. Fix any $i \in \{1, 2, \ldots, n\}$. Thanks to (B) we have $\operatorname{reg}(A_i + F)(0,0) = 1/\operatorname{sur}(A_i + F)(0,0) \leq 1/(\beta + \varrho) < 1/(\beta + \varrho' + 3\nu) := \kappa$. Find an $a_i > 0$ such that (8) holds with $a := a_i$, $G := A_i + F$, $\bar{x} := 0$, and $\bar{y} := 0$. Let $\kappa' := 1/(\beta + \varrho' + \nu)$. Then

$$\kappa'(1 - \kappa\nu) = \frac{1 - \frac{\nu}{\beta + \varrho' + 3\nu}}{\beta + \varrho' + \nu} = \frac{\beta + \varrho' + 2\nu}{(\beta + \varrho' + \nu)(\beta + \varrho' + 3\nu)} > \frac{1}{\beta + \varrho' + 3\nu} = \kappa.$$

Pick positive α_i and β_i such that $2\kappa'\beta_i + \alpha_i \leq a_i$ and $2\beta_i + \nu(2\kappa'\beta_i + \alpha_i) \leq a_i$. Let $\delta_i := \min\{\alpha_i, \beta_i/(\beta + \varrho + 1)\}$. We claim that for each $A \in B(A_i, \nu)$, each $t \in (0, \delta_i)$, and each $(x, w) \in \operatorname{gph}(A + F) \cap \big(\delta_i B_X \times \delta_i B_Y\big)$

$$(A + F)\big(B(x,t)\big) \supset B(w, (\beta + \varrho' + \nu)t).$$

To see this, fix any such A, t, and (x,w). Pick any $y' \in B(w,(\beta+\varrho'+\nu)t)$. Let $g := A - A_i$. Then $g(0) = 0$, $A + F = g + G$, and g satisfies (10). Thus $w \in (g+G)(x)$, and noting that $\|x\| \le \delta_i \le \alpha_i$, we get that $x \in (g+G)^{-1}(w) \cap (\alpha_i B_X)$. Further $\|w\| \le \delta_i < \beta_i$ and

$$\|y'\| \le \|y' - w\| + \|w\| \le (\beta + \varrho' + \nu)t + \delta_i < (\beta + \varrho + 1)\delta_i \le \beta_i.$$

By Theorem 2.3 (with $y := w$), there is an $x' \in (g+G)^{-1}(y') = (A+F)^{-1}(y')$ satisfying

$$\|x - x'\| \le \kappa' \|w - y'\| \le \frac{(\beta+\varrho'+\nu)t}{\beta+\varrho'+\nu} = t.$$

This means that $y' \in (A+F)(B(x,t))$ as claimed.
Put $r' := \frac{1}{2}\min\{r, \delta_1, \delta_2, \ldots, \delta_n\}$. As $\mathcal{A} \subset \mathcal{B} + \nu B_{\mathcal{L}(X,Y)}$, we proved that for each $A \in \mathcal{A}$, each $t' \in (0, r')$, and each $(x,w) \in \operatorname{gph}(A+F) \cap \big(2r'B_X \times 2r'B_Y\big)$ we have

$$(A+F)\big(B(x,t')\big) \supset B(w,(\beta+\varrho'+\nu)t').$$

Fix any A, t, and (x,w) as in the premise of our Lemma. The latter inclusion with $t' := t(\beta + \varrho')/(\beta+\varrho'+\nu)$ $(< t < r')$ implies that

$$(A+F)\big(\mathring{B}(x,t)\big) \supset (A+F)\big(B(x,t')\big) \supset B(w,(\beta+\varrho'+\nu)t') = B(w,(\beta+\varrho')t).$$

We proved (18). □

Fix any $(x,v) \in \operatorname{gph}(f+F) \cap \big(r'B_X \times r'B_Y\big)$ and any $y \in Y$ distinct from v. Pick some $\alpha \in \big(0, \min\{r', \|y-v\|/(\beta+\varrho')\big)$ and let

$$z := \alpha(\beta+\varrho')\frac{y-v}{\|y-v\|} \quad (\neq 0).$$

Define the set-valued mapping $\Phi : \mathcal{A} \rightrightarrows X$ by

$$\mathcal{A} \ni A \longmapsto \big[(A+F)^{-1}(z + Ax - f(x) + v) - x\big] \cap \big(\alpha B_X\big) =: \Phi(A) \subset X.$$

Lemma 1. *The mapping Φ is closed-convex-valued and lower semi-continuous on $\mathcal{A}$.*

Proof. Let $\mu := \sup_{A\in\mathcal{A}} \|A\| < \infty$. First, we show that $\Phi(A) \cap (\alpha \mathring{B}_X)$ is non-empty for each $A \in \mathcal{A}$. Fix any $A \in \mathcal{A}$ and let $w := Ax - f(x) + v$. Then $w \in Ax - f(x) + f(x) + F(x) = (A+F)(x)$ and, by (A), we have

$$\|w\| \le \|Ax\| + \|f(0) - f(x)\| + \|v\| \le \mu r' + (\mu+\beta)r' + r' < 2r'(1+\beta+\mu+\varrho).$$

As $\|z\| = \alpha(\beta+\varrho')$, we have $z + w \in B(w, \alpha(\beta+\varrho'))$. Remembering that $\alpha < r'$ and setting $t := \alpha$ in (18), we find $u \in \mathring{B}(x,\alpha)$ such that $u \in (A+F)^{-1}(z+w)$. Then $h := u - x \in \Phi(A) \cap (\alpha \mathring{B}_X)$.

In particular, we showed that $\operatorname{dom}\Phi = \mathcal{A}$. By (C), Φ has convex values. As $\operatorname{gph} F$ is closed, so are the values of Φ. It remains to prove that Φ is lower semi-continuous. Fix any $\overline{A} \in \mathcal{A}$ and any open set $\Omega \subset X$ such that $\Phi(\overline{A}) \cap \Omega \neq \emptyset$. Pick an $\overline{h} \in \Phi(\overline{A}) \cap \Omega$. We are going to distinguish two cases:

Suppose that $\|\overline{h}\| < \alpha$. Let $\overline{w} := z + \overline{A}x - f(x) + v$. Then $x + \overline{h} \in (\overline{A} + F)^{-1}\big(\overline{w}\big)$, that is, $\overline{w} \in (\overline{A} + F)(x + \overline{h})$. Define $G : X \rightrightarrows Y$ by $G(\cdot) := (\overline{A} + F)(x + \cdot) - \overline{w}$. Thus $G(\overline{h}) \ni 0$. Further, define $g : \mathcal{L}(X,Y) \times X \longrightarrow Y$ by $g(A,h) := (A - \overline{A})(h)$ for each $(A,h) \in \mathcal{L}(X,Y) \times X$. Hence $g(\overline{A}, \overline{h}) = 0$. Consider the solution mapping

$$\mathcal{L}(X,Y) \ni A \longmapsto S(A) := \{h \in X : \ g(A,h) + G(h) \ni 0\}.$$

Put $Q := \overline{A} + \beta B_{\mathcal{L}(X,Y)}$ and $V := \overline{h} + B_X$. Then (12) holds with $\nu := \beta$ and $\gamma := \|\overline{h}\| + 1$. On the other hand, by (18) we know that $\operatorname{sur} G(\overline{h}, 0) = \operatorname{sur}(\overline{A} + F)(x + \overline{h}, \overline{w}) \geq \beta + \varrho'$ since $\|x + \overline{h}\| < r' + \alpha < 2r'$ and

$$\|\overline{w}\| \leq \|z\| + \|\overline{A}x\| + \|f(0) - f(x)\| + \|v\| \leq \alpha(\beta + \varrho') + \mu r' + (\mu + \beta)r' + r' < 2r'(1 + \beta + \mu + \varrho).$$

Pick an $\varepsilon \in (0, \varrho')$. By Theorem 2.4 (with $P := \mathcal{L}(X,Y)$, $\bar{x} := \overline{h}$, $\bar{p} := \overline{A}$, and $\tau := \beta + \varrho' - \varepsilon$), there are neighborhoods Q' of $\overline{A}$ and V' of $\overline{h}$ such that

$$S(A_1) \cap V' \subset S(A_2) + \frac{\gamma}{\varrho' - \varepsilon}\|A_1 - A_2\| B_X \quad \text{for all} \quad A_1, A_2 \in Q'.$$

In particular, for every $A \in Q'$ there is an $h_A \in S(A)$ such that $\overline{h} \in h_A + \frac{\gamma}{\varrho' - \varepsilon}\|\overline{A} - A\| B_X$. Find $\delta > 0$ so small that $B(\overline{h}, \delta) \subset \Omega \cap \big(\alpha \overset{\circ}{B}_X\big)$, which is possible since the open set Ω contains $\overline{h}$ and $\|\overline{h}\| < \alpha$. Then for every $A \in Q' \cap \big(\overline{A} + \frac{\delta(\varrho' - \varepsilon)}{\gamma} B_{\mathcal{L}(X,Y)}\big)$ we have $h_A \in \Omega$ as well as $h_A \in \alpha \overset{\circ}{B}_X$. On the other hand, once we know that h_A belongs to $S(A)$, then $(A - \overline{A})(h_A) + (\overline{A} + F)(x + h_A) - \overline{w} \ni 0$, that is, $z - f(x) + v + Ax \in (A + F)(x + h_A)$, that is, $x + h_A \in (A + F)^{-1}\big(z + Ax - f(x) + v\big)$, and so, h_A lies in $\Phi(A)$. Therefore $\Phi(A) \cap \Omega \neq \emptyset$ for every $A \in Q' \cap \big(\overline{A} + \frac{\delta(\varrho' - \varepsilon)}{\gamma} B_{\mathcal{L}(X,Y)}\big)$.

Now, suppose that $\|\overline{h}\| = \alpha$. Pick any $\widehat{h} \in \Phi(\overline{A}) \cap (\alpha \overset{\circ}{B}_X)$ (which exists as we have seen at the very beginning of the proof of this Lemma). Since the set $\Phi(\overline{A})$ is convex and contains both $\widehat{h}$ and $\overline{h}$, there exists $\widetilde{h} \in \Phi(\overline{A}) \cap \Omega$ such that $\|\widetilde{h}\| < \alpha$. By the previous case, there is a neighborhood Q of $\overline{A}$ such that $\Phi(A) \cap \Omega \neq \emptyset$ for every $A \in Q$. □

Now, the set $\mathcal{A}$ being compact, it is automatically paracompact. Therefore, by Michael's selection theorem [12, Theorem 7.53], there exists a continuous mapping $\varphi : \mathcal{A} \longrightarrow X$ such that $\varphi(A) \in \Phi(A)$ for every $A \in \mathcal{A}$. Set $M := \varphi(\mathcal{A})$. Then M is closed. Define the set-valued mapping $\Psi : M \rightrightarrows \mathcal{A}$ by

$$M \ni h \longmapsto \big\{A \in \mathcal{A} \colon \|f(x + h) - f(x) - Ah\| \leq \beta\|h\|\big\} =: \Psi(h) \subset \mathcal{A}. \tag{19}$$

Lemma 2. *The composition mapping $\Psi \circ \varphi$ acting from $\mathcal{A}$ into itself has a fixed point.*

Proof. Fix any $h \in M$. Clearly, the set $\Psi(h)$ is convex (possibly empty). By the very definition of Φ, we get that $\|x + h\| \leq \|x\| + \|h\| \leq r' + \alpha < 2r' < r$ and that $x + h \in \operatorname{dom} F \subset \operatorname{dom} f$. The assumption (A) then guarantees that $\Psi(h) \neq \emptyset$. Therefore $\operatorname{dom}(\Psi \circ \varphi) = \mathcal{A}$. Since f and $\|\cdot\|$ are continuous and both M and $\mathcal{A}$ are closed, the graph of Ψ is closed, and thus so is the graph of $\Psi \circ \varphi$ thanks to the continuity of φ. Since $\mathcal{A}$ is compact and convex, we can apply Gliksberg's extension of Kakutani's fixed point theorem [15] to finish the proof of Lemma 2. □

Let $A \in \mathcal{A}$ be a fixed point of $\Psi \circ \varphi$. Put $h := \varphi(A)$. Then $h \in \Phi(A)$, and hence, $x + h \in (A+F)^{-1}(z + Ax - f(x) + v)$, that is, $z - Ah - f(x) + v \in F(x+h)$. Thus for

$$x' := x + h \quad \text{and} \quad v' := z + v + f(x+h) - f(x) - Ah$$

we have $v' \in (f+F)(x')$. Since $\alpha(\beta + \varrho') < \|y - v\|$ we have $\|y - v - z\| = \|y - v\| - \alpha(\beta + \varrho')$. As $A \in \Psi(h)$, we get that $\|f(x+h) - f(x) - Ah\| \leq \beta\|h\| \leq \alpha\beta$. Combing the previous facts, we get that

$$\begin{aligned} \|y - v'\| &= \|y - v - z + (f(x) - f(x+h) + Ah)\| \leq \|y - v - z\| + \|f(x+h) - f(x) - Ah\| \\ &\leq \|y - v\| - \alpha(\beta + \varrho') + \alpha\beta = \|y - v\| - \varrho'\alpha < \|y - v\| - c\alpha. \end{aligned}$$

Note that $\|x' - x\| = \|h\| \leq \alpha$ and

$$\|v' - v\| = \|z + f(x+h) - f(x) - Ah\| \leq \alpha(\beta + \varrho') + \alpha\beta < 2\alpha(\beta + \varrho').$$

Thus $\|y - v'\| < \|y - v\| - c \max\big\{\|x' - x\|, \|v' - v\|/(2(\beta + \varrho'))\big\}$. Proposition 2.2 with $\alpha := 1/(2(\beta + \varrho'))$ says that $\operatorname{sur}(f+F)(0,0) \geq c$ and letting $c \uparrow \varrho$ we conlude the proof Theorem 3.6. $\square$

Remark 3.7. Assume that F in Theorem 3.6 is such that $F(x)$ is either 0 or $\emptyset$ for every $x \in X$. Then (C) means that for every $y \in Y$ and every $A \in \mathcal{A}$ the set $A^{-1}\big(y + A\bar{x} - f(\bar{x})\big) \cap B(\bar{x}, r) \cap \operatorname{dom} F$ is convex. Hence, if such an F has convex domain, then (C) is satisfied and we obtain [6, Theorem 3.9].

Given a nonempty subset S of X, we put $\operatorname{cone} S := [0, \infty)S$ and call it the *cone generated* by S. Now, we derive a generalization of [6, Theorem 3.4] to set-valued mappings.

Theorem 3.8. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $F : X \rightrightarrows Y$ be a mapping with closed graph and convex domain, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Assume that there are positive constants ϱ, β, and r along with a positively homogeneous set-valued mapping $H : X \rightrightarrows Y$ such that $H(B_X \setminus \{0\})$ is bounded, that $\operatorname{dom} H = X$, that*

$$(20) \quad F(x) \cap B(\bar{y}, r) \subset F(x') + H(x - x') + \beta\|x - x'\| B_Y \quad \textit{for each} \quad x, x' \in \operatorname{dom} F \cap B(\bar{x}, r),$$

and that

$$(21) \quad \sup_{h \in \operatorname{cone}(\operatorname{dom} F - x) \cap B_X} \ \inf_{w \in -H(-h)} \langle y^*, w \rangle \geq \beta + \varrho \quad \textit{whenever} \quad x \in \operatorname{dom} F \cap B(\bar{x}, r) \ \textit{and} \ y^* \in S_{Y^*}.$$

Assume finally that one of the following two conditions holds:

(a) H has relatively norm compact values, and Y is separable or the norm $\|\cdot\|$ on Y is Gateaux smooth;

(b) Y is reflexive or the norm $\|\cdot\|$ on Y is Fréchet smooth.

Then $\operatorname{sur} F(\bar{x}, \bar{y}) \geq \varrho$.

Proof. First, assume that (a) holds. If Y is separable, then there is an equivalent Gateaux smooth norm arbitrarily close to the original one; see [6]. Hence, we may assume without any loss of generality that the norm $\|\cdot\|$ on Y is Gateaux differentiable off the origin. Find $\mu > 0$ so big that $H(B_X \setminus \{0\}) \subset \mu B_Y$. Take any $\varepsilon \in (0, \varrho/2)$. Pick $\gamma \in (0, r/2)$. Fix any $(x, v) \in \operatorname{gph} F \cap \big(B(\bar{x}, \gamma) \times B(\bar{y}, \gamma)\big)$ and any $y \in B(\bar{y}, \gamma)$ distinct from v. Let y^* denote the derivative of $\|\cdot\|$ at $y - v$. Then

$$\lim_{0 \neq t \to 0} t^{-1}\big(\|y - v + tw\| - \|y - v\|\big) - \langle y^*, w\rangle = 0 \quad \text{for every} \quad w \in Y. \tag{22}$$

By (21), there is an $h \in \operatorname{cone}(\operatorname{dom} F - x) \cap B_X$ such that

$$\langle y^*, w\rangle > \beta + \varrho - \varepsilon > 0 \quad \text{for all} \quad w \in -H(-h). \tag{23}$$

Note that h cannot be zero. Fix $\delta > 0$ such that $h \in \delta\big(\operatorname{dom} F - x\big)$. Since the set $H(-h)$ is non-empty, relatively compact, and the limit in (22) is uniform with respect to w's from any fixed compact set, we get that there is a $t \in \big(0, \min\{1/\delta, \gamma\}\big)$ such that

$$\|y - v - tw\| - \|y - v\| + \langle y^*, tw\rangle < t\varepsilon \quad \text{for all} \quad w \in -H(-h).$$

This and (23) imply that

$$\|y - v - tw\| < \|y - v\| - \langle y^*, tw\rangle + \varepsilon t < \|y - v\| - t(\beta + \varrho - 2\varepsilon) \quad \text{for all} \quad w \in -H(-h). \tag{24}$$

Let $x' := x + th$. Noting that $t \in (0, 1/\delta)$, we have $x' \in x + t\delta\big(\operatorname{dom} F - x\big) \subset \operatorname{dom} F$ by convexity of $\operatorname{dom} F$. As $2t < 2\gamma < r$, we have $\|x' - x\| = \|th\| \leq t < r/2$. Thus $x' \in \operatorname{dom} F \cap B(\bar{x}, r)$. Since H is positively homogeneous, we have $H(x - x') = H(-th) = tH(-h)$. As $v \in F(x) \cap B(\bar{y}, \gamma)$, (20) says that there are $v' \in F(x')$ and $w \in -H(-h)$ such that

$$\|v - (v' - tw)\| \leq \beta t\|h\| \leq \beta t. \tag{25}$$

Using (25) and (24) we infer that

$$\|y - v'\| \;\; \leq \;\; \|y - v - tw\| + \|v - v' + tw\| < \|y - v\| - (\beta + \varrho - 2\varepsilon)t + \beta t = \|y - v\| - (\varrho - 2\varepsilon)t.$$

As $w \in -H(-h) \subset -H(B_X \setminus \{0\}) \subset \mu B_Y$, (25) implies that $\|v - v'\| \leq (\beta + \mu)t < (\varrho + \beta + \mu)t$. Remembering that $\|x' - x\| \leq t$ we obtain that

$$\|y - v'\| < \|y - v\| - (\varrho - 2\varepsilon)\max\big\{\|x' - x\|, \|v' - v\|/(\varrho + \beta + \mu)\big\}.$$

Now, Proposition 2.2 with $\alpha := 1/(\varrho + \beta + \mu)$ and $c := \varrho - 2\varepsilon$ says that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq \varrho - 2\varepsilon$. And, letting $\varepsilon \downarrow 0$, we get the conclusion.

Suppose that (b) holds. If $(Y, \|\cdot\|)$ is reflexive, then there is a dual locally uniformly rotund norm on Y^*, see [7, Corollary VII.1.13]. By [7, Theorem II.4.1(ii)] there is a norm $|\cdot|$ on Y which is equivalent and arbitrarily close to $\|\cdot\|$; moreover, the dual norm of $|\cdot|$ is locally uniformly rotund on Y^*. Hence [7, Proposition II.1.5] implies that the norm $|\cdot|$ on Y is Fréchet smooth. Hence, we may assume without any loss of generality that the norm $\|\cdot\|$ on Y is Fréchet differentiable off the origin. Therefore we only have to take into account that under (b) the limit in (22) is uniform for w's from any bounded subset of Y. □

We would like to point out that we assume neither that $\|H\|^+ < \infty$ nor that $H(0) = 0$. (Note that $H(0) = 0$ whenever $\|H\|^+ < \infty$).

Consider a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. A positively homogeneous set-valued mapping $H : X \rightrightarrows Y$ is called a *pseudo strict prederivative* of F at $(\bar{x}, \bar{y})$ if for each $\varepsilon > 0$ there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(x') + H(x - x') + \varepsilon \|x - x'\| B_Y \quad \text{for each} \quad x, x' \in U. \tag{26}$$

Then, in the terminology of C.H.J. Pang [22], F is called *pseudo strictly H–differentiable* at $\bar{x}$ for $\bar{y}$. This notion, for single-valued mappings $f : X \to Y$, was introduced by A.D. Ioffe in [18] and called the *strict Fréchet prederivative* of f at $\bar{x}$, that is, the positively homogeneous set-valued mapping $H : X \rightrightarrows Y$ has to be such that for each $\varepsilon > 0$ there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$f(x) \in f(x') + H(x - x') + \varepsilon \|x - x'\| B_Y \quad \text{for each} \quad x, x' \in U.$$

If $H(h) := Ah$, $h \in X$, for some $A \in \mathcal{L}(X, Y)$, then F is called *quasi-peridifferentiable* at $(\bar{x}, \bar{y})$ in [21]. Note that such a mapping A may not be unique when $F(\bar{x}) \neq \{\bar{y}\}$. Similarly, if H is the strict Fréchet prederivative of f at $\bar{x}$ then so is $\widetilde{H} := -H(-\,\cdot)$. Using the notions above, we get the following statement.

Corollary 3.9. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $(\bar{x}, \bar{y}) \in X \times Y$ be given, and let $F : X \rightrightarrows Y$ be a mapping, with closed graph, and such that $\bar{x} \in \operatorname{int} \operatorname{dom} F$. Assume that F has a pseudo strict prederivative $H : X \rightrightarrows Y$ at $(\bar{x}, \bar{y})$ such that $\|H\|^+ < \infty$ and*

$$\inf_{y^* \in S_{Y^*}} \sup_{h \in B_X} \inf_{w \in -H(-h)} \langle y^*, w \rangle > \varrho > 0 \quad \textit{for some constant} \quad \varrho.$$

If either (a) or (b) in Theorem 3.8 is satisfied then $\operatorname{sur} F(\bar{x}, \bar{y}) > \varrho$.

Proof. As $\|H\|^+ < \infty$, the set $H(B_X)$ is bounded and $H(0) = 0$. Find a $\beta > 0$ such that

$$\inf_{y^* \in S_{Y^*}} \sup_{h \in B_X} \inf_{w \in -H(-h)} \langle y^*, w \rangle > 2\beta + \varrho.$$

Find $r > 0$ such that (20) holds and $B(\bar{x}, 2r) \subset \operatorname{dom} F$. Then necessarily $\operatorname{dom} H = X$. Indeed, for any $h \in rB_X$, we have $\bar{x} - h \in B(\bar{x}, r)$ and (20) with $x := \bar{x}$ and $x' := \bar{x} - h$ implies that $H(h) \neq \emptyset$. As $B(\bar{x}, r) \subset \operatorname{int} \operatorname{dom} F$, we have that $\operatorname{cone}(\operatorname{dom} F - x) = X$ for each $x \in B(\bar{x}, r)$. Hence Theorem 3.8 (with $\varrho := \varrho + \beta$) implies that $\operatorname{sur} F(\bar{x}, \bar{y}) \geq \beta + \varrho > \varrho$. □

If $\mathcal{A}$ is a collection of linear operators from X to Y, then the set-valued mapping $X \ni x \longmapsto \mathcal{A}x := \{Ax : \ A \in \mathcal{A}\}$ is of course homogeneous, that is, $\mathcal{A}(\lambda x) = \lambda \mathcal{A}(x)$ for each $x \in X$ and each $\lambda \in \mathbb{R}$. We will consider set-valued mappings which can be approximated by such a set $\mathcal{A}$. We need [6, Proposition 3.7]:

Proposition 3.10. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $0 \in K \subset X$ be a convex closed set, let $y^* \in S_{Y^*}$, let $\alpha > 0$, and let $\mathcal{A} \subset \mathcal{L}(X, Y)$ be a convex set such that*

$$A(B_X \cap K) \supset \alpha B_Y \quad \textit{for every} \quad A \in \mathcal{A}. \tag{27}$$

(a) If the set $C := \{A^*y^* : \ A \in \mathcal{A}\}$ is weak* closed and $\mathcal{A}$ is bounded in $\mathcal{L}(X,Y)$, then for each $\alpha' \in (0,\alpha)$ there exists a non-zero $h \in K$ such that

$$\langle y^*, Ah\rangle \geq \alpha'\|h\| \quad \text{for every} \quad A \in \mathcal{A};$$

(b) If X is reflexive, then there exists a non-zero $h \in K$ such that

$$\langle y^*, Ah\rangle \geq \alpha\|h\| \quad \text{for every} \quad A \in \mathcal{A}.$$

Given two Banach spaces X and Y, then we introduce a *strong operator topology (SOT)* on $\mathcal{L}(X,Y)$ as that where a net (A_γ) converges to A if for every $x \in X$ we have that $\|A_\gamma x - Ax\| \longrightarrow 0$. A *weak operator topology (WOT)* on $\mathcal{L}(X,Y)$ is the topology given by the convergence $A_\gamma \longrightarrow A$ if and only if $A_\gamma x \longrightarrow Ax$ weakly for every $x \in X$. Now, we are ready to prove another generalization of [6, Theorem 3.9]. Note that even for single-valued mappings the statement below is more general due to (c2). Under (c3), we obtain a generalization of the openness part of [21, Theorem 4.1] where the set-valued mapping is supposed to have $\bar{x}$ in the interior of the domain and to be quasi-peridifferentiable at $(\bar{x},\bar{y})$ in such a way that some derivative is invertible.

Theorem 3.11. *Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be Banach spaces, let $F : X \rightrightarrows Y$ be a mapping with closed graph and closed convex domain, and let $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Suppose that $\mathcal{A} \subset \mathcal{L}(X,Y)$ is a convex set such that for some positive constants r, β, ε, and ϱ we have*

$$F(x) \cap B(\bar{y},r) \subset F(x') + \mathcal{A}(x-x') + \beta\|x-x'\|B_Y \quad \text{for each} \quad x,x' \in \operatorname{dom} F \cap B(\bar{x},r), \tag{28}$$

and also

$$\varepsilon(\varrho+\beta)B_Y \subset A((\varepsilon B_X) \cap (\operatorname{dom} F - \bar{x})) \quad \text{for each} \quad A \in \mathcal{A}. \tag{29}$$

Then $\operatorname{sur} F(\bar{x},\bar{y}) \geq \varrho$ provided that one of the following six assumptions holds:

- *(c1) F is upper semi-continuous and compact-convex-valued on $\operatorname{dom} F \cap B(\bar{x},r)$, and the set $\mathcal{A}$ is relatively compact in the norm topology on $\mathcal{L}(X,Y)$;*
- *(c2) F is upper semi-continuous and compact-convex-valued on $\operatorname{dom} F \cap B(\bar{x},r)$, and the set $\mathcal{A}$ is compact in the strong operator topology on $\mathcal{L}(X,Y)$;*
- *(c3) $\mathcal{A}$ is singleton;*
- *(c4) X is reflexive and $\mathcal{A}$ is norm-bounded;*
- *(c5) Y is separable, $\mathcal{A}$ is norm-bounded and compact in the weak operator topology on $\mathcal{L}(X,Y)$, and $\mathcal{A}x$ is compact for each $x \in X$;*
- *(c6) X is separable, $\mathcal{A}$ is norm-bounded, and $\mathcal{A}x$ is compact for each $x \in X$.*

Proof. Assume, without any loss of generality, that $\bar{x} = 0$ and $\bar{y} = 0$. Note that in all the cases (c1)–(c6) the number $\mu := \sup_{A\in\mathcal{A}} \|A\|$ is finite. Indeed, to show this for (c2) observe that $\mathcal{A}x$ is bounded for any $x \in X$ by the compactness of $\mathcal{A}$ in the strong operator topology. Therefore $X = \bigcup_{n=1}^{\infty} \Omega_n$, where

$$\Omega_n := \{x \in X : \ \sup_{A\in\mathcal{A}} \|Ax\| \leq n\}, \quad n \in \mathbb{N}.$$

Hence there is an $n \in \mathbb{N}$ such that Ω_n has a non-empty interior. As Ω_n is convex and symmetric, there is $\delta > 0$ such that $\delta B_X \subset \Omega_n$. Fix any $A \in \mathcal{A}$. For any non-zero $h \in X$, we have

$$\|Ah\| = \frac{\|h\|}{\delta}\|A(\delta h/\|h\|)\| \leq \frac{n}{\delta}\|h\|.$$

Hence $\|A\| \leq n/\delta$ for each $A \in \mathcal{A}$.

Let $\varrho' \in (0, \varrho)$ be arbitrary. It suffices to show that $\operatorname{sur} F(0,0) \geq \varrho'$. Indeed, taking $\varrho' \uparrow \varrho$, we conclude the proof. Pick a $\gamma \in (0,1)$ such that $\beta + \varrho' + \gamma < (1-\gamma)^2(\beta + \varrho)$ and denote by $\mathcal{B}$ the norm-closure of $\mathcal{A}$. Applying Graves' theorem we get from (29) that

$$\varepsilon(1-\gamma)(\beta+\varrho)B_Y \subset A((\varepsilon B_X) \cap \operatorname{dom} F) \quad \text{for each} \quad A \in \mathcal{B}. \tag{30}$$

As $\mathcal{A}$ is norm-bounded, so is $\mathcal{B}$. By shrinking r, if necessary, we get that for each $x \in rB_X$ and each $A \in \mathcal{B}$ we have

$$(1-\gamma)B_X - \varepsilon^{-1}x \subset B_X \quad \text{and} \quad \|Ax\| \leq \varepsilon\big[(1-\gamma)^2(\beta+\varrho) - \beta - \varrho' - \gamma\big]. \tag{31}$$

We will show a relaxation of (30) when x is not far from 0, that is,

$$A\big((\varepsilon B_X) \cap (\operatorname{dom} F - x)\big) \supset \varepsilon(\beta + \varrho' + \gamma)B_Y \quad \text{whenever } x \in (rB_X) \cap \operatorname{dom} F \text{ and } A \in \mathcal{B}. \tag{32}$$

To this end, fix any such x and A. Then (31), (30), and the convexity of $\operatorname{dom} F$ imply that

$$\begin{aligned} A\big((\varepsilon B_X) \cap (\operatorname{dom} F - x)\big) &\supset A\big([(\varepsilon(1-\gamma)B_X) \cap \operatorname{dom} F] - x\big) \\ &\supset A\big((1-\gamma)[(\varepsilon B_X) \cap \operatorname{dom} F]\big) - Ax \\ &\supset (1-\gamma)^2\varepsilon(\beta+\varrho)B_Y - Ax \supset \varepsilon(\beta+\varrho'+\gamma)B_Y \end{aligned}$$

and (32) is verified.

To provide the proof under (c4) and (c5), let $H(x) := \{Ax : A \in \mathcal{A}\}$, $x \in X$. Then dom $H = X$, $H(B_X) \subset \mu B_Y$, and $-H(-h) = H(h)$ for each $h \in X$. It suffices to show that (32) implies (21) with ϱ replaced by ϱ'. So, fix any $x \in (rB_X) \cap \operatorname{dom} F$ and any $y^* \in S_{Y^*}$.

Assume that (c4) holds. Since X is reflexive, (29) implies that so is Y. Indeed, fix any $A \in \mathcal{A}$. As A is surjective, [12, Corollary 2.26 (iii)] says that Y is isomorphic to $X/A^{-1}(0)$. The continuity of A implies that $A^{-1}(0)$ is the closed subspace of X. Thus $X/A^{-1}(0)$, and hence Y is reflexive by [12, Exercises 3.114 and 3.112]. Proposition 3.10 (b) implies that there is a non-zero $h \in \varepsilon^{-1}(\operatorname{dom} F - x)$ such that

$$\inf_{A \in \mathcal{A}} \langle y^*, Ah \rangle \geq (\beta + \varrho' + \gamma)\|h\|.$$

Apply then Theorem 3.8 (b) to get that $\operatorname{sur} F(0,0) > \varrho'$.

Assume that (c5) holds. As $\mathcal{A}$ is compact in the weak operator topology, the set $C := \{A^*y^* : A \in \mathcal{A}\}$ is weak* closed. Proposition 3.10 (a) implies that there is a non-zero $h \in \varepsilon^{-1}(\operatorname{dom} F - x)$ such that

$$\inf_{A \in \mathcal{A}} \langle y^*, Ah \rangle \geq (\beta + \varrho')\|h\|.$$

Now, apply Theorem 3.8 (a) to finish the proof of this case.

Assume that (c6) holds. Under this assumption, $\mathcal{A}$ is compact in the strong operator topology, thus also in the weak operator topology. Moreover, Y is separable due to (32) as a continuous image of a separable space. Thus the conclusion follows from the previous case.

Note that the result under (c1) will follow once we establish it under (c2). Indeed, by (c1) the set $\mathcal{B}$, being norm-compact, is compact also in the strong operator topology. Moreover, (30) holds.

To furnish the proof under (c2) or (c3), fix any $(x,v) \in \operatorname{gph} F \cap \big(\frac{r}{2}B_X \times \frac{r}{2}B_Y\big)$ and any $y \in \frac{r}{2}B_Y$ distinct from v. Proposition 2.2 with $c := \varrho'$ will yield the desired estimate on the surjectivity modulus provided that we find a pair $(x',v') \in \operatorname{gph} F$ and a constant $\alpha \in (0, 1/\varrho')$ such that

$$\|y - v'\| < \|y - v\| - \varrho' \max\big\{\|x'-x\|, \alpha\|v'-v\|\big\}. \tag{33}$$

Pick some $\lambda \in \big(0, \min\{\varepsilon, r/2, \|y-v\|/(\beta+\varrho'+\gamma)\}\big)$ and put

$$z := \lambda(\beta+\varrho'+\gamma)\frac{y-v}{\|y-v\|} \quad \text{and} \quad K := \operatorname{dom} F - x.$$

First, assume that (c2) holds and let us agree that $\mathcal{L}(X,Y)$ will be considered with the strong operator topology until the end of the proof. We will construct a continuous mapping $\tilde{h} : \mathcal{A} \to K$ such that

$$\|\tilde{h}(A)\| \le \lambda \quad \text{and} \quad \|A(\tilde{h}(A)) - z\| < \gamma\lambda \quad \text{for each} \quad A \in \mathcal{A}. \tag{34}$$

Note that $(\varepsilon/\lambda)z \in \varepsilon(\beta+\varrho'+\gamma)B_Y$ and $(\lambda/\varepsilon)K \subset K$, because K is convex and contains 0. Using (32), for every $A \in \mathcal{A}$, we find an $h_A \in K$ such that $A(h_A) = z$ and $\|h_A\| \le \lambda$. Let

$$U(A) := \big\{A' \in \mathcal{A} : \ \|A'(h_A) - z\| < \gamma\lambda\big\}, \quad A \in \mathcal{A};$$

these sets are clearly open. For every $A \in \mathcal{A}$, we have $A(h_A) = z$; hence $U(A) \ni A$. The union of $U(A)$ when A ranges through $\mathcal{A}$ covers $\mathcal{A}$. As $\mathcal{A}$ is compact, we can choose a finite subcover $\{U(A_1), \dots, U(A_k)\}$ of it. Put $h_i := h_{A_i}$, $i = 1, \dots, k$. Using, [F∼, Theorem 17.21], we find a partition of unity subordinated to this subcover, that is, for every $i = 1, \dots, k$ there is a continuous function $\alpha_i : \mathcal{A} \to [0,1]$ such that $\alpha_i(\cdot) \ge 0$, $\alpha_i(A) = 0$ if $A \in \mathcal{A} \setminus U(A_i)$, and that $\alpha_1(\cdot) + \cdots + \alpha_i(\cdot) = 1$. (An elemenary construction, of the α_i's is as follows: $U(A_i)$ being an open and convex set containing A_i, let p_i be the Minkowski functional of the set $U(A_i) - A_i$. Put then $\gamma_i(\cdot) := \big(1 - p_i(\cdot)\big) \vee 0$, and finally $\alpha_i(\cdot) := \gamma_i(\cdot)/\big(\gamma_1(\cdot) + \cdots + \gamma_k(\cdot)\big)$.) Define

$$\mathcal{A} \ni A \longmapsto \tilde{h}(A) := \sum_{i=1}^{k} \alpha_i(A) h_i.$$

Clearly, $\tilde{h}$ is a continuous mapping with values in K (as this set is convex). Fix any $A \in \mathcal{A}$. The first inequality in (34) follows immediately from the definition of $\tilde{h}(A)$. To verify the second one, we observe that $\alpha_i(A) > 0$ only if $A \in \mathcal{A}$ and $\|A(h_{A_i}) - z\| < \gamma\lambda$. Therefore

$$\|A(\tilde{h}(A)) - z\| = \Big\|\sum_{i=1}^{k} \alpha_i(A)(Ah_i - z)\Big\| \le \sum_{i=1}^{k} \alpha_i(A)\|A(h_{A_i}) - z\| < \gamma\lambda$$

for every $A \in \mathcal{A}$, and the second inequality in (34) follows.

Next, define the set-valued mapping Ψ from $\mathcal{A}$ into itself by

$$\mathcal{A} \ni A \longmapsto \Psi(A) := \big\{B \in \mathcal{A} : \ v + B(\tilde{h}(A)) \in F\big(x + \tilde{h}(A)\big) + \beta\|\tilde{h}(A)\|B_Y\big\}.$$

Fix any $A \in \mathcal{A}$. By (34), we have $\|\tilde{h}(A)\| \leq \lambda < r/2$ and $x + \tilde{h}(A) \in x + K = \operatorname{dom} F$. Since $v \in F(x) \cap rB_Y$, the inclusion (28), with $x' := x + \tilde{h}(A) \in (rB_X) \cap \operatorname{dom} F$, yields a $B \in \mathcal{A}$ such that

$$v \in F\big(x + \tilde{h}(A)\big) + B(-\tilde{h}(A)) + \beta\|\tilde{h}(A)\|B_Y,$$

which means that $B \in \Psi(A) \neq \emptyset$. By assumption, the set $F\big(x + \tilde{h}(A)\big)$ is convex. Hence so is $F\big(x + \tilde{h}(A)\big) + \beta\|\tilde{h}(A)\|B_Y$, and therefore the set $\Psi(A)$ is also convex.

Further, we will show that the graph of Ψ is closed. Indeed, take any net $\big((A_\alpha, B_\alpha)\big)$ in $\operatorname{gph}\Psi$ converging to $(A, B) \in \mathcal{A} \times \mathcal{A}$. The continuity of $\tilde{h}$ implies that $\tilde{h}(A_\alpha)$ converges to $\tilde{h}(A) \in K$, the latter being true because $\operatorname{dom} F$ is closed. Moreover

$$\begin{aligned} \|B_\alpha(\tilde{h}(A_\alpha)) - B(\tilde{h}(A))\| &\leq \|B_\alpha(\tilde{h}(A_\alpha)) - B_\alpha(\tilde{h}(A))\| + \|B_\alpha(\tilde{h}(A)) - B(\tilde{h}(A))\| \\ &\leq \|B_\alpha\|\,\|\tilde{h}(A_\alpha) - \tilde{h}(A)\| + \|(B_\alpha - B)\tilde{h}(A)\| \longrightarrow 0, \end{aligned}$$

where we used that $\mathcal{A}$ is norm-bounded. Suppose, on the contrary, that for each $w' \in F(x+\tilde{h}(A))$ we have $\|v + B(\tilde{h}(A)) - w'\| > \beta\|\tilde{h}(A)\|$. Thus, by the norm-compactness of $F(x + \tilde{h}(A))$, there is a $\nu > 0$ such that

$$d\big(v + B(\tilde{h}(A)), F(x + \tilde{h}(A))\big) > \beta\|\tilde{h}(A)\| + 2\nu.$$

Since $\tilde{h}(A_\alpha) \longrightarrow \tilde{h}(A)$, the upper semi-continuity of F yields an index α_0 such that for each $\alpha \geq \alpha_0$ we have

$$F(x + \tilde{h}(A_\alpha)) \subset F(x + \tilde{h}(A)) + \nu B_Y.$$

As $B_\alpha(\tilde{h}(A_\alpha)) \to B(\tilde{h}(A))$ and $v + B_\alpha(\tilde{h}(A_\alpha)) \in F\big(x + \tilde{h}(A_\alpha)\big) + \beta\|\tilde{h}(A_\alpha)\|B_Y$, for each α, there is an $\alpha \geq \alpha_0$ and a $w'_\alpha \in F(x + \tilde{h}(A_\alpha))$ with $\|v + B(\tilde{h}(A)) - w'_\alpha\| < \beta\|\tilde{h}(A)\| + \nu$. Thus

$$\begin{aligned} \beta\|\tilde{h}(A)\| + 2\nu &< d\big(v + B(\tilde{h}(A)), F(x + \tilde{h}(A))\big) \leq d\big(w'_\alpha, F(x + \tilde{h}(A))\big) + \|v + B(\tilde{h}(A)) - w'_\alpha\| \\ &< \nu + \beta\|\tilde{h}(A)\| + \nu, \end{aligned}$$

a contradiction. Therefore $B \in \Psi(A)$ and the closeness of $\operatorname{gph}\Psi$ is verified.

Now, applying Gliksberg's extension of Kakutani's fixed point theorem [15] for the mapping Ψ, we conclude that there is an $\widehat{A} \in \mathcal{A}$ such that $\Psi(\widehat{A}) \ni \widehat{A}$. Set $\hat{h} := \tilde{h}(\widehat{A})$ and $x' := x + \hat{h}$. Then $\|x' - x\| = \|\hat{h}\| \leq \lambda$. The definition of Ψ implies that there is a $v' \in F(x')$ with $\|v + \widehat{A}\hat{h} - v'\| \leq \beta\|\hat{h}\| \leq \beta\lambda$. The definition of z reveals that $\|y - v - z\| = \|y - v\| - \lambda(\beta + \varrho' + \gamma)$. The second inequality in (34) implies that $\|\widehat{A}\hat{h} - z\| < \gamma\lambda$. Summarizing the previous inequalities we get that

$$\begin{aligned} \|y - v'\| &\leq \|y - v - z\| + \|v + \widehat{A}\hat{h} - v'\| + \|z - \widehat{A}\hat{h}\| \\ &< \|y - v\| - \lambda(\beta + \varrho' + \gamma) + \beta\lambda + \gamma\lambda = \|y - v\| - \varrho'\lambda. \end{aligned}$$

Noting that $\|\widehat{A}\hat{h}\| \leq \mu\lambda$, we see that $\|v - v'\| \leq (\beta + \mu)\lambda < (\varrho + \beta + \mu)\lambda$. Since $\|x' - x\| \leq \lambda$ we obtain (33) with $\alpha := 1/(\varrho + \beta + \mu)$, which finishes the proof of the case (c2).

Finally, assume that (c3) holds. Let A be the only element of $\mathcal{A}$. Now, (32) reads as

$$A\big((\varepsilon B_X) \cap K\big) \supset \varepsilon(\beta + \varrho' + \gamma)B_Y \quad \text{whenever } x \in (rB_X) \cap \operatorname{dom} F.$$

This inclusion implies that there is an $h \in K$ with $\|h\| \leq \lambda$ such that $Ah = z$. Let $x' = x + h$. Then $\|x' - x\| = \|h\| \leq \lambda < r/2$, hence (28) yields a $v' \in F(x')$ with $\|v + Ah - v'\| \leq \beta\|h\| \leq \beta\lambda$. Since $\|y - v - z\| = \|y - v\| - \lambda(\beta + \varrho' + \gamma)$ and $Ah = z$, we have

$$\|y - v'\| \quad \leq \quad \|y - v - z\| + \|v + Ah - v'\| \leq \|y - v\| - \lambda(\beta + \varrho' + \gamma) + \beta\lambda < \|y - v\| - \varrho'\lambda.$$

As before, we obtain (33) with $\alpha := 1/(\varrho + \beta + \mu)$. This (last) case is proved. □

Remark 3.12. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces and $\mathcal{A}$ be a subset of $\mathcal{L}(X, Y)$. Put $\operatorname{sur}\mathcal{A} := \inf\big\{\operatorname{sur} A : A \in \mathcal{A}\big\}$. Then $\operatorname{sur}\mathcal{A} = \operatorname{sur}\overline{\mathcal{A}}^{\|\cdot\|}$ as we have seen in the poof of Theorem 3.11. This leads to the question *whether* $\operatorname{sur}\mathcal{A}$ *is equal to* $\operatorname{sur}\overline{\mathcal{A}}^{SOT}$ *if not even to* $\operatorname{sur}\overline{\mathcal{A}}^{WOT}$.

Consider a bounded convex subset $\mathcal{A}$ of $\mathcal{L}(X, Y)$. If X is reflexive or the set $C := \{A^*y^* : A \in \mathcal{A}\}$ is weak* closed for each $y^* \in S_{Y^*}$ then $\operatorname{sur}\mathcal{A} = \operatorname{sur}\overline{\mathcal{A}}^{WOT}$. Indeed, fix any $\alpha \in (0, \operatorname{sur}\mathcal{A})$ and then any $\varepsilon > 0$ such that $\alpha + 2\varepsilon < \operatorname{sur}\mathcal{A}$. Pick an arbitrary $B \in \overline{\mathcal{A}}^{WOT}$. Find a net (A_γ) in $\mathcal{A}$ which converges to B in WOT. Fix any $y^* \in S_{Y^*}$. By Proposition 3.10 with $K := X$, there is an $\bar{h} \in S_X$ such that $\langle y^*, A_\gamma \bar{h}\rangle \geq \alpha + \varepsilon$ for every γ. The definition of WOT yields a γ_0 such that for each $\gamma \geq \gamma_0$ we have

$$\sup_{h \in S_X} \langle y^*, Bh\rangle \geq \langle y^*, B\bar{h}\rangle > \langle y^*, A_\gamma \bar{h}\rangle - \varepsilon \geq \alpha + \varepsilon - \varepsilon = \alpha.$$

As $y^* \in S_{Y^*}$ was chosen arbitrary, [24, Proposition 1.106] implies that

$$\operatorname{sur} B = \inf_{y^* \in S_{Y^*}} \|B^*y^*\| = \inf_{y^* \in S_{Y^*}} \sup_{h \in S_X} \langle B^*y^*, h\rangle = \inf_{y^* \in S_{Y^*}} \sup_{h \in S_X} \langle y^*, Bh\rangle \geq \alpha.$$

Thus $\operatorname{sur}\overline{\mathcal{A}}^{WOT} \geq \alpha$, and consequently $\operatorname{sur}\overline{\mathcal{A}}^{WOT} \geq \operatorname{sur}\mathcal{A}$.

Note that if $Y := \mathbb{R}$, then $\mathcal{L}(X, Y) = X^*$ and SOT, WOT and the weak* topology on X^* coincide. Moreover, $\operatorname{sur} x^* = \|x^*\|$ for every $x^* \in X^*$. If X is infinite-dimensional, for $\mathcal{A} := S_{X^*}$ we thus have that $\operatorname{sur}\mathcal{A} = 1$ while $\operatorname{sur}\overline{\mathcal{A}}^{w^*} = 0$ as 0 lies in the weak* closure of S_{X^*}. Note however that this $\mathcal{A}$ is not convex. In what follows, we will focus on finding convex sets $\mathcal{A}$ in X^*.

If X is non-reflexive, we do not know the answer to the question raised above in general. For dual spaces X the answer is negative. Indeed, assume that $(Z, \|\cdot\|)$ is a Banach space such that its dual is X. Pick $\varepsilon \in (0, 1)$. By Riesz' lemma [12, Lemma 1.37], there is a $z^{**} \in B_{Z^{**}}$ such that $d\big(z^{**}, Z\big) > 1 - \varepsilon$. Put then $\mathcal{A} := z^{**} + B_Z$; this is a convex, norm-closed set. For every $x^* \in \mathcal{A}$ we can write $x^* = z^{**} + z$, with a suitable $z \in B_Z$, and so $\|x^*\| \geq d\big(z^{**}, B_Z\big) > 1 - \varepsilon$. Therefore $\operatorname{sur}\mathcal{A} \geq 1 - \varepsilon$ (> 0). However, $0 \in \overline{\mathcal{A}}^{w^*}$, and so $\operatorname{sur}\overline{\mathcal{A}}^{w^*} = 0$. (More terestrially, take $Z := c_0$. Then $X = \ell_1$ and $X^* = \ell_\infty$. Put $\mathcal{A} := (1, 1, \ldots) + B_{c_0}$. Then $\operatorname{sur}\mathcal{A} = 1$ while $0 \in \overline{\mathcal{A}}^{w^*}$.)

Finally, we discuss several particular non-dual spaces X:

(a) Let $X := c_0$. Put

$$\mathcal{A} := S_{\ell_1^+} := \big\{(x_n)^{\mathbb{N}} : \ x_n \geq 0 \ \text{ for every } \ n \in \mathbb{N} \ \text{ and } \ \sum x_n = 1\big\} \quad (\subset X^*).$$

Then, clearly $\|x^*\|_{\ell_1} = 1$ for every $x^* \in \mathcal{A}$ while $\overline{\mathcal{A}}^{w^*} \ni 0$.

(b) Let $X := C([0,1])$ and equip it with the maximum norm $\|\cdot\|$. Put

$$\mathcal{A} := \lambda + \operatorname{co}\big\{\delta_t :\ t \in [0,1]\big\} \quad (\subset X^*),$$

where λ is Lebesgue measure on $[0,1]$ and δ_t's are Diracs. It is not difficult to calculate that $\|\mu\| = 2$ for every $\mu \in \mathcal{A}$ and to check that $\overline{\mathcal{A}}^{w^*} \ni 0$.

(c) Let $X := L_1([0,1])$, for $n \in \mathbb{N}$ let $f_n(t) := (1 - nt)^+,\ t \in [0,1]$, and then put

$$\mathcal{A} := \operatorname{co}\big\{f_1, f_2, \dots\big\} \quad (\subset X^*).$$

Clearly, for every $f \in \mathcal{A}$ we have $\|f\|_{L_\infty} = 1$ and for every $n \in \mathbb{N}$ and every measurable set $A \subset [0,1]$ we have that

$$0 \le \int_A f_n \mathrm{d}\lambda \le \int_0^1 f_n \mathrm{d}\lambda = \frac{1}{2n} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thus $\overline{\mathcal{A}}^{w^*} \ni 0$.

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