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SEPARABLE REDUCTIONS AND RICH FAMILIES IN THEORY OF FRÉCHET SUBDIFFERENTIALS

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Lecture Notes

Abstract Consider the following phenomena: Given a metric space X a function $f : X \to \mathbb{R}$ and $x \in X$, we can study the continuity of f at x, to calculate $\sup f$; if X is a Banach space, we can study non-emptiness of a subdifferential $\partial f(x)$, or ask for $f_1, f_2 : X \to \mathbb{R}$ if $\partial f_1(x_1) + \partial f_2(x_2)$ contains 0 provided that the sum $f_1 + f_2$ attains infimum at x and x_1, x_2 are close to x; given two metric spaces X, Y, a mapping $f : X \to 2^Y$, and $x \in X$, we want to calculate a modulus of surjectivity $\operatorname{sur} f(x)$ of f at x, or a slope of f at x; ... Such questions are important to study, no doubts. On the other hand, countable/separable objects are easier to manipulate with than uncountable/nonseparable ones, no doubts. Separable reduction is a procedure which transforms uncountable/nonseparable settings into countable/separable ones and thus enables to tackle them more easily. We plan to show that behind many uncountable/nonseparable phenomena (like those mentioned above, but actually behind many other ones, often going far beyond variational analysis) there are "rich families" of separable subspaces of a space in question, which are good/big/small enough to focus on the corresponding separable cases only. In order, just to get a taste, a main and most important property of rich families is that the intersection of two, or even of countably many rich families is non-empty, it is even a rich family again. Amazing, isn't? The rich families were first articulated some 15 years ago by J.M. Borwein and W. Moors, though, in set theory a similar concept existed for several decades. Now a definition follows. Given a non-separable metric space X, a family $\mathcal{R}$ consisting of (some) closed subspaces of X is called *rich* if it is "big enough" and moreover, whenever $Y_1, Y_2, \ldots$ is an increasing sequence of elements of $\mathcal{R}$, then the closure of $Y_1 \cup Y_2 \cup \cdots$ also belongs to $\mathcal{R}$.

Keywords Asplund space, separable reduction, cofinal family, rich family, Fréchet differentiability, Fréchet subdifferential, Fréchet normal cone, fuzzy calculus, extremal principle

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Contents

Motto *"The Asplund spaces form a right framework for variational analysis."* Boris Mordukhovich around 2000

1 Motivation for performing separable reductions

A central theme of our investigation will be the concept of Fréchet differentiability and subdifferentiability.

Definition 1.1. Let $(X, \|\cdot\|)$ be a Banach space, let $f : X \longrightarrow (-\infty, +\infty]$ be a proper function, i.e. $f \not\equiv +\infty$, and let x be any element of the domain of f, which means that $f(x) < +\infty$. We say that f is *Fréchet differentiable* at x if there are an element x^* of the dual space X^* and a function $o : [0, +\infty) \longrightarrow [0, +\infty]$ such that $\frac{o(t)}{t} \to 0$ as $t \downarrow 0$, and

$$o(\|h\|) > f(x+h) - f(x) - \langle x^*, h\rangle > -o(\|h\|)$$

holds for every non-zero $h \in X$. In this case the x^* is called the *Fréchet derivative* of f at x and it is denoted by the symbol $f'(x)$. We say that f is *Fréchet subdifferentiable* at x if there are $x^* \in X^*$ and a function $o : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\frac{o(t)}{t} \to 0$ as $t \downarrow 0$, and

$$f(x+h) - f(x) - \langle x^*, h\rangle > -o(\|h\|) \tag{1.1}$$

holds for every $0 \neq h \in X$. The (possibly empty) set of all x^*'s for which (1.1) holds with a suitable function $o(\cdot)$ is called the *Fréchet subdifferential* of f at x and is denoted by $\partial_F f(x)$.

Of course, if $f'(x)$ exists, then $\partial_F f(x) = \{f'(x)\}$. If f as well as $-f$ are Fréchet subdifferentiable at x, then an easy reasoning reveals that the function f is Fréchet differentiable at x. We also observe that $x^* \in \partial_F f(x)$ if and only if, for every $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{1}{\|h\|}\big(f(x+h) - f(x) - \langle x^*, h\rangle\big) > -\varepsilon$ whenever $h \in X$ and $0 < \|h\| < \delta$. Finally, for convex functions, $\partial_F f(x)$ coincides with the well known Moreau-Rockafellar subdifferential $\partial f(x)$; see [Ph, page 6].

Now, consider a "big", that is non-separable Banach space X, e.g. $X := \ell_2(\Gamma)$, where Γ is a"big" set, say $\Gamma := \mathbb{R}$; the set of real numbers. Let $f : X \to \mathbb{R}$ be a convex continuous function. We want to focus on points of Fréchet differentiability of f. Because X is big, we are facing a problem how to tackle this question. Indeed, dealing with uncountable objects can be difficult, once we have at hand just ten fingers. This "tool" can help us in considering problems with all finite numbers, and at the best case, to manipulate with countable objects, having cardinality ω — the first infinite number (right after all millions, billions, trillions, ...). Now, assume that we can find points of Fréchet differentiability of convex continuous functions defined on separable spaces, that is, on those Banach spaces which possess a countable dense subset. For our concrete $X := \ell_2(\Gamma)$, the restriction of f as above to any separable subspace $Y \subset X$, denoted by $f|_Y$, has points of Fréchet differentiability; this true by Preiss-Zajíček theorem [Ph, page 22]. Can we deduct from this "separable" information that the whole f, defined on X, is somewhere

Fréchet differentiable? The answer is affirmative and this is caused by availability of a suitable "separable reduction" of the phenomenon of Fréchet differentiability. More generally, let $f : X \longrightarrow (-\infty, +\infty]$ be a proper function and assume that we know that for every (or at least for many) separable subspaces Y of X, the restriction $f|_Y$ has points of Fréchet subdifferentiability. Then we look for a suitable separable reduction allowing us to prove that the whole f has the same property. Let us first present (and prove) a meaningful and very useful "separable" statement.

Proposition 1.2. *Let $(X, \|\cdot\|)$ be a (separable) Banach space whose dual X^* is separable and let $f : X \longrightarrow (-\infty, +\infty]$ be a lower semi-continuous function. Then the set of all $x \in X$ where the Fréchet subdifferential $\partial_F f(x)$ is non-empty is dense in the domain of f.*

Proof. We realize that X admits an equivalent Fréchet smooth (off the origin) norm. Indeed, let $\{x_1, x_2, \ldots\}$ be a countable dense subset of the unit sphere S_X of X and $\{\xi_1, \xi_2, \ldots\}$ be a countable dense subset of the unit sphere S_{X^*} of the dual. Then the assignment

$$X^* \ni x^* \longmapsto \sqrt{\|x^*\|^2 + \sum_{n=1}^\infty 2^{-n}\langle x^*, x_n\rangle^2 + \sum_{n=1}^\infty 2^{-n}\mathrm{dist}\left(x^*, \mathrm{sp}\{\xi_1, \ldots, \xi_n\}\right)^2} =: |x^*|$$

is an equivalent weak* lower semi-continuous norm. A small effort reveals that this norm is locally uniformly rotund, and hence, by Šmulyan test, the predual norm $|\cdot|$ on X defined by

$$X \ni x \longmapsto \sup\left\{\langle x^*, x\rangle : \ x^* \in X^*, \ |x^*| \le 1\right\} =: |x|$$

is Fréchet differentiable at every nonzero point of X. For more details, see [DGZ, page 43].

Next, let $\overline{x} \in \mathrm{dom}\, f$ and $\varepsilon > 0$ be given. We shall find an $x \in X$ such that $|x - \overline{x}| < \varepsilon$ and $\partial_F f(x)$ is non-empty. From the lower semi-continuity of f find $\varepsilon' \in (0, \varepsilon)$ so small that f is bounded below on the open ball $B(\overline{x}, \varepsilon)$. Define

$$\varphi(x) := \begin{cases} \left(\tan(\frac{\pi}{2\varepsilon'}|x - \overline{x}|)\right)^2 & \text{if } x \in B(\overline{x}, \varepsilon') \\ +\infty & \text{if } x \in X \setminus B(\overline{x}, \varepsilon'). \end{cases}$$

Then $\varphi : X \longrightarrow [0, +\infty]$ is easily seen to be proper and lower semi-continuous. Now, Borwein-Preiss variational principle [Ph, Theorem 4.20] provides a Fréchet smooth function $\theta : X \longrightarrow [0, +\infty)$ such that the sum $f + \varphi + \theta$ attains infimum at some $v \in X$; clearly, $v \in B(\overline{x}, \varepsilon') \cap \mathrm{dom}\, f$. We thus have

$$f(v + h) + \varphi(v + h) + \theta(v + h) \ge f(v) + \varphi(v) + \theta(v) \quad \text{for every} \quad h \in X,$$

and so

$$f(v + h) - f(v) + \langle \varphi'(v) + \theta'(v), h\rangle \ge -o(|h|) \quad \text{for every} \quad h \in X$$

where o is a function such that $\frac{o(t)}{t} \to 0$ as $t \downarrow 0$. We proved that $-\varphi'(v) - \theta'(v) \in \partial_F f(v)$, and hence $\partial_F f(v) \neq \emptyset$. $\square$

Joke. If X^* is separable and $g : X \to \mathbb{R}$ is convex continuous, then putting $f := -g$ in the proposition above, we conclude that g is Fréchet differentiable at points of a dense subset of X.

Well, the separable case is done. Now, consider non-separable spaces. Here one should be careful, since, for instance, ℓ_∞ admits an equivalent norm which is nowhere (even) Gateaux differentiable: $\ell_\infty \ni (x_n) \longmapsto \|(x_n)\|_\infty + \limsup_{n\to\infty} |x_n|$ is such a norm. In what follows, we shall restrict ourselves to Banach spaces with the property that every separable subspace of it has separable dual. Such spaces are called *Asplund spaces.* (The original definition is that X is Asplund if every convex continuous function on it has points of Fréchet differentiability.) The reason why we cannot go beyond Asplund spaces can bee seen from the fact that (even) the canonical norm of the "innocent" ℓ_1 is nowhere Fréchet differentiable; actually every non-Asplund space admits an equivalent norm which is nowhere Fréchet differentiable, see [M, page 197]. (The situation is even worse. According to R. Haydon, there is a non-separable Asplund space having no Gateaux smooth norm [H].) Now a main question which concerns us arises: Is it possible to extend Proposition 1.2 to non-separable Asplund spaces? The answer is affirmative. Just the proof is not easy. It will occupy more or less the rest of this text. The technology used goes back to [F, FZ, Pr] and D. Gregory [Ph, Theorem 2.14] (in the reverse chronological order). In order not to get lost, we shall first restrict ourselves to Fréchet differentiability of convex functions. And this is right D. Gregory's theorem below, with (now) a bit simplified proof.

Proposition 1.3. (D. Gregory) *Let $(X, \|\cdot\|)$ be a non-separable Banach space, $f : X \to \mathbb{R}$ a* `convex` *continuous function, and Z a separable subspace of X. Then there exists a separable subspace Y of X, containing Z, and such that, if the restriction $f|_Y$ of f to Y is Fréchet differentiable at some $x \in Y$, then the "whole" f is Fréchet differentiable at x.*

Proof. First, we need a `translation` of Fréchet differentiability (of convex functions) completely to the terms of the space X: *f is Fréchet differentiable at $x \in X$ if and only if*

$$S(x,t) := \sup_{h\in B_X} \big(f(x+th) + f(x-th)\big) = 2f(x) + o(t) \quad \text{as} \quad \mathbb{Q}_+ \ni t \downarrow 0;$$

this is easy to check. For any $x \in X$ and any $t > 0$, if $S(x,t) < +\infty$, we find a vector $u(x,t) \in B_X$ such that

$$f\big(x + t\,u(x,t)\big) + f\big(x - t\,u(x,t)\big) > S(x,t) - t^2. \tag{1.2}$$

(This $u(x,t)$ is almost "the worst possible" as regards the Fréchet differentiability of f at x.) Let C_0 be a countable dense subset of Z. We shall construct countable sets $C_0 \subset C_1 \subset C_2 \subset \cdots \subset X$ as follows. Let $m \in \mathbb{N}$ be given and assume that C_{m-1} was already found. Find a countable set C_m in X such that it is stable under making all finite linear combinations with rational coefficients, and that it contains C_{m-1} as well as the set $\big\{u(x,t) : \; x \in C_{m-1}, \; t \in \mathbb{Q}_+\big\}$; clearly, C_m is again countable. Do so for every $m \in \mathbb{N}$, and put finally $Y := \overline{C_1 \cup C_2 \cup \cdots}$. Clearly, $Y \supset Z$.

We claim that this Y has the desired property. So, assume that $f|_Y$ is Fréchet differentiable at some $x \in Y$. We shall show that the whole f is Fréchet differentiable at x as well. (If $x \in C_1 \cup C_2 \cup \cdots$, then it is rather easy to proceed. So, we have to find an argument working also for $x \in Y \setminus C_1 \cup C_2 \cup \cdots$.) Let L denote a Lipschitz constant of f in a vicinity of x; see [Ph, Proposition 1.6]. Pick any $t \in \mathcal{Q}_+$ small enough (so that we can profit below from the L-Lipschitz property of f around x). Find then $c \in \bigcup_{m=1}^{\infty} C_m$ such that $\|c - x\| < t^2$. We can now subsequently estimate for all sufficiently small $t \in \mathbb{Q}_+$ (so that we can profit from the Lipschitz property of f)

$$
\begin{aligned}
2f(x) &\leq S(x,t) < S(c,t) + 2Lt^2 \\
&< f\big(c + t\,u(c,t)\big) + f\big(c - t\,u(c,t)\big) + t^2 + 2Lt^2 \\
&< f\big(x + t\,u(c,t)\big) + f\big(x - t\,u(c,t)\big) + t^2 + 4Lt^2 \\
&\leq \sup_{k \in B_Y} \big(f(x+tk) + f(x-tk)\big) + t^2 + 4Lt^2 \\
&= o(t) + 2f(x) \quad \text{as} \quad \mathbb{Q}_+ \ni t \downarrow 0
\end{aligned}
$$

since $f|_Y$ is Fréchet differentiable at x. Therefore, the "whole" f is Fréchet differentiable at x. $\square$

Homework. Show that the (semi)-continuity of a function is also separable reducible in the spirit of Proposition1.3. Hint. Realize that the continuity of f at x can be characterized via "oscillation", that is diameter of $f\big(\overset{\circ}{B}(x,t)\big),\ t > 0$ and that the latter quantity can be calculated via a suitable countable subset of $\overset{\circ}{B}(x,t)$; see [FI2].

We finish this section by one observation. Let $\mathcal{S}(X)$ denote the family of all closed separable subspaces of a non-separable Banach space X. We actually showed that the subfamily $\mathcal{C}$ of all $Y \in \mathcal{S}(X)$ such that the conclusion of Proposition 1.3 holds is *cofinal/dominating/saturating* in $\mathcal{S}(X)$, that is, it has the property that for every $Z \in \mathcal{S}(X)$ there is $Y \in \mathcal{C}$ containing Z.

2 Motivation for introducing rich families

Going back to the previous section, we can say that Fréchet differentiability of a convex continuous function is "separably reducible" via a suitable cofinal subfamily of $\mathcal{S}(X)$. Consider now two convex continuous functions f_1, f_2. We would like to separably reduce Fréchet differentiability simultaneously for both functions. A natural way how to do this would be to consider the family $\mathcal{C} := \mathcal{C}_1 \cap \mathcal{C}_2$. However, we, or may be just the author of this text, does not see if $\mathcal{C}$ is cofinal, not even if it is non-empty. Then it remains to cultivate a bit the argument above and thus get a cofinal family working simultaneously for both functions. And what about if somebody brings one more functions, etc? This trouble can be remedied with help the concept of *rich family.* It was first articulated in a joint paper by J.M. Borwein and W. Moors [BM]; see also [LPT, Section 3.6] and [FI2].

Definition 2.1. Let X be a (rather) non-separable Banach space [or just a metrizable space]. Let $\mathcal{S}(X)$ denote the family of all closed separable subspaces [closed separable subsets] of X. A family $\mathcal{R} \subset \mathcal{S}(X)$ is called *rich* if
(i) it is cofinal, and
(ii) it is *σ-complete*, that is, whenever $Y_1, Y_2, \ldots$ are in $\mathcal{R}$ and $Y_1 \subset Y_2 \subset \cdots$, then $Y := \overline{Y_1 \cup Y_2 \cup \cdots} \in \mathcal{R}$.

The power of rich families is demonstrated by the following fundamental fact; see [BM] and also [LPT, page 37].

Proposition 2.2. *The intersection of two, even of countably many, rich families of a given space is (not only non-empty but even) rich again.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ be two rich families in $\mathcal{S}(X)$. Let $Z \in \mathcal{S}(X)$ be arbitrary. From the cofinality of $\mathcal{R}_1, \mathcal{R}_2$ we find, alternatively, two sequences $Y_1^1,\ Y_2^1,\ \ldots$ in $\mathcal{R}_1$ and $Y_1^2,\ Y_2^2,\ \ldots$ in $\mathcal{R}_2$ such that $Z \subset Y_1^1 \subset Y_1^2 \subset Y_2^1 \subset Y_2^2 \subset Y_3^1 \subset \cdots$. Then $Y := \overline{Y_1^1 \cup Y_2^1 \cup \cdots} = \overline{Y_1^2 \cup Y_2^2 \cup \cdots}$ belongs to $\mathcal{R}$ by (ii), and the cofinality of $\mathcal{R}$ is proved. The proof that $\mathcal{R}$ is σ-complete is simple, and is left to a potential reader.

Okay, once we know that the intersection of two rich families is rich, we can conclude, via a simple induction, that the intersection of any finite number of rich families is again rich. Finally, let $\mathcal{R}_1, \mathcal{R}_2, \ldots$ be a sequence of rich families and denote by $\mathcal{R}$ the intersection of all them. Then, for sure, and immediately, $\mathcal{R}$ is σ-complete. Let $Z \in \mathcal{S}(X)$ be any. Find subsequently $Y_1 \in \mathcal{R}_1$ so that $Y_1 \supset Z$. Find $Y_2 \in \mathcal{R}_1 \cap \mathcal{R}_2$ so that $Y_2 \supset Y_1$. Find $Y_3 \in \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$ so that $Y_3 \supset Y_2$. (Fed up already?) ... Profiting from what was already proved, we get that $Y := \overline{Y_1^1 \cup Y_2^1 \cup \cdots}$ belongs to $\mathcal{R}$, and the cofinality of $\mathcal{R}$ is also verified. $\square$

Next we shall strengthen the conclusion of Proposition 1.3 by showing that, behind the separable reduction of Fréchet differentiability of (for this moment only) `convex` functions, there is a suitable rich family.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a non-separable Banach space and $f : X \to \mathbb{R}$ be a convex,* `not necessarily continuous,` *function. Then there exists a rich family $\mathcal{R}$ in $\mathcal{S}(X)$ such that for every $Y \in \mathcal{R}$ and every $x \in Y$ the whole function f is Fréchet differentiable at x if (and only if) the restriction $f|_Y$ is Fréchet differentiable at x.*

Proof. Search for a suitable $\mathcal{R}$ is not so obvious. Indeed, it is not clear to us if the "natural" candidate, the family of all $Y \in \mathcal{S}(X)$ such that for every $x \in Y$, the function f is Fréchet differentiable at x if (and only if) the restriction $f|_Y$ is Fréchet differentiable at x, works. We shall proceed as follows. For every subspace Y of X, every $x \in X$, and every $t > 0$ we put

$$S_Y(x,t) := \sup\left\{f(x+th) + f(x-th) :\ h \in \overset{\circ}{B}_Y\ \right\};$$

here, and below $\overset{\circ}{B}_Y$ denotes the open unit ball in Y. (Here, and in what follows, we shall frequently profit from the boon coming from using open balls.) Now, we define

$$\mathcal{R} := \left\{Y \in \mathcal{S}(X) :\ S_Y(x,t) = S_X(x,t) \ \text{ for every } \ x \in Y \ \text{ and every } \ t > 0\right\}.$$

We shall show that this $\mathcal{R}$ is rich.

Let us prove that $\mathcal{R}$ is cofinal in $\mathcal{S}(X)$. For every $x \in X$ and every $t > 0$, $\gamma > 0$, if $S_X(x,t) < +\infty$, we find a vector $u(x,t,\gamma) \in \overset{\circ}{B}_X$ such that

$$f\big(x + tu(x,t,\gamma)\big) + f\big((x - tu(x,t,\gamma)\big) > S_X(x,t) - \gamma.$$

Now, fix any $Z \in \mathcal{S}(X)$. Pick a countable dense subset C_0 in Z. Let $m \in \mathbb{N}$ be fixed for a short while and assume we have already found C_{m-1}. Find then a rationally linear countable set C_m in X such that it contains C_{m-1} as well as the set $\big\{u(x,t,\gamma) :\ x \in C_{m-1},\ t,\gamma \in \mathbb{Q}_+\big\}$. Doing so for every $m \in \mathbb{N}$, put finally $Y := \overline{C_0 \cup C_1 \cup C_2 \cup \cdots}$. Clearly, Y lies in $\mathcal{S}(X)$ and contains Z. It remains to verify that Y does belong to $\mathcal{R}$. So, fix any $h \in \overset{\circ}{B}_X$ and any $t > 0$. We have to show that $S_X(x,t) \le S_Y(x,t)$. Find a rational $t' \in (0,t)$ such that $\frac{t}{t'}\|h\| < 1$. Find then $x' \in C_1 \cup C_2 \cup \cdots$ such that $t\|h\| + \|x' - x\| < t'$ and $t' + \|x' - x\| < t$. Find then $m \in \mathbb{N}$ so big that C_{m-1} contains x'. Now, for every $k \in \overset{\circ}{B}_Y$ we have

$$f(x'+t'k) + f(x'-t'k) = f\Big(x + t\Big(\frac{t'}{t}k + \frac{x'-x}{t}\Big)\Big) + f\Big(x - t\Big(\frac{t'}{t}k + \frac{x'-x}{t}\Big)\Big) \le S_Y(x,t);$$

here we profited from the fact that $\big\|\frac{t'}{t}k + \frac{x'-x}{t}\big\| < \frac{t'}{t} + \frac{\|x'-x\|}{t} < 1$. Thus, recalling that k was an arbitrary element of $\overset{\circ}{B}_Y$, we have that $S_Y(x',t') \le S_Y(x,t)$ $\ (\le S_X(x,t) < +\infty)$.

Next, fix for a while any $\gamma \in \mathbb{Q}_+$. We can estimate:

$$\begin{aligned} f(x+th) + f(x-th) &= f\Big(x' + t'\Big(\frac{t}{t'}h + \frac{x-x'}{t'}\Big)\Big) + f\Big(x' - t'\Big(\frac{t}{t'}h + \frac{x-x'}{t'}\Big)\Big) \\ &\le S_X(x,t') < \gamma + f\big(x' + t'u(x',t',\gamma)\big) + f\big(x' - t'u(x',t',\gamma)\big) \\ &\le \gamma + S_Y(x',t') \le \gamma + S_Y(x,t); \end{aligned}$$

here we profited from the inequality $\|\frac{t}{t'}h + \frac{x-x'}{t'}\| \le \frac{t}{t'}\|h\| + \frac{\|x-x'\|}{t'} < 1$. Thus we have that $f(x+th) + f(x-th) \le S_Y(x,t) + \gamma$. And, as $\gamma \in \mathbb{Q}_+$ and $h \in \overset{\circ}{B}_X$ were arbitrary, we obtain that $S_X(x,t) \le S_Y(x,t)$ for every $x \in Y$ and $t > 0$. This means that $Y \in \mathcal{R}$.

As regards the σ-completeness of $\mathcal{R}$, consider an increasing sequence $Y_1, Y_2, \ldots$ in it and put $Y := \overline{Y_1 \cup Y_2 \cup \cdots}$. Fix for a while any $x \in Y$ and any $t > 0$. (If $x \in Y_1 \cup Y_2 \cup \cdots$, then a reader can easily verify that $S_Y(x,t) = S_X(x,t)$. Further, we take into account that this may not be so.) Fix any $h \in \overset{\circ}{B}_X$ and any $t > 0$. We have to show that $f(x+th) + f(x-th) \le S_Y(x,t)$. Find $t' \in (0,t)$ such that $\frac{t}{t'}\|h\| < 1$. Find then $x' \in Y_1 \cup Y_2 \cup \cdots$ such that $t\|h\| + \|x'-x\| < t'$ and $t' + \|x'-x\| < t$. Finally find $m \in \mathbb{N}$ so big that Y_m contains x'. Now, we can estimate:

$$\begin{aligned} f(x+th) + f(x-th) &= f\Big(x' + t'\Big(\frac{t}{t'}h + \frac{x-x'}{t'}\Big)\Big) + f\Big(x' - t'\Big(\frac{t}{t'}h + \frac{x-x'}{t'}\Big)\Big) \\ &\le S_X\big(x',t'\big) = S_{Y_m}(x',t') \le S_Y(x',t'); \end{aligned} \tag{2.1}$$

here we used that $Y_m \in \mathcal{R}$ and that $\|\frac{t}{t'}h + \frac{x-x'}{t'}\| < 1$. On the other hand, for every $k \in \overset{\circ}{B}_Y$ we have

$$f(x'+t'k) + f(x'-t'k) = f\Big(x + t\Big(\frac{t'}{t}k + \frac{x'-x}{t}\Big)\Big) + f\Big(x - t\Big(\frac{t'}{t}k + \frac{x'-x}{t}\Big)\Big) \le S_Y(x,t);$$

here we used that $\big\|\frac{t'}{t}k + \frac{x'-x}{t}\big\| < 1$. Thus, recalling that k was an arbitrary element of $\overset{\circ}{B}_Y$, we have that $S_Y(x',t') \le S_Y(x,t)$. Now putting together this inequality with (2.1), we can conclude that $f(x+th) + f(x-th) \le S_Y(x,t)$. And, as $h \in \overset{\circ}{B}_X$ were arbitrary, we have that $S_X(x,t) \le S_Y(x,t)$ for every $x \in Y$ and $t > 0$. This means that $Y \in \mathcal{R}$. We verified that our $\mathcal{R}$ is σ-complete.

It remains to check that our $\mathcal{R}$ "works". So, take any Y in it and any x in Y such that the restriction $f|_Y$ is Fréchet differentiable at x (if there is any such). This means that

$$\big(S_Y(x,t) - 2f(x) = \big)\ \sup\big\{f(x+th) + f(x-th) :\ h \in \overset{\circ}{B}_Y\ \big\} - 2f(x) = o(t) \quad \text{as} \quad t \downarrow 0\,.$$

But the definition of $\mathcal{R}$ guarantees that $S_Y(x,t) = S_X(x,t)$ for every $t > 0$. Therefore

$$\sup\big\{f(x+th) + f(x-th) :\ h \in \overset{\circ}{B}_X\ \big\} - 2f(x) = o(t) \quad \text{as} \quad t \downarrow 0\,,$$

that is, the whole f is Fréchet differentiable at x. □

There are many other separable reducible statements, and practically, behind any, such there is a suitable rich family.

Homework. Given a real-valued function f defined on a Banach space (more generally, on a metric space), show that the continuity of it is separable reducible via a rich family. Hint: Define $\mathcal{R}$ as that consisting of all $Y \in \mathcal{S}(X)$ such that for every $x \in Y$ and every $t > 0$ we have $\operatorname{diam} f\big(\overset{\circ}{B}(x,t)\big) = \operatorname{diam} f\big(\overset{\circ}{B}(x,t) \cap Y\big)$; here $\overset{\circ}{B}(x,t)$ means the open ball around x with radius t.

3 Separable reduction of Fréchet subdifferentiability via rich families in general Banach spaces

Consider a convex function $\varphi : X \longrightarrow -\infty, +\infty]$, with $\varphi(0)$ finite, and let $c \geq 0$ be given. We start with proving a simple but basic **Fact:** *$\partial\varphi(0) \cap cB_{X^*}$ is non-empty if and only if $\varphi(h) \geq \varphi(0) - c\|h\|$ for every $h \in X$.* (We note that the subdifferential $\partial\varphi(0)$ can be empty, for instance, when φ is a linear non-continuous functional on X.) Indeed, if $x^* \in \partial\varphi(0) \cap cB_{X^*}$, then for every $h \in X$ we have $\varphi(h) \geq \varphi(0) + \langle x^*, h\rangle \geq \varphi(0) - c\|h\|$. Reversely, having this inequality at hand, we know that the function $\varphi + \|\cdot\|$ attains infimum at $h := 0$. Hence $0 \in \partial(\varphi + \|\cdot\|)(0) = \partial\varphi(0) + \partial\|\cdot\|(0) = \partial\varphi(0) - cB_{X^*}$, by Moreau-Rockafellar theorem [Ph, page 47]. Thus $\partial\varphi(0) \cap cB_{X^*} \neq \emptyset$.

The next statement translates the non-emptiness of Fréchet subdifferential of any, not necessarily convex or continuous, function purely into terms of the space X. To do so, we need more notation. Namely, we denote

- by Δ the collection of all sequences $\delta = (\delta_n) \in (0, +\infty)^\omega$ such that $\delta_1 \geq \delta_2 \geq \cdots$;
- by Λ the collection of all sequences $\lambda = (\lambda_n) \in [0, +\infty)^\omega$ such that the set $\{n \in \mathbb{N} : \lambda_n > 0\}$ is finite and $\sum_{n=1}^\infty \lambda_n = 1$;
- by Υ the collection of all $(\nu_n) \in \mathbb{N}^\omega$ such that the set $\{n \in \mathbb{N} :\ \nu_n \neq 1\}$ is finite, and
- given $\nu = (\nu_n) \in \Upsilon$ and $\delta = (\delta_n) \in \Delta$, we denote by $\mathcal{H}(\nu, \delta)$ the collection of all $H = (h_n) \in X^\omega$ such that $\|h_n\| < \delta_{\nu_n}$ for every $n \in \mathbb{N}$.

Proposition 3.1. *Let $(X, \|\cdot\|)$ be a general Banach space, consider a proper function $f : X \longrightarrow (-\infty, +\infty]$, let $x \in \operatorname{dom} f$, and let $c \geq 0$ be given. Then $\partial_F f(x) \cap cB_{X^*}$ is non-empty if and only if there is a sequence $\delta = (\delta_n) \in \Delta \cap \mathbb{Q}^\omega$ so that*

$$\sum_{n=1}^\infty \lambda_n\Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + c\Big\|\sum_{n=1}^\infty \lambda_n h_n\Big\| \geq f(x) \tag{3.1}$$

for all $(\lambda_n) \in \Lambda$, all $\nu = (\nu_n) \in \Upsilon$, and all $(h_n) \in \mathcal{H}(\nu, \delta)$.

Proof. Sufficiency. For $h \in X$ define

$$\begin{aligned}\varphi(h) := \inf\Bigg\{ &\sum_{i=1}^\infty \lambda_n\big(f(x+h_n) + \tfrac{1}{\nu_n}\|h_n\|\big) - f(x) : \\ &\qquad (\lambda_n) \in \Lambda,\ (\nu_n) \in \Upsilon,\ (h_n) \in \mathcal{H}(\nu, \delta),\ \sum_{n=1}^\infty \lambda_n h_n = h\Bigg\}\end{aligned} \tag{3.2}$$

if $\|h\| \leq \delta_1$, and $\varphi(h) := +\infty$ if $\|h\| > \delta_1$. It is clear from (3.1) that $\varphi(h) \geq -c\|h\| > -\infty$ for all $h \in X$. We shall verify that $\varphi : X \longrightarrow (-\infty, +\infty]$ is a convex function. Fix any $\alpha \in (0,1)$ and any $h, h' \in \operatorname{dom}\varphi$; then $\|h\| \leq \delta_1$, $\|h'\| \leq \delta_1$. Fix any $t > 0$ and find (λ_n), $(\lambda'_n) \in \Lambda$, $\nu = (\nu_n) \in \Upsilon$, $\nu' = (\nu'_n) \in \Upsilon$, and $(h_n) \in \mathcal{H}(\nu, \delta)$, $(h'_n) \in \mathcal{H}(\nu', \delta)$ such that

$$\sum_{n=1}^\infty \lambda_n h_n = h, \quad \varphi(h) + t > \sum_{n=1}^\infty \lambda_n\big(f(x+h_n) + \tfrac{1}{\nu_n}\|h_n\|\big) - f(x)$$

and that the same holds if we replace λ_n, ν_n, h_n and h, respectively, by λ'_n, ν'_n, h'_n and h'. Take an $\bar{n} \in \mathbb{N}$ so big that $\lambda_n = \lambda'_n = 0$ for $n > \bar{n}$ and set

$$\begin{array}{rrrl} \lambda''_n = \alpha\lambda_n, & \nu''_n = \nu_n, & h''_n = h_n, & \text{if } 1 \le n \le \bar{n}; \\ \lambda''_n = (1-\alpha)\lambda'_{n-\bar{n}}, & \nu''_n = \nu'_{n-\bar{n}}, & h''_n = h'_{n-\bar{n}}, & \text{if } \bar{n} < n \le 2\bar{n}; \\ \lambda''_n = 0, & \nu''_n = 1, & h''_n = 0, & \text{if } n > 2\bar{n}. \end{array}$$

We note that $(\lambda''_n) \in \Lambda$, $\nu'' := (\nu''_n) \in \Upsilon$, and $\|h''_n\| < \delta_{\nu''_n}$ for every $n \in \mathbb{N}$; so $(h''_n) \in \mathcal{H}(\nu'', \delta)$. Then

$$\begin{aligned} & \alpha\varphi(h) + (1-\alpha)\varphi(h') + t + f(x) \\ > & \sum_{n=1}^{\bar{n}} \Big[\alpha\lambda_n\Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + (1-\alpha)\lambda'_n\Big(f(x+h'_n) + \frac{1}{\nu'_n}\|h'_n\|\Big)\Big] \\ = & \sum_{n=1}^{\bar{n}} \lambda''_n\Big(f(x+h''_n) + \frac{1}{\nu''_n}\|h''_n\|\Big) + \sum_{n=1}^{\bar{n}} \lambda''_{\bar{n}+n}\Big(f(x+h''_{\bar{n}+n}) + \frac{1}{\nu''_{\bar{n}+n}}\|h''_{\bar{n}+n}\|\Big) \\ = & \sum_{n=1}^{\infty} \lambda''_n\Big(f(x+h''_n) + \frac{1}{\nu''_n}\|h''_n\|\Big) \ge \varphi\big(\alpha h + (1-\alpha)h'\big) + f(x). \end{aligned}$$

which proves the convexity of φ as $t > 0$ could be taken arbitrarily small.

Now, it follows from (3.1) and (3.2) that $0 \ge \varphi(0) \ge 0$. Thus $\varphi(h) \ge \varphi(0) - c\|h\|$ for every $h \in X$. By the Fact above, there is a $x^* \in \partial\varphi(0)$ such that $\|x^*\| \le c$. We shall show that $x^* \in \partial_F f(x)$. So, consider an arbitrary $\varepsilon > 0$. Find $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Take any fixed $h \in X$ such that $\|h\| < \delta_m$. Put $\lambda_m = 1$, $h_m = h$ and $\nu_m = m$, and for $n \in \mathbb{N} \setminus \{m\}$ put $\lambda_n = 0$, $h_n = 0$, and $\nu_n = 1$; then $(\lambda_n) \in \Lambda$, $\nu := (\nu_n) \in \Upsilon$, and $(h_n) \in \mathcal{H}(\nu, \delta)$. Hence, by (3.2) we get that

$$f(x+h) + \varepsilon\|h\| - f(x) \ge f(x+h) + \tfrac{1}{m}\|h\| - f(x) \ge \varphi(h) \ge \varphi(0) + \langle x^*, h\rangle = \langle x^*, h\rangle$$

as $x^* \in \partial\varphi(0)$. Therefore $x^* \in \partial_F f(x) \cap cB_{X^*}$.

Conversely, assume there is x^* in $\partial_F f(x) \cap cB_{X^*}$. For every $n \in \mathbb{N}$ we find $\delta_n \in \mathbb{Q}_+$ such that $f(x+h) + \frac{1}{n}\|h\| - f(x) > \langle x^*, h\rangle$ whenever $h \in X$ and $\|h\| \le \delta_n$. We may arrange that $\delta_1 \ge \delta_2 \ge \cdots$. Then $\delta := (\delta_n) \in \Delta \cap \mathbb{Q}^\omega$. For $h \in X$ define $\varphi(h)$ by the formula (3.2) with this δ if $\|h\| \le \delta_1$, and $\varphi(h) := +\infty$ if $\|h\| > \delta_1$. As above, we can check that this φ is a convex function on X. We certainly have that $\varphi(0) \le 0$. Fix for a while any $h \in X$, with $\|h\| < \delta_1$. Consider any $(\lambda_n) \in \Lambda$, any $\nu = (\nu_n) \in \Upsilon$, and any $(h_n) \in \mathcal{H}(\Upsilon, \delta)$ such that $\sum_{n=1}^\infty \lambda_n h_n = h$. We observe that for every $n \in \mathbb{N}$ we have $f(x+h_n) + \frac{1}{\nu_n}\|h_n\| - f(x) > \langle x^*, h_n\rangle$. It follows that

$$\sum_{n=1}^{\infty} \lambda_n\big(f(x+h_n) + \tfrac{1}{\nu_n}\|h_n\|\big) - f(x) \ge \sum_{n=1}^{\infty} \lambda_n\langle x^*, h_n\rangle = \langle x^*, h\rangle.$$

Therefore, by (3.2), $\varphi(h) \ge \langle x^*, h\rangle \ \big(\ge \varphi(0) + \langle x^*, h\rangle\big)$ whenever $h \in X$ and $\|h\| < \delta_1$, and hence $x^* \in \partial\varphi(0)$. And as $\|x^*\| \le c$, applying the Fact above again, we get that (3.1) holds. $\square$

Now we are ready to perform separable reduction of the nonemptiness of Fréchet subdifferential via a rich family. For every subspace Y of X, every $x \in Y$, every $\lambda = (\lambda_n) \in \Lambda$, every $\nu = (\nu_n) \in \mathbb{N}^\omega$, every $\delta = (\delta_n) \in \Delta$, and every $c \geq 0$ we denote by $I(x, \lambda, \nu, \delta, c, Y)$ the following (possibly infinite) quantity

$$\inf \Big\{ \sum_{n=1}^{\infty} \lambda_n \Big(f(x+h_n) + \frac{1}{\nu_n} \|h_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n h_n \Big\| : \ (h_n) \in \mathcal{H}(\nu, \delta) \cap Y^\omega \Big\}. \tag{3.3}$$

Clearly, with this notation, (3.1) reads as $I(x, \lambda, \nu, \delta, c, X) \geq f(x)$.

For $\lambda = (\lambda_n) \in \Lambda$, $\nu = (\nu_n) \in \mathbb{N}^\omega$, and $c \geq 0$, we define the family $\mathcal{R}_{\lambda,\nu,c}$ as that consisting of all $Y \in \mathcal{S}(X)$ such that for every $x \in Y$ and every $\delta \in \Delta \cap \mathbb{Q}^\omega$ we have

$$I(x, \lambda, \nu, \delta, c, X) = I(x, \lambda, \nu, \delta, c, Y). \tag{3.4}$$

Proposition 3.2. *For any $\lambda = (\lambda_n) \in \Lambda$, $\nu = (\nu_n) \in \mathbb{N}^\omega$, and $c \geq 0$ the family $\mathcal{R}_{\lambda,\nu,c}$ defined above is rich.*

Proof. Fix any λ, ν and c as above. We re-denote $I(x, \lambda, \nu, \delta, c, X)$ and $I(x, \lambda, \nu, \delta, c, Y)$, respectively, by $I(x, \delta, X)$ and $I(x, \delta, Y)$. Now, for every $x \in X$, every $\delta \in \Delta$, and for every $m, n \in \mathbb{N}$, we find vectors $g_n(x, \delta, m) \in X$, with $\|g_n(x, \delta, m)\| < \delta_{\nu_n}$, such that

$$I(x, \delta, X) + \frac{1}{m} \geq \sum_{n=1}^{\infty} \lambda_n \Big(f\big(x + g_n(x, \delta, m)\big) + \frac{1}{\nu_n} \|g_n(x, \delta, m)\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n g_n(x, \delta, m) \Big\| \tag{3.5}$$

if $I(x, \delta, X) > -\infty$, and

$$-m > \sum_{n=1}^{\infty} \lambda_n \Big(f\big(x + g_n(x, \delta, m)\big) + \frac{1}{\nu_n} \|g_n(x, \delta, m)\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n g_n(x, \delta, m) \Big\| \tag{3.6}$$

if $I(x, \delta, X) = -\infty$. Here, we choose the vectors $g_n(x, \delta, m)$ in such a way that $g_n(x, \delta, m) = g_n(x, \delta', m)$ whenever $\delta, \delta' \in \Delta$ and $\delta_{\nu_j} = \delta'_{\nu_j}$ for every $j \in \mathbb{N}$ such that $\lambda_j > 0$. Using this policy, we guarantee that for every $x \in X$ and every $m \in \mathbb{N}$ the set $\{g_n(x, \delta, m) : \ n \in \mathbb{N}, \ \delta \in \Delta \cap \mathbb{Q}^\omega\}$ is countable.

We first show that $\mathcal{R}_{\lambda,\nu,c}$ is cofinal in $\mathcal{S}(X)$. So, fix any $Z \in \mathcal{S}(X)$. Choose a countable dense subset C_0 in Z. Assume further that for some $m \in \mathbb{N}$ we have already constructed countable sets $C_0 \subset C_1 \subset \cdots \subset C_{m-1} \subset X$. Define then C_m as the $\mathbb{Q}$-linear span of the set $C_{m-1} \cup \big\{ g_n(x, \delta, m) : \ n \in \mathbb{N}, \ x \in C_{m-1}, \ \delta \in \Delta \cap \mathbb{Q}^\omega \big\}$. Clearly, C_m is again countable.

Put $Y := \overline{C_0 \cup C_1 \cup \cdots}$. Clearly, $Y \in \mathcal{S}(X)$ and $Y \supset Z$. We have to show that Y belongs to $\mathcal{R}$, that is, that (3.4) holds. So, fix any $x \in Y$ and any $\delta \in \Delta \cap \mathbb{Q}^\omega$. Clearly, it is enough to prove that $I(x, \delta, X) \geq I(x, \delta, Y)$. Consider any $(h_n) \in \mathcal{H}(\nu, \delta)$. Put $N := \{n \in \mathbb{N} : \ \lambda_n > 0\}$; this is a finite set. Take an arbitrary $r \in \mathbb{Q}_+$ so small that $\|h_n\| < \delta_{\nu_n} - 2r$ for every $n \in N$. Find then $\delta' = (\delta'_i) \in \Delta \cap \mathbb{Q}^\omega$ such that $\delta'_i \leq \delta_i$ for every $i \in \mathbb{N}$ and $\delta'_{\nu_n} = \delta_{\nu_n} - r$ if $n \in N$. Find $m \in \mathbb{N}$ so big that $\operatorname{dist}(x, C_{m-1}) < r$; pick then

$y_m \in C_{m-1}$ such that $\|x - y_m\| < r$. We are now ready to estimate

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n \Big(f\big(y_m + g_n(y_m, \delta', m)\big) + \frac{1}{\nu_n} \|g_n(y_m, \delta', m)\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n g_n(y_m, \delta', m) \Big\|, \\
&\geq \sum_{n=1}^{\infty} \lambda_n \Big(f\big(x + (y_m - x + g_n(y_m, \delta', m))\big) + \frac{1}{\nu_n} \|y_m - x + g_n(y_m, \delta', m)\| \Big) \\
&+ c \Big\| \sum_{n=1}^{\infty} \lambda_n \big(y_m - x + g_n(y_m, \delta', m)\big) \Big\| - r - cr \geq I(x, \delta, Y) - r - cr,
\end{aligned}
\tag{3.7}
$$

the last inequality being true because $\big(y_m - x + g_n(y_m, \delta', m) : \; n \in \mathbb{N}\big) \in \mathcal{H}(\nu, \delta) \cap Y^{\omega}$.

If $I(y_m, \delta', X) = -\infty$ for infinitely many $m \in \mathbb{N}$, then (3.6) and (3.7) imply together that $-m > I(x, \delta, Y) - r - cr$ for all such m; hence $I(x, \delta, Y) = -\infty$, and thus $I(x, \delta, X) \geq -\infty = I(x, \delta, Y)$.

Assume now that $I(y_m, \delta', X) > -\infty$ for all sufficiently large $m \in \mathbb{N}$. Fix one such m, big enough to guarantee that $m > \frac{1}{r}$. Define $h'_n := h_n + x - y_m$ if $n \in N$, and $h'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h'_n) \in \mathcal{H}(\nu, \delta')$ and we can estimate

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n \Big(f(x + h_n) + \frac{1}{\nu_n} \|h_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n h_n \Big\| \\
&= \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m + h'_n) + \frac{1}{\nu_n} \|h_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n h_n \Big\| \\
&\geq \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m + h'_n) + \frac{1}{\nu_n} \|h'_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n h_n \Big\| - r - cr \\
&\geq I(y_m, \delta', X) - r - cr \\
&\geq \sum_{n=1}^{\infty} \lambda_n \Big(f\big(y_m + g_n(y_m, \delta', m)\big) + \frac{1}{\nu_n} \|g_n(y_m, \delta', m)\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n g_n(y_m, \delta', m) \| \\
&- \frac{1}{m} - r - cr \geq I(x, \delta, Y) - 3r - 2cr,
\end{aligned}
\tag{3.8}
$$

by (3.5) and (3.7). Since $r \in \mathbb{Q}_+$ could be arbitrarily small, this proves that $I(x, \delta, X) \geq I(x, \delta, Y)$. Therefore $Y \in \mathcal{R}_{\lambda,\nu,c}$ and so the cofinality of $\mathcal{R}_{\lambda,\nu,c}$ is verified.

To prove that $\mathcal{R}_{\lambda,\nu,c}$ is σ-complete, we have to further elaborate the construction above. Let $Y_1, \; Y_2, \; \ldots,$ be an increasing sequence of elements of $\mathcal{R}_{\lambda,\nu,c}$. Put $Y := \overline{Y_1 \cup Y_2 \cup \cdots}$. We have to show that Y belongs to $\mathcal{R}_{\lambda,\nu,c}$. This means that we have to verify (3.4). So, fix any $x \in Y$ and any $\delta \in \Delta \cap \mathbb{Q}^{\omega}$. We have to prove that $I(x, \delta, X) \geq I(x, \delta, Y)$. Take any $(h_n) \in \mathcal{H}(\nu, \delta)$. Let again $N := \{n \in \mathbb{N} : \; \lambda_n > 0\}$; this is a finite set. Take an arbitrary $r \in \mathbb{Q}_+$ so small that $\|h_n\| < \delta_{\nu_n} - 2r$ for every $n \in N$. Find then $\delta' = (\delta'_i) \in \Delta \cap \mathbb{Q}^{\omega}$ such that $\delta'_i \leq \delta_i$ for every $i \in \mathbb{N}$ and $\delta'_{\nu_n} = \delta_{\nu_n} - r$ if $n \in N$. Take $m \in \mathbb{N}$ so big that $\operatorname{dist}(x, Y_m) < r$; pick then $y_m \in Y_m$ so that $\|x - y_m\| < r$. Define $h'_n := h_n + x - y_m$ if $n \in N$, and $h'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h'_n) \in \mathcal{H}(\nu, \delta')$ and from the first half of (3.8) (valid also now) we have

$$
\begin{aligned}
r + cr + & \quad \sum_{n=1}^{\infty} \lambda_n \Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n h_n \Big\| \\
\geq & \quad I(y_m, \delta', X) = I(y_m, \delta', Y_m) \geq I(y_m, \delta', Y)
\end{aligned} \tag{3.9}
$$

since $y_m \in Y_m$, $Y_m \in \mathcal{R}_{\lambda,\nu,c}$, and $Y_m \subset Y$.

Now, consider any $(k_n) \in \mathcal{H}(\nu, \delta') \cap Y^\omega$. Set $k_n' := k_n + y_m - x$ if $n \in N$, and $k_n' := 0$ if $n \in \mathbb{N} \setminus N$. Then $(k_n') \in \mathcal{H}(\nu, \delta) \cap Y^\omega$ and we can estimate

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m + k_n) + \frac{1}{\nu_n}\|k_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n k_n \Big\| \\
= \ & \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k_n') + \frac{1}{\nu_n}\|k_n\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n k_n \Big\| \\
\geq \ & \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k_n') + \frac{1}{\nu_n}\|k_n'\| \Big) + c \Big\| \sum_{n=1}^{\infty} \lambda_n k_n \Big\| - r - cr \\
\geq \ & I(x, \delta, Y) - r - cr.
\end{aligned}
$$

Hence $I(y_m, \delta', Y) \geq I(x, \delta, Y) - r - cr$. Therefore, combining the latter inequality with (3.9) and recalling that $r \in \mathbb{Q}_+$ was arbitrarily small, we conclude that $I(x, \delta, X) \geq I(x, \delta, Y)$. This verifies (3.4) for our Y and hence guarantees that $Y \in \mathcal{R}_{\lambda,\nu,c}$. We proved that $\mathcal{R}_{\lambda,\nu,c}$ is σ-complete. □

Theorem 3.3. ([FZ], [F]) *Let X be a non-separable Banach space and f an extended-real-valued function on X. Then there is a rich family $\mathcal{R}$ in $\mathcal{S}(X)$ such that for every $Y \in \mathcal{R}$, every $x \in Y$, and every $c \geq 0$ we have that: $\partial_F f(x) \cap cB_{X^*}$ is non-empty if (and only if) $\partial_F \big(f|_Y\big)(x) \cap cB_{Y^*}$ is non-empty.*

Proof. Assume that $\partial_F \big(f|_Y\big)(x) \cap cB_{Y^*} \neq \emptyset$. By Proposition 3.1, there is $\delta \in \Delta \cap \mathbb{Q}^\omega$ such that $I(x, \lambda, \nu, \delta, c, Y) \geq f(x)$ whenever $\lambda \in \Lambda \cap \mathbb{Q}^\omega$ and $\nu \in Y$. Hence, by Proposition 3.2, for all these λ's and ν's, we have $I(x, \lambda, \nu, \delta, c, X) \geq f(x)$. And using Proposition 3.1 again, we can conclude that $\partial_F f(x) \cap cB_{X^*}$ is nonempty. The necessity statement is obvious. □

Corollary 3.4 (Preiss-Zajíček; see [LPT]). *Let X be a Banach space and f an extended-real-valued function on X. Then there is a rich family $\mathcal{R}$ of separable subspaces of X such that for every $Y \in \mathcal{R}$ and every $x \in Y$ we have that f is Fréchet differentiable at x, with $\|f'(x)\| \leq c$, if (and only if) $f|_Y$ is Fréchet differentiable at x, with and $\|(f|_Y)'(x)\| \leq c$.*

Proof. Applying Theorem 3.3 to our f, we get a rich family $\mathcal{R}_+$ in $\mathcal{S}(X)$ such that for any $Y \in \mathcal{R}_+$ and any $x \in Y$ we can be sure that $\partial f(x)$ contains an element with norm not greater than c if the same is true for $\partial(f|_Y)(x)$. Likewise, applying the theorem to $-f$, we find a rich family $\mathcal{R}_-$ for $-f$ with similar properties. It remains to set $\mathcal{R} = \mathcal{R}_+ \cap \mathcal{R}_-$ and to apply Theorem 2.2, taking into account that, f is Fréchet differentiable at x if and only if both $\partial f(x)$ and $\partial(-f)(x)$ are nonempty. This proves that the sufficiency. The necessity is obvious. □

Okay, we have separable reduction, via rich family for Fréchet (sub)differentiability of functions. Unfortunately, this is not enough for more complicated statements with Fréchet subdifferentials: in particular for fuzzy calculus, non-zerones of Fréchet normal cones and extremal principle. This will be a content of the next section.

4 Primal representation of more complex statements involving ∂_F

This section is a broad enhancement of the situation considered in Section 3. We wish here to construct rich families for separable reductions of various statements associated with Fréchet subdifferentiability. In pursuing this goal we shall follow the traditional approach going back to [Pr], [FZ] (see also [F], [P], [I2], and [FI1]), whose first step is "primal" (not involving anything associated with the dual space) characterization of the desired property.

Let $k \in \mathbb{N}$, let $X, X_1, \dots, X_k$ be (rather) non-separable Banach spaces, and let $A_i : X_i \to X,\ i = 1, \dots, k$, be bounded linear operators. The statement below is an extension of the Fact from the beginning of Section 3.

Proposition 4.1. *Let $c \geq 0,\ \varepsilon_1 > 0,\ \dots, \varepsilon_k > 0,\ \rho_1 \geq 0,\ \dots,\ \rho_k \geq 0$ be given constants and let $\varphi : X \longrightarrow (-\infty, +\infty]$ be a convex function, with $\varphi(0) < +\infty$. Then the following two assertions are equivalent:*

(i) There exist $\varepsilon_i' \in (0, \varepsilon_i),\ i = 1, \dots, k$, and $(w_1, \dots w_k) \in S_{X_1} \times \cdots \times S_{X_k}$ such that

$$\varphi(x) \geq \varphi(0) - c\big\|x - \textstyle\sum A_i x_i\big\| - \sum \varepsilon_i' \|x_i\| - \sum \rho_i \|x_i - w_i\| + \sum \rho_i$$

holds for all $(x, x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$.

(ii) There exists $x^* \in \partial\varphi(0)$ such that $\|x^*\| \leq c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ for every $i = 1, \dots, k$.

For better understanding of this proposition, we can consider several special cases of it. For instance: $k = 1$ and $\rho_1 = 0$; or $k = 1$, $A_1 = 0$ and $\rho_1 \neq 0$; etc. For more examples we refer to the end of Section 5.

Proof. (Above and below, $\sum$ means $\sum_{i=1}^k$.) Assume (ii) holds. Find $\varepsilon_i' \in (0, \varepsilon_i)$ so that $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i'$ for every $i = 1, \dots, k$. For each i find a norm attaining $w_i^* \in X_i^*$ such that $\|w_i^*\| = \rho_i$ and $\|A_i^* x^* - w_i^*\| < \varepsilon_i'$. Take finally a $w_i \in S_{X_i}$ so that $\|w_i^*\| = \langle w_i^*, w_i \rangle$. Then for all $(x, , x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$ we have

$$\begin{aligned}\varphi(x) \geq \varphi(0) + \langle x^*, x\rangle &= \varphi(0) + \big\langle x^*, x - \textstyle\sum A_i x_i \big\rangle \\ &+ \textstyle\sum \big\langle A_i^* x^* - w_i^*, x_i \big\rangle + \sum \big\langle w_i^*, x_i - w_i \big\rangle + \sum \langle w_i^*, w_i \rangle \\ &\geq \varphi(0) - c\Big\|x - \textstyle\sum A_i x_i\Big\| - \sum \varepsilon_i' \|x_i\| - \sum \rho_i \|x_i - w_i\| + \sum \rho_i.\end{aligned}$$

Assume that (i) holds. Set

$$\psi(x, x_1, \dots, x_k) := \varphi(x) + c\big\|x - \textstyle\sum A_i x_i\big\| + \sum \varepsilon_i' \|x_i\| + \sum \rho_i \|x_i - w_i\| - \sum \rho_i.$$

Then

$$\psi(x, x_1, \dots, x_k) \geq \varphi(0) = \psi(0, 0, \dots, 0)$$

for all $x \in X$ and for all $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$. Thus, by Moreau-Rockafellar theorem [Ph, page 47], there are $x^* \in \partial\varphi(0),\ \xi \in cB_{X^*}$, and further, for $i = 1, \dots, k$, there are $\xi_i \in \varepsilon_i' B_{X_i^*}$ and $w_i^* \in X_i^*$, with $\langle w_i^*, w_i \rangle = \|w_i^*\| = \rho_i$, such that

$$(0, 0, \dots, 0) = (x^*, 0, \dots, 0) + (\xi, -A_1^* \xi, \dots, -A_i^* \xi) + (0, \xi_1, \dots, \xi_k) + (0, w_1^*, \dots, w_k^*).$$

Hence, $0 = x^* + \xi$ and $A_i^*\xi = \xi_i + w_i^*$ for $i = 1, \dots, k$. Therefore, $\|x^*\| \le c$ and

$$\big|\|A_i^*\xi\| - \rho_i\big| = \big|\|A_i^*\xi\| - \|w_i^*\|\big| \le \big\|A_i^*\xi - w_i^*\big\| = \|\xi_i\| \le \varepsilon_i' < \varepsilon_i$$

for every $i = 1, \dots, k$. $\square$

The proposition above gives us the key instrument for finding the necessary primal characterization of Fréchet subdifferentiability and several associated properties.

Let us call *data* any triple $d = (c, \varepsilon, \rho)$ such that $c \ge 0$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in (0, +\infty)^k$, and $\rho = (\rho_1, \dots, \rho_k) \in [0, +\infty)^k$. To begin with, we define for any given data d and any $w = (w_1, \dots, w_k) \in S_{X_1} \times \cdots \times S_{X_k}$ the function

$$p_{d,w}(h, x_1, \dots, x_k) := c\big\|h - \textstyle\sum A_i x_i\big\| + \sum \varepsilon_i \|x_i\| + \sum \rho_i \|x_i - w_i\| - \sum \rho_i,$$

where $(h, x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$ are the arguments of the function and and d and w are parameters changing within the indicated limits. For any fixed d and w this is a convex continuous function, equal to zero at $(0, 0, \dots, 0)$. Moreover for $u = (u_1, \dots, u_k) \in S_{X_1} \times \cdots \times S_{X_k}$ we have

$$p_{d,w}(h, x_1, \dots, x_k) - p_{d,u}(h, x_1, \dots, x_k) \le \textstyle\sum \rho_i \|w_i - u_i\|. \tag{4.1}$$

We shall need again the notation introduced in Section 3, that is the symbols $\Delta, \Lambda, \Upsilon$, and $\mathcal{H}(\nu, \delta)$. The next proposition offers the desired primal characterization. It translates the non-emptiness of Fréchet subdifferential (even a subtler fact) completely into terms of the space X. The proof of the proposition repeats word for word the proof of [FI1, Lemma 2.2] if we replace reference to [FI1, Lemma 2.1] by the reference to Proposition 4.1. Hence we omit this proof.

Proposition 4.2. *Consider a proper function $f : X \longrightarrow (-\infty, +\infty]$ and fix $x \in X$ such that $f(x) < +\infty$. Then, given data $d = (c, \varepsilon, \rho)$, the following two assertions are equivalent:*

(i) There exist $\varepsilon_i' \in (0, \varepsilon_i)$, $i = 1, \dots, k$, $w := (w_1, \dots, w_k) \in S_{X_1} \times \ldots \times S_{X_k}$, and a sequence $\delta := (\delta_1, \delta_2, \dots) \in \Delta \cap \mathbb{Q}^\omega$ such that for $d' := (c, \varepsilon', \rho)$ the inequality

$$\sum_{n=1}^{\infty} \lambda_n \Big(f(x + h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + p_{d',w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n, x_1, \dots, x_k\Big) \ge f(x) \tag{4.2}$$

holds whenever $x_1, \dots, x_k \in X_1 \times \cdots \times X_k$, $(\lambda_n) \in \Lambda$, $\nu \in \Upsilon$, and $(h_n) \in \mathcal{H}(\nu, \delta)$.

(ii) There exists $x^* \in \partial_F f(x)$ such that $\|x^*\| \le c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ for every $i = 1, \dots, k$.

Our aim is to find a rich family that could be used for separable reduction of (ii). It is the first property (i) of the proposition that equips us with a suitable instrument for constructing such family. Let $k, X, X_1, \dots, X_k, A_1, \dots A_k$ have the same meaning as before. By a *block* we understand any product $Y \times Y_1 \times \cdots \times Y_k$ where $Y, Y_1, \dots, Y_k$ are subspaces of $X, X_1, \dots, X_k$, respectively. Any $\mathcal{F} \subset \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ whose elements are blocks shall be called a *block-family*. For every block $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k$, every $x \in Y$, every $\lambda = (\lambda_n) \in \Lambda$, every $\nu = (\nu_n) \in \mathbb{N}^\omega$, every $\delta = (\delta_n) \in \Delta$, every data

$d = (c, \varepsilon, \rho) \in [0, +\infty) \times (0, +\infty)^k \times [0, +\infty)^k$ and every $w \in S_{X_1} \times \cdots \times S_{X_k}$ we denote by $I(x, \lambda, \nu, \delta, d, w, \mathcal{Y})$ the following (possibly infinite) quantity

$$\begin{aligned} \inf\Big\{\sum_{n=1}^{\infty} \lambda_n \Big(f(x+h_n)+\frac{1}{\nu_n}\|h_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n, x_1, \ldots, x_k\Big) : \\ (h_n) \in \mathcal{H}(\nu, \delta) \cap Y^\omega,\ (x_1, \ldots, x_k) \in Y_1 \times \cdots \times Y_k\Big\}. \end{aligned} \tag{4.3}$$

If $Y = X$ and $Y_i = X_i$ for all $i = 1, \ldots, k$, we write just $I(x, \lambda, \nu, \delta, d, w)$. With this notation, (4.2) reads as $I(x, \lambda, \nu, \delta, d, w) \geq f(x)$.

For $\lambda = (\lambda_n) \in \Lambda,\ \nu = (\nu_n) \in \mathbb{N}^\omega$, and $d = (c, \varepsilon, \rho) \in [0, +\infty) \times (0, +\infty)^k \times [0, +\infty)^k$ we define the block-family $\mathcal{R}_{\lambda,\nu,d}$ as that consisting of all blocks $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k \in \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ such that

$$A_1(Y_1) \subset Y,\ \ldots,\ A_k(Y_k) \subset Y \tag{4.4}$$

and that for all $x \in Y,\ \delta \in \Delta \cap \mathbb{Q}^\omega$, and $w \in S_{Y_1} \times \cdots \times S_{Y_k}$

$$I(x, \lambda, \nu, \delta, d, w) = I(x, \lambda, \nu, \delta, d, w, \mathcal{Y}), \tag{4.5}$$

Proposition 4.3. *For any $\lambda = (\lambda_n) \in \Lambda,\ \nu = (\nu_n) \in \mathbb{N}^\omega$ and $d = (c, \varepsilon, \rho) \in [0, +\infty) \times (0, +\infty)^k \times [0, +\infty)^k$, the family $\mathcal{R}_{\lambda,\nu,d}$ defined above is rich.*

Proof. Fix any λ, ν and d as above and put, for simplicity, $\mathcal{R} := \mathcal{R}_{\lambda,\nu,d}$. We redenote $I(x, \lambda, \nu, \delta, d, w)$ and $I(x, \lambda, \nu, \delta, d, w, \mathcal{Y})$, respectively, by $I(x, \delta, w)$ and $I(x, \delta, w, \mathcal{Y})$. Now, for every $x \in X$, every $w = (w_1, \ldots, w_k) \in S_{X_1} \times \ldots \times S_{X_k}$, every $\delta \in \Delta$, and for every $m, n \in \mathbb{N}$ we find vectors $v_1(x, \delta, w, m) \in X_1, \ldots,\ v_k(x, \delta, w, m) \in X_k$, and vectors $g_n(x, \delta, w, m) \in X$, with $\|g_n(x, \delta, w, m)\| < \delta_{\nu_n}$, such that

$$\begin{aligned} I(x, \delta, w) + \frac{1}{m} \geq \sum_{n=1}^{\infty} \lambda_n \Big(f\big(x + g_n(x, \delta, w, m)\big) + \frac{1}{\nu_n}\|g_n(x, \delta, w, m)\|\Big) \\ + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n g_n(x, \delta, w, m), v_1(x, \delta, w, m), \ldots, v_k(x, \delta, w, m)\Big) \end{aligned} \tag{4.6}$$

if $I(x, \delta, w) > -\infty$, and

$$\begin{aligned} -m > \sum_{n=1}^{\infty} \lambda_n \Big(f\big(x + g_n(x, \delta, w, m)\big) + \frac{1}{\nu_n}\|g_n(x, \delta, w, m)\|\Big) \\ + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n g_n(x, \delta, w, m), v_1(x, \delta, w, m), \ldots, v_k(x, \delta, w, m)\Big) \end{aligned} \tag{4.7}$$

if $I(x, \delta, w) = -\infty$. Here, we choose the vectors $v_i(x, \delta, w, m)$ and $g_n(x, \delta, w, m)$ in such a way that $v_i(x, \delta, w, m) = v_i(x, \delta', w, m)$ and $g_n(x, \delta, w, m) = g_n(x, \delta', w, m)$ whenever $\delta, \delta' \in \Delta$ and $\delta_{\nu_j} = \delta'_{\nu_j}$ for every $j \in \mathbb{N}$ such that $\lambda_j > 0$. By this we guarantee that for every $x \in X$, every $w \in S_{X_1} \times \cdots \times S_{X_k}$, and every $m \in \mathbb{N}$ the set

$$\{v_i(x,\delta,w,m):\ i=1,\dots,k,\ \delta\in\Delta\cap\mathbb{Q}^\omega\}\bigcup\{g_n(x,\delta,w,m):\ n\in\mathbb{N},\ \delta\in\Delta\cap\mathbb{Q}^\omega\}$$

is countable.

We first show that $\mathcal{R}$ is cofinal in $\mathcal{S}(X\times X_1\times\cdots\times X_k)$. To begin with, fix any $Z\in\mathcal{S}(X)$ and any $Z_i\in\mathcal{S}(X_i)$, $i=1,\dots,k$. Choose countable dense subsets C_0 in Z, C_0^1 in $Z_1,\dots,$ and C_0^k in Z_k. Assume further that for some $m\in\mathbb{N}$ we have already constructed countable sets $C_0\subset C_1\subset\cdots\subset C_{m-1}\subset X$ and $C_0^i\subset C_1^i\subset\cdots\subset C_{m-1}^i\subset S_{X_i}$, $i=1,\dots,k$. Define then C_m as the $\mathbb{Q}$-linear span of the union of C_{m-1}, $A_i(C_{m-1}^i)$, $i=1,\dots,k$, and the set

$$\big\{g_n(x,\delta,w,m):\ n\in\mathbb{N},\ x\in C_{m-1},\ \delta\in\Delta\cap\mathbb{Q}^\omega,\ w\in C_{m-1}^1\times\cdots\times C_{m-1}^k\big\}$$

Likewise, for any $i=1,\dots,k$ define the set C_m^i as the $\mathbb{Q}$-linear span of the union of C_{m-1}^i and

$$\{v_i(x,\delta,w,m):\ i=1,\dots,k,\ x\in C_{m-1},\ \delta\in\Delta\cap\mathbb{Q}^\omega,\ w\in C_{m-1}^1\times\cdots\times C_{m-1}^k\}.$$

augmented with normalized versions of its elements (that is, vectors of the form $\xi/\|\xi\|$). Clearly, all these sets are still countable.

Set $Y:=\overline{C_0\cup C_1\cup\cdots}$ and $Y_i:=\overline{C_0^i\cup C_1^i\cup\cdots}$ for every $i=1,\dots,k$. Clearly, these are closed separable subspaces and $\mathcal{Y}:=Y\times Y_1\times\cdots\times Y_k\supset Z\times Z_1\times\cdots\times Z_k$. We have to show that $\mathcal{Y}$ belongs to $\mathcal{R}$, that is, that (4.4) and (4.5) hold. The verification of (4.4) is easy. As regards (4.5), fix any $x\in Y$, $\delta\in\Delta\cap\mathbb{Q}^\omega$, and $w=(w_1,\dots,w_k)\in S_{Y_1}\times\cdots\times S_{Y_k}$. Clearly, it is enough to prove that $I(x,\delta,w)\ge I(x,\delta,w,\mathcal{Y})$. Uniform continuity of the assignment $u\mapsto p_{d,u}(\cdots)$ (see(4.1)) allows us to assume that w_i belongs to $C_0^i\cup C_1^i\cup\cdots$ for every $i=1,\dots,k$. Now, consider any $(h_n)\in\mathcal{H}(\nu,\delta)$ and any $(x_1,\dots,x_k)\in X_1\times\cdots\times X_k$. Put $N:=\{n\in\mathbb{N}:\ \lambda_n>0\}$; this is a finite set. Take an arbitrary $r\in\mathbb{Q}_+$ so small that $\|h_n\|<\delta_{\nu_n}-2r$ for every $n\in N$. Find then $\delta'=(\delta'_n)\in\Delta\cap\mathbb{Q}^\omega$ such that $\delta'_n\le\delta_n$ for every $n\in\mathbb{N}$ and $\delta'_n=\delta_n-r$ if $n\in N$. Find $m\in\mathbb{N}$ so big that $w_1\in C_{m-1}^1,\dots,\ w_k\in C_{m-1}^k$, and that $\operatorname{dist}(x,C_{m-1})<r$; pick then $y_m\in C_{m-1}$ such that $\|x-y_m\|<r$.

We are now ready to estimate

$$\begin{aligned}
&\sum_{n=1}^\infty\lambda_n\Big(f\big(y_m+g_n(y_m,\delta',w,m)\big)+\frac{1}{\nu_n}\|g_n(y_m,\delta',w,m)\|\Big)\\
&+p_{d,w}\Big(\sum_{n=1}^\infty\lambda_n g_n(y_m,\delta',w,m),v_1(y_m,\delta',w,m),\dots,v_k(y_m,\delta',w,m)\Big)\\
&\ge\sum_{n=1}^\infty\lambda_n\Big(f\big(x+(y_m-x+g_n(y_m,\delta',w,m))\big)+\frac{1}{\nu_n}\|y_m-x+g_n(y_m,\delta',w,m)\|\Big)\\
&+p_{d,w}\Big(\sum_{n=1}^\infty\lambda_n\big(y_m-x+g_n(y_m,\delta',w,m)\big),v_1(y_m,\delta',w,m),\dots,v_k(y_m,\delta',w,m)\Big)\\
&-r-cr\ge I(x,\delta,w,\mathcal{Y})-r-cr,
\end{aligned}\tag{4.8}$$

the last inequality being true because $v_i(y_m,\delta',w,m)\in C_m\subset Y$ and $\big(y_m-x+g_n(y_m,\delta',w,m):\ n\in\mathbb{N}\big)\in\mathcal{H}(\nu,\delta)\cap Y^\omega$.

If $I(y_m,\delta',w)=-\infty$ for infinitely many $m\in\mathbb{N}$, then (4.7) and (4.8) imply together that $-m>I(x,\delta,w,\mathcal{Y})-r-cr$ for all such m; hence $I(x,\delta,w,\mathcal{Y})=-\infty$, and thus $I(x,\delta,w)\ge-\infty=I(x,\delta,w,\mathcal{Y})$.

Assume now that $I(y_m,\delta',w) > -\infty$ for all sufficiently large $m \in \mathbb{N}$. Fix one such m, big enough to guarantee that $m > \frac{1}{r}$. Define $h'_n := h_n + x - y_m$ if $n \in N$, and $h'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h'_n) \in \mathcal{H}(\nu,\delta')$ and we can estimate

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_n\Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^{\infty}\lambda_n h_n, x_1,\dots,x_k\Big)\\
&= \sum_{n=1}^{\infty} \lambda_n\Big(f(y_m+h'_n) + \frac{1}{\nu_n}\|h_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^{\infty}\lambda_n h_n, x_1,\dots,x_k\Big)\\
&\geq \sum_{n=1}^{\infty} \lambda_n\Big(f(y_m+h'_n) + \frac{1}{\nu_n}\|h'_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^{\infty}\lambda_n h'_n, x_1,\dots,x_k\Big) - r - cr\\
&\geq I(y_m,\delta',w) - r - cr\\
&\geq \sum_{n=1}^{\infty} \lambda_n\Big(f\big(y_m + g_n(y_m,\delta',w,m)\big) + \frac{1}{\nu_n}\|g_n(y_m,\delta',w,m)\|\Big)\\
&+ p_{d,w}\Big(\sum_{n=1}^{\infty}\lambda_n g_n(y_m,\delta',w,m), v_1(y_m,\delta',w,m),\dots,v_k(y_m,\delta',w,m)\Big)\\
&- \frac{1}{m} - r - cr \geq I(x,\delta,w,\mathcal{Y}) - 3r - 2cr,
\end{aligned} \tag{4.9}$$

by (4.6) and (4.8). Since $r \in \mathbb{Q}_+$ could be arbitrarily small, this proves that $I(x,\delta,w) \geq I(x,\delta,w,\mathcal{Y})$. Therefore, $\mathcal{R}$ is cofinal in $\mathcal{S}(X \times X_1 \times \cdots \times X_k)$.

To prove that $\mathcal{R}$ is σ-complete, we have to somewhat elaborate on the reasoning above. Let $\mathcal{Y}_1 = Y_1 \times Y_1^1 \times \cdots \times Y_1^k$, $\mathcal{Y}_2 = Y_2 \times Y_2^1 \times \cdots \times Y_2^k$, $\dots$, be an increasing sequence of elements of $\mathcal{R}$. Put $\mathcal{Y} := Y \times Y^1 \times \cdots \times Y^k$ where

$$Y := \overline{Y_1 \cup Y_2 \cup \cdots}, \quad Y^1 := \overline{Y_1^1 \cup Y_2^1 \cup \cdots}, \quad \dots, \quad Y^k := \overline{Y_1^k \cup Y_2^k \cup \cdots}.$$

We have to show that $\mathcal{Y}$ belongs to $\mathcal{R}$. This means to verify (4.4) and (4.5).

The proof of (4.4) is straightforward. As regards (4.5), fix some $x \in Y$, $\delta \in \Delta \cap \mathbb{Q}^\omega$ and $w = (w_1,\dots,w_k) \in S_{Y^1} \times \cdots \times S_{Y^k}$. We have to prove that $I(x,\delta,w) \geq I(x,\delta,w,\mathcal{Y})$. Because the assignment $u \mapsto p_{d,u}(\cdots)$ is uniformly continuous, we may and do assume that $w \in S_{Y_j^1} \times \cdots \times S_{Y_j^k}$ for some $j \in \mathbb{N}$. Now, take any $(h_n) \in \mathcal{H}(\nu,\delta)$ and any $(x_1,\dots,x_k) \in X_1 \times \cdots \times X_k$. Let again $N := \{n \in \mathbb{N} :\ \lambda_n > 0\}$; this is a finite set. Take an arbitrary $r \in \mathbb{Q}_+$ so small that $\|h_n\| < \delta_{\nu_n} - 2r$ for every $n \in N$. Find then $\delta' = (\delta'_n) \in \Delta \cap \mathbb{Q}^\omega$ such that $\delta'_n \leq \delta_n$ for every $n \in \mathbb{N}$ and $\delta'_n = \delta_n - r$ if $n \in N$. Take $m \in \mathbb{N}$ so big that $m > j$ and $\operatorname{dist}(x, Y_m) < r$; pick then $y_m \in Y_m$ so that $\|x - y_m\| < r$. Define $h'_n := h_n + x - y_m$ if $n \in N$, and $h'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h'_n) \in \mathcal{H}(\nu,\delta')$ and from the first half of (4.9) (valid also now) we have

$$\begin{aligned}
r + cr + \sum_{n=1}^{\infty} \lambda_n\Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^{\infty}\lambda_n h_n, x_1,\dots,x_k\Big)\\
\geq I(y_m,\delta',w) = I(y_m,\delta',w,\mathcal{Y}_m) \geq I(y_m,\delta',w,\mathcal{Y})
\end{aligned} \tag{4.10}$$

since $y_m \in Y^m$, $\mathcal{Y}_m \in \mathcal{R}$, and $\mathcal{Y}_m \subset \mathcal{Y}$.

Now, consider any $(k_n) \in \mathcal{H}(\nu, \delta') \cap Y^\omega$ and any $(x'_1, \dots, x'_k) \in Y_1 \times \dots \times Y_k$. Set $k'_n := k_n + y_m - x$ if $n \in N$, and $k'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(k'_n) \in \mathcal{H}(\nu, \delta) \cap Y^\omega$ and we can estimate

$$\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m + k_n) + \frac{1}{\nu_n} \|k_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k_n, x'_1, \dots, x'_k \Big) \\
= \;& \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k'_n) + \frac{1}{\nu_n} \|k_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k_n, x'_1, \dots, x'_k \Big) \\
\geq \;& \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k'_n) + \frac{1}{\nu_n} \|k'_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k'_n, x'_1, \dots, x'_k \Big) - r - cr \\
\geq \;& I(x, \delta, w, \mathcal{Y}) - r - cr.
\end{aligned}$$

Hence $I(y_m, \delta', w, \mathcal{Y}) \geq I(x, \delta, w, \mathcal{Y}) - r - cr$. Therefore, combining this inequality with (4.10), and recalling that $r \in \mathbb{Q}_+$ was arbitrarily small, we conclude that $I(x, \delta, w) \geq I(x, \delta, w, \mathcal{Y})$. This verifies (4.5) for our $\mathcal{Y}$ and hence guarantees that $\mathcal{Y} \in \mathcal{R}$. We proved that $\mathcal{R}$ is σ-complete. $\square$

5 Umbrella theorem for separable reduction of many statements dealing with ∂_F

We can now state and prove one of the main results of the whole text.

Theorem 5.1. *Let $k \in \mathbb{N}$, let $X, X_1, \ldots, X_k$ be* `general` *Banach spaces, let $A_i : X_i \to X$, $i = 1, \ldots, k$, be bounded linear operators, and let f be a proper extended real-valued function on X. Let finally $c \geq 0$, $\varepsilon_1 > 0, \ldots, \varepsilon_k > 0$, $\rho_1 \geq 0, \ldots, \rho_k \geq 0$ be given constants. Then there exists a rich block-family $\mathcal{R} \subset \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ such that for every $Y \times Y_1 \times \cdots \times Y_k \in \mathcal{R}$ we have $A_1(Y_1) \subset Y, \ldots, A_k(Y_k) \subset Y$, and for every $x \in Y$ the following holds:*
There is an $x^* \in \partial_F f(x)$ such that $\|x^*\| \leq c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$, $i = 1, \ldots, k$, whenever there is a $y^* \in \partial_F(f|_Y)(x)$ such that $\|y^*\| \leq c$ and $\big|\|(A_i|_{Y_i})^* y^*\| - \rho_i\big| < \varepsilon_i$, $i = 1, \ldots, k$.

Proof. Put $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_k)$, $\rho := (\rho_1, \ldots, \rho_k)$, and $d := (c, \varepsilon, \rho)$. For every $\lambda \in \Lambda \cap \mathbb{Q}^\omega$ and every $\nu \in \Upsilon$ let $\mathcal{R}_{\lambda,\nu,d}$ be the corresponding rich family from Proposition 4.3. As there are countably many such λ and ν, the intersection $\mathcal{R}$ of all such families over λ and ν is also a rich family by Proposition 2.2. This is precisely the family we need.

Indeed, take any $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k \in \mathcal{R}$. Take any $x \in Y$ and assume that there is $y^* \in \partial_F(f|_Y)(x)$, with $\|y^*\| \leq c$ and $\big|\|(A_i|_{Y_i})^* y^*\| - \rho_i\big| < \varepsilon_i$, $i = 1, \ldots, k$. By Proposition 4.2, there are $\varepsilon'_i \in (0, \varepsilon_i) \cap \mathbb{Q}$, $i = 1, \ldots, k$, $w \in S_{Y_1} \times \cdots \times S_{Y_k}$ and $\delta \in \Delta \cap \mathbb{Q}^\omega$ such that, when putting $d' := (c, (\varepsilon'_1, \ldots, \varepsilon'_k), \rho)$, we have $I(x, \lambda, \nu, \delta, d', w, \mathcal{Y}) \geq f(x)$ for every $\lambda \in \Lambda$ and for every $\nu \in \Upsilon$. But then, by the definition of our $\mathcal{R}$ and by (7.1), we have that $I(x, \lambda, \nu, \delta, d', w) \geq f(x)$ for every $\lambda \in \Lambda \cap \mathbb{Q}^\omega$ and $\nu \in \Upsilon$. Applying again Proposition 4.2, we conclude that there exists $x^* \in \partial_F f(x)$, with $\|x^*\| \leq c$ and $\big|\|(A_i|_{Y_i})^* x^*\| - \rho_i\big| < \varepsilon_i$ for every $i = 1, \ldots, k$. □

All the results to follow are consequences of the theorem above. It is suitable for separable reductions of various statements on Fréchet subdifferential of one function. As a very particular case of it we get the existence of a rich family of separable subspaces that guarantees separable reduction of the non-emptiness of Fréchet subdifferential. But Theorem 5.1 allows to say more.

Corollary 5.2. *Given a Banach space X, a proper function $f : X \longrightarrow (-\infty, +\infty]$, and constants $0 \leq \delta < c$, then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X)$ such that $\delta < \|x^*\| < c$ for some $x^* \in \partial_F f(x)$ whenever $Y \in \mathcal{R}$, $x \in Y$, and $\delta < \|y^*\| < c$ for some $y^* \in \partial_F(f|_Y)f(x)$.*

Proof. Let $k := 1$, $X_1 := X$, and let A_1 be the identity operator on X. For every $\varepsilon_1 > 0$, $\rho_1 > 0$ (and our given c) let $\mathcal{R}_{\varepsilon_1,\rho_1}$ be the corresponding rich block-family in $\mathcal{S}(X \times X)$ found in Theorem 5.1. Put $\mathcal{R}_0 := \bigcap \big\{\mathcal{R}_{\varepsilon_1,\rho_1} : \ \varepsilon_1, \rho_1 \in \mathbb{Q}_+\big\}$; this is again a rich block-family in $\mathcal{S}(X \times X)$ by Proposition 2.2. Put $\mathcal{R}_1 := \{Y \times Y : \ Y \in \mathcal{S}(X)\}$; clearly this is a rich family in $\mathcal{S}(X \times X)$. Put $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; this is a rich family by Proposition 2.2. Define finally $\mathcal{R} := \{Y \in \mathcal{S}(X) : \ Y \times Y \in \mathcal{R}_2\}$; it is easy to show that this is a rich family in $\mathcal{S}(X)$.

It remains to verify that this $\mathcal{R}$ "works". So take any Y in it, any $x \in Y$, and assume there is $y^* \in \partial_F(f|_Y)(x)$ satisfying that $\delta < \|y^*\| < c$. Find $\varepsilon, \rho \in \mathbb{Q}_+$ such that

$\delta < \rho - \varepsilon < \|y^*\| < \rho + \varepsilon < c$. By Theorem 5.1, as $Y \in \mathcal{R}_{\varepsilon,\rho}$, there is $x^* \in \partial_F f(x)$ such that $\|x^*\| \le c$ and $\big|\|x^*\| - \rho\big| < \varepsilon$. It then follows that $\delta < \rho - \varepsilon < \|x^*\| < \rho + \varepsilon < c$. $\square$

If the f is an indicator function of a closed subset Ω of X, then we get separable reduction (via a rich family) of non-zeroness of the Fréchet normal cone of Ω.

We can make a one step further and apply Theorem 5.1 to get the existence of rich families for separable reduction of Fréchet subdifferentiability of composite functions obtained by means of one or another functional operation with various quantitative requirements on elements of Fréchet subdifferentials. The following umbrella theorem is a gateway to many results of this sort.

Theorem 5.3. *Let $m \in \mathbb{N}$, let $Z, Z_1, \dots, Z_m$ be Banach spaces, and let constants $c \ge 0$, $\gamma > 0$, $\varepsilon_i > 0$, $\rho_i \ge 0$, proper functions $f_i : Z_i \longrightarrow (-\infty, +\infty]$, and linear bounded operators $\Lambda_i : Z \to Z_i$, $i = 1, \dots, m$, be given. Then there exists a rich block-family $\mathcal{R} \subset \mathcal{S}(Z \times Z_1 \times \cdots \times Z_m)$ such that for every $V \times V_1 \times \cdots \times V_m \in \mathcal{R}$ we have $\Lambda_1(V) \subset V_1, \dots, \Lambda_m(V) \subset V_m$, and for every $(z_1, \dots, z_m) \in V_1 \times \cdots \times V_m$, the following holds: There are $z_1^* \in \partial_F f_1(z_1), \dots, z_m^* \in \partial_F f_m(z_m)$ such that*

$$\textstyle\sum_{i=1}^m \|z_i^*\| \le c, \quad \|\sum_{i=1}^m \Lambda_i^* z_i^*\| < \gamma, \quad \big|\|\Lambda_i^* z_i^*\| - \rho_i\big| < \varepsilon_i, \;\; i = 1, \dots, m,$$

whenever there are $v_1^* \in \partial_F(f_1|_{V_1})(z_1), \dots, v_m^* \in \partial_F(f_m|_{V_m})(z_m)$ such that

$$\textstyle\sum_{i=1}^m \|v_i^*\| \le c, \quad \|\sum_{i=1}^m (\Lambda_i|_V)^* v_i^*\| < \gamma, \quad \big|\|(\Lambda_i|_V)^* v_i^*\| - \rho_i\big| < \varepsilon_i, \;\; i = 1, \dots, m.$$

Proof. Set $X := Z_1 \times \cdots \times Z_m$, and endow it with the ℓ_∞-norm, so that for $x = (z_1, \dots, z_m) \in X$ and $x^* = (z_1^*, \dots, z_m^*) \in X^*$ we have $\|x\| = \max\{\|z_1\|, \dots, \|z_m\|\}$ and $\|x^*\| = \|z_1^*\| + \cdots + \|z_m^*\|$. For every subspace U of Z we denote $\Delta U := \{(z, \dots, z) : \; z \in U\}$. Set further $X_0 := \Delta Z$, $X_1 := Z, \dots, X_m := Z$, and define operators $A_i : X_i \to X$, $i = 0, 1, \dots, m$, as follows: $A_0(z, \dots, z) := (\Lambda_1 z, \dots, \Lambda_m z)$ and, for $i = 1, \dots, m$, $A_i(z) := (0, \dots, 0, \Lambda_i z, 0, \dots 0)$ with $\Lambda_i z$ at the i-th place. An elementary calculation reveals that for $z_1^* \in Z_1^*, \dots, z_m^* \in Z_m^*$ we have

$$\|A_0^*(z_1^*, \dots, z_m^*)\| = \|\Lambda_1^* z_1^* + \cdots + \Lambda_m^* z_m^*\|; \quad \|A_i^*(z_1^*, \dots, z_m^*)\| = \|\Lambda_i^* z_i^*\|, \;\; i = 1, \dots, m. \tag{5.1}$$

More generally, if $V \in \mathcal{S}(Z)$, $V_i \in \mathcal{S}(Z_i)$, and $v_i^* \in V_i^*$, $i = 1, \dots, k$, we have

$$\big\|\big(A_0|_{\Delta V}\big)^*(v_1^*, \dots, v_m^*)\big\| = \big\|\big(\Lambda_1|_V\big)^* v_1^* + \cdots + \big(\Lambda_m|_V\big)^* v_m^*\big\| \tag{5.2}$$

$$\big\|\big(A_i|_{V_i}\big)^*(v_1^*, \dots, v_m^*)\big\| = \big\|\big(\Lambda_i|_V\big)^* v_i^*\big\|, \quad i = 1, \dots, m. \tag{5.3}$$

Let now $f : X \longrightarrow (-\infty, +\infty]$ be defined by

$$f(z_1, \dots, z_m) = f_1(z_1) + \cdots + f_m(z_m), \quad (z_1, \dots, z_m) \in X.$$

Clearly, this is a proper function. Moreover, this is a "separable" function, i.e., the sum of functions depending on mutually distinct arguments; so

$$\partial_F f(z_1, \dots, z_m) = \partial_F f_1(z_1) \times \cdots \times \partial_F f_m(z_m). \tag{5.4}$$

Finally, we put $\varepsilon_0 = \gamma$, $\rho_0 = 0$.

Let $\mathcal{R}_0 \subset \mathcal{S}(X \times X_0 \times X_1 \times \cdots \times X_m)$ be the rich block-family found in Theorem 5.1 for our constants, $c,\ \varepsilon_i,\ \rho_i,\ i = 0, 1, \ldots, m$, and for our operators $A_0, A_1, \ldots, A_m$. Consider the block-family

$$\mathcal{R}_1 := \big\{V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V :\ V_1 \in \mathcal{S}(Z_1), \ldots, V_m \in \mathcal{S}(Z_m),\ V \in \mathcal{S}(Z)\big\};$$

clearly, it is rich in $\mathcal{S}\big(X \times X_0 \times X_1 \times \cdots \times X_m\big)$. Put $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; it is also rich by Proposition 2.2. Finally, put

$$\mathcal{R} := \big\{V \times V_1 \times \cdots \times V_m :\ V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V \in \mathcal{R}_2\big\};$$

this block-family is also rich, now in $\mathcal{S}\big(Z \times Z_1 \times \cdots \times Z_m\big)$.

We shall show that $\mathcal{R}$ has the desired properties. So, fix any $V \times V_1 \times \cdots \times V_m \in \mathcal{R}$. Then $V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V \in \mathcal{R}_0$. Now, apply Theorem 5.1 where we plug $k := m + 1,\ Y := V_1 \times \cdots \times V_m,\ Y_0 := \Delta V,\ Y_1 := V,\ \ldots,\ Y_m := V$, and get that $A_0(\Delta V) \subset V_1 \times \cdots \times V_m,\ A_1(V) \subset V_1 \times \cdots \times V_m, \ldots,\ A_m(V) \subset V_1 \times \cdots \times V_m$. Thus, using the definition of A_i's, we get that $\Lambda_1(V) \subset V_1, \ldots,\ \Lambda_m(V) \subset V_m$.

Take now any $x = (z_1, \ldots, z_m) \in V_1 \times \cdots \times V_m$. Then the statement: "there is an $x^* \in \partial_F f(x)$ such that $\|x^*\| \le c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ for $i = 0, 1, \ldots, m$" means, by (5.1), that $x^* = (z_1^*, \ldots, z_m^*)$ for some $z_i^* \in \partial_F f_i(z_i)$ and $\|z_1^*\| + \cdots + \|z_m^*\| \le c$, $\|\Lambda_1^* z_1^* + \cdots + \Lambda_m^* z_m^*\| < \varepsilon_0 = \gamma$, and $\big|\|\Lambda_i^* z_i^*\| - \rho_i\big| < \varepsilon_i$, $i = 1, \ldots, m$.

Likewise, the statement: "there is $v^* \in \partial_F\big(f|_{V_1 \times \cdots \times V_m}\big)(x)$ such that $\|v^*\| \le c$ and $\big|\|\big(A_i|_V\big)^* v^*\| - \rho_i\big| < \varepsilon_i$ for $i = 0, \ldots, m$" means by (5.2) and (5.3), that $v^* = (v_1^*, \ldots, v_m^*)$ for some $v_i^* \in \partial_F\big(f_i|_{V_i}\big)(z_i)$, $\|v_1^*\| + \cdots + \|v_m^*\| \le c$, $\|(\Lambda_1|_V)^* v_1^* + \cdots + (\Lambda_m|_V)^* v_m^*\| < \varepsilon_0 = \gamma$ and $\big|\|(\Lambda_i|_V)^* v_i^*\| - \rho_i\big| < \varepsilon_i,\ i = 1, \ldots, m$.

Now, by Theorem 5.1, the first statement holds at $x = (z_1, \ldots, z_m) \in V_1 \times \cdots \times V_m$ if the second statement holds at the point. This completes the proof. □

As consequences of Theorem 5.3, we get quantitative versions of separable reductions (via suitable rich families) for a fuzzy calculus and an extremal principle for Fréchet subdifferentials and Fréchet normal cones, respectively. In the following corollaries we consider (as simple examples) the operations of composition with a linear operator and sum of functions.

Corollary 5.4. *Let X and Y be Banach spaces, let f be a proper function on Y, let $A : X \to Y$ be a bounded linear operator, and let $x^* \in X^*$. Given an $\varepsilon > 0$ and $c > \|x^*\|$, then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X \times Y)$ such that for every $U \times V \in \mathcal{R}$ we have $A(U) \subset V$ and for every $y \in V$ the following holds:*
There is $y^* \in \partial_F f(y)$ such that $\|y^*\| + \|x^*\| \le c$ and $\|A^* y^* - x^*\| < \varepsilon$ whenever there is $v^* \in \partial_F(f|_V)(y)$ such that $\|v^*\| + \big\|x^*|_U\big\| \le c$ and $\big\|(A|_U)^* v^* - x^*|_U\big\| < \varepsilon$.

Proof. Applying Theorem 5.3 to $m := 2,\ \gamma := \varepsilon$, to any $\varepsilon_1 > 0, \varepsilon_2 > 0, \rho_1 > 0, \rho_2 > 0$, and to $Z := X,\ Z_1 := Y,\ Z_2 := X,\ f_1 := f,\ f_2 := -x^*,\ A_1 := A$, to A_2 being the identity operator on Z_2, we get a rich block-family $\mathcal{R}_{\varepsilon, \varepsilon_2, \rho_1, \rho_2} \in \mathcal{S}(X \times Y \times X)$. Put then $\mathcal{R}_0 := \bigcap \big\{\mathcal{R}_{\varepsilon_1, \varepsilon_2, \rho_1, \rho_2} :\ \varepsilon_1, \varepsilon_2, \rho_1, \rho_2 \in \mathbb{Q}_+\big\}$; this is a rich block-family. Further put $\mathcal{R}_1 := \{U \times V \times U :\ U \in \mathcal{S}(X),\ V \in \mathcal{S}(Y)\}$ and then $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; we again got a rich block-family. Finally, define $\mathcal{R} := \{U \times V :\ U \times V \times U \in \mathcal{R}_2\}$; it is easy to check that this is also a rich block-family. Now, the verification that our $\mathcal{R}$ has the desired properties is routine. □

As an immediate consequence, we get a statement on the Fréchet subdifferential of composition with a linear operator.

Proposition 5.5. *In addition to the assumptions of Corollary 5.4, suppose that Y is an* `Asplund` *space, f is a lower semicontinuous proper function on Y and $x^* \in \partial_F(f \circ A)(x)$. Then for any $\varepsilon > 0$ there are $y \in Y$ and $y^* \in \partial_F f(y)$ such that $\|y - Ax\| < \varepsilon$ and $\|A^* y^* - x^*\| < \varepsilon$.*

Proof. In view of the preceding corollary, we only need to verify that the result is true if Y is separable. To this end we have to take into account that in a separable Asplund space there is an equivalent Fréchet smooth norm (see [DGZ, pages 48, 43] or the proof of Proposition 1.2) and then use the standard arguments based on minimization of the function

$$X \times Y \ni (u, y) \longmapsto f(y) + r\|y - Au\|^2 + \|u - x\|^2 - \langle x^*, u - x\rangle$$

with sufficiently large r. $\square$

The second consequence of Theorem 5.3 is related to sums of functions.

Corollary 5.6. *Let Z be a Banach space, consider constants $c \geq 0$, $\varepsilon > 0$, $\rho_1 \geq 0, \ldots, \rho_m \geq 0$, and let proper functions $f_i : Z \longrightarrow (-\infty, +\infty]$, $i = 1, \ldots, m$, be given. Then there exists a rich family $\mathcal{R} \in \mathcal{S}(Z)$ such that for every $V \in \mathcal{R}$ and every $z_1, \ldots, z_m \in V$ the following holds: There are $z_1^* \in \partial_F f_1(z_1), \ \ldots, \ z_m^* \in \partial_F f_m(z_m)$ such that*

$$\|z_1^*\| + \cdots + \|z_m^*\| \leq c, \quad \|z_1^* + \cdots + z_m^*\| < \varepsilon, \text{ and } \big|\|z_i^*\| - \rho_i\big| < \varepsilon, \ i = 1, \ldots, m$$

whenever there are $v_1^* \in \partial_F(f_1|_V)(z_1), \ \ldots, \ v_1^* \in \partial_F(f_1|_V)(z_1)$ such that

$$\|v_1^*\| + \cdots + \|v_m^*\| \leq c, \quad \|v_1^* + \cdots + v_m^*\| < \varepsilon, \text{ and } \big|\|v_i^*\| - \rho_i\big| < \varepsilon, \ i = 1, \ldots, m$$

Proof. Apply Theorem 5.3, with $Z_1 := \cdots = Z_m := Z$, Λ_i being identities and $\gamma := \varepsilon_1 := \cdots = \varepsilon_m := \varepsilon$, and get a rich block-family $\mathcal{R}_0 \subset \mathcal{S}(Z^{m+1})$. Using a simple gymnastics like in the proof of Corollary 5.4, we produce a rich family $\mathcal{R}$ in $\mathcal{S}(Z)$ with the desired property. $\square$

The latter corollary, in turn, provides a direct access to the fuzzy sum rule in Asplund spaces which, in the simplest form, is stated as follows.

Proposition 5.7. *Let X be an* `Asplund` *space, and let f_1 and f_2 be two lower semicontinuous functions on X, with one of them being Lipschitz, or at least uniformly continuous, near a certain $x \in X$. If $x^* \in \partial_F(f + g)(x)$, then for any $\varepsilon > 0$ there are $x_1, x_2 \in X$ and $x_1^* \in \partial_F f_1(x_1)$, $x_2^* \in \partial_F f_2(x_2)$, such that $\|x_1 - x\| < \varepsilon$, $\|x_2 - x\| < \varepsilon$, and $\|x_1^* + x_2^* - x^*\| < \varepsilon$.*

Proof. If X is separable, first find an equivalent Fréchet smooth norm (see [DGZ, pages 48, 43] or the proof of Proposition 1.2) and then proceed as in [I1]. Further assume that X is non-separable. Put $Z := X$, $n := 2$, $f_1 := f$, $f_2 := g - x^*$, $\varepsilon_1 := \varepsilon_2 := \varepsilon$, and let $c, \rho_1, \rho_2 \in \mathbb{Q}_+$ be any. For these data find the corresponding rich family $\mathcal{R}_{c,\rho_1,\rho_2}$ by Corollary 5.6. Put $\mathcal{R} := \bigcap \big\{\mathcal{R}_{c,\rho_1,\rho_2} : \ c, \rho_1, \rho_2 \in \mathbb{Q}_+\big\}$; this family is again rich. Find $V \in \mathcal{R}$ so that $V \ni x$. From the separable case find $x_1, \ x_2 \in V$ and $v_1^* \in \partial_F(f|_V)(x_1)$, $v_2^* \in \partial_F\big(g|_V - x^*|_V\big)(x_2)$ such that $\|x_1 - x\| < \varepsilon$, $\|x_2 - x\| < \varepsilon$, and $\|v_1^* + v_2^*\| < \varepsilon$. Now, applying Corollary 5.6, we get the result. $\square$

Remark 5.8. Performing intersections of countably many suitable rich families in $\mathcal{S}(Z)$ we can replace the conclusion of Corollary 5.6 by

Then there exists a rich family $\mathcal{R} \in \mathcal{S}(Z)$ such that for every $V \in \mathcal{R}$, every $\varepsilon > 0$, and every $z_1, \ldots, z_m \in V$ the following holds: There are $z_1^* \in \partial_F f_1(z_1)$, $\ldots$, $z_m^* \in \partial_F f_m(z_m)$ such that

$$\|z_1^* + \cdots + z_m^*\| < \varepsilon \quad \textit{and} \quad 1 - \varepsilon < \|z_1^*\| + \cdots + \|z_m^*\| < 1 + \varepsilon$$

whenever there are $v_1^* \in \partial_F(f_1|_V)(z_1)$, $\ldots$, $v_1^* \in \partial_F(f_1|_V)(z_1)$ such that

$$\|v_1^* + \cdots + v_m^*\| < \varepsilon \quad \textit{and} \quad \|v_1^*\| + \cdots + \|v_m^*\| = 1.$$

Amazing, isn't it? We believe that this observation will be appreciated by sympathizers with extremal principles, see [M, Chapter 2].

6 Rich families in Asplund spaces

Material in this and next section comes from the forthcoming paper [CF2]. First we shall present a structural result characterizing Asplund spaces, which will prove to be useful later and could be of broad interest.

Let P be a set and let $\prec$ be a *partial order* on it, i.e. $\prec$ is a subset of $P \times P$ which is reflexive, symmetric and transitive, see [E, page 21]. We agree that, instead of "$s, t \in \prec$" we rather write "$s \prec t$". Assume moreover that P is *(up)-directed by* $\prec$, i.e., for every $t_1, t_2 \in P$ there is $t_3 \in P$ such that $t_1 \prec t_3$ and $t_2 \prec t_3$. A subset $R \subset P$ is called *cofinal/dominating/saturating* if for every $t \in P$ there is $r \in P$ such that $t \prec r$. R is called *σ-complete/closed* if, whenever $r_1 \prec r_2 \prec \cdots$ is an increasing sequence in R, then there is $r \in R$ such that $r_i \prec r$ for every $i \in \mathbb{N}$ and $r \prec t$ whenever $t \in P$ and $r_i \prec t$ for every $i \in \mathbb{N}$. The set $R \subset P$ is called *rich/a club set* if it is both cofinal and σ-complete. (Note that the whole P is rich if it is σ-complete.)

Now, we are ready to provide concrete examples of the poset $(P, \prec)$ that emerge in the framework of Banach spaces. Let Z be a (rather non-separable) Banach space. By $\mathcal{S}(Z)$ we denote the family of all separable closed subspaces of Z and we endow it by the partial order "$\subset$". Thus, we can consider rich families in the poset $(\mathcal{S}(Z), \subset)$. We then also simply say that they are rich in Z. Now, let X be a Banach space and apply the above to the product $Z := X \times X^*$. Then we can speak about *rectangle-families* lying in $\mathcal{S}_{\sqsubset\!\sqsupset}(X \times X^*)$. (The definition of the latter symbol is left to the fantasy of a reader, if there is any.) More generally, let $k \in \mathbb{N}$ be greater that 1, and let $X_1, \ldots, X_k$ be Banach spaces. By a *block* we understand any product $Y_1 \times \cdots \times Y_k$. The symbol $\mathcal{S}_{\boxplus}(X_1 \times \cdots \times X_k)$ will denote the (rich) family of all blocks $Y_1 \times \cdots \times Y_k$ such that $Y_1 \in \mathcal{S}(X_1), \ \ldots, \ Y_k \in \mathcal{S}(X_k)$. Any subset of $\mathcal{S}_{\boxplus}(X_1 \times \cdots \times X_k)$ will be called a *block-family* in $\mathcal{S}(X_1 \times \cdots \times X_k)$ or just in $X_1 \times \cdots \times X_k$.

We conclude by one warning. If $\mathcal{R}$ is a rich rectangle-family in $\mathcal{S}_{\sqsubset\!\sqsupset}(X \times X^*)$, we do not know if, the "projection" of it on, say, the second coordinate, that is, the family $\{A_2 : \ A_1 \times A_2 \in \mathcal{R}$ for some $A_1 \in \mathcal{S}(X)\}$ is rich in $\mathcal{S}(X^*)$. Fortunately, in one important case, the "projection" of $\mathcal{R}$ to the first coordinate is again rich; see Theorems 6.2 and 7.1 below.

The power of rich families in Banach spaces is demonstrated by the fundamental Proposition 2.2 (see also [BM] and [LPT, page 37]) saying that they are stable under countable intersections.

Let X be a Banach space. If $A \subset X$, the symbols $\overline{\mathrm{sp}}\, A$ and $\mathrm{sp}_{\mathbb{Q}} A$ mean the closed linear span of A and the set consisting of all finite linear combinations of elements in A with rational coefficients, respectively. For $A \subset X$ and $B \subset X^*$ we put $B|_A := \{x^*|_A : x^* \in B\}$; hence, if A is a subspace of X, then $B|_A$ is a subset of the dual space A^*. Let $\mathcal{C}(X)$ and $\mathcal{C}(X^*)$ denote the families of all countable subsets of X and X^* respectively.

A Banach space is called *Asplund* if every convex continuous function on it is Fréchet differentiable at a point (equivalently, at the points of a dense set, yet equivalently, at the points of a dense G_δ set). An important, and widely used, equivalent condition for the Asplund property of a Banach space is that *every separable subspace of it has separable dual,* see [Ph, Theorem 2.34].

Now, we introduce a concept which serves as a link between X and X^* (and exists right if and only if X is Asplund).

Definition 6.1. By an *Asplund generator* in a Banach space X we understand any correspondence $G : \mathcal{C}(X) \longrightarrow \mathcal{C}(X^*)$ such that
(a) $\left(\overline{\mathrm{sp}}\, C\right)^* \subset \overline{G(C)|_{\overline{\mathrm{sp}}\, C}}$ for every $C \in \mathcal{C}(X)$;
(b) if $C_1,\ C_2,\ \ldots$ is an increasing sequence in $\mathcal{C}(X)$, then $G(C_1 \cup C_2 \cup \cdots) = G(C_1) \cup G(C_2) \cup \cdots$;
(c) $\bigcup\{G(C) :\ C \in \mathcal{C}(X)\}$ is a dense subset in X^*; and
(d) if $C_1, C_2 \in \mathcal{C}(X)$ are such that $\overline{\mathrm{sp}}\, C_1 = \overline{\mathrm{sp}}\, C_2$, then $\overline{\mathrm{sp}}\, G(C_1) = \overline{\mathrm{sp}}\, G(C_2)$.

In the next section, we shall frequently profit from the following basic structural statement.

Theorem 6.2. *Let $(X, \|\cdot\|)$ be a (rather non-separable) Banach space. Then the following assertions are mutually equivalent.*

(i) X is an Asplund space.

(ii) X admits an Asplund generator.

(iii) There exists a rich rectangle-family $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ such that $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2 \in \mathcal{A}$ and $V_1 \subset V_2$, and for every $V \times Y \in \mathcal{A}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry.

(iv) There exists a cofinal rectangle-family $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ such that for every $V \times Y \in \mathcal{A}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjection.

Proof. (i)$\Longrightarrow$(ii). In order not to get lost in the case of general Asplund space, assume first that the norm $\|\cdot\|$ on X is Fréchet smooth, or more generally, that there exists a smooth function $f : X \to \mathbb{R}$, with continuous derivative f' such that $f'(V)|_V$ is dense in V^* for every subspace V of X; note that this easily implies that X is Asplund. Define then $G : \mathcal{C}(X) \longrightarrow \mathcal{C}(X^*)$ by

$$\mathcal{C}(X) \ni C \longmapsto f'\big(\mathrm{sp}_{\mathbb{Q}} C\big) =: G(C) \in \mathcal{C}(X^*).$$

It remains to verify the properties (a), (b), (c), and (d) in Definition 6.1. As regards (a), fix any $C \in \mathcal{C}(X)$ and any non-zero v^* in $(\overline{\mathrm{sp}}\, C)^*$. Let any $\varepsilon > 0$ be given. The properties of f provide a $v \in \mathrm{sp}_{\mathbb{Q}}\, C$ such that $\big\|v^* - f'(v)|_{\overline{\mathrm{sp}}\, C}\big\| < \varepsilon$. But $f'(v)$ belongs to $G(C)$. And, as $\varepsilon > 0$ was arbitrary, we get that v^* belongs $\overline{G(C)|_{\overline{\mathrm{sp}}\, C}}$. Thus (a) is verified. As regards (b), let $C_1, C_2, \ldots$ be as in the premise. Because our generator G is "monotone", it is enough to prove the inclusion "$\subset$". So, pick any x^* in $G(C_1 \cup C_2 \cup \cdots)$. Since $C_1 \subset C_2 \subset \cdots$, there is $m \in \mathbb{N}$ so big that x^* belongs to $G(C_m)$. We thus verified (b). The claim (c) follows immediately from the fact that $f'(X)$ is dense in X^* and from the definition of G. The last property (d) is guaranteed by the continuity of f'.

If we are facing a general Asplund space (and we do not have at hand a function as above), we must work harder. Either, we use [CF1, Propositions 1 and 2], based on Ch. Stegall's ideas (and proved without use of Simons' lemma), or we exploit an information from [FG]; see also [CF1, Remark 2] (where Simons' lemma is needed!). More concretely, using symbols Λ and $\mathcal{L}$ from [CF1], define a generator $G : \mathcal{C}(X) \longrightarrow \mathcal{C}(X^*)$ by

$$\mathcal{C}(X) \ni C \longmapsto \Lambda\big(\mathcal{L}\big(\mathrm{sp}_{\mathbb{Q}}\, C \cap B_X\big)\big) =: G(C) \in \mathcal{C}(X^*).$$

Now (a) in Definition 6.1 follows from [CF1, Proposition 1] and the proof of it. (We actually get a stronger inclusion that $\big(\overline{\mathrm{sp}}\, C\big)^* \subset \overline{G(C)}\,|_{\overline{\mathrm{sp}}\, C}$.) (b) follows immediately from the very definition of our G, the definition of $\Lambda, \mathcal{L}$, and from the monotonicity of the sequence $C_1, C_2, \ldots$ (c) follows immediately from [CF1, Proposition 1]. (d) follows easily from the properties of Λ and $\mathcal{L}$ and from the definition of G. We thus proved that (ii) holds in a general Asplund space.

(ii)$\Longrightarrow$(iii). Let $G : \mathcal{C}(X) \longrightarrow \mathcal{C}(X^*)$ be a generator in X. Define $\mathcal{A} \subset \mathcal{S}_{\Box}(X \times X^*)$ as the family consisting of all "rectangles" $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C)$, with $C \in \mathcal{C}(X)$, such that the assignment

$$\overline{\mathrm{sp}}\, G(C) \ni x^* \longmapsto x^*|_{\overline{\mathrm{sp}}\, C} \in (\overline{\mathrm{sp}}\, C)^* \tag{6.1}$$

is a surjective isometry. We shall show that $\mathcal{A}$ is a rich family.

As regards the cofinality of $\mathcal{A}$, fix any $V \times Y \in \mathcal{S}_{\Box}(X \times X^*)$. Since G is a generator, the condition (c) guarantees that there is $C_0 \in \mathcal{C}(X)$ so big that $\overline{C_0} \supset V$ and $\overline{G(C_0)} \supset Y$. Assume that for some $m \in \mathbb{N}$ we already found countable sets $C_0 \subset C_1 \subset \cdots \subset C_{m-1} \subset X$. Realizing that $\mathrm{sp}_{\mathbb{Q}}\, G(C_{m-1})$ is countable, we find $C_m \in \mathcal{S}(X)$ such that $C_m \supset C_{m-1}$ and that $\|x^*\| = \sup \langle x^*, C_m \cap B_X\rangle$ for every $x^* \in \mathrm{sp}_{\mathbb{Q}}\, G(C_{m-1})$. Do so for every $m \in \mathbb{N}$ and put finally $C := C_0 \cup C_1 \cup \cdots$. Clearly $C \in \mathcal{C}(X)$ and also $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C) \supset V \times Y$. It remains to show that the assignment (6.1), with our just constructed C, is a surjective isometry.

Take any $x^* \in \mathrm{sp}_{\mathbb{Q}}\, G(C)$. Realizing that x^* is a linear combination of finitely many elements from $G(C)$ and that $C_0 \subset C_1 \subset \cdots$, the property (b) of G provides an $m \in \mathbb{N}$ so big that x^* belongs to $\mathrm{sp}_{\mathbb{Q}}\, G(C_{m-1})$. But then, from the construction above,

$$\|x^*\| = \sup \langle x^*, C_m \cap B_X\rangle \le \sup\langle x^*, \overline{\mathrm{sp}}\, C \cap B_X\rangle = \big\|x^*|_{\overline{\mathrm{sp}}\, C}\big\| \le \|x^*\|.$$

And, as $\overline{\mathrm{sp}}\, G(C) = \overline{\mathrm{sp}_{\mathbb{Q}}\, G(C)}$, we get that $\big\|x^*|_{\overline{\mathrm{sp}}\, C}\big\| = \|x^*\|$ for every $x^* \in \overline{\mathrm{sp}}\, G(C)$. We proved that the assignment (6.1) with our C is isometrical.

Now, fix any $v^* \in (\overline{\mathrm{sp}}\, C)^*$. By (a) from Definition 6.1, there is a sequence (x_n^*) in $G(C)$ so that $\big\|v^* - x_i^*|_{\overline{\mathrm{sp}}\, C}\big\| \longrightarrow 0$ as $n \to \infty$. By the isometric property of (6.1) just proved, we have that $\|x_i^* - x_j^*\| = \big\|x_i^*|_{\overline{\mathrm{sp}}\, C} - x_j^*|_{\overline{\mathrm{sp}}\, C}\big\| \longrightarrow 0$ as $i, j \to \infty$. Put $x^* := \lim_{i\to\infty} x_i^*$; then $x^* \in \overline{G(C)} \subset \overline{\mathrm{sp}}\, G(C)$ and $v^* = x^*|_{\overline{\mathrm{sp}}\, C}$. This shows the surjectivity of the assignment (6.1) with our C. This way, we proved that $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C)$ belongs to $\mathcal{A}$, and hence, the family $\mathcal{A}$ is cofinal.

For checking the σ-completeness of $\mathcal{A}$, consider any increasing sequence $V_1 \times Y_1,\ V_2 \times Y_2,\ \ldots$ of elements in $\mathcal{A}$. Then, clearly, $\overline{V_1 \times Y_1 \cup V_2 \times Y_2 \cup \cdots}$ is of form $V \times Y$ and this is an element of $\mathcal{S}_{\Box}(X \times X^*)$. Also, clearly, $V = \overline{V_1 \cup V_2 \cup \cdots}$ and $Y = \overline{Y_1 \cup Y_2 \cup \cdots}$. From the definition of $\mathcal{A}$, for every $i \in \mathbb{N}$ find $C_i \in \mathcal{C}(X)$ such that $V_i = \overline{\mathrm{sp}}\, C_i$ and $Y_i = \overline{\mathrm{sp}}\, G(C_i)$. Put $C := C_1 \cup C_2 \cup \cdots$; then $C \in \mathcal{C}(X)$. Since $V_1 \subset V_2 \subset \cdots$ and $Y_1 \subset Y_2 \subset \cdots$, some rather boring reasoning, profiting from the properties (b) and (d) of G in Definition 6.1, guarantees that $V = \overline{\mathrm{sp}}\, C$ and $Y = \overline{\mathrm{sp}}\, G(C)$. (Hint: Replace the sequence $C_1, C_2, \ldots$ by the increasing one $C_1,\ C_1 \cup C_2,\ C_1 \cup C_2 \cup C_3,\ \ldots$) Hence, by (a), $V^* \subset \overline{Y|_V}$.

It remains to verify that the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry. As regards the isometric property, we recall that for every $i \in \mathbb{N}$ the rectangle $V_i \times Y_i$ belongs to $\mathcal{A}$, and so for every $x^* \in Y_i$ we have

$$\|x^*\| = \big\|x^*|_{V_i}\big\| \leq \big\|x^*|_V\big\| \leq \|x^*\|.$$

It then follows, using the density of $Y_1 \cup Y_2 \cup \cdots$ in Y, that $\|x^*\| = \big\|x^*|_V\big\|$ for every $x^* \in Y$. Now, once having the information just proved, we have that $\overline{Y|_V} = Y|_V$ $(\subset V^*)$, and hence $V^* = Y|_V$. Therefore, summarizing all the above, we are sure that our $\mathcal{A}$ is a rich family.

Finally, consider any $V_1 \times Y_1,\ V_2 \times Y_2$ in $\mathcal{A}$ such that $V_1 \subset V_2$. From the very definition of $\mathcal{A}$ we find $C_1, C_2 \in \mathcal{C}(X)$ such that $\overline{\mathrm{sp}}\, C_1 = V_1$ and $\overline{\mathrm{sp}}\, C_2 = V_2$. Then

$$C_2 \subset C_1 \cup C_2 \subset \overline{\mathrm{sp}}\, C_1 \cup \overline{\mathrm{sp}}\, C_2 = V_1 \cup V_2 = V_2 = \overline{\mathrm{sp}}\, C_2,$$

and so $\overline{\mathrm{sp}}\, C_2 \subset \overline{\mathrm{sp}}\,(C_1 \cup C_2) \subset \overline{\mathrm{sp}}\, C_2$. Now (d) in Definition 6.1 gives that $\overline{\mathrm{sp}}\, G(C_2) = \overline{\mathrm{sp}}\, G(C_1 \cup C_2)$, and so

$$Y_2 = \overline{\mathrm{sp}}\, G(C_1 \cup C_2) \overset{(b)}{=} \overline{\mathrm{sp}}\, \big(G(C_1) \cup G(C_1 \cup C_2)\big) \supset \overline{\mathrm{sp}}\, G(C_1) = Y_1.$$

We completely proved (iii).

(iii)$\Longrightarrow$(iv) is trivial.

(iv)$\Longrightarrow$(i). Assume (iv) holds. Let $Z \in \mathcal{S}(X)$ be arbitrary. From the cofinality of $\mathcal{A}$, find $V \times Y \in \mathcal{A}$ such that $V \times Y \supset Z \times \{0\}$. Then V^*, being the image of Y $(\in \mathcal{S}(X^*))$, is itself separable. It then follows that Z^*, the quotient of V^*, must be also separable. Now it remains to use the aforementioned characterization of the Asplund property, and thus (i) follows. $\square$

Remark 6.3. Assume that the norm $\|\cdot\|$ on X is Fréchet smooth and define $f := \|\cdot\|^2$. Then for every subspace $V \subset X$ we get that $V^* \subset \overline{f'(V)|_V}$ but not $V^* \subset \overline{f'(V)}\,|_V$. Indeed, this stronger inclusion seems to be a privilege of only some V's; we can find them by playing a suitable "volleyball" with countably many steps, see the proof of (ii)$\Rightarrow$(iii) above. (Fortunately, these "selected/better" V's form a rich family in $\mathcal{S}(X)$.) From this, and from the proof of implication (i)$\Rightarrow$(ii) above, it follows that the Stegall's approach is somehow stronger, see [CF1, Proposition 1]. Likewise, the Stegall's approach is stronger than that from [FG], see [CF1, Remark 2].

It can be useful to extend Theorem 6.2 to the following statement.

Theorem 6.4. *Let $(Z, \|\cdot\|)$ be a Banach space, $(X, \|\cdot\|)$ an Asplund space, and $T : Z \to X$ a bounded linear operator. Then there exists a rich block-family $\mathcal{A}_T$ in $Z \times X \times X^*$ such that $Y_1 \subset Y_2$ whenever $U_1 \times V_1 \times Y_1,\ U_2 \times V_2 \times Y_2 \in \mathcal{A}_T$ and $U_1 \times V_1 \subset U_2 \times V_2$, and that for every $U \times V \times Y$ in $\mathcal{A}_T$ we have $T(U) \subset V$, the restriction assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry, and $\|T^* x^*\| = \big\|(T|_U)^*(x^*|_V)\big\|$ for every $x^* \in Y$.*

Proof. It is easy (and left to a reader) to check that the rectangle-family $\mathcal{R}_T$ consisting of all $U \times V \in \mathcal{S}_{\square}(Z \times X)$ such that $T(U) \subset V$ is rich in $Z \times X$. Denote

$$\mathcal{R}_1 := \big\{U \times V \times Y :\ U \times V \in \mathcal{R}_T \ \text{ and } \ Y \in \mathcal{S}(X^*)\big\},$$

$$\mathcal{R}_2 := \big\{U\times V\times Y:\ U\in\mathcal{S}(Z)\ \text{ and }\ V\times Y\in\mathcal{A}\big\}$$

where $\mathcal{A}$ is from Theorem 6.2. Clearly, both these families are rich, and therefore $\mathcal{R} := \mathcal{R}_1\cap\mathcal{R}_2$ is a rich block-family in $\mathcal{S}_{\boxminus}(Z\times X\times X^*)$. Clearly, every triple $U\times V\times Y$ in $\mathcal{R}$ possesses the first two properties from the conclusion of our theorem. Now, define the family

$$\mathcal{A}_T := \big\{U\times V\times Y\in\mathcal{R}:\ \|T^*x^*\| = \big\|(T|_U)^*(x^*|_V)\big\|\ \text{ for every }\ x^*\in Y\big\}.$$

This family has all the three required properties. Thus, it remains to check that $\mathcal{A}_T$ is rich.

As regards the cofinality of $\mathcal{A}_T$, consider any $M\in\mathcal{S}(Z\times X\times X^*)$. From the cofinality of $\mathcal{R}$, find $U_0\times V_0\times Y_0$ in $\mathcal{R}$ such that $U_0\times V_0\times Y_0\supset M$. We shall construct an increasing sequence $U_m\times V_m\times Y_m,\ m\in\mathbb{N}$, in $\mathcal{R}$ as follows. Let $m\in\mathbb{N}$ and assume that we have already found $U_{m-1}\times V_{m-1}\times Y_{m-1}$. Using the separability of Y_{m-1} find $C_{m-1}\subset\mathcal{C}(Z)$ such that $\overline{C_{m-1}}\supset U_{m-1}$ and $\|T^*x^*\| = \sup\langle T^*x^*, C_{m-1}\cap B_Z\rangle$ for every $x^*\in Y_{m-1}$. Find $U_m\times V_m\times Y_m$ in $\mathcal{R}$ so big that it contains $(U_{m-1}\cup C_{m-1})\times V_{m-1}\times Y_{m-1}$. Doing so for every $m\in\mathbb{N}$, put finally $U := \overline{\bigcup U_m}$, $V := \overline{\bigcup V_m}$, and $Y := \overline{\bigcup Y_m}$. Clearly, $U\times V\times Y = \overline{\bigcup U_m\times V_m\times Y_m}\supset M$. The σ-completeness of $\mathcal{R}$ guarantees that $U\times V\times Y$ lies in $\mathcal{R}$. Now fix any $m\in\mathbb{N}$ and any $x^*\in Y_{m-1}$. We can estimate

$$\begin{aligned}\|T^*x^*\| &= \sup\big\langle T^*x^*, C_{m-1}\cap B_Z\big\rangle \le \sup\big\langle T^*x^*, B_U\big\rangle = \sup\big\langle x^*, T(B_U)\big\rangle\\ &= \sup\big\langle (x^*|_V), (T|_U)(B_U)\big\rangle = \big\|(T|_U)^*(x^*|_V)\big\| \le \|T^*x^*\|\,.\end{aligned}$$

Thus $\|T^*x^*\| = \big\|(T|_U)^*(x^*|_V)\big\|$ for every x^* from $\bigcup Y_m$, and finally, for every x^* from Y. We verified $\mathcal{A}_T$ is cofinal.

As regards the σ-completeness of $\mathcal{A}_T$, consider any increasing sequence $U_1\times V_1\times Y_1,\ U_2\times V_2\times Y_2,\dots$ in $\mathcal{A}_T$. Put $U := \overline{\bigcup U_i}$, $V := \overline{\bigcup V_i}$, and $Y := \overline{\bigcup Y_i}$. Clearly, $U\times V\times Y = \overline{\bigcup U_i\times V_i\times Y_i}$. As $\mathcal{R}$ was rich, our $U\times V\times Y$ belongs to it. Take any $i\in\mathbb{N}$ and any $x^*\in Y_i$. We can estimate

$$\begin{aligned}\|T^*x^*\| &= \big\|(T|_{U_i})^*(x^*|_{V_i})\big\| = \sup\big\langle (T|_{U_i})^*(x^*|_{V_i}), B_{U_i}\big\rangle = \sup\big\langle x^*|_{V_i}, T(B_{U_i})\big\rangle\\ &\le \sup\big\langle x^*|_V, T(B_U)\big\rangle = \sup\big\langle x^*|_V, (T|_U)(B_U)\big\rangle = \big\|(T|_U)^*(x^*|_V)\big\| \le \|T^*x^*\|.\end{aligned}$$

Thus $\|T^*x^*\| = \big\|(T|_U)^*(x^*|_V)\big\|$ holds for every x^* from $\bigcup Y_i$, and finally for every x^* from Y. We proved that $\mathcal{A}_T$ is σ-complete.

The other properties of $\mathcal{A}_T$ follow from similar properties of $\mathcal{A}$ proclaimed in Theorem 6.2. □

Remark 6.5. Of course, Theorem 6.4 can be easily extended to several spaces $Z_1,\dots,Z_k$ and to operators $T_i: Z_i\to X,\ i=1,\dots,k$.

7 Separable reduction of Fréchet subdifferentiability in Asplund spaces

This section brings a new approach. The novelty is that, under the (small) price of restricting to the framework of Asplund spaces, for separable reductions of statements involving Fréchet subdifferentials `we do not need to translate these statements into terms of the primal space` X. This is a drastic simplification when comparing with the so far existing proofs; see [FI2]. In addition we get "isometric" statements, which again substantially improve those from [FI2].

Theorem 7.1. *Let $(X, \|\cdot\|)$ be a (rather non-separable) Asplund space and let $f : X \longrightarrow (-\infty, +\infty]$ be any proper function. Then there exists a rich rectangle-family $\mathcal{R} \subset \mathcal{S}_{\Box}(X \times X^*)$ such that $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subset V_2$, with further properties that for every $V \times Y \in \mathcal{R}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is an* `isometry` *from $Y|_V$ onto V^* and for every $v \in V$ we have that*

$$\big(\partial_F f(v) \cap Y\big)|_V = \big(\partial_F f(v)\big)|_V = \partial_F (f|_V)(v)\,.$$

Proof. We obviously have that

$$\big(\partial_F f(v) \cap Y\big)|_V \subset \big(\partial_F f(v)\big)|_V \subset \partial_F (f|_V)(v).$$

It remains to prove that $\partial_F(f|_V)(v) \subset \big(\partial_F f(v) \cap Y\big)|_V$ holds for every $v \in V$. For $x \in X,\ x^* \in X^*,\ r \in \mathbb{R},\ 0 < \delta_1 < \delta_2$, and $V \subset X$ we define

$$I_V(x, x^*, r, \delta_1, \delta_2) := \inf\left\{\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) :\ h \in V \ \text{ and } \ \delta_1 < \|h\| < \delta_2\right\};$$

if $V = X$, we omit the index V. Further for each such cortege $x, x^*, r, \delta_1, \delta_2$ and each $\gamma > 0$, if $I(x, x^*, r, \delta_1, \delta_2) > -\infty$, we find a vector $h(x, x^*, r, \delta_1, \delta_2, \gamma) \in X$ such that

$$\begin{aligned}&\frac{1}{\|h(x, x^*, r, \delta_1, \delta_2, \gamma)\|}\big(f(x + h(x, x^*, r, \delta_1, \delta_2, \gamma)) - r - \langle x^*, h(x, x^*, r, \delta_1, \delta_2, \gamma)\rangle\big)\\ &< I(x, x^*, r, \delta_1, \delta_2) + \gamma.\end{aligned} \tag{7.1}$$

Let $\mathcal{A} \subset \mathcal{S}_{\Box}(X \times X^*)$ be the rich family found in Theorem 6.2. We define a family $\mathcal{R}$ as that consisting of all $V \times Y \in \mathcal{A}$ satisfying

$$I(x, x^*, r, \delta_1, \delta_2) = I_V(x, x^*, r, \delta_1, \delta_2) \ \text{ whenever } \ x \in V,\ x^* \in Y,\ r \in \mathbb{R}, \ \text{ and } \ 0 < \delta_1 < \delta_2. \tag{7.2}$$

We shall prove that $\mathcal{R}$ is cofinal in $\mathcal{S}(X \times X^*)$. Let $\mathbb{Q}$ denote the set of all rational numbers and put $\mathbb{Q}_+ = \mathbb{Q} \cap (0, +\infty)$. Fix any $Z \in \mathcal{S}(X \times X^*)$. Since $\mathcal{A}$ is rich, there is $V_0 \times Y_0 \in \mathcal{A}$ such that $V_0 \times Y_0 \supset Z$. Find countable sets C_0, D_0 contained and dense in V_0 and Y_0, respectively. We shall construct increasing sequences $Y_0 \times V_0,\ V_1 \times Y_1,\ V_2 \times Y_2,\ \ldots$ in $\mathcal{A}$, and $C_0 \times D_0,\ C_1 \times D_1,\ C_2 \times D_2,\ \ldots$ in $\mathcal{C}_{\Box}(X \times X^*)$ such that $\overline{C_i} = V_i,\ \overline{D_i} = Y_i$ for every $i \in \mathbb{N}$, and having some extra properties described below. Let $m \in \mathbb{N}$ be arbitrary and assume that we have already found $V_{m-1}, Y_{m-1}, C_{m-1}, D_{m-1}$. From the cofinality of $\mathcal{A}$ we find $V_m \times Y_m \in \mathcal{A}$ such that V_m contains the (countable) set

$$\widetilde{C} := C_{m-1} \cup \big\{ h(x, x^*, q, \delta_1, \delta_2, \gamma) : \ x \in C_{m-1}, \ x^* \in D_{m-1}, \ q \in \mathbb{Q}, \ \delta_1, \delta_2, \gamma \in \mathbb{Q}_+, \text{ and } \delta_1 < \delta_2 \big\}$$

and $Y_m \supset Y_{m-1}$. Find then a countable set $\widetilde{C} \subset C_m \subset V_m$ such that $\overline{C_m} = V_m$ and a countable set $D_{m-1} \subset D_m \subset Y_m$ so that $\overline{D_m} = Y_m$. Do so subsequently for every $m \in \mathbb{N}$. Put $V := \overline{V_0 \cup V_1 \cup V_2 \cup \cdots}$ and $Y := \overline{Y_0 \cup Y_1 \cup Y_2 \cup \cdots}$. The σ-completeness of $\mathcal{A}$ guarantees that $V \times Y$ belongs to $\mathcal{R}$.

We shall show that $V \times Y \in \mathcal{R}$. This means that we have to verify (7.2). So, fix any cortege $x, x^*, r, \delta_1, \delta_2$ as there. Consider any $h \in X$ such that $\delta_1 < \|h\| < \delta_2$. We have to show that $\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. This inequality is trivially satisfied if $I_V(x, x^*, r, \delta_1, \delta_2) = -\infty$ Further assume that this is not so. Pick some $\delta_1', \delta_2' \in \mathbb{Q}$ such that $\delta_1 < \delta_1' < \|h\| < \delta_2' < \delta_2$. It is easy to check that $V = \overline{C_0 \cup C_1 \cup \cdots}$ and $Y = \overline{D_0 \cup D_1 \cup \cdots}$. Find $x_0 \in C_0, \ x_1 \in C_1, \ \ldots$ and $x_0^* \in D_0, \ x_1^* \in D_1, \ \ldots$ such that $\|x_i - x\| \longrightarrow 0$ and $\|x_i^* - x^*\| \longrightarrow 0$ as $i \to \infty$. Consider any fixed $\gamma \in \mathbb{Q}_+$. Pick $q \in \mathbb{Q}$ such that $|q - r| < \gamma\|h\|$. Denote $N_1 := \big\{ i \in \mathbb{N} : \ \|x_i - x\| < \min\{\delta_1' - \delta_1, \delta_2 - \delta_2'\}\big\}$; this is a co-finite set in $\mathbb{N}$. Now, take any $k \in V$, with $\delta_1' < \|k\| < \delta_2'$. For $i \in N_1$ we have $\delta_1 < \|x_i - x + k\| < \delta_2$ and then we can estimate

$$\begin{aligned}
&\frac{1}{\|k\|}\big(f(x_i + k) - q - \langle x_i^*, k\rangle\big) \\
&= \frac{\|k + x_i - x\|}{\|k\|} \cdot \frac{1}{\|k + x_i - x\|}\big(f(x + (x_i - x + k)) - r - \langle x^*, x_i - x + k\rangle\big) \\
&+ \frac{1}{\|k\|}\big(\langle x^*, x_i - x + k\rangle - \langle x_i^*, k\rangle\big) + \frac{r - q}{\|k\|} \\
&\geq \Big(1 + s_i \frac{\|x_i - x\|}{\|k\|}\Big) I_V(x, x^*, r, \delta_1, \delta_2) - \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big) - \gamma\frac{\delta_2}{\delta_1'}
\end{aligned} \tag{7.3}$$

where $s_i = 1$ if $I_V(x, x^*, r, \delta_1, \delta_2) \leq 0$ and $s_i = -1$ otherwise. It then follows that

$$\begin{aligned}
I_V(x_i, x_i^*, q, \delta_1', \delta_2') &\geq \Big(1 + s_i \frac{\|x_i - x\|}{\delta_1}\Big) I_V(x, x^*, r, \delta_1, \delta_2) \\
&- \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big) - \gamma\frac{\delta_2}{\delta_1'} > -\infty\,,
\end{aligned} \tag{7.4}$$

and, in particular $I_V(x_i, x_i^*, q, \delta_1', \delta_2') > -\infty$, holds for every $i \in N_1$.

Now, put

$$N_2 := \big\{ i \in N_1 : \ \delta_1' < \|h + x - x_i\| < \delta_2' \text{ and } \langle x_i^*, x - x_i\rangle + \langle x_i^* - x^*, h\rangle > -\|h\|\gamma \big\}; \tag{7.5}$$

this is still a co-finite set in $\mathbb{N}$. Using (7.1), for every $i \in N_2$ we can estimate,

$$\begin{aligned}
&\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \\
&= \frac{\|x - x_i + h\|}{\|h\|} \cdot \frac{1}{\|x - x_i + h\|}\big(f(x_i + (x - x_i + h)) - q - \langle x_i^*, x - x_i + h\rangle\big) \\
&+ \frac{1}{\|h\|}\big(\langle x_i^*, x - x_i\rangle + \langle x_i^* - x^*, h\rangle\big) + \frac{q - r}{\|h\|} \\
&> \frac{\|x - x_i + h\|}{\|h\|} I(x_i, x_i^*, q, \delta_1', \delta_2') - \gamma - \gamma \\
&\geq \frac{\|x - x_i + h\|}{\|h\|}\left[\frac{1}{\|h(x_i, x_i^*, q, \delta_1', \delta_2', \gamma)\|}\big(f\big(x_i + h(\cdots)\big) - q - \big\langle x_i^*, h(\cdots)\big\rangle\big) - \gamma\right] - 2\gamma \\
&\geq \frac{\|x - x_i + h\|}{\|h\|}\big[I_V\big(x_i, x_i^*, q, \delta_1', \delta_2',\big) - \gamma\big] - 2\gamma;
\end{aligned} \tag{7.6}$$

here $\cdots$ meant the cortege $x_i, x_i^*, q, \delta_1', \delta_2', \gamma$. Now, plugging here (7.4), and then letting $N_2 \ni i \to \infty$, we get that

$$\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2) - 3\gamma - \gamma\frac{\delta_2}{\delta_1'}\,,$$

Finally, realizing that $\gamma \in \mathbb{Q}_+$ could be arbitrarily small, we get that $\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. This, of course, implies that $I(x, x^*, r, \delta_1, \delta_2) \geq I_V(x, x^*, r, \delta_1, \delta_2)$.

The proof of σ-completeness of $\mathcal{R}$ is very similar to (but a bit different from) the proof of cofinality. Let $V_1, \times Y_1,\ V_2 \times Y_2,\ \ldots$ be an increasing sequence of elements in our $\mathcal{R}$. We have to verify that $\overline{V_1 \times Y_1 \cup V_2 \times Y_2 \cup \cdots}$ also belongs to $\mathcal{R}$. Clearly, this set is of form $V \times Y$. As $\mathcal{A}$ is σ-complete, $V \times Y \in \mathcal{A}$. It remains to verify (7.2). So, fix any cortege x, x^*, r, δ_1, and δ_2 as there. Consider any $h \in X$ such that $\delta_1 < \|h\| < \delta_2$. We have to show that $\frac{1}{\|h\|}(f(x+h) - r - \langle x^*, h\rangle) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. This inequality is trivially satisfied if $I_V(x, x^*, r, \delta_1, \delta_2) = -\infty$. Further assume that this is not so. Pick some $\delta_1', \delta_2' \in \mathbb{Q}$ such that $\delta_1 < \delta_1' < \|h\| < \delta_2' < \delta_2$. It is easy to check that $V = \overline{V_1 \cup V_2 \cup \cdots}$ and $Y = \overline{Y_1 \cup Y_2 \cup \cdots}$. Find $x_1 \in V_1,\ x_2 \in V_2,\ \ldots$ and $x_1^* \in Y_1,\ x_2^* \in Y_2,\ \ldots$ such that $\|x_i - x\| \longrightarrow 0$ and $\|x_i^* - x^*\| \longrightarrow 0$ as $i \to \infty$. Consider any fixed $\gamma \in \mathbb{Q}_+$. Pick $q \in \mathbb{Q}$ such that $|q - r| < \gamma\|h\|$. Denote $N_1 := \big\{i \in \mathbb{N} :\ \|x_i - x\| < \min\{\delta_1' - \delta_1, \delta_2 - \delta_2'\}\big\}$; this is a co-finite set in $\mathbb{N}$. Now, take any $k \in V$, with $\delta_1' < \|k\| < \delta_2'$. For $i \in N_1$ we have $\delta_1 < \|x_i - x + k\| < \delta_2$ and then we can estimate (This chain is exactly as (7.3).)

$$\begin{aligned}
&\frac{1}{\|k\|}\big(f(x_i + k) - q - \langle x_i^*, k\rangle\big) \\
= \quad &\frac{\|k + x_i - x\|}{\|k\|} \cdot \frac{1}{\|k + x_i - x\|}\big(f(x + (x_i - x + k)) - r - \langle x^*, x_i - x + k\rangle\big) \\
&+ \frac{1}{\|k\|}\big(\langle x^*, x_i - x + k\rangle - \langle x_i^*, k\rangle\big) + \frac{r - q}{\|k\|} \\
\geq \quad &\Big(1 + s_i\frac{\|x_i - x\|}{\|k\|}\Big) I_V(x, x^*, r, \delta_1, \delta_2) - \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big) - \gamma\frac{\delta_2}{\delta_1'}
\end{aligned}$$

where $s_i = 1$ if $I_V(x, x^*, r, \delta_1, \delta_2) \le 0$ and $s_i = -1$ otherwise. It then follows that (This is exactly as (7.4).)

$$\begin{aligned} I_V(x_i, x_i^*, q, \delta_1', \delta_2') &\ge \Big(1 + s_i \frac{\|x_i - x\|}{\delta_1}\Big) I_V(x, x^*, r, \delta_1, \delta_2) \\ &- \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big) - \gamma\frac{\delta_2}{\delta_1'} > -\infty, \end{aligned} \tag{7.7}$$

and, in particular $I_V(x_i, x_i^*, q, \delta_1', \delta_2') > -\infty$, holds for every $i \in N_1$.

Now, put (This N_2 is defined exactly as in (7.5).)

$$N_2 := \big\{ i \in N_1 :\ \delta_1' < \|h + x - x_i\| < \delta_2' \ \text{ and } \ \langle x_i^*, x - x\rangle + \langle x_i^* - x^*, h\rangle > -\|h\|\gamma \big\};$$

this is still a co-finite set in $\mathbb{N}$. Using (7.7), for every $i \in N_2$ we can estimate (The following chain is a bit different from (7.6).)

$$\begin{aligned} & \frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \\ = \ & \frac{\|x - x_i + h\|}{\|h\|} \cdot \frac{1}{\|x - x_i + h\|}\big(f(x_i + (x - x_i + h)) - q - \langle x_i^*, x - x_i + h\rangle\big) \\ & + \frac{1}{\|h\|}\big(\langle x_i^*, x - x_i\rangle + \langle x_i^* - x^*, h\rangle\big) + \frac{q - r}{\|h\|} \\ > \ & \frac{\|x - x_i + h\|}{\|h\|} I(x_i, x_i^*, q, \delta_1', \delta_2') - \gamma - \gamma \\ = \ & \frac{\|x - x_i + h\|}{\|h\|} I_{V_i}(x_i, x_i^*, q, \delta_1', \delta_2') - 2\gamma \quad (\text{as } (x_i, x_i^*) \in V_i \times Y_i \in \mathcal{R} \text{ and (7.2) holds}) \\ \ge \ & \frac{\|x - x_i + h\|}{\|h\|} I_V(x_i, x_i^*, q, \delta_1', \delta_2') - 2\gamma. \end{aligned}$$

Now, plugging here (7.7), and then letting $N_2 \ni i \to \infty$, we get that

$$\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \ge I_V(x, x^*, r, \delta_1, \delta_2) - 2\gamma - \gamma\frac{\delta_2}{\delta_1'}\,,$$

Finally, realizing that $\gamma \in \mathbb{Q}_+$ could be arbitrarily small, we get that $\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \ge I_V(x, x^*, r, \delta_1, \delta_2)$. This, of course, implies that $I(x, x^*, r, \delta_1, \delta_2) \ge I_V(x, x^*, r, \delta_1, \delta_2)$. We proved that $\mathcal{R}$ is σ-complete, and therefore $\mathcal{R}$ is a rich rectangle family in $X \times X^*$.

That $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subset V_2$, follows immediately from the same property shared by $\mathcal{A}$.

It remains to prove that our $\mathcal{R}$ "works". So, pick any $V \times Y \in \mathcal{R}$. We know from Theorem 6.2 that $Y \ni x^* \longmapsto x^*|_V \in V^*$ is (an isometry) onto. Assume there is $(v, v^*) \in V \times V^*$ such that $v^* \in \partial_F(f|_V)(v)$. Find (a unique) $x^* \in Y$ such that $x^*|_V = v^*$. We shall show that $x^* \in \partial_F f(v)$. So, fix any $\varepsilon > 0$. Find $\delta > 0$ such that $f(v+k) - f(v) - \langle v^*, k\rangle > -\varepsilon\|k\|$ whenever $k \in V$ and $0 < \|k\| < \delta$; then $I_V(v, v^*, f(v), \delta_1, \delta) \ge -\varepsilon$. Now, let $h \in X$ be any vector such that $0 < \|h\| < \delta$. Pick $\delta_1 \in (0, \|h\|)$. Then we have

$$\frac{1}{\|h\|}\big(f(v+h) - f(v) - \langle x^*, h\rangle\big) \ge I(v, x^*, f(v), \delta_1, \delta) = I_V(v, v^*, f(v), \delta_1, \delta) \ge -\varepsilon$$

by (7.2). We proved that x^* belongs to $\partial f(v)$, and so v^* belongs to $\big(\partial_F f(v) \cap Y\big)|_V$. The reverse inclusion is obvious. $\square$

Corollary 7.2. *Let $(X, \|\cdot\|)$ be a (rather non-separable) Asplund space and let $f : X \longrightarrow (-\infty, +\infty]$ be any proper function. Then there exists a rich family $\mathcal{Q} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{Q}$ and for every $v \in V$ we have:*

(i) $\partial_F f(v) \neq \emptyset$ if (and only if) $\partial_F(f|_V)(v) \neq \emptyset$.

(ii) $\partial_F f(v) \setminus \{0\} \neq \emptyset$ if (and only if) $\partial_F(f|_V)(v) \setminus \{0\} \neq \emptyset$.

(iii) f is Fréchet differentiable at v if (and only if) $f|_V$ is Fréchet differentiable at v; and in this case $\|f'(v)\| = \big\|(f|_V)'(v)\big\|$.

Proof. Let $\mathcal{R}$ and $\mathcal{R}'$ be rich rectangle-families found in Theorem 7.1 for the functions f and $-f$, respectively. Let $\mathcal{Q}$ be the "projection" of $\mathcal{R} \cap \mathcal{R}'$ on the first coordinate, that is, put

$$\mathcal{Q} := \big\{V \in \mathcal{S}(X) : \ V \times Y \in \mathcal{R} \cap \mathcal{R}' \ \text{ for some Y} \in \mathcal{S}(\text{X}^*)\big\}.$$

It is easy check that $\mathcal{Q}$ is rich. It works. Indeed, take any $V \in \mathcal{Q}$ and any $v \in V$. Find $Y \in \mathcal{S}(X^*)$ so that $V \times Y$ is in $\mathcal{R} \cap \mathcal{R}'$. Then (i) and (ii) immediately follow from Theorem 7.1. Further, assume that $f|_V$ is Fréchet differentiable at v and put $v^* := (f|_V)'(v)$. This implies that $v^* \in \partial_F(f|_V)(v)$ and $-v^* \in \partial_F((-f)|_V)(v)$. Find (the unique) $x^* \in Y$ such that $x^*|_V = v^*$ and $\|x^*\| = \|v^*\|$; then $(-x^*)|_V = -v^*$. Now, by Theorem 7.1, $x^* \in \partial_F f(v)$ and $-x^* \in \partial_F(-f)(v)$. It then easily follows that f is Fréchet differentiable at v, with $f'(v) = x^*$ and $\|f'(v)\| = \|x^*\| = \|v^*\| = \|(f|_V)'(v)\|$. $\square$

Corollary 7.3. (see, e.g. [FM]) *Let $(X, \|\cdot\|)$ be an Asplund space, let $f : X \longrightarrow (-\infty, +\infty]$ be a lower semicontinuous function, and $g : X \longrightarrow (-\infty, +\infty]$ be a function uniformly continuous in a vicinity of a certain $\overline{x} \in X$. Then:*

(i) The set $\{x \in X : \ \partial_F f(x) \neq \emptyset\}$ is dense in the domain of f.

(ii) If $x^* \in \partial_F(f+g)(\overline{x})$, then for every $\varepsilon > 0$ there are $x_1, x_2 \in X$, $x_1^* \in \partial_F f(x_1)$, and $x_2^* \in \partial_F g(x_2)$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$, and $\|x_1^* + x_2^* - x^*\| < \varepsilon$.

Proof. Assume first that X is separable. Find an equivalent Fréchet smooth norm $|\cdot|$, see e.g. [DGZ, pages 48, 43] arbitrarily close to $\|\cdot\|$. Then proceed as in [I1] and [M, Section 2.2], using Borwein-Preiss or Deville-Godefroy-Zizler smooth variational principles [Ph, Section 4].

Second, assume that X is non-separable. As regards (i), combine the just proved separable statement with Corollary 7.2 (i). To prove (ii), assume that $x^* \in \partial_F(f+g)(\overline{x})$ and let $\varepsilon > 0$ be given. By Theorem 7.1, find rich families $\mathcal{R}_1$, $\mathcal{R}_2$ corresponding to f, g, respectively, and put $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$. Find $V \times Y \in \mathcal{R}$ so big that it contains $(\overline{x}, x^*)$. Using the validity of the separable statement, find $x_1, x_2 \in V$, $v_1^* \in \partial_F(f|_V)(x_1)$, and $v_2^* \in \partial_F(g|_V)(x_2)$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$, and $\|v_1^* + v_2^* - x^*|_V\| < \varepsilon$. Now, the conclusion of Theorem 7.1 provides unique $x_1^* \in \partial_F f(x_1) \cap Y$ and $x_2^* \in \partial_F g(x_2) \cap Y$ such that $x_i^*|_V = v_i^*$, $i = 1, 2$. Hence, using the isometric property of the restriction mapping $Y \ni \xi \longmapsto \xi|_V$, we conclude that $\|x_1^* + x_2^* - x^*\| = \|v_1^* + v_2^* - x^*|_V\| < \varepsilon$. $\square$

Let $(X, \|\cdot\|)$ be a Banach space, let $\Omega \subset X$, and let $\overline{x} \in \Omega$. The Fréchet normal cone $N_F(\overline{x}, \Omega)$ of Ω at $\overline{x}$ is defined as the Fréchet subdifferential of the indicator function ι_Ω at $\overline{x}$; note that $N_F(\overline{x}, \Omega)$ always contains 0. By an *extremal system* in X we understand any triple $(\Omega_1, \Omega_2, \overline{x})$ such that Ω_1, Ω_2 are subsets of X, the point $\overline{x}$ lies in $\Omega_1 \cap \Omega_2$, and there are $\varepsilon > 0$ and sequences (a_n^1), (a_n^2) in X satisfying that $(a_n^1 + \Omega_1) \cap (a_n^2 + \Omega_2) \cap (\overline{x} + \varepsilon B_X) = \emptyset$ for every $n \in \mathbb{N}$.

Corollary 7.4. (see, e.g., [FM]) *Let $(X, \|\cdot\|)$ be an Asplund space and let $(\Omega_1, \Omega_2, \overline{x})$ be an extremal system of closed sets in X. Then:*

(i) The set $\{x \in X : \ N_F(\overline{x}, \Omega_1) \neq \{0\}\}$ is dense in the boundary of Ω_1.

(ii) The Fréchet extremal principle for the triple $(\Omega_1, \Omega_2, \overline{x})$ holds, that is, for every $\varepsilon > 0$ there are $x_1, x_2 \in X$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$ and there are $x_i^* \in N_F(x_i, \Omega_i) + \varepsilon B_{X^*}$, $i = 1, 2$, such that $\|x_1^*\| + \|x_2^*\| = 1$, and $x_1^* + x_2^* = 0$.

The proof is very similar to that of Corollary 7.3, once we have at hand the "separable" statements.

Now, we present a strengthening of the main result of the paper [FI2] provided that the space in question is Asplund; see Theorem 5.1. It should be noted that the requirement of Asplund property is not a big restriction, once we realize that Fréchet (sub)differentiability is not always guaranteed in non-Asplund spaces, see [M, page 197].

Theorem 7.5. *Let $k \in \mathbb{N}$, let X be a non-separable* `Asplund` *space, let $Z_1, \ldots, Z_k$ be Banach spaces, let $T_i : Z_i \to X$, $i = 1, \ldots, k$, be bounded linear operators, and let f be a proper extended real-valued function on X. Then there exists a rich block-family $\mathcal{R} \subset \mathcal{S}_{\boxplus}(Z_1 \times \cdots \times Z_k \times X)$ such that, for every $U_1 \times \cdots \times U_k \times V \in \mathcal{R}$ we have $T_1(U_1) \subset V, \ \ldots, \ T_k(U_k) \subset V$ and there is $Y \in \mathcal{S}(X^*)$ such that:*

(i) The assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is an isometry `onto` *V^*;*

(ii) $\partial_F(f|_V)(v) = \big(\partial_F f(v) \cap Y\big)|_V = \big(\partial_F f(v)\big)|_V$ for every $v \in V$; and

(iii) $\big\|T_i^* x^*\big\| = \big\|\big(T_i|_{U_i}\big)^*(x^*|_V)\big\|$ for every $x^* \in Y$ and $i = 0, 1, \ldots, k$ (where $Z_0 := X$ and T_0 is the identity operator on X).

Proof. Putting together Theorem 6.4 and Remark 6.5, we find a rich block-family $\mathcal{A}_{T_1,\ldots,T_k}$ in $Z_1 \times \cdots \times Z_k \times X \times X^*$ with the properties similar as the family $\mathcal{A}_T$ in Theorem 6.4 has. Let $\mathcal{R}'$ be the rich family in $X \times X^*$ found in Theorem 7.1. Define

$$\mathcal{R} := \big\{U_1 \times \cdots \times U_k \times V : \ U_1 \times \cdots \times U_k \times V \times Y \in \mathcal{A}_{T_1,\ldots,T_k} \text{ and } V \times Y \in \mathcal{R}' \text{ for some } Y \in \mathcal{S}(X^*)\big\}.$$

Clearly, $\mathcal{R}$ is cofinal. And from the "monotonicity" property of $\mathcal{A}_{T_1,\ldots,T_k}$ we easily get that $\mathcal{R}$ is σ-complete. The properties (i), (ii), (iii) are clearly satisfied. $\square$

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