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Abstract. There are two basic ways of weakening the definition of the well-known metric regularity property by fixing one of the points involved in the definition. The first resulting property is called metric subregularity and has attracted a lot of attention during the last decades. On the other hand, the latter property which we call semiregularity can be found under several names and the corresponding results are scattered in the literature. We provide a self-contained material gathering and extending the existing theory on the topic. We demonstrate a clear relationship with other regularity properties, for example, the equivalence with the so-called openness with a linear rate at the reference point is shown. In particular cases, we derive necessary and/or sufficient conditions of both primal and dual type. As an application we study an inexact Newton-type scheme for generalized equations with not necessarily differentiable single-valued part.

Key Words. open mapping theorem, linear openness, metric semiregularity, set-valued perturbation

AMS Subject Classification (2010) 49J53, 49J52, 49K40, 90C31.

1 Introduction

The concept of *regularity* of a set-valued mapping F acting from a metric space (X, d) into (subsets of) another metric space (Y, ϱ), denoted by $F : X \rightrightarrows Y$, around a given reference point $(\bar{x}, \bar{y})$ in its graph $\operatorname{gph} F$ plays a fundamental role in modern variational analysis and non-smooth optimization, see, for example, a recent survey [18] by Ioffe or books [4, 13, 23, 33]. By regularity we mean that one of the three equivalent properties – metric regularity, openness with a linear rate around the reference point, and pseudo-Lipschitz property[4] of the inverse F^{-1} – holds for the mapping under consideration. First, the mapping F is said to be *metrically regular*[5] around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$(1) \qquad \operatorname{dist}\big(x, F^{-1}(y)\big) \le \kappa \operatorname{dist}\big(y, F(x)\big) \quad \text{for every} \quad (x, y) \in U \times V,$$

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[4]Often also called Lipschitz-like or Aubin property.

[5]In [13], this property is called metric regularity at $\bar{x}$ for $\bar{y}$ under an additional assumption that the graph of F is locally closed at the reference point.

where $\operatorname{dist}(u, C)$ is the distance from a point u to a set C and the space $X \times Y$ is equipped with the product (box) topology. The infimum of $\kappa > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1) holds is called the *regularity modulus* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{reg} F(\bar{x}, \bar{y})$.

Second, the mapping F is called *open with a linear rate*[6] around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$(2) \qquad I\!\!B[y, ct] \subset F(I\!\!B[x, t]) \quad \text{whenever} \quad (x, y) \in U \times V, \quad y \in F(x) \quad \text{and} \quad t \in (0, \varepsilon),$$

where $I\!\!B[u, r]$ denotes the closed ball centered at u with a radius $r > 0$. The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (2) holds is called the *modulus of surjection* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur} F(\bar{x}, \bar{y})$ [7]. Finally, the mapping $F : X \rightrightarrows Y$ is said to be *pseudo-Lipschitz* around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there is a constant $\mu > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$(3) \qquad \operatorname{dist}\big(y, F(x)\big) \leq \mu\, d(x, x') \quad \text{whenever} \quad x, x' \in U \quad \text{and} \quad y \in F(x') \cap V.$$

The infimum of $\mu > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (3) holds is called the *Lipschitz modulus* of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$.

A fundamental well-known fact is that

$$(4) \qquad \operatorname{sur} F(\bar{x}, \bar{y}) \cdot \operatorname{reg} F(\bar{x}, \bar{y}) = 1 \quad \text{and} \quad \operatorname{reg} F(\bar{x}, \bar{y}) = \operatorname{lip} F^{-1}(\bar{y}, \bar{x}),$$

under the convention that $0 \cdot \infty = \infty \cdot 0 = 1$, $\inf \emptyset = \infty$, and, as we work with nonnegative quantities, that $\sup \emptyset = 0$.

Fixing one of the components of (x, y) in (1), that is letting either $x := \bar{x}$ or $y := \bar{y}$, one gets two different, weaker than regularity, concepts. Of course, one can reformulate both of them in terms of openness and continuity of the inverse, respectively.

Definition 1.1. *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in X \times Y$.*

(A1) *F is said to be* metrically subregular at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$(5) \qquad \operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \leq \kappa \operatorname{dist}\big(\bar{y}, F(x)\big) \quad \textit{for every} \quad x \in U.$$

The infimum of $\kappa > 0$ for which there exists a neighborhood U of $\bar{x}$ in X such that (5) *holds is called the* subregularity modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{subreg} F(\bar{x}, \bar{y})$*;*

(A2) *F is said to be* pseudo-open with a linear rate at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood U of $\bar{x}$ in X such that*

$$(6) \qquad \bar{y} \in F(I\!\!B[x, t]) \quad \textit{whenever} \quad x \in U \cap F^{-1}(I\!\!B[\bar{y}, ct]) \quad \textit{and} \quad t \in (0, \varepsilon).$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood U of $\bar{x}$ in X such that (6) holds is called the modulus of pseudo-openness *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{popen} F(\bar{x}, \bar{y})$*;*

[6] There are other equivalent definitions in the literature. Also note that in [13] the constant c appears on the right-hand side of (2).

[7] Clearly, we can replace the closed balls in (2) with the open ones.

(A3) *F is said to be* calm at $(\bar{x}, \bar{y})$ *when* $\bar{y} \in F(\bar{x})$ *and there is a constant* $\mu > 0$ *along with a neighborhood* $U \times V$ *of* $(\bar{x}, \bar{y})$ *in* $X \times Y$ *such that*

$$\text{(7)} \qquad \operatorname{dist}\big(y, F(\bar{x})\big) \le \mu\, d(\bar{x}, x) \quad \textit{whenever} \quad x \in U \quad \textit{and} \quad y \in F(x) \cap V.$$

The infimum of $\mu > 0$ *for which there exists a neighborhood* $U \times V$ *of* $(\bar{x}, \bar{y})$ *in* $X \times Y$ *such that* (7) *holds is called the* calmness modulus *of* F *at* $(\bar{x}, \bar{y})$ *and is denoted by* $\operatorname{calm} F(\bar{x}, \bar{y})$*.*

Properties in (A1) and (A3) are entrenched in the literature [33, 13] and the metric subregularity of a mapping is known to be equivalent to the calmness of its inverse. (A2) is defined and proved to be equivalent with the remaining ones in [2]. More precisely, the following analogue of (4) holds true

$$\text{(8)} \qquad \operatorname{popen} F(\bar{x}, \bar{y}) \cdot \operatorname{subreg} F(\bar{x}, \bar{y}) = 1 \quad \text{and} \quad \operatorname{subreg} F(\bar{x}, \bar{y}) = \operatorname{calm} F^{-1}(\bar{y}, \bar{x}).$$

The case when $x := \bar{x}$ in (1), being the same as letting $(x, y) := (\bar{x}, \bar{y})$ in (2), is known under several names. In this note we provide a self-contained material gathering and extending results on this property scattered in the literature and illustrate possible applications.

Definition 1.2. *Consider a mapping* $F : X \rightrightarrows Y$ *between metric spaces* (X, d) *and* (Y, ϱ) *and a point* $(\bar{x}, \bar{y}) \in X \times Y$*.*

(B1) *F is said to be* metrically semiregular at $(\bar{x}, \bar{y})$ *when* $\bar{y} \in F(\bar{x})$ *and there is a constant* $\kappa > 0$ *along with a neighborhood* V *of* $\bar{y}$ *in* Y *such that*

$$\text{(9)} \qquad \operatorname{dist}\big(\bar{x}, F^{-1}(y)\big) \le \kappa\, \varrho(\bar{y}, y) \quad \textit{for every} \quad y \in V.$$

The infimum of $\kappa > 0$ *for which there exists a neighborhood* V *of* $\bar{y}$ *in* Y *such that* (9) *holds is called the* semiregularity modulus *of* F *at* $(\bar{x}, \bar{y})$ *and is denoted by* $\operatorname{semireg} F(\bar{x}, \bar{y})$*;*

(B2) *F is said to be* open with a linear rate at $(\bar{x}, \bar{y})$ *when* $\bar{y} \in F(\bar{x})$ *and there are positive constants* c *and* ε *such that*

$$\text{(10)} \qquad I\!\!B[\bar{y}, ct] \subset F(I\!\!B[\bar{x}, t]) \quad \textit{for each} \quad t \in (0, \varepsilon).$$

The supremum of $c > 0$ *for which there exists a constant* $\varepsilon > 0$ *such that (10) holds is called the* modulus of openness *of* F *at* $(\bar{x}, \bar{y})$ *and is denoted by* $\operatorname{lopen} F(\bar{x}, \bar{y})$*;*

(B3) *F is said to* recede from $\bar{y}$ at $(\bar{x}, \bar{y})$ at a linear rate *when* $\bar{y} \in F(\bar{x})$ *and there is a constant* $\mu > 0$ *along with a neighborhood* U *of* $\bar{x}$ *in* X *such that*

$$\text{(11)} \qquad \operatorname{dist}\big(\bar{y}, F(x)\big) \le \mu\, d(\bar{x}, x) \quad \textit{for each} \quad x \in U.$$

The infimum of $\mu > 0$ *for which there exists a neighborhood* U *of* $\bar{x}$ *in* X *such that* (11) *holds is called the* speed of recession *of* F *at* $(\bar{x}, \bar{y})$ *and is denoted by* $\operatorname{recess} F(\bar{x}, \bar{y})$*.*

Properties (B1) and (B2) were studied by the third author in [29] (see also [31]), where their equivalence was established (see Proposition 2.1 below) and the term *semiregularity* was suggested for property (B1). This property has been later used in [3, 14, 37] under the name *hemiregularity.*

Following [10], property (B2) was referred to in [29] as *c-covering*, while in the earlier paper [28] it was called simply *regularity*. This property can be found also in [14, 13]. In the recent survey by Ioffe [18], the property is called *controllability*, the concept stemming from the control theory. The explicit definition of $\operatorname{lopen} F(\bar{x}, \bar{y})$ can be found in [26, 27], while its main components are present already in [24, 25]. Note that thanks to the Robinson-Ursescu theorem, if F has a closed convex graph, the openness (with a linear rate) at a point is equivalent to the openness around this point.

To the best of our knowledge, property (B3) first appeared in [23, p. 34] under the name *Lipschitz lower semicontinuity*. It was defined for F^{-1} via inequality (9). In [14], this property is called *pseudo-calmness*. The terminology in (B3) above comes from [18].

A (graphical) localization of a set-valued mapping $F : X \rightrightarrows Y$ around the reference point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is any mapping $\widetilde{F} : X \rightrightarrows Y$ such that $\operatorname{gph} \widetilde{F} = \operatorname{gph} F \cap (U \times V)$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$. Using this notion we can define "stronger" versions of the properties mentioned above.

Definition 1.3. *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. Then F is said to be*

(S) strongly metrically regular around $(\bar{x}, \bar{y})$ *when F is metrically regular at $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued;*

(SA) strongly metrically subregular at $(\bar{x}, \bar{y})$ *when F is metrically subregular at $(\bar{x}, \bar{y})$ and F^{-1} has no localization around $(\bar{y}, \bar{x})$ that is multivalued at $\bar{y}$;*

(SB) strongly metrically semiregular at $(\bar{x}, \bar{y})$ *when F is metrically semiregular at $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued.*

Clearly, (S)–(SA) are connected with (and can be defined by) the properties of the inverse F^{-1}. Indeed, (S) means that for each $\ell > \operatorname{reg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that the localization $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued and Lipschitz continuous on V with the constant ℓ [13, Proposition 3G.1]. While (SA) means that for each $\ell > \operatorname{subreg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$d(\bar{x}, x) \leq \ell\, \varrho(\bar{y}, y) \quad \text{whenever} \quad x \in U \quad \text{and} \quad y \in F(x) \cap V.$$

Finally, (SB) means that for each $\ell > \operatorname{semireg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that the localization $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued and calm on V with the constant ℓ.

In Section 2, we recall that, similarly to (4) and (8), we have

$$\text{(12)} \qquad \operatorname{lopen} F(\bar{x}, \bar{y}) \cdot \operatorname{semireg} F(\bar{x}, \bar{y}) = 1 \quad \text{and} \quad \operatorname{semireg} F(\bar{x}, \bar{y}) = \operatorname{recess} F^{-1}(\bar{y}, \bar{x}),$$

and we provide a comparison with the other properties defined above. As in the case of regularity, we omit the word "metrically" in the rest of the note, that is, we say that F is subregular (semiregular, strongly regular, etc.) at/around $(\bar{x}, \bar{y})$.

Note that the validity of both the weaker point-based properties does not imply the stronger one, that is, if F satisfies $(A1)$ and $(B1)$ then F does not need to be regular around the reference point (see Example 2.3). Let us point out that if $f : X \to Y$ is a single-valued mapping then we do not mention the point $\bar{y} = f(\bar{x})$ in all the above definitions, that is, we write $\operatorname{sur} f(\bar{x})$, $\operatorname{reg} f(\bar{x})$, etc., instead of $\operatorname{sur} f(\bar{x}, f(\bar{x}))$, $\operatorname{reg} f(\bar{x}, f(\bar{x}))$, etc.; and if the corresponding modulus is independent of $\bar{x}$ then we omit $\bar{x}$ as well.

Suppose that X and Y are Banach spaces and $A : X \to Y$ is a continuous linear operator. Then the Banach-Schauder open mapping theorem and the linearity of A imply (c.f. [33, Theorem 1.104 and Proposition 1.106], [3, Proposition 5.2]) that: *A is regular around any point $\Leftrightarrow$ A is semiregular at any point $\Leftrightarrow$ A is surjective; moreover*

$$\operatorname{semireg} A = \operatorname{reg} A \quad \textit{and} \quad \operatorname{sur} A = \sup\{\varrho > 0 : A(\mathbb{B}_X) \supset \varrho \mathbb{B}_Y\} = \inf\{\|A^* y^*\| : y^* \in \mathbb{S}_{Y^*}\},$$

where A^* is the adjoint (dual) operator to A acting between the dual spaces Y^* and X^* of Y and X, respectively. This is a particular case of Proposition 2.2 (iv). If A is invertible, then $\operatorname{sur} A = 1/\|A^{-1}\|$. For a real m-by-n matrix $A \in \mathbb{R}^{m \times n}$, $\operatorname{sur} A$ equals to the least singular value of A. Using the Banach-Schauder theorem again, if A has closed range, then it is subregular at any point; and if, in addition, A is injective then it is strongly subregular everywhere. Note that *both* the statements fail without the closedness assumption (see [8, Example 2.7]). In general, A is strongly subregular everywhere if any only if $\kappa := \inf_{h \in \mathbb{S}_X} \|Ah\| > 0$; moreover $\operatorname{subreg} A = 1/\kappa$. If the dimension of X is finite, then $\kappa > 0$ if and only if $A^{-1}(0) = \{0\}$, that is, A is injective.

Using the above notation, for a non-linear mapping we have the following result:

Theorem 1.4. *Consider a mapping $f : X \to Y$ defined around a point $\bar{x} \in X$ and a continuous linear mapping $A : X \to Y$.*

(i) Then $\operatorname{sur} f(\bar{x}) \geq \operatorname{sur} A - \operatorname{lip}(f - A)(\bar{x})$. *If, in addition, the mapping A is invertible and* $\operatorname{lip}(f - A)(\bar{x}) < \operatorname{sur} A$, *then f is strongly regular at $\bar{x}$ and* $\operatorname{sur} f(\bar{x}) \geq 1/\|A^{-1}\| - \operatorname{lip}(f - A)(\bar{x})$.

(ii) If A is strongly subregular (everywhere) and $\operatorname{calm}(f - A)(\bar{x}) < \operatorname{popen} A$, *then f is strongly subregular at $\bar{x}$ and* $\operatorname{popen} f(\bar{x}) \geq \operatorname{popen} A - \operatorname{calm}(f - A)(\bar{x})$ *(> 0).*

Theorem 1.4 is a particular case of the well known fact that (strong) regularity as well as strong subregularity are stable with respect to a single-valued perturbation (see Theorem 1.9 below). Part (i) was proved by Graves [16] and Graves-Hildebrand [17]. More precisely, Graves proved that $\operatorname{lopen} f(\bar{x}) \geq \operatorname{sur} A - \operatorname{lip}(f - A)(\bar{x}) > 0$, which is weaker. As observed in [11] a slight modification of the original proof yields the (stronger) version above. If A is the strict derivative[8] of f at $\bar{x}$, that is, when $\operatorname{lip}(f - A)(\bar{x}) = 0$, then we have $\operatorname{sur} f(\bar{x}) = \operatorname{sur} A$. This is the case, for example, if f is (Gateaux) differentiable in a vicinity of $\bar{x}$ and the derivative mapping $X \ni x \longmapsto Df(x) \in \mathcal{L}(X, Y)$ is continuous at $\bar{x}$. In fact, the weak Gateaux differentiability is enough. In particular, the Lyusternik theorem [32], proved before the Graves theorem, follows from Theorem 1.4. On the other hand, assume that $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$. If f is strictly differentiable at $\bar{x}$, then there is a neighborhood U of $\bar{x}$ such that f is Lipschitz continuous on U. Let $D \subset U$ be the set of all $x \in U$ such that f is Fréchet differentiable at x. Then D has full Lebesgue measure by the Rademacher theorem. Moreover, the Jacobian mapping $D \ni x \longmapsto \nabla f(x) \in \mathbb{R}^{m \times n}$ is continuous at $\bar{x}$ [34, Lemma 5.1]. However, this does not imply that f is differentiable on any neighborhood of $\bar{x}$ ([34, p. 324] or [13, p.35]). If f is differentiable in a vicinity of $\bar{x}$ then f is strictly differentiable at $\bar{x}$ if

[8]Sometimes called strong derivative [34].

and only if ∇f is continuous at $\bar{x}$ [13, Proposition 1D.7]. Theorem 1.4 (ii) which can be found as [8, Theorem 2.1], for example, fails when (non-strong) subregularity is considered [13, p. 201]. Let us mention examples where (strong) (sub)regularity holds.

Example 1.5. Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

(i) Suppose that a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (not necessarily symmetric) and F is maximal monotone, for example, $F := \partial\varphi$, a subdifferential in the sense of convex analysis of a proper convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. Then $A + F$ is strongly regular around any point with $U = V = \mathbb{R}^n$ [38, Lemma 2.2].

(ii) Suppose that F has a monotone localization around a point $(\bar{x}, \bar{y})$. Then F is strongly regular around $(\bar{x}, \bar{y})$ if and only if it is regular around the same point [13, Theorem 3G.5].

(iii) Suppose that $\operatorname{gph} F$ is the union of finitely many (convex) polyhedra. Then F is subregular at any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ [13, Proposition 3H.1]. Moreover, F is strongly subregular at $(\bar{x}, \bar{y})$ if and only if $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$ [13, Proposition 3I.1].

To check the regularity of the mapping in question we have the following regularity criterion [15, Corollary 1], [19, Theorem 1b], [9, Proposition 2.1].

Proposition 1.6. *Let (X, d) be a complete metric space and (Y, ϱ) be a metric space, let $\bar{x} \in X$ be given, and let $g : X \to Y$ be a continuous mapping, whose domain is all of X. Then* $\operatorname{sur} g(\bar{x})$ *equals to the supremum of all $c > 0$ for which there is $r > 0$ such that for all $(x, y) \in I\!B(\bar{x}, r) \times \big(I\!B(g(\bar{x}), r) \setminus \{g(x)\}\big)$ there is a point $x' \in X$ satisfying*

$$c\, d(x', x) < \varrho(g(x), y) - \varrho(g(x'), y). \tag{13}$$

More precisely, Fabian and Preiss [15] proved only a sufficient condition guaranteeing that $\operatorname{lopen} g(\bar{x}) > 0$. The full version (for set-valued mappings) was shown independently by Ioffe [19]. As in the case of Theorem 1.4, only a tiny modification of the original proof from [15] yields the statement above (see [9]). Although Proposition 1.6 is formulated for a single-valued function, it is well-known that the study of regularity properties for a set-valued mapping $F : X \rightrightarrows Y$ can always be reduced to the study of the corresponding property for a simple single-valued mapping, namely, the restriction of the canonical projection from $X \times Y$ onto Y, that is, the assignment $\operatorname{gph} F \ni (x, y) \longmapsto y \in Y$ (e.g., see [19, Proposition 3]). Using this, one gets the following statement for set-valued mappings.

Theorem 1.7. *Let (X, d) and (Y, ϱ) be metric spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping having a localization around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with a complete graph. Then* $\operatorname{sur} F(\bar{x}, \bar{y})$ *equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $(x, v) \in \operatorname{gph} F \cap \big(I\!B(\bar{x}, r) \times I\!B(\bar{y}, r)\big)$ and any $y \in I\!B(\bar{y}, r) \setminus \{v\}$ there is a pair $(x', v') \in \operatorname{gph} F$ satisfying*

$$c \max\{d(x, x'), \alpha\varrho(v, v')\} < \varrho(v, y) - \varrho(v', y). \tag{14}$$

It follows directly from the definition that a mapping $F : X \rightrightarrows Y$ is subregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if its *subregularity constant* at $(\bar{x}, \bar{y})$ defined in [30] as[9]

$$\operatorname{SuR}[F](\bar{x}, \bar{y}) := \liminf_{x \to \bar{x},\ x \notin F^{-1}(\bar{y})} \frac{\operatorname{dist}(\bar{y}, F(x))}{\operatorname{dist}(x, F^{-1}(\bar{y}))} \tag{15}$$

[9] In this article, we use notations $\operatorname{SuR}[F](\bar{x}, \bar{y})$ and $\operatorname{SeR}[F](\bar{x}, \bar{y})$ for the subregularity and semiregularity constants, respectively, cf. (15) and (24).

is positive (with the convention that the limit in (15) is ∞ when $\bar{x}$ is an internal point in $F^{-1}(\bar{y})$). When $\bar{x}$ is an isolated point in $F^{-1}(\bar{y})$, then $\mathrm{SuR}\,[F](\bar{x},\bar{y})$ coincides with the *steepest displacement rate* at $(\bar{x},\bar{y})$ defined by Uderzo in [36] as

$$|F|^{\downarrow}(\bar{x},\bar{y}) := \liminf_{x\to\bar{x}} \frac{\mathrm{dist}\,(\bar{y},F(x))}{d(\bar{x},x)} \tag{16}$$

(with the convention that the limit in (16) is ∞ when $\bar{x}$ is an isolated point in $\mathrm{dom}\,F$). The inequality $|F|^{\downarrow}(\bar{x},\bar{y}) > 0$ is equivalent to the strong subregularity of F at $(\bar{x},\bar{y})$. It is elementary to check that

$$\mathrm{SuR}\,[F](\bar{x},\bar{y}) \cdot \mathrm{subreg}\,F(\bar{x},\bar{y}) = 1, \tag{17}$$

and hence $\mathrm{SuR}\,[F](\bar{x},\bar{y}) = \mathrm{popen}\,F(\bar{x},\bar{y})$.

There is a similar statement to Theorem 1.7 guaranteeing the (strong) subregularity. The next theorem combines a portion of [30, Corollary 5.8] (with condition (d)) and [8, Theorem 5.3]. The latter one was formulated in [8] for Banach spaces, but its proof remains valid in the present setting.

Theorem 1.8. *Let (X,d) and (Y,ϱ) be metric spaces and let $F : X \rightrightarrows Y$ be a set-valued mapping having a localization around $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$ with a complete graph. Then $\mathrm{SuR}\,[F](\bar{x},\bar{y})$ (respectively, $|F|^{\downarrow}(\bar{x},\bar{y})$) equals to the supremum of $c > 0$ for which there exists $r > 0$ such that for any $(x,y) \in \mathrm{gph}\,F$ with $x \notin F^{-1}(\bar{y})$ and $d(x,\bar{x}) < r$ (respectively, $0 < d(x,\bar{x}) < r$) and $\varrho(y,\bar{y}) < r$, there is a pair $(u,v) \in \mathrm{gph}\,F \setminus \{(x,y)\}$ satisfying*

$$c\max\{d(u,x), r\varrho(v,y)\} < \varrho(y,\bar{y}) - \varrho(v,\bar{y}). \tag{18}$$

Theorems 1.7 and 1.8 can be used to get short proofs of various regularity statements in the literature [18, 9].

In Section 3, we will discuss conditions guaranteeing (strong) semiregularity and derive primal and dual derivative-type conditions. Note that (sufficient) conditions for (non-strong) subregularity and semiregularity are much more involved because of their instability with respect to calm (or Lipschitz) single-valued perturbations (see counterexamples [13, pp. 200–201]). More precisely, for these two properties, the analogues of the following statement (see [13, Theorems 5E.1 and 5F.1] and [8, Corollary 2.2]) fail without additional assumptions.

Theorem 1.9. *Let (X,d) be a complete metric space, (Y,ϱ) be a linear metric space with a shift-invariant metric, and $(\bar{x},\bar{y}) \in X \times Y$. Consider a mapping $g : X \to Y$ defined around $\bar{x}$ and a mapping $F : X \rightrightarrows Y$ such that $\bar{y} \in F(\bar{x})$.*

(i) If F is (strongly) regular around $(\bar{x},\bar{y})$ and $\mathrm{lip}\,g(\bar{x}) < \mathrm{sur}\,F(\bar{x},\bar{y})$, then so is $g+F$ around $(\bar{x}, g(\bar{x})+\bar{y})$ and

$$\mathrm{sur}\,(g+F)(\bar{x}, g(\bar{x})+\bar{y}) \geq \mathrm{sur}\,F(\bar{x},\bar{y}) - \mathrm{lip}\,g(\bar{x}) > 0.$$

(ii) If F is strongly subregular at $(\bar{x},\bar{y})$ and $\mathrm{calm}\,g(\bar{x}) < \mathrm{popen}\,F(\bar{x},\bar{y})$, then so is $g+F$ at $(\bar{x}, g(\bar{x})+\bar{y})$ and

$$\mathrm{popen}(g+F)(\bar{x}, g(\bar{x})+\bar{y}) \geq \mathrm{popen}\,F(\bar{x},\bar{y}) - \mathrm{calm}\,g(\bar{x}) > 0.$$

The above statement fails if a perturbation is set-valued (see [13, Example 5I.1] and [8, p. 5]). However, in Section 4, we prove that the sum of two set-valued mappings is semiregular provided that one is regular while the other is pseudo-Lipschitz.

In Section 5, as an application of the theoretical results, we provide a local convergence analysis for Newton-type iterative schemes for solving a generalized equation, introduced by Robinson in [35], which reads as:

$$\text{Find} \quad x \in X \quad \text{such that} \quad f(x) + F(x) \ni 0, \tag{19}$$

where X and Y are (real) Banach spaces, $f : X \to Y$ is a single-valued (possibly nonsmooth) mapping, and $F : X \rightrightarrows Y$ is a set-valued mapping with closed graph. This model has been used to describe in a unified way various problems such as equations (when $F \equiv 0$), inequalities (when $Y = \mathbb{R}^n$ and $F \equiv \mathbb{R}^n_+$), variational inequalities (when $Y = X^*$ and F is the normal cone mapping corresponding to a closed convex subset of X or more broadly the subdifferential mapping of a convex function on X).

The Newton iteration for (19) with a smooth function f, also known as the Josephy-Newton method [22], has the form

$$f(x_k) + f'(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \ni 0 \quad \text{for each} \quad k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \text{ and a given } x_0 \in X. \tag{20}$$

From the numerical point of view, it is clear that the auxiliary inclusions above cannot be solved exactly because of the finite precision arithmetic and rounding errors. Moreover, it can be much quicker to find an inexact solution at each step which has a sufficiently small residual. Various (in)exact methods were proposed in the literature (see [20] for an in-depth study and a vast bibliography, or [23] and references therein). In order to represent inexactness, Dontchev and Rockafellar proposed in [12] an inexact version of the iteration (20) in which, for given $k \in \mathbb{N}_0$ and $x_k \in X$, the next iterate $x_{k+1} \in X$ is determined as a *coincidence point* of the mapping on the left-hand side of (20) and a mapping $R_k : X \times X \rightrightarrows Y$ which models inexactness, that is,

$$\big(f(x_k) + f'(x_k)(x_{k+1} - x_k) + F(x_{k+1})\big) \cap R_k(x_k, x_{k+1}) \neq \emptyset. \tag{21}$$

We are going to analyze an inexact Newton-type iteration for the case when the function f in (19) is not necessarily differentiable. Specifically, we introduce a mapping $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ viewed as a *generalized set-valued derivative* of the function f, and consider the following iteration: *Given an index $k \in \mathbb{N}_0$ and a point $x_k \in X$, choose any $A_k \in \mathcal{H}(x_k)$ and then find $x_{k+1} \in X$ satisfying*

$$\big(f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1})\big) \cap R_k(x_k, x_{k+1}) \neq \emptyset. \tag{22}$$

The case when the mappings R_k depend on the current iterate x_k only, was studied in [7].

Notation and terminology. When we write $f : X \to Y$ we mean that f is a (single-valued) mapping acting from X into Y while $F : X \rightrightarrows Y$ is a mapping from X into Y which may be set-valued. The set $\operatorname{dom} F := \{x \in X : \ F(x) \neq \emptyset\}$ is the *domain* of F, the *graph* of F is the set $\operatorname{gph} F := \{(x, y) \in X \times Y : \ y \in F(x)\}$ and the *inverse* of F is the mapping $Y \ni y \longmapsto \{x \in X : y \in F(x)\} =: F^{-1}(y) \subset X$; thus $F^{-1} : Y \rightrightarrows X$. In any metric space, $\mathbb{B}[x, r]$ denotes the closed ball centered at x with a radius $r > 0$ and $\mathbb{B}(x, r)$ is the corresponding open ball. $\mathbb{B}_X$ and $\mathbb{S}_X$

are respectively the *closed unit ball* and the *unit sphere* in a normed space X. The *distance from a point* x to a subset C of a metric space (X,d) is $\operatorname{dist}(x,C) := \inf\{d(x,y) : \ y \in C\}$. We use the convention that $\inf \emptyset := \infty$ and as we work with non-negative quantities we set $\sup \emptyset := 0$. If a set is a singleton we identify it with its only element, that is, we write a instead of $\{a\}$. The symbol $\mathcal{L}(X,Y)$ denotes the space of all linear bounded operators from a Banach space X into a Banach space Y. Then $\mathbb{R}^{m\times n} := \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $X^* := \mathcal{L}(X,\mathbb{R})$. Given $A \in \mathcal{L}(X,Y)$, the operator $A^* : Y^* \to X^*$ denotes the adjoint (dual, transpose) operator to A. The transpose of a matrix $A \in \mathbb{R}^{m\times n}$ is $A^T \in \mathbb{R}^{n\times m}$. Given a set $\mathcal{A}$ in $\mathcal{L}(X,Y)$, the *measure of noncompactness* $\chi(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}) := \inf\big\{r > 0 : \ \mathcal{A} \subset \mathcal{F} + r\mathbb{B}_{\mathcal{L}(X,Y)} \text{ for some finite } \mathcal{F} \subset \mathcal{A}\big\}.$$

Given an extended real-valued function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ and a point $x \in X$, the *limes inferior* of φ at x is defined by

$$\liminf_{u\to x} \varphi(u) := \sup_{r>0} \inf_{u\in B(x,r)} \varphi(u).$$

2 Relationship among regularity concepts

Let us start with a simple observation [14, Proposition 2.4] and [29, Theorem 6(i)]:

Proposition 2.1. *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X,d) and (Y,ϱ) and a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Then (12) holds, that is,*

$$\operatorname{lopen} F(\bar{x},\bar{y}) \cdot \operatorname{semireg} F(\bar{x},\bar{y}) = 1 \quad \textit{and} \quad \operatorname{semireg} F(\bar{x},\bar{y}) = \operatorname{recess} F^{-1}(\bar{y},\bar{x}).$$

Proof. First, we show that $\operatorname{semireg} F(\bar{x},\bar{y}) \le 1/\operatorname{lopen} F(\bar{x},\bar{y})$. If $\operatorname{lopen} F(\bar{x},\bar{y}) = 0$ we are done. Suppose that this is not the case. Fix any $c \in (0, \operatorname{lopen} F(\bar{x},\bar{y}))$. Find $\varepsilon > 0$ such that $\mathbb{B}[\bar{y}, ct] \subset F(\mathbb{B}[\bar{x}, t])$ for each $t \in (0,\varepsilon)$. Let $V := \mathbb{B}(\bar{y}, c\varepsilon)$. Pick any $y \in V \setminus \{\bar{y}\}$. Then $t := \varrho(\bar{y}, y)/c \in (0,\varepsilon)$ and $y \in \mathbb{B}[\bar{y}, ct]$. Hence there is $x \in X$ such that $y \in F(x)$ and $d(\bar{x}, x) \le t = c^{-1}\varrho(\bar{y}, y)$. If $y = \bar{y}$, then we take $x := \bar{x}$ and the latter inequality holds. Letting $c \uparrow \operatorname{lopen} F(\bar{x},\bar{y})$ we get the desired estimate. Clearly, the opposite inequality holds when $\operatorname{semireg} F(\bar{x},\bar{y}) = \infty$. Assume that this is not the case and pick any $\kappa > \operatorname{semireg} F(\bar{x},\bar{y})$. Find $\gamma > 0$ such that $\operatorname{dist}\big(\bar{x}, F^{-1}(y)\big) \le \kappa\varrho(\bar{y}, y)$ for each $y \in \mathbb{B}(\bar{y}, \gamma)$. Fix any $c \in (0, 1/\kappa)$. Let $\varepsilon := \gamma\kappa$. Given $t \in (0,\varepsilon)$ and $y \in \mathbb{B}[\bar{y}, ct]$, we have $y \in \mathbb{B}(\bar{y}, \gamma)$, and consequently $\operatorname{dist}\big(\bar{x}, F^{-1}(y)\big) \le \kappa\varrho(\bar{y}, y) < t$. This implies that $\mathbb{B}[\bar{y}, ct] \subset F(\mathbb{B}(\bar{x}, t)) \subset F(\mathbb{B}[\bar{x}, t])$. Letting $c \uparrow 1/\kappa$, we conclude that $\operatorname{lopen} F(\bar{x},\bar{y}) \ge 1/\kappa > 0$, that is, $\kappa \ge 1/\operatorname{lopen} F(\bar{x},\bar{y})$. Letting $\kappa \downarrow \operatorname{semireg} F(\bar{x},\bar{y})$, we get $\operatorname{semireg} F(\bar{x},\bar{y}) \ge 1/\operatorname{lopen} F(\bar{x},\bar{y})$. The first equality in (12) is proved. The latter statement in (12) is obvious from the definitions (B1) and (B3). $\square$

Proposition 2.2. *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X,d) and (Y,ϱ) and a point $(\bar{x},\bar{y}) \in X \times Y$. Then*

(i) $\operatorname{lopen} F(\bar{x},\bar{y}) \ge \liminf_{(x,y)\to(\bar{x},\bar{y}),\ y\in F(x)} \operatorname{lopen} F(x,y) \ge \operatorname{sur} F(\bar{x},\bar{y})$.

(ii) Suppose that X and Y are normed spaces and that F has a locally star-shaped graph at $(\bar{x}, \bar{y})$, that is, there is $a \in (0,1]$ such that $(1-t)\,(\bar{x}, \bar{y}) + t \operatorname{gph} F \subset \operatorname{gph} F$ for each $t \in [0,a]$. If there are positive constants α and β such that

$$\mathbb{B}[\bar{y}, \beta] \subset F(\mathbb{B}[\bar{x}, \alpha]), \tag{23}$$

then $\operatorname{lopen} F(\bar{x}, \bar{y}) \geq \beta/\alpha$.

(iii) If X and Y are normed spaces and F has a convex graph then $\operatorname{lopen} F(\bar{x}, \bar{y}) = \operatorname{sur} F(\bar{x}, \bar{y})$.

(iv) If X and Y are Banach spaces and F is a closed convex process, that is, $\operatorname{gph} F$ *is a closed convex cone in $X \times Y$, then*

$$\operatorname{lopen} F(0,0) = \operatorname{sur} F(0,0) = \sup\{\varrho > 0 : F(\mathbb{B}_X) \supset \varrho \mathbb{B}_Y\} = \inf\{\|x^*\| : x^* \in F^*(\mathbb{S}_{Y^*})\},$$

where $F^* : Y^* \to X^*$ is the adjoint process to F defined by

$$F^*(y^*) = \{x^* \in X^* : \langle x^*, x\rangle \leq \langle y^*, y\rangle \text{ for each } (x,y) \in \operatorname{gph} F\}.$$

Proof. Statement (i) follows immediately from the definitions of $\operatorname{sur} F(\bar{x}, \bar{y})$ and the limes inferior, while (iv) is [18, Theorem 7.9]. Assume without any loss of generality that $\bar{x} = 0$ and $\bar{y} = 0$.

(ii) By assumption, there is $a \in (0,1]$ such that $\tau \operatorname{gph} F \subset \operatorname{gph} F$ for each $\tau \in [0,a]$. Then (23) implies that

$$\tau\beta\mathbb{B}_Y \subset F(\tau\alpha\mathbb{B}_X) \quad \text{for each} \quad \tau \in [0,a].$$

Indeed, fix any such τ. Pick an arbitrary $y \in \tau\beta\mathbb{B}_Y$. Then $v := y/\tau \in \beta\mathbb{B}_Y$. By (23), there is $u \in X$ such that $v \in F(u)$ and $\|u\| \leq \alpha$. Then $x := \tau u \in \tau\alpha\mathbb{B}_X$. Moreover, $(x,y) = \tau(u,v) \in \tau \operatorname{gph} F \subset \operatorname{gph} F$. Thus $y \in F(x)$.
Set $c := \beta/\alpha$ and $\varepsilon := \alpha a$. Fix any $t \in (0, \varepsilon)$. Then $\tau := t/\alpha \in (0,a)$, and consequently,

$$F(t\mathbb{B}_X) = F(\tau\alpha\mathbb{B}_X) \supset \tau\beta\mathbb{B}_Y = ct\mathbb{B}_Y.$$

(iii) By (i), it suffices to show that $\operatorname{lopen} F(0,0) \leq \operatorname{sur} F(0,0)$. Fix arbitrary $c, \tilde{c} \in (0, \operatorname{lopen} F(0,0))$ with $c < \tilde{c}$. Find $\alpha \in (0,1)$ such that $\tilde{c}\alpha\mathbb{B}_Y \subset F(\alpha\mathbb{B}_X)$, and then $r > 0$ such that $c(\alpha+r)+r < \tilde{c}\alpha$. Fix any $(x,y) \in \operatorname{gph} F$ with $\|x\| \leq r$ and $\|y\| \leq r$. Then

$$\mathbb{B}[y, c(\alpha+r)] \subset (c(\alpha+r)+r)\mathbb{B}_Y \subset \tilde{c}\alpha\mathbb{B}_Y \subset F(\alpha\mathbb{B}_X) \subset F(\mathbb{B}[x, \alpha+r]).$$

As in the proof of (ii), with $a := 1$, $\beta := c(\alpha+r)$, and $(\bar{x}, \bar{y}, \alpha)$ replaced by $(x, y, \alpha+r)$, we conclude that for any $t \in (0, \alpha+r)$ we have $\mathbb{B}[y, ct] \subset F(\mathbb{B}[x,t])$. Since α and r are independent of (x,y), we obtain that $\operatorname{sur} F(0,0) \geq c$. Letting $c \uparrow \operatorname{lopen} F(0,0)$ we get the desired estimate. $\square$

To illustrate the difference between the regularity properties we provide the following examples.

Example 2.3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x + \frac{x^3}{|x|}\left|\sin\left(\frac{1}{x}\right)\right| & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then f is locally Lipschitz around 0, Fréchet differentiable at 0 (and almost everywhere) but not strictly differentiable at 0, and there is no neighborhood U of 0 such that f is differentiable on U. Moreover, f is semiregular (not strongly), strongly subregular at 0, and $\operatorname{sur} f(0) = \liminf\limits_{x\to 0} \operatorname{lopen} f(x) = 0$, while $f'(0) = \operatorname{lopen} f(0) = \operatorname{popen} f(0) = 1$. In particular, the first inequality in Proposition 2.2 (i) is strict.

Example 2.4. Consider a function $f : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$f(x) := \begin{cases} x, & \text{if } x \le 0, \\ x - \frac{1}{n}, & \text{if } \frac{1}{n} < x \le \frac{1}{n-1}, \quad n = 3, 4, \dots, \\ x - \frac{1}{2}, & \text{if } x > \frac{1}{2}, \end{cases}$$

and its epigraphical mapping $F(x) := \{y \in \mathbb{R} : y \ge f(x)\}$, $x \in \mathbb{R}$. It is easy to check that $\operatorname{lopen} F(x, y) = \infty$ if $y > f(x)$ and $\operatorname{lopen} F(x, y) = 1$ if $y = f(x)$. Hence,

$$\liminf_{r\downarrow 0} \{\operatorname{lopen} F(x, y) : \ (x, y) \in \operatorname{gph} F \cap \big(I\!\!B(0, r) \times I\!\!B(0, r)\big)\} = 1.$$

Take any $r > 0$ and $\varepsilon > 0$, and choose an index $n \in \mathbb{N}$ such that $x_n := \frac{1}{n} + \frac{1}{n^2} < r$ and $t_n := \frac{1}{n} < \varepsilon$. Then $y_n := f(x_n) = \frac{1}{n^2} < r$ and

$$\sup \{c > 0 : I\!\!B[y_n, ct_n] \subset F(I\!\!B[x_n, t_n])\} = \frac{1}{n}.$$

Hence,

$$\inf_{(x,y)\in \operatorname{gph} F\cap \big(B(0,r)\times B(0,r)\big)} \ \inf_{t\in(0,\varepsilon)} \sup\{c > 0 : I\!\!B[y, ct] \subset F(I\!\!B[x, t])\} = 0,$$

and therefore $\operatorname{sur} F(0, 0) = 0$. Consequently, the second inequality in Proposition 2.2 (i) is strict.

3 Primal and dual conditions

It follows directly from Definition 1.2(B1) that a mapping $F : X \rightrightarrows Y$ is metrically semiregular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if and only if the quantity

$$\text{(24)} \qquad \operatorname{SeR}[F](\bar{x}, \bar{y}) := \liminf_{y\to\bar{y},\, y\notin F(\bar{x})} \frac{\varrho(y, \bar{y})}{\operatorname{dist}(\bar{x}, F^{-1}(y))}$$

is positive (with the convention that the limit in (24) is ∞ when $\bar{y} \in \operatorname{int} F(\bar{x})$). It is easy to check that $\operatorname{SeR}[F](\bar{x}, \bar{y})$ coincides with the reciprocal of the exact lower bound of all $\kappa > 0$ such that (9) holds true for some neighborhood V of $\bar{y}$, and hence, $\operatorname{SeR}[F](\bar{x}, \bar{y}) = \operatorname{lopen} F(\bar{x}, \bar{y})$.

Theorem 3.1. *Let (X, d) be a metric space, (Y, ϱ) a complete metric space, and let $F : X \rightrightarrows Y$ be a set-valued mapping such that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and the function $y \longmapsto \operatorname{dist}(\bar{x}, F^{-1}(y))$ is upper semicontinuous near $\bar{y}$. Set*

$$\text{(25)} \qquad \varphi(y) := \begin{cases} \dfrac{\varrho(y, \bar{y})}{\operatorname{dist}(\bar{x}, F^{-1}(y))}, & \text{if } y \ne \bar{y}, \\ 0, & \text{otherwise}, \end{cases}$$

(26) $$\overline{|\nabla F|}^{\diamond}_{\rm SeR}(\bar x,\bar y) := \liminf_{y\to\bar y,\; y\notin F(\bar x)} \varrho(y,\bar y)\sup_{v\ne y}\frac{\varphi(y)-\varphi(v)}{\varrho(y,v)}.$$

Then

(27) $$\frac{1}{2}\overline{|\nabla F|}^{\diamond}_{\rm SeR}(\bar x,\bar y) \le {\rm SeR}[F](\bar x,\bar y) \le \overline{|\nabla F|}^{\diamond}_{\rm SeR}(\bar x,\bar y).$$

In particular, if numbers $c>0$ and $r>0$ are such that, for any $y\in I\!\!B[\bar y,r]\setminus F(\bar x)$, there is a vector $v\in Y$ satisfying

$$\varrho(y,\bar y)\,(\varphi(y)-\varphi(v)) > c\,\varrho(y,v),$$

then ${\rm SeR}[F](\bar x,\bar y)\ge c/2$.

Proof. In view of (24),

(28) $${\rm SeR}[F](\bar x,\bar y) = \liminf_{y\to\bar y,\; y\notin F(\bar x)} \varphi(y).$$

We prove the first inequality in (27). If ${\rm SeR}[F](\bar x,\bar y)=\infty$, the inequality holds trivially. Let ${\rm SeR}[F](\bar x,\bar y)<\gamma<\infty$. We are going to show that $\overline{|\nabla F|}^{\diamond}_{\rm SeR}(\bar x,\bar y)\le 2\gamma$. Note that φ is lower semicontinuous near $\bar y$ and $\varphi(y)\ge 0$ for all $y\in Y$. Choose a number $\delta>0$ such that φ is lower semicontinuous on $I\!\!B[\bar y,3\delta]$. By (28), there exists a point $y'\in I\!\!B[\bar y,\delta]$ such that $y'\notin F(\bar x)$ and $\varphi(y')<\gamma$. Set $\delta':=\varrho(y',\bar y)$. Then $0<\delta'\le\delta$. Employing the Ekeland variational principle, we find a point $\hat y\in I\!\!B(y',\delta')$ such that $\varphi(\hat y)\le\varphi(y')$ and

(29) $$\varphi(\hat y)\le\varphi(v)+\frac{\gamma}{\delta'}\varrho(\hat y,v)$$

for all $v\in I\!\!B[\bar y,3\delta]$. Since $\varphi(\hat y)\le\varphi(y')<\infty$, in view of (25), we have either $\hat y\notin F(\bar x)$ or $\hat y=\bar y$. At the same time,

$$\varrho(\hat y,\bar y)\ge\varrho(y',\bar y)-\varrho(\hat y,y')>0.$$

Thus, $\hat y\ne\bar y$, and consequently, $\hat y\notin F(\bar x)$. Note that

$$\varrho(\hat y,\bar y)\le\varrho(\hat y,y')+\varrho(y',\bar y)<2\delta'.$$

If $v\notin I\!\!B[\bar y,3\delta]$, then

$$\varphi(\hat y)\le\varphi(y')<\gamma\le\frac{\gamma}{\delta'}(3\delta-2\delta')<\frac{\gamma}{\delta'}(\varrho(v,\bar y)-\varrho(\hat y,\bar y))\le\frac{\gamma}{\delta'}\varrho(\hat y,v)\le\varphi(v)+\frac{\gamma}{\delta'}\varrho(\hat y,v).$$

Hence, inequality (29) holds true for all $v\in Y$, and consequently,

$$\varrho(\hat y,\bar y)\sup_{v\ne\hat y}\frac{\varphi(\hat y)-\varphi(v)}{\varrho(\hat y,v)}<2\delta'\frac{\gamma}{\delta'}=2\gamma.$$

Thus,

$$\inf_{y\in B(\bar y,2\delta)\setminus F(\bar x)}\varrho(y,\bar y)\sup_{v\ne y}\frac{\varphi(y)-\varphi(v)}{\varrho(y,v)}<2\gamma.$$

Passing to the limit as $\delta\downarrow 0$, we obtain $\overline{|\nabla F|}^{\diamond}_{\rm SeR}(\bar x,\bar y)\le 2\gamma$. Since $\gamma>{\rm SeR}[F](\bar x,\bar y)$ is arbitrary, the first inequality in (27) is proved. Given any $y\ne\bar y$, we have

$$\varrho(y,\bar y)\sup_{v\ne y}\frac{\varphi(y)-\varphi(v)}{\varrho(y,v)}\ge\varrho(y,\bar y)\frac{\varphi(y)-\varphi(\bar y)}{\varrho(y,\bar y)}=\varphi(y).$$

In view of the representations (26) and (28), this proves the second inequality in (27). □

Remark 3.2. The second inequality in (27) is valid without the assumptions of the completeness of Y and upper semicontinuity of the function $y \longmapsto \operatorname{dist}(\bar{x}, F^{-1}(y))$. The last property holds, for example, if F^{-1} is lower semicontinuous, that is, when F is open at the corresponding reference point.

Let X and Y be normed spaces. Given a set $\Omega \subset X$ and a point $\bar{x} \in \Omega$, the *Fréchet normal cone* to Ω at $\bar{x}$, denoted by $\widehat{N}_\Omega(\bar{x})$, is the set of all $x^* \in X^*$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \varepsilon \|x - \bar{x}\| \quad \text{whenever} \quad x \in \Omega \cap I\!B(\bar{x}, \delta).$$

For a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, the *Fréchet coderivative* of F at $(\bar{x}, \bar{y})$ acts from Y^* to the subsets of X^* and is defined as

$$Y^* \ni y^* \longmapsto \widehat{D}^* F(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in X^* : \ (x^*, -y^*) \in \widehat{N}_{\operatorname{gph} F}(\bar{x}, \bar{y}) \right\}.$$

We have the following dual necessary condition for semiregularity [29, Theorem 6 (iv)].

Theorem 3.3. *Consider a mapping $F : X \rightrightarrows Y$ between normed spaces X and Y and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$\operatorname{SeR}[F](\bar{x}, \bar{y}) \le \inf_{y^* \in \mathbb{S}_{Y^*}} \{\|x^*\| : \ x^* \in \widehat{D}^* F(\bar{x}, \bar{y})(y^*)\}.$$

Hence, if F is semiregular at $(\bar{x}, \bar{y})$ then

$$\widehat{D}^* F^{-1}(\bar{y}, \bar{x})(0) = \{0\}.$$

In finite dimensions, using Brouwer's fixed point theorem, we get:

Theorem 3.4. *Consider a point $\bar{x} \in \mathbb{R}^n$ along with a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ which is both defined and continuous in a vicinity of $\bar{x}$. Suppose that there is a surjective linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ such that* $\operatorname{calm}(f - A)(\bar{x}) < \operatorname{sur} A$. *Then $n \ge m$ and*

$$\operatorname{lopen} f(\bar{x}) \ge \operatorname{sur} A - \operatorname{calm}(f - A)(\bar{x}) > 0.$$

Proof. Clearly, if $n < m$, there is no chance to have a linear surjection from $\mathbb{R}^n$ onto $\mathbb{R}^m$. Therefore $n \ge m$. Without any loss of generality assume that $\bar{x} = 0$ and $f(\bar{x}) = 0$. Let us identify a linear mapping A with its matrix representation in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Then $A \in \mathbb{R}^{m\times n}$ has a full rank m. Hence the (symmetric) matrix $AA^T \in \mathbb{R}^{m\times m}$ is non-singular. Let $B := A^T(AA^T)^{-1} \in \mathbb{R}^{n\times m}$. Note that $\operatorname{sur} A$ is equal to the smallest singular value of A and $\|B\|$ is equal to the largest singular value of B. As

$$B^T B = \left(A^T(AA^T)^{-1}\right)^T A^T(AA^T)^{-1} = \left((AA^T)^{-1}\right)^T = \left((AA^T)^T\right)^{-1} = (AA^T)^{-1},$$

the singular values of A and B are reciprocal. Therefore $\|B\| = 1/\operatorname{sur} A$. Pick any $c \in (0, \operatorname{sur} A - \operatorname{calm}(f - A)(0))$. Let $\gamma > 0$ be such that $\operatorname{calm}(f - A)(0) + c + \gamma < \operatorname{sur} A$. By the assumptions, there is $\varepsilon > 0$ such that f is continuous on $I\!B(0, 2\varepsilon)$ and

$$\|f(x) - Ax\| \le (\operatorname{calm}(f - A)(0) + \gamma)\, \|x\| \quad \text{whenever} \quad x \in I\!B(0, 2\varepsilon). \tag{30}$$

Fix any $t \in (0, \varepsilon)$. Pick an arbitrary $y \in I\!\!B[0, ct]$. Define the mapping $h_y : I\!\!B_{\mathbb{R}^n} \to \mathbb{R}^m$ by

$$h_y(u) := \frac{1}{t} B\left(A(tu) - f(tu) + y\right), \quad u \in I\!\!B_{\mathbb{R}^n}. \tag{31}$$

Note that, for every $u \in I\!\!B[0, 2]$, we have $tu \in I\!\!B(0, 2\varepsilon)$. In particular, h_y is well defined and continuous on $I\!\!B_{\mathbb{R}^n}$. Given $u \in I\!\!B_{\mathbb{R}^n}$, inequality (30) with $x := tu$ implies that

$$\begin{aligned}
\|h_y(u)\| &\leq \frac{1}{t}\|B\| \left\|(A(tu) - f(tu)) + y\right\| \\
&\leq \frac{\|B\|}{t}\big((\operatorname{calm}(f - A)(0) + \gamma)\,\|tu\| + \|y\|\big) \leq \frac{\|B\|}{t}\big((\operatorname{calm}(f - A)(0) + \gamma)t + ct\big) \\
&= \|B\|(\operatorname{calm}(f - A)(0) + c + \gamma) < \|B\| \operatorname{sur} A = 1.
\end{aligned}$$

Therefore h_y maps $I\!\!B_{\mathbb{R}^n}$ into itself. Using Brouwer's fixed point theorem, we find $u_y \in I\!\!B_{\mathbb{R}^n}$ such that $h_y(u_y) = u_y$. Hence $Ah_y(u_y) = Au_y$. As $AB = I_{\mathbb{R}^m}$, the definition of h_y implies that

$$A(tu_y) - f(tu_y) + y = tA(u_y) = A(tu_y).$$

Then $x_y := tu_y$ is such that $f(x_y) = y$ and $\|x_y\| \leq t$. Hence $y \in f(I\!\!B[0, t])$. Since $y \in I\!\!B[0, ct]$ was chosen arbitrarily, we have $I\!\!B[0, ct] \subset f(I\!\!B[0, t])$. Therefore $\operatorname{lopen} f(\bar{x}) \geq c$. Letting $c \uparrow (\operatorname{sur} A - \operatorname{calm}(f - A)(0))$, we finish the proof. □

The above statement is quite similar to Theorem 1.4 with one important difference. If, in addition to the assumptions of Theorem 3.4, the mapping A is invertible, then $n = m$ and $\operatorname{sur} A = 1/\|A^{-1}\|$. Consequently,

$$\operatorname{lopen} f(\bar{x}) \geq 1/\|A^{-1}\| - \operatorname{calm}(f - A)(\bar{x}).$$

However, Example 2.3 shows that one cannot conclude that f is *strongly* semiregular at $\bar{x}$, that is, that the mapping f^{-1} has a single-valued localization around $(\bar{x}, f(\bar{x}))$. This example also shows that we can have $\operatorname{sur} f(\bar{x}) = 0$ although all the assumptions of Theorem 3.4 hold.

We immediately obtain that the surjectivity of the Fréchet derivative at the reference point implies the openness with a linear rate of the mapping in question at this point. The following result improves [13, Corollary 1G.6] where a weaker property of openness is shown. This statement was motivated by a discussion of the second author with V. Kaluža, who suggested a proof using Borsuk-Ulam theorem.

Corollary 3.5. *Consider a point $\bar{x} \in \mathbb{R}^n$ along with a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ which is both defined and continuous in a vicinity of $\bar{x}$ and Fréchet differentiable at $\bar{x}$. If $f'(\bar{x})$ is surjective, then $n \geq m$ and* $\operatorname{lopen} f(\bar{x}) \geq \operatorname{sur} f'(\bar{x}) > 0$.

We also obtain an extension of [13, Theorem 1G.3].

Theorem 3.6. *Suppose that the assumptions of Theorem 3.4 hold and denote by Σ the set of all selections for f^{-1} defined in a vicinity of $\bar{y} := f(\bar{x})$. Then*

$$\inf_{\sigma \in \Sigma} \operatorname{calm} \sigma(\bar{y}) \leq \frac{1}{\operatorname{sur} A - \operatorname{calm}(f - A)(\bar{x})}$$

and

$$\inf_{\sigma\in\Sigma} \operatorname{calm}(\sigma - A^T(AA^T)^{-1})(\bar{y}) \leq \frac{\operatorname{calm}(f-A)(\bar{x})}{\operatorname{sur} A\,(\operatorname{sur} A - \operatorname{calm}(f-A)(\bar{x}))}\ .$$

In particular, if f is Fréchet differentiable at $\bar{x}$, then there is $\sigma \in \Sigma$ which is Fréchet differentiable at $\bar{y}$ and

$$\sigma'(\bar{y}) = [f'(\bar{x})]^*(f'(\bar{x})\,[f'(\bar{x})]^*)^{-1}).$$

Proof. Let B, c, γ, ε, and t be as in the proof of Theorem 3.4. Consider the mapping

$$V := I\!B[0, ct] \ni y \longmapsto \sigma(y) := x_y \in I\!B[0, t] =: U,$$

where x_y is such that $h_y(x_y/t) = x_y/t$ with h_y defined in (31). We already know that $f(\sigma(y)) = y$ for each $y \in V$. Moreover, given $y \in V$, we have by (31) and (30)

$$\begin{aligned}\|\sigma(y)\| &= \|th_y(\sigma(y)/t)\| = \|B\,(A(\sigma(y)) - f(\sigma(y)) + y)\,\| \\ &\leq \|B\|\big((\operatorname{calm}(f-A)(0)+\gamma)\,\|\sigma(y)\| + \|y\|\big).\end{aligned}$$

As $\|B\| = 1/\operatorname{sur} A$ and $\operatorname{calm}(f-A)(0)+\gamma < \operatorname{sur} A$, the above estimate implies that

$$\|\sigma(y)\| \leq \frac{1}{\operatorname{sur} A - \operatorname{calm}(f-A)(0) - \gamma}\,\|y\| \quad \text{whenever} \quad y \in V. \tag{32}$$

Moreover, for a fixed $y \in V$, we have by (31) and (30)

$$\begin{aligned}\|\sigma(y) - By\| &= \|th_y(\sigma(y)/t) - By\| = \|B\,(A(\sigma(y)) - f(\sigma(y)))\,\| \\ &\leq \|B\|(\operatorname{calm}(f-A)(0)+\gamma)\,\|\sigma(y)\|.\end{aligned}$$

Using (32), we get

$$\|\sigma(y) - By\| \leq \frac{\operatorname{calm}(f-A)(0)+\gamma}{\operatorname{sur} A\,(\operatorname{sur} A - \operatorname{calm}(f-A)(0) - \gamma)}\,\|y\| \quad \text{whenever} \quad y \in V. \tag{33}$$

As $\gamma > 0$ can be arbitrarily small, (32) and (33), respectively, imply the desired estimates.

To prove the second part, it suffices to observe that if f is Fréchet differentiable at $\bar{x}$ then $\operatorname{calm}(f - f'(\bar{x}))(\bar{x}) = 0$. □

A similar approach as in the proof of Theorem 3.4, but applying Kakutani's fixed point theorem instead of Brouwer's theorem, yields a sufficient condition for openness with a linear rate of a set-valued mapping satisfying certain "strong monotonicity/ellipticity" assumptions.

Theorem 3.7. *Consider positive constants ℓ and r, a point $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, and a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Assume that F has a closed graph and convex values, the set $F(I\!B[\bar{x}, r])$ is bounded, and that one of the following conditions holds:*

(C1) for each $x \in I\!B[\bar{x}, r]$ there is $y \in F(x)$ such that $\langle y - \bar{y}, x - \bar{x}\rangle \geq \ell\|x - \bar{x}\|^2$;

(C2) for each $x \in I\!B[\bar{x}, r]$ there is $y \in F(x)$ such that $\langle \bar{y} - y, x - \bar{x}\rangle \geq \ell\|x - \bar{x}\|^2$.

Then $\operatorname{lopen} F(\bar{x},\bar{y}) \geq \ell$*; more precisely,*

$$\mathbb{B}[\bar{y},\ell t] \subset F(\mathbb{B}[\bar{x},t]) \quad \textit{for each} \quad t \in (0,r]. \tag{34}$$

Proof. Note that (34) for F satisfying (C2) follows by considering the reference point $(\bar{x},-\bar{y})$ and the mapping $-F$, which necessarily satisfies (C1). Suppose that (C1) holds. Assume without any loss of generality that $(\bar{x},\bar{y}) = (0,0)$. Find $m > 0$ such that $F(\mathbb{B}[0,r]) \subset \mathbb{B}[0,m]$.

First, we show that

$$\mathbb{B}[0,ct] \subset F(\mathbb{B}[0,t]) \quad \text{for each} \quad c \in (0,\ell) \quad \text{and each} \quad t \in (0,r]. \tag{35}$$

Let c and t be as in (35). Fix an arbitrary (non-zero) $y \in \mathbb{B}[0,ct]$. Pick $\alpha > 0$ such that

$$2\alpha\ell < 1 \quad \text{and} \quad \alpha(m+cr)^2 < 2(\ell-c)t^2.$$

Define the mapping $H : \mathbb{B}[0,t] \rightrightarrows \mathbb{B}[0,t]$, depending on the choice of (y,c,t,α), by

$$H(u) := \big(u + \alpha(y - F(u))\big) \cap \mathbb{B}[0,t], \quad u \in \mathbb{B}[0,t].$$

Fix any $u \in \mathbb{B}[0,t]$. Using (C1), we find a point $v \in F(u)$ such that $\langle v,u\rangle \geq \ell\|u\|^2$. Let $z := u + \alpha(y-v)$. Then

$$\begin{aligned}
\|z\|^2 &= \|u\|^2 + 2\alpha\langle u, y-v\rangle + \alpha^2\|y-v\|^2 = \|u\|^2 - 2\alpha\langle v,u\rangle + 2\alpha\langle u,y\rangle + \alpha^2\|y-v\|^2 \\
&\leq (1-2\alpha\ell)\|u\|^2 + 2\alpha\|u\|\|y\| + \alpha^2(\|v\|+\|y\|)^2 \\
&\leq (1-2\alpha\ell)t^2 + 2\alpha t(ct) + \alpha^2(m+cr)^2 < \big(1+2\alpha(c-\ell)\big)t^2 + 2\alpha(\ell-c)t^2 = t^2.
\end{aligned}$$

Hence $z \in H(u)$. Consequently, the domain of H is equal to $\mathbb{B}[0,t]$, which is a non-empty compact convex set. Since F has closed graph and convex values, we conclude that H has the same properties. Applying Kakutani's fixed point theorem, we find $u \in \mathbb{B}[0,t]$ such that $u \in H(u)$. This implies that $y \in F(u) \subset F(\mathbb{B}[0,t])$. As $y \in \mathbb{B}[0,ct]$, and also $(c,t) \in (0,\ell)\times(0,r]$ are arbitrary, (35) is proved.

To show (34), fix any $t \in (0,r]$. Pick an arbitrary $y \in \mathbb{B}[0,\ell t]$. Let $y_n := (1-1/n)y$ for each $n \in \mathbb{N}$. Then (y_n) converges to y. For each $n \geq 2$, using (35) with $c := (1-1/n)\ell$, we find $x_n \in \mathbb{R}^n$ such that $y_n \in F(x_n)$ and $\|x_n\| \leq t$. Passing to a subsequence, if necessary, we may assume that (x_n) converges to, say, $x \in \mathbb{R}^n$. Then $\|x\| \leq t$ and $y \in F(x)$ because $\operatorname{gph} F$ is closed. So $F(\mathbb{B}[0,t])$ contains y, which is an arbitrary point in $\mathbb{B}[0,\ell t]$. □

The above statement implies [6, Theorem 1 and Corollary 1] under slightly weaker assumptions and the above proof also shows that there is no need to extend the locally defined mapping under consideration on the whole space.

Corollary 3.8. *Consider positive constants ℓ and r, a point $\bar{x} \in \mathbb{R}^n$, and a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with* $\operatorname{dom} F = \mathbb{B}[\bar{x},r]$*. Assume that F is upper semicontinuous, has compact convex values, and*

$$\forall\, x \in \mathbb{B}[\bar{x},r]\ \forall \bar{y} \in F(\bar{x})\ \exists y \in F(x) : \langle \bar{y}-y, x-\bar{x}\rangle \geq \ell\|x-\bar{x}\|^2. \tag{36}$$

Then, for each $y \in \mathbb{R}^n$ such that $\operatorname{dist}(y,F(\bar{x})) \leq r\ell$*, there is $x \in \mathbb{B}[\bar{x},r]$ satisfying*

$$y \in F(x) \quad \textit{and} \quad \|x-\bar{x}\| \leq \frac{1}{\ell}\operatorname{dist}\big(y,F(\bar{x})\big).$$

Proof. Since F is upper semicontinuous and has compact values, using a standard compactness argument we conclude that the set $F(\mathbb{B}[\bar{x},r])$ is bounded. Moreover, $\operatorname{gph} F$ is closed since F is upper semicontinuous with closed values, closed domain, and bounded range. Fix any $y \in \mathbb{R}^n$ with $r\ell \geq \operatorname{dist}(y, F(\bar{x}))$ (> 0). As $F(\bar{x})$ is a compact set, there is $\bar{y} \in \mathbb{R}^n$ such that $\|y - \bar{y}\| = \operatorname{dist}(y, F(\bar{x}))$. Now (36) implies that (C2) is satisfied. By (34) with $t := \|y - \bar{y}\|/\ell \leq r$, there is $x \in \mathbb{B}[\bar{x}, \|y - \bar{y}\|/\ell] = \mathbb{B}[\bar{x}, \operatorname{dist}(y, F(\bar{x}))/\ell] \subset \mathbb{B}[\bar{x}, r]$ such that $y \in F(x)$. □

We also get:

Corollary 3.9. *Consider positive constants ℓ and r, a point $\bar{x} \in \mathbb{R}^n$, and a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with* $\operatorname{dom} F = \mathbb{B}[\bar{x}, 2r]$. *Assume that F is upper semicontinuous, has compact convex values, and*

$$\forall x, x' \in \mathbb{B}[\bar{x}, 2r] \ \forall y \in F(x) \ \exists y' \in F(x') : \langle y - y', x' - x\rangle \geq \ell\|x' - x\|^2. \tag{37}$$

Then $\operatorname{sur} F(\bar{x}, \bar{y}) \geq \ell$*; more precisely,*

$$\mathbb{B}[y, \ell t] \subset F(\mathbb{B}[x, t]) \quad \textit{whenever} \quad (x, y) \in (\mathbb{B}[\bar{x}, r] \times \mathbb{B}[\bar{y}, r]) \cap \operatorname{gph} F \ \textit{and} \ t \in (0, r]. \tag{38}$$

Proof. Fix any (x, y) and t as in (38). Then $\mathbb{B}[x, r] \subset \mathbb{B}[\bar{x}, 2r]$. Hence, (37) implies that for each $x' \in \mathbb{B}[x, r]$ there is $y' \in F(x')$ such that $\langle y - y', x' - x\rangle \geq \ell\|x' - x\|^2$, which is (C2) with $(\bar{x}, \bar{y}, x, y)$ replaced by (x, y, x', y'). As in the proof of Corollary 3.8, we conclude that all the assumptions of Theorem 3.7 with $(\bar{x}, \bar{y}) := (x, y)$ are satisfied. □

Remark 3.10. Given $\ell > 0$, condition (36) holds, in particular, if F is relaxed one-sided Lipschitz (ROSL) on $\mathbb{B}[\bar{x}, r]$ with the constant $-\ell$ in the sense of [6, Definition 1], that is,

$$\forall x, x' \in \mathbb{B}[\bar{x}, r] \ \forall y \in F(x) \ \exists y' \in F(x') : \langle y - y', x - x'\rangle \leq -\ell\|x - x'\|^2.$$

Condition (37) means that F is ROSL on $\mathbb{B}[\bar{x}, 2r]$ with the constant $-\ell$. Up to minor changes in notation, Corollary 3.9 seems to be the statement which the authors tried to formulate and prove in [6, Corollary 2 (ii)] under an additional assumption that F is (Hausdorff) continuous. However, their formulation seems to be not completely correct, since (local) metric regularity at $(\bar{x}, \bar{y})$ presumes the reference point *to lie in* $\operatorname{gph} F$. So the assumption in [6, Corollary 2 (ii)] that $\operatorname{dist}(\bar{y}, F(\bar{x}))$ is small enough holds trivially. Also note that "a slightly generalized definition of metric regularity" in [6] is nothing else but the *usual* definition of this property because $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in [13] means neither that $\operatorname{dom} F = \mathbb{R}^n$ nor that $\bar{x}$ is an interior point of $\operatorname{dom} F$.

Remark 3.11. A sufficient condition for semiregularity of a continuous (possibly nonsmooth) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ by using equi-invertibility of a pseudo-Jacobian can be found in [21, Theorem 3.2.1].

4 Semiregularity of the sum

In this section, we prove that the sum of two set-valued mappings is metrically semiregular provided that one is metrically regular while the other is pseudo-Lipschitz. This statement, in Banach spaces, was published in [1] but without a detailed proof. We present an essentially simplified proof here. Note that the proof can be easily modified for X being a metric space and Y being a linear metric space with a shift-invariant metric.

Theorem 4.1. *Consider normed spaces X and Y, points $\bar{x} \in X$ and $\bar{y}$, $\bar{z} \in Y$, and set-valued mappings F, $G : X \rightrightarrows Y$ such that $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{z}) \in \operatorname{gph} G$. Suppose that a, b, κ, and ℓ are positive real numbers such that $\kappa\ell < 1$, the set $\operatorname{gph} F \cap \big(\mathbb{B}[\bar{x}, a] \times \mathbb{B}[\bar{y}, 2a]\big)$ is closed, the set $\operatorname{gph} G \cap \big(\mathbb{B}[\bar{x}, a] \times \mathbb{B}[\bar{z}, a]\big)$ is complete,*

$$\operatorname{dist}\big(x, F^{-1}(y)\big) \leq \kappa \operatorname{dist}\big(y, F(x)\big) \quad \text{for all} \quad (x, y) \in \mathbb{B}(\bar{x}, a) \times \mathbb{B}(\bar{y}, a), \quad \text{and} \tag{39}$$

$$G(x_1) \cap \mathbb{B}(\bar{z}, a) \subset G(x_2) + \ell\|x_1 - x_2\|\mathbb{B}_Y \quad \text{for all} \quad x_1, x_2 \in \mathbb{B}(\bar{x}, a). \tag{40}$$

Then, for any $\beta > 0$ such that $2\beta \max\{1, \kappa\} < a(1 - \kappa\ell)$, we have

$$\operatorname{dist}\big(\bar{x}, (F + G)^{-1}(y)\big) \leq \frac{\kappa}{1 - \kappa\ell} \operatorname{dist}\big(y, F(\bar{x}) + \bar{z}\big) \quad \text{for all} \quad y \in \mathbb{B}(\bar{y} + \bar{z}, \beta). \tag{41}$$

Proof. Fix any $y \in \mathbb{B}(\bar{y} + \bar{z}, \beta)$. If $y - \bar{z} \in F(\bar{x})$ then (41) holds trivially. Assume that

$$y - \bar{z} \notin F(\bar{x}). \tag{42}$$

Set $W := \operatorname{gph} G \cap \big(\mathbb{B}[\bar{x}, a] \times \mathbb{B}[\bar{z}, a]\big)$ and let $\varrho : W \longrightarrow [0, \infty)$ be defined by

$$\varrho((x_1, z_1), (x_2, z_2)) := \max\{\|x_1 - x_2\|, \kappa\|z_1 - z_2\|\}, \quad (x_1, z_1), (x_2, z_2) \in W.$$

Then (W, ϱ) is a complete metric space. Let $\varphi : W \longrightarrow [0, \infty]$ be defined by

$$\varphi(x, z) \;:=\; \liminf_{u \to x,\ u \in B[\bar{x}, a]} \operatorname{dist}\,(y, F(u) + z), \quad (x, z) \in W. \tag{43}$$

The function φ is lower semicontinuous. Indeed, fix any $(x, z) \in W$. Let $\eta < \varphi(x, z)$ be an arbitrary number. Pick $\xi > 0$ such that $\eta + \xi < \varphi(x, z)$. Using (43), we find $r \in (0, \kappa\xi)$ such that

$$\eta + \xi < \inf_{u \in B(x, 2r) \cap B[\bar{x}, a]} \operatorname{dist}\,(y, F(u) + z).$$

Fix any $(x', z') \in W$ with $\varrho((x', z'), (x, z)) < r$. Then $\mathbb{B}(x', r) \subset \mathbb{B}(x, 2r)$ and $\|z' - z\| < r/\kappa < \xi$. Hence,

$$\begin{aligned} \eta + \xi \;&<\; \inf_{u \in B(x, 2r) \cap B[\bar{x}, a]} \operatorname{dist}\,(y, F(u) + z) \leq \inf_{u \in B(x', r) \cap B[\bar{x}, a]} \operatorname{dist}\,(y, F(u) + z) \\ &\leq\; \inf_{u \in B(x', r) \cap B[\bar{x}, a]} \operatorname{dist}\,(y, F(u) + z') + \|z - z'\| \leq \varphi(x', z') + \xi. \end{aligned}$$

We showed that $\eta < \varphi(x', z')$ for each $(x', z') \in W$ with $\varrho((x', z'), (x, z)) < r$. Therefore φ is lower semicontinuous at (x, z), which is an arbitrary point in W.

Set $S := \varphi^{-1}(0)$. Then

$$S = \{(x, z) \in \mathbb{B}[\bar{x}, a] \times \mathbb{B}[\bar{z}, a] :\ z \in G(x) \text{ and } y - z \in F(x)\}. \tag{44}$$

Indeed, for every $(x, z) \in \mathbb{B}[\bar{x}, a] \times \mathbb{B}[\bar{z}, a]$ such that $z \in G(x)$ and $y - z \in F(x)$ (if there is any) we have $(x, z) \in W$ and $0 \leq \varphi(x, z) \leq \operatorname{dist}\,(y, F(x) + z) = 0$; thus $(x, z) \in S$. Conversely, fix an arbitrary $(x, z) \in S$ (if there is any). Then $(x, z) \in W$, in particular, $z \in G(x)$, and $\varphi(x, z) = 0$. Let (u_n) be a sequence in $\mathbb{B}[\bar{x}, a]$ converging to x such that, for each $n \in \mathbb{N}$, we have

$\operatorname{dist}(y, F(u_n) + z) < 1/n$. For each $n \in \mathbb{N}$, find $v_n \in F(u_n)$ with $\|y - z - v_n\| < 1/n$. Then (v_n) converges to $y - z$. We have $\|y - z - \bar{y}\| \le \|y - (\bar{y} + \bar{z})\| + \|\bar{z} - z\| < \beta + a < 2a$. Hence, for $n \in \mathbb{N}$ large enough, we have $(u_n, v_n) \in \operatorname{gph} F \cap \big(I\!B[\bar{x}, a] \times I\!B[\bar{y}, 2a]\big)$. Since the latter set is closed, we conclude that $y - z \in F(x)$. We have proved (44).

Taking into account (42) and (44), we have that $(\bar{x}, \bar{z}) \notin S$, that is, $\varphi(\bar{x}, \bar{z}) > 0$. And, as $\bar{y} \in F(\bar{x})$, we get that

$$0 < \varphi(\bar{x}, \bar{z}) \le \operatorname{dist}(y, F(\bar{x}) + \bar{z}) \le \|y - \bar{y} - \bar{z}\| < \beta. \tag{45}$$

In particular, the function φ is proper. Set $\tau := \kappa/(1 - \kappa\ell)$. By the assumptions, we have

$$2\beta\tau < a \quad \text{and} \quad \beta(1 + \tau/\kappa) < a. \tag{46}$$

We `claim` that, *for every* $(u, v) \in W$ *satisfying*

$$\varrho((u, v), (\bar{x}, \bar{z})) \le \beta\tau \quad \textit{and} \quad 0 < \varphi(u, v) \le \varphi(\bar{x}, \bar{z}), \tag{47}$$

we have

$$\sup_{(u', v') \in W \setminus \{(u, v)\}} \frac{\varphi(u, v) - \varphi(u', v')}{\varrho((u, v), (u', v'))} \ge \frac{1}{\tau}\ . \tag{48}$$

To prove this, consider any pair $(u, v) \in W$ satisfying (47). Using (43), we find a sequence (u_n) in $I\!B[\bar{x}, a]$ converging to u such that

$$\lim_{n \to \infty} \operatorname{dist}(y, F(u_n) + v) = \varphi(u, v) \in (0, \beta) \qquad \text{(by (45) and (47))}. \tag{49}$$

By (47) and (46), we have $\|u - \bar{x}\| \le \beta\tau < a$ and $\|(y - v) - \bar{y}\| \le \|y - (\bar{y} + \bar{z})\| + \|\bar{z} - v\| < \beta + \beta\tau/\kappa < a$. In view of (49), we can assume without any loss of generality that, for each $n \in \mathbb{N}$, $u_n \in I\!B(\bar{x}, a) \cap \operatorname{dom} F$ and $y - v \notin F(u_n)$. For each $n \in \mathbb{N}$, we get from (39) with u_n and $y - v$ in place of x and y, respectively, that

$$\operatorname{dist}\big(u_n, F^{-1}(y - v)\big) \le \kappa \operatorname{dist}\big(y - v, F(u_n)\big) \quad (< \infty),$$

and consequently, there is a point $u'_n \in F^{-1}(y - v)$ such that

$$\|u_n - u'_n\| < \big(1 + (n\kappa)^{-1}\big) \operatorname{dist}\big(u_n, F^{-1}(y - v)\big) \le (\kappa + n^{-1}) \operatorname{dist}\big(y - v, F(u_n)\big). \tag{50}$$

As a consequence, we have

$$\limsup_{n \to \infty} \|u - u'_n\| = \limsup_{n \to \infty} \|u_n - u'_n\| \overset{(50)}{\le} \lim_{n \to \infty} (\kappa + n^{-1}) \operatorname{dist}\big(y - v, F(u_n)\big) \overset{(49)}{=} \kappa\varphi(u, v) \overset{(49)}{<} \kappa\beta \tag{51}$$

and

$$\limsup_{n \to \infty} \|\bar{x} - u'_n\| \ \le \ \|\bar{x} - u\| + \limsup_{n \to \infty} \|u - u'_n\| \overset{(47),(51)}{<} \beta\tau + \kappa\beta < 2\beta\tau \overset{(46)}{<} a.$$

As $\varphi(u, v) > 0$ and $u \in G(v)$, we have $u \notin F^{-1}(y - v)$ thanks to (44), and hence,

$$\liminf_{n \to \infty} \|u - u'_n\| = \liminf_{n \to \infty} \|u_n - u'_n\| \ge \lim_{n \to \infty} \operatorname{dist}\big(u_n, F^{-1}(y - v)\big) = \operatorname{dist}\big(u, F^{-1}(y - v)\big) > 0.$$

In view of the above consideration, neglecting several starting terms and relabeling, if necessary, we may assume that, for each $n \in \mathbb{N}$, we have $u \neq u'_n \in I\!B(\bar{x}, a)$. For each $n \in \mathbb{N}$, using (40), with $x_1 := u$ and $x_2 := u'_n$, we find $v'_n \in G(u'_n)$ such that

$$\|v - v'_n\| \leq \ell \|u - u'_n\| < \|u - u'_n\|/\kappa. \tag{52}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} \|v'_n - \bar{z}\| &\leq \|v - \bar{z}\| + \limsup_{n\to\infty} \|v - v'_n\| \overset{(52)}{\leq} \|v - \bar{z}\| + \kappa^{-1} \limsup_{n\to\infty} \|u - u'_n\| \\ &\overset{(47),(51)}{\leq} \beta\tau/\kappa + \kappa^{-1}(\kappa\beta) = \beta(\tau/\kappa + 1) \overset{(46)}{<} a. \end{aligned}$$

For $n \in \mathbb{N}$ sufficiently large, we have $v'_n \in I\!B[\bar{z}, a]$, hence $(u'_n, v'_n) \in W$, and so

$$\varphi(u'_n, v'_n) \leq \operatorname{dist}\big(y, F(u'_n) + v'_n\big) \leq \|y - (y - v) - v'_n\| = \|v - v'_n\| \leq \ell \|u - u'_n\|. \tag{53}$$

From (52) we get $\varrho\big((u, v), (u'_n, v'_n)\big) = \|u - u'_n\|$ for each $n \in \mathbb{N}$. Summarizing, we conclude that

$$\begin{aligned} \sup_{(u',v') \in W \setminus \{(u,v)\}} \frac{\varphi(u, v) - \varphi(u', v')}{\varrho((u, v), (u', v'))} &\geq \limsup_{n\to\infty} \frac{\varphi(u, v) - \varphi(u'_n, v'_n)}{\varrho((u, v), (u'_n, v'_n))} \\ &\overset{(49),(53)}{\geq} \limsup_{n\to\infty} \frac{\operatorname{dist}(y - v, F(u_n)) - \ell\|u - u'_n\|}{\|u - u'_n\|} \\ &\overset{(50)}{\geq} \limsup_{n\to\infty} \frac{\|u_n - u'_n\|}{(\kappa + n^{-1})\|u - u'_n\|} - \ell \\ &\geq \lim_{n\to\infty} \frac{\|u - u'_n\| - \|u - u_n\|}{\kappa\|u - u'_n\|} - \ell = \frac{1}{\kappa} - \ell = \frac{1}{\tau}\,. \end{aligned}$$

Inequality (48) is proved and so is our claim.

Recall from (45) that $\varphi(\bar{x}, \bar{z}) \in (0, \beta)$. Take an arbitrary $\varepsilon > 0$ such that $(\tau + \varepsilon)\varphi(\bar{x}, \bar{z}) < \tau\beta$. The Ekeland variational principle yields a point $(u, v) \in W$, satisfying

$$\varrho((u, v), (\bar{x}, \bar{z})) \leq (\tau + \varepsilon)\varphi(\bar{x}, \bar{z}) \tag{54}$$

and $\varphi(u, v) \leq \varphi(\bar{x}, \bar{z})$, such that

$$\varphi(u', v') + \frac{1}{\tau + \varepsilon}\varrho((u, v), (u', v')) \geq \varphi(u, v) \quad \text{for all} \quad (u', v') \in W. \tag{55}$$

Then $v \in G(u)$. Supposing that $y - v \notin F(u)$, we have $\varphi(u, v) > 0$ by (44), and our claim would imply that

$$\frac{1}{\tau + \varepsilon} \overset{(55)}{\geq} \sup_{(u',v') \in W \setminus \{(u,v)\}} \frac{\varphi(u, v) - \varphi(u', v')}{\varrho((u, v), (u', v'))} \overset{(48)}{\geq} \frac{1}{\tau}\,,$$

a contradiction. Hence $y - v \in F(u)$, which means that $u \in (F + G)^{-1}(y)$. Therefore

$$\begin{aligned} \operatorname{dist}\big(\bar{x}, (F + G)^{-1}(y)\big) &\leq \|\bar{x} - u\| \leq \varrho((\bar{x}, \bar{z}), (u, v)) \overset{(54)}{\leq} (\tau + \varepsilon)\varphi(\bar{x}, \bar{z}) \\ &\leq (\tau + \varepsilon) \operatorname{dist}(y, F(\bar{x}) + \bar{z}). \end{aligned}$$

Letting $\varepsilon \downarrow 0$ and noting that $y \in I\!B(\bar{y} + \bar{z}, \beta)$ is arbitrary, we conclude the proof. □

We immediately get the following consequence of the above statement.

Theorem 4.2. *Consider normed spaces* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$, *points* $\bar{x} \in X$ *and* $\bar{y}$, $\bar{z} \in Y$, *and set-valued mappings* F, $G : X \rightrightarrows Y$. *Suppose that* F *has a localization around* $(\bar{x}, \bar{y})$ *with a closed graph and* G *has a localization around* $(\bar{x}, \bar{z})$ *with a complete graph. Then*

$$\operatorname{lopen}(F+G)(\bar{x}, \bar{y}+\bar{z}) \geq \operatorname{sur} F(\bar{x}, \bar{y}) - \operatorname{lip} G(\bar{x}, \bar{z}).$$

Proof. If $\operatorname{lip} G(\bar{x}, \bar{z}) \geq \operatorname{sur} F(\bar{x}, \bar{y})$, then the conclusion holds trivially. Assume that $\operatorname{lip} G(\bar{x}, \bar{z}) < \operatorname{sur} F(\bar{x}, \bar{y})$. Take any $\ell > \operatorname{lip} G(\bar{x}, \bar{z})$ and $\kappa > 1/\operatorname{sur} F(\bar{x}, \bar{y})$ such that $\kappa\ell < 1$. Apply Theorem 4.1 to get that $\operatorname{lopen}(F+G)(\bar{x}, \bar{y}+\bar{z}) \geq 1/\kappa - \ell$. Letting $\ell \downarrow \operatorname{lip} G(\bar{x}, \bar{z})$ and $\kappa \downarrow 1/\operatorname{sur} F(\bar{x}, \bar{y})$ we finish the proof. $\square$

5 Convergence of the Newton-type methods

In this section, we study inexact iterative methods of Newton type for solving the generalized equation (19). We focus on a local convergence analysis of (22) around a reference solution.

Theorem 5.1. *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be Banach spaces. Consider a point* $\bar{x} \in X$ *along with a continuous mapping* $f : X \to Y$ *and a set-valued mapping* $F : X \rightrightarrows Y$ *with closed graph such that* $f(\bar{x}) + F(\bar{x}) \ni 0$. *Suppose that there is* $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$ *which is upper semicontinuous at* $\bar{x} \in \operatorname{int} \operatorname{dom} \mathcal{H}$ *with* $\chi(\mathcal{H}(\bar{x})) < \infty$, *and such that, for each* $A \in \mathcal{H}(\bar{x})$, *the mapping* $G_A : X \rightrightarrows Y$ *defined by*

$$G_A(x) := f(\bar{x}) + A(x - \bar{x}) + F(x), \quad x \in X, \tag{56}$$

is regular around $(\bar{x}, 0)$, *and*

$$\lim_{x \to \bar{x},\ x \neq \bar{x}} \frac{\sup_{A \in \mathcal{H}(x)} \|f(x) - f(\bar{x}) - A(x - \bar{x})\|}{\|x - \bar{x}\|} = 0. \tag{57}$$

Let (R_k) *be a sequence of mappings* $R_k : X \times X \rightrightarrows Y$, $k \in \mathbb{N}_0$, *with closed graphs such that* $(\bar{x}, \bar{x}) \in \operatorname{int}\big(\bigcap_{k \in \mathbb{N}_0} \operatorname{dom} R_k\big)$ *and* $0 \in R_k(\bar{x}, \bar{x})$ *for each* $k \in \mathbb{N}_0$, *and assume that there are positive constants* a, γ, *and* ℓ *satisfying*

$$\chi(\mathcal{H}(\bar{x})) + \ell + \gamma < \inf_{A \in \mathcal{H}(\bar{x})} \operatorname{sur} G_A(\bar{x}, 0) \tag{58}$$

such that

$$\limsup_{x \to \bar{x},\ x \neq \bar{x}} \frac{\sup_{k \in \mathbb{N}_0} \operatorname{dist}\big(0, R_k(x, \bar{x})\big)}{\|x - \bar{x}\|} < \gamma, \tag{59}$$

and that, for all x, u, $u' \in \mathbb{B}(\bar{x}, a)$ *and all* $k \in \mathbb{N}_0$, *we have*

$$R_k(x, u) \cap \mathbb{B}(0, a) \subset R_k(x, u') + \ell\|u - u'\|\mathbb{B}_Y. \tag{60}$$

Then there exist $t \in (0, 1)$ *and* $r > 0$ *such that, for any starting point* $x_0 \in \mathbb{B}(\bar{x}, r)$, *there exists a sequence* (x_k) *in* $\mathbb{B}(\bar{x}, r)$ *generated by* (22) *such that*

$$\|x_{k+1} - \bar{x}\| \leq t\|x_k - \bar{x}\| \quad \textit{for each} \quad k \in \mathbb{N}_0, \tag{61}$$

that is, (x_k) *converges q-linearly to* $\bar{x}$.

Proof. Shrink a, if necessary, to guarantee that

$$I\!B(\bar{x},a) \subset \operatorname{dom}\mathcal{H} \quad \text{and} \quad I\!B(\bar{x},a)\times I\!B(\bar{x},a) \subset \operatorname{dom} R_k \quad \text{for all} \quad k\in\mathbb{N}_0.$$

Let $c := \chi(\mathcal{H}(\bar{x}))$ and $m := \sup_{A\in\mathcal{H}(\bar{x})} \operatorname{reg} G_A(\bar{x},0)$. By (58), there are $\mu > c$, $\kappa > m$, $\varepsilon > 0$, and $t\in(0,1)$ satisfying

$$(\mu+\ell+\gamma+\varepsilon)\kappa < 1, \quad c+2\varepsilon<\mu \quad \text{and} \quad \kappa(\gamma+\varepsilon) < t(1-(\mu+\ell)\kappa). \tag{62}$$

Step 1. *There exist $b\in(0,a)$ and $\theta\in(0,\kappa/(1-\mu\kappa))$ such that, for every $A\in\mathcal{H}(I\!B(\bar{x},b))$ and for every $(x,y)\in I\!B(\bar{x},b)\times I\!B(0,b)$, we have*

$$\operatorname{dist}\big(x, G_A^{-1}(y)\big) \le \theta \operatorname{dist}\big(y, G_A(x)\big).$$

As $\mathcal{H}$ is upper semicontinuous at $\bar{x}$, there is $\delta\in(0,a)$ such that

$$\mathcal{H}(x)\subset\mathcal{H}(\bar{x})+\varepsilon I\!B_{\mathcal{L}(X,Y)} \quad \text{for each} \quad x\in I\!B(\bar{x},\delta). \tag{63}$$

From the definition of measure of noncompactness, we find a finite subset $\mathcal{A}$ of $\mathcal{H}(\bar{x})$ such that

$$\mathcal{H}(\bar{x})\subset\mathcal{A}+\big(c+\varepsilon\big)I\!B_{\mathcal{L}(X,Y)}.$$

Therefore, given $x\in I\!B(\bar{x},\delta)$, we have

$$\mathcal{H}(x) \overset{(63)}{\subset} \mathcal{A}+\big(c+\varepsilon\big)I\!B_{\mathcal{L}(X,Y)}+\varepsilon I\!B_{\mathcal{L}(X,Y)} = \mathcal{A}+(c+2\varepsilon)I\!B_{\mathcal{L}(X,Y)}.$$

The second inequality in (62) implies that

$$\mathcal{H}(x)\subset\mathcal{A}+\mu I\!B_{\mathcal{L}(X,Y)} \quad \text{for every} \quad x\in I\!B(\bar{x},\delta). \tag{64}$$

Choose θ to satisfy

$$m/(1-\mu m)<\theta<\kappa/(1-\mu\kappa),$$

and then choose $\tau\in(m,\kappa)$ with $\tau/(1-\mu\tau)<\theta$. Pick any $\bar{A}\in\mathcal{A}$ and $A\in\mu I\!B_{\mathcal{L}(X,Y)}$. There exists $\alpha>0$ such that

$$\operatorname{dist}\big(x, G_{\bar{A}}^{-1}(y)\big) \le \tau \operatorname{dist}\big(y, G_{\bar{A}}(x)\big) \quad \text{for all} \quad (x,y)\in I\!B(\bar{x},\alpha)\times I\!B(0,\alpha).$$

The mapping $G_{\bar{A}}$ has closed graph, because so does F. Let $g(x) := A(x-\bar{x})$, $x\in X$; then $G_{\bar{A}+A} = G_{\bar{A}}+g$. Observe that g is single-valued, Lipschitz continuous with the constant μ such that $\mu\tau<1$, and $g(\bar{x})=0$. We can apply [13, Theorem 5G.3] with $F:=G_{\bar{A}}$, $\bar{y}=0$, $a=b:=\alpha$, $\kappa:=\tau$, and $\kappa':=\theta$, obtaining that there is $\beta=\beta(\bar{A})>0$, independent of A, such that the following `claim` holds: *for each y, $y'\in I\!B[0,\beta]$ and each $x\in\big(G_{\bar{A}+A}\big)^{-1}(y')\cap I\!B[\bar{x},2\theta\beta]$, there is $x'\in\big(G_{\bar{A}+A}\big)^{-1}(y)$ satisfying $\|x-x'\|\le\theta\|y-y'\|$.*

We show that, for each $(x,y)\in I\!B(\bar{x},\theta\beta/3)\times I\!B(0,\beta/3)$, we have

$$\operatorname{dist}\big(x, \big(G_{\bar{A}+A}\big)^{-1}(y)\big) \le \theta \operatorname{dist}\big(y, G_{\bar{A}+A}(x)\big). \tag{65}$$

To see this, fix any such a pair (x, y). Pick an arbitrary $y' \in G_{\bar{A}+A}(x)$ (if there is any). If $\|y'\| \le \beta$, then the claim yields $x' \in \big(G_{\bar{A}+A}\big)^{-1}(y)$ with $\|x - x'\| \le \theta\|y - y'\|$, and consequently,

$$\operatorname{dist}\big(x, \big(G_{\bar{A}+A}\big)^{-1}(y)\big) \;\le\; \|x - x'\| \le \theta\,\|y - y'\|.$$

On the other hand, assuming that $\|y'\| > \beta$, we have $\|y' - y\| > \beta - \beta/3 = 2\beta/3$. Then, using the claim with (x, y') replaced by $(\bar{x}, 0)$, we find $x' \in \big(G_{\bar{A}+A}\big)^{-1}(y)$ such that $\|\bar{x} - x'\| \le \theta\|y\|$. Consequently,

$$\begin{aligned}\operatorname{dist}\big(x, \big(G_{\bar{A}+A}\big)^{-1}(y)\big) &\le \|x - \bar{x}\| + \operatorname{dist}\big(\bar{x}, \big(G_{\bar{A}+A}\big)^{-1}(y)\big) \le \|x - \bar{x}\| + \|\bar{x} - x'\| \\ &< \theta\beta/3 + \theta\beta/3 = \theta(2\beta/3) < \theta\|y - y'\|.\end{aligned}$$

Since $y' \in G_{\bar{A}+A}(x)$ is arbitrary, (65) is proved.

Summarizing, given $\bar{A} \in \mathcal{A}$, there exists $\beta := \beta(\bar{A}) > 0$ such that, for each $A \in \mu I\!B_{\mathcal{L}(X,Y)}$ and each $(x, y) \in I\!B(\bar{x}, \theta\beta/3) \times I\!B(0, \beta/3)$, inequality (65) holds. Taking into account (64), one has $\mathcal{H}(I\!B(\bar{x}, \delta)) \subset \mathcal{A} + \mu I\!B_{\mathcal{L}(X,Y)}$. Letting $b = \min_{\bar{A}\in\mathcal{A}}\{\delta, \beta(\bar{A})/3, \theta\beta(\bar{A})/3\}$, we finish the proof of this step.

Step 2. *There exists $r > 0$ such that, for each $x \in I\!B(\bar{x}, r)$, each $A \in \mathcal{H}(x)$, and each $k \in \mathbb{N}_0$, there is $x' \in I\!B(\bar{x}, r)$ such that*

$$\big(f(x) + A(x' - x) + F(x')\big) \cap R_k(x, x') \ne \emptyset \quad \textit{and} \quad \|x' - \bar{x}\| \le t\|x - \bar{x}\|.$$

Let b and θ be the constants found in Step 1. Using (57) and (59), we find a constant $\delta \in (0, b/(1 + \gamma))$ such that, for every $x \in I\!B(\bar{x}, \delta) \setminus \{\bar{x}\}$ and every $k \in \mathbb{N}_0$, we have

$$\sup_{A\in\mathcal{H}(x)} \|f(x) - f(\bar{x}) - A(x - \bar{x})\| < \varepsilon\|x - \bar{x}\| \quad \text{and} \quad \operatorname{dist}\big(0, R_k(x, \bar{x})\big) < \gamma\|x - \bar{x}\|. \tag{66}$$

The first inequality in (62) implies that $\theta\ell < \kappa\ell/(1 - \mu\kappa) < 1$. Let $r \in (0, \delta)$ be such that

$$r < \frac{\delta(1 - \theta\ell)}{2(\varepsilon + \gamma)\max\{1, \theta\}}\,.$$

Fix an arbitrary $x \in I\!B(\bar{x}, r)$. Choose any $A \in \mathcal{H}(x)$ and $k \in \mathbb{N}_0$. If $x = \bar{x}$, then, setting $x' := \bar{x}$, we are done because $0 \in R_k(\bar{x}, \bar{x})$ and $0 \in f(\bar{x}) + F(\bar{x})$. Assume that $x \ne \bar{x}$. By (66) we find $\bar{z} \in -R_k(x, \bar{x})$ such that $\|\bar{z}\| < \gamma\|x - \bar{x}\|$. Then

$$I\!B(\bar{z}, \delta) \subset I\!B(0, (1 + \gamma)\delta) \subset I\!B(0, b) \subset I\!B(0, a).$$

Consequently, for all u, $u' \in I\!B(\bar{x}, \delta)$, we have

$$\begin{aligned}(-R_k(x, u)) \cap I\!B(\bar{z}, \delta) &\subset (-R_k(x, u)) \cap I\!B(0, a) = -\big(R_k(x, u) \cap I\!B(0, a)\big) \\ &\overset{(60)}{\subset} -\big(R_k(x, u') + \ell\|u - u'\|I\!B_Y\big) = -R_k(x, u') + \ell\|u - u'\|I\!B_Y.\end{aligned}$$

From Step 1 we get

$$\operatorname{dist}\big(u, G_A^{-1}(v)\big) \le \theta \operatorname{dist}\big(v, G_A(u)\big) \quad \text{for all} \quad (u, v) \in I\!B(\bar{x}, \delta) \times I\!B(0, \delta).$$

As $\theta\ell < 1$, applying Theorem 4.1 with $(F, G, \bar{y}, a, \kappa, \beta)$ replaced by $(G_A, -R_k(x, \cdot), 0, \delta, \theta, (\varepsilon+\gamma)r)$, we get

$$\text{(67)} \qquad \operatorname{dist}\big(\bar{x}, (G_A - R_k(x, \cdot))^{-1}(y)\big) \le \frac{\theta}{1-\theta\ell}\|y - \bar{z}\| \quad \text{for all} \quad y \in I\!B(\bar{z}, (\varepsilon+\gamma)r).$$

Set

$$\text{(68)} \qquad y := f(\bar{x}) - f(x) + A(x - \bar{x}).$$

If $y = \bar{z}$, then $f(x) + A(\bar{x} - x) - f(\bar{x}) \in R_k(x, \bar{x}) \cap (f(x) + A(\bar{x} - x) + F(\bar{x}))$, and setting $x' := \bar{x}$ we are done. Assume that $y \neq \bar{z}$. The first inequality in (66) and the choice of $\bar{z}$ imply that

$$0 < \|y - \bar{z}\| \le \|f(x) - f(\bar{x}) - A(x - \bar{x})\| + \|\bar{z}\| < (\varepsilon + \gamma)\|x - \bar{x}\| < (\varepsilon + \gamma)r.$$

Remembering that $\theta < \kappa/(1 - \mu\kappa)$ and $\kappa\ell/(1 - \mu\kappa) < 1$, and using the last inequality in (62), we get

$$\frac{\theta}{1-\theta\ell} < \frac{\frac{\kappa}{1-\mu\kappa}}{1 - \frac{\kappa\ell}{1-\mu\kappa}} = \frac{\kappa}{1-(\mu+\ell)\kappa} < \frac{t}{\gamma+\varepsilon}.$$

This and (67) imply that there is $x' \in (G_A - R_k(x, \cdot))^{-1}(y)$ such that

$$\|x' - \bar{x}\| \ < \ \frac{t}{\gamma+\varepsilon}\|y - \bar{z}\| < \frac{t}{\varepsilon+\gamma}(\varepsilon+\gamma)\|x - \bar{x}\| = t\|x - \bar{x}\|.$$

Hence, $\|x' - \bar{x}\| < r$ because $t \in (0, 1)$ and $x \in I\!B(\bar{x}, r)$. The choice of y implies that

$$f(\bar{x}) - f(x) + A(x - \bar{x}) \in G_A(x') - R_k(x, x') = f(\bar{x}) + A(x' - \bar{x}) + F(x') - R_k(x, x').$$

Therefore $0 \in f(x) + A(x' - x) + F(x') - R_k(x, x')$, which means that $\big(f(x) + A(x' - x) + F(x')\big) \cap R_k(x, x') \neq \emptyset$. The proof of Step 2 is finished.

To conclude the proof, let $r > 0$ be the constant found in Step 2. Consider the iteration (22) and choose any $k \in \mathbb{N}_0$, $x_k \in I\!B(\bar{x}, r)$ and $A_k \in \mathcal{H}(x_k)$. Apply Step 2 with $A := A_k$ and $x := x_k$, and set $x_{k+1} := x'$. Then x_{k+1} satisfies (22) and (61). It remains to choose any $x_0 \in I\!B(\bar{x}, r)$ to obtain this way an infinite sequence (x_k) in $I\!B(\bar{x}, r)$ generated by (22) and satisfying (61) for all $k \in \mathbb{N}_0$. Since $t \in (0, 1)$, (x_k) converges linearly to $\bar{x}$. □

Remark 5.2. If (59) is replaced by a stronger condition

$$\lim_{x \to \bar{x},\ x \neq \bar{x}} \frac{\sup_{k \in \mathbb{N}_0} \operatorname{dist}\big(0, R_k(x, \bar{x})\big)}{\|x - \bar{x}\|} = 0,$$

then there is $r > 0$ such that, for any starting point $x_0 \in I\!B(\bar{x}, r)$, there exists a sequence (x_k) in $I\!B(\bar{x}, r)$ generated by (22) such that (x_k) converges q-super-linearly to $\bar{x}$, that is, if there is $k_0 \in \mathbb{N}$ such that $x_k \neq \bar{x}$ for all $k > k_0$ then $\lim_{k\to\infty} \|x_{k+1} - \bar{x}\|/\|x_k - \bar{x}\| = 0$. Indeed, in (62) both the constants ε and γ, and consequently, also t can be chosen arbitrarily small.

Suppose that $X := \mathbb{R}^n$, $Y := \mathbb{R}^m$, and f is locally Lipschitz continuous. We can take, for example, Clarke's generalized Jacobian or Bouligand's limiting Jacobian as $\mathcal{H}$. Then $\mathcal{H}$ is upper semicontinuous and condition (57) is satisfied when f is semismooth at $\bar{x}$ (with respect to the corresponding Jacobian). Moreover, $\chi(\mathcal{H}(\bar{x})) = 0$. If, in addition, $F \equiv 0$ and $R_k \equiv 0$ for each $k \in \mathbb{N}_0$, then the assumption of regularity of all mappings G_A in (56) is nothing else but the requirement that all matrices in $\mathcal{H}(\bar{x})$ have full-rank m, and we arrive at the classical result for semismooth Newton-type methods (see, for example, [5, 39, 20, 1, 8, 7]).

In [1], the following iterative process was studied: *Choose a sequence of set-valued mappings* $A_k : X \times X \rightrightarrows Y$ *and a starting point* $x_0 \in X$, *and generate a sequence* (x_k) *in* X *by taking* x_{k+1} *to be a solution to the auxiliary inclusion*

$$0 \in A_k(x_{k+1}, x_k) + F(x_{k+1}) \quad \text{for each} \qquad k \in \mathbb{N}_0. \tag{69}$$

Theorem 4.1 therein for iteration (69) is quite similar to Theorem 5.1 above with one important difference. We assume that all the "partial linearizations" G_A in (56) are regular around $(\bar{x}, 0)$, while in [1] the mapping $f + F$ is assumed to be such. Clearly, our assumption is weaker. Indeed take, for example, $f(x) := |x|$, $x \in \mathbb{R}$, $F \equiv 0$, and $\mathcal{H}(x) := x/|x|$ if $x \neq 0$ and $\mathcal{H}(0) := \{-1, 1\}$. Then f is not even semiregular at 0 while $\mathcal{H}$ satisfies all the assumptions in Theorem 5.1.

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