INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Matching polytons

Martin Doležal
Jan Hladký

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MATCHING POLYTONS

MARTIN DOLEŽAL AND JAN HLADKÝ

Abstract. Hladký, Hu, and Piguet [Tilings in graphons, preprint] introduced the notions of matching and fractional vertex covers in graphons. These are counterparts to the corresponding notions in finite graphs.

Combinatorial optimization studies the structure of the matching polytope and the fractional vertex cover polytope of a graph. Here, in analogy, we initiate the study of the structure of the set of all matchings and of all fractional vertex covers in a graphon. We call these sets the matching polyton and the fractional vertex cover polyton.

We also study properties of matching polytons and fractional vertex cover polytons along convergent sequences of graphons.

1. Introduction

Hladký, Hu, and Piguet in [4] translated the concept of vertex-disjoint copies of a fixed finite graph F in a (large) host graph to graphons. Following preceding literature on this topic, they use the name *F-tiling* (in a graph or in a graphon). This allows them to introduce the *F-tiling ratio* of a graphon. They also translate the closely related concept of (fractional) F-covers in finite graphs to graphons which is a dual concept to F-tilings. The case when F is an edge, $F = K_2$, is the most important. Then F-tilings are exactly matchings, the F-tiling ratio is just the matching ratio[1], and (fractional) F-covers are exactly (fractional) vertex covers.

In this paper we deal exclusively with the case $F = K_2$ and from now on we specialize our description to this case. Some of our results, however, generalize to other F-tilings. We discuss the possible generalizations in Section 5.3.

Hladký, Hu, and Piguet give transference statements between the finite (i.e., graph) and limit (i.e., graphon) versions of the said notions. They mostly study the numerical quantities provided by the theory they develop, that is, the matching ratio and the fractional vertex cover ratio. Given a graphon W, we denote these two quantities (which we define in Section 2.3) by $\mathsf{match}(W)$ and $\mathsf{fcov}(W)$. One of the main results from [4] is the counterpart of the prominent linear programming duality between the fractional matching number of a

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[1] the *matching ratio* is just the matching number divided by the number of vertices

graph and its fractional vertex cover number. Since — as Hladký, Hu, and Piguet argue — in the graphon world there is no distinction between matchings and fractional matchings, their LP duality has the form $\mathsf{match}(W) = \mathsf{fcov}(W)$. In [4] and [3] they give applications of this LP duality in extremal graph theory.[2]

In this paper, on the other hand, we study the sets of all fractional matchings, and of all fractional vertex covers. In the case of a finite graph G, these sets are known as the fractional matching polytope and the fractional vertex cover polytope. We shall denote them by $\mathrm{MATCH}(G)$ and $\mathrm{FCOV}(G)$. Study of $\mathrm{MATCH}(G)$ and $\mathrm{FCOV}(G)$ (and study of related polytopes such as the (integral) matching polytope and the perfect matching polytope) is central in polyhedral combinatorics and in combinatorial optimization. From the numerous results on the geometry of these polytopes, let us mention integrality of the fractional matching polytope and fractional vertex cover polytope of a bipartite graph, or the Edmonds' perfect matching polytope theorem. Here, we initiate a parallel study in the context of graphons. While in the finite case we have $\mathrm{MATCH}(G) \subseteq \mathbb{R}^{E(G)}$ and $\mathrm{FCOV}(G) \subseteq \mathbb{R}^{V(G)}$, given a graphon $W : \Omega^2 \to [0,1]$, for the corresponding objects $\mathrm{MATCH}(W)$ and $\mathrm{FCOV}(W)$ it turns out that we have $\mathrm{MATCH}(W) \subseteq L^1(\Omega^2)$ and $\mathrm{FCOV}(W) \subseteq L^\infty(\Omega)$. So, while $\mathrm{MATCH}(G)$ and $\mathrm{FCOV}(G)$ is studied using tools from linear algebra, in order to study $\mathrm{MATCH}(W)$ and $\mathrm{FCOV}(W)$ we need to use the language of functional analysis. We employ the *-on* word ending used among others for graph*on*s and for permut*on*s and call the limit counterparts to polytopes (such as $\mathrm{MATCH}(W)$ and $\mathrm{FCOV}(W)$) polyt*on*s.

1.1. **Overview of the paper.**

In Section 2 we recall the necessary background concerning graphons and the theory of matchings/tilings in graphons developed in [4].

In Section 3 we treat (half)-integrality of the extreme points of the fractional vertex cover polyton of a graphon. As an application, we deduce a graphon version of the Erdős–Gallai theorem on matchings in dense graphs. In Section 4 we show that if a sequence of graphons $(W_n)_n$ converges to a graphon W then "$\mathrm{MATCH}(W_n)$ asymptotically contains $\mathrm{MATCH}(W)$". This result is dual to results from [4] on the relation between $\mathrm{FCOV}(W_n)$ and $\mathrm{FCOV}(W)$, which we recall in Section 4.1.

Section 5 contains some concluding remarks.

2. Notation and preliminaries

2.1. **Measure theory.**

Lemma 1. *Let (Ω, ν) be a probability space and let $A, B \subseteq \Omega$ be given. Let $D \subseteq A \times B$ be a set of positive $\nu^{\oplus 2}$ measure. Then for every $\varepsilon > 0$ there is a measurable rectangle $R \subseteq A \times B$ such that $\nu^{\oplus 2}(R \setminus D) < \varepsilon \nu^{\oplus 2}(R)$.*

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[2] These applications become particularly interesting when general F-tilings are considered.

Proof. Let us fix $\varepsilon > 0$. By the definition of the product measure, we can find measurable rectangles $R_1, R_2, \ldots \subseteq A \times B$ such that $D \subseteq \bigcup_{i=1}^{\infty} R_i$ and

$$\nu^{\oplus 2}\left(\bigcup_{i=1}^{\infty} R_i \setminus D\right) < \varepsilon \nu^{\oplus 2}(D) \leq \varepsilon \nu^{\oplus 2}\left(\bigcup_{i=1}^{\infty} R_i\right) .$$

Then there is a natural number m such that

$$\nu^{\oplus 2}\left(\bigcup_{i=1}^{m} R_i \setminus D\right) < \varepsilon \nu^{\oplus 2}\left(\bigcup_{i=1}^{m} R_i\right) . \tag{1}$$

Now the finite union $\bigcup_{i=1}^{m} R_i$ can obviously be decomposed into finitely many *pairwise disjoint* measurable rectangles $S_1, \ldots, S_l$. Then the inequality (1) can be rewritten as

$$\sum_{i=1}^{l} \nu^{\oplus 2}(S_i \setminus D) < \varepsilon \sum_{i=1}^{l} \nu^{\oplus 2}(S_i) .$$

Thus there is some $i \in \{1, \ldots, l\}$ such that $\nu^{\oplus 2}(S_i \setminus D) < \varepsilon \nu^{\oplus 2}(S_i)$. The corresponding S_i is the wanted measurable rectangle R. $\square$

2.2. **Graphon basics.** Our notation follows [6]. Throughout the paper we shall assume that Ω is an atomless Borel probability space equipped with a measure ν (defined on an implicit σ-algebra). We denote by $\nu^{\oplus k}$ the product measure on Ω^k.

Let us recall that a graphon $W : \Omega^2 \to [0, 1]$ is *bipartite* if there exists a partition $\Omega = \Omega_A \sqcup \Omega_B$ into two sets of positive measure such that W is zero almost everywhere on $(\Omega_A \times \Omega_A) \cup (\Omega_B \times \Omega_B)$.

Suppose that F is a graph on vertex set $[k]$. Then the *density* of F in a graphon W is defined as

$$t(F, W) = \int_{x_1} \int_{x_2} \cdots \int_{x_k} \prod_{\substack{ij \in E(F) \\ i<j}} W(x_i, x_j) .$$

Recall that the *cut-norm* $\| \cdot \|_\square$ and the *cut-distance* $\mathrm{dist}_\square(\cdot, \cdot)$ are defined by

$$\|U\|_\square = \sup_{S,T \subseteq \Omega} \left| \int_{S \times T} U \right| , \; U \in L^1(\Omega^2), \text{ and}$$
$$\mathrm{dist}_\square(U, W) = \inf_{\phi} \|U - W^\phi\|_\square , \; U, W \in L^1(\Omega^2) ,$$

where the infimum in the definition of the cut-distance ranges over all measure-preserving bijections on Ω, and W^ϕ is defined by $W^\phi(x, y) = W(\phi(x), \phi(y))$.

2.3. **Introducing matchings and vertex covers in graphons.** We introduce the notion of matchings in a graphon. Our definitions follow [4], where they were given in the more general context of F-tilings.

Definition 2. *Suppose that* $W : \Omega^2 \to [0, 1]$ *is a graphon. We say that a function* $\mathfrak{m} \in L^1(\Omega^2)$ *is a* matching *in* W *if*

(1) $\mathfrak{m} \geq 0$ *almost everywhere,*
(2) $\operatorname{supp} \mathfrak{m} \subseteq \operatorname{supp} W$ *up to a null-set, and*
(3) for almost every $x \in \Omega$*, we have* $\int_y \mathfrak{m}(x, y) + \int_y \mathfrak{m}(y, x) \leq 1$.

In [4] we argued in detail why this is "the right" notion of matchings. We do not repeat this discussion here and only briefly mention that the requirements in Definition 2 are counterparts to fractional matchings in finite graphs. Namely, a fractional matching in a graph G can be represented as a function $f : V(G)^2 \to \mathbb{R}$ such that

(1) $f \geq 0$,
(2) if $f(x, y) > 0$ then $xy \in E(G)$, and
(3) for every $x \in V(G)$, we have $\sum_y f(x, y) + \sum_y f(y, x) \leq 1$.

(Note that usually fractional matchings are represented using symmetric functions. This is however only a notational matter.[3])

Remark 3. *As said already in the Introduction, even though Definition 2 is inspired by fractional matchings in finite graphs, the resulting graphon concept is referred to as "matchings". This is because in the graphon world every function* $\mathfrak{m}$ *from Definition 2 behaves in many ways as an integral matching.*

Given a matching $\mathfrak{m}$ in a graphon W we define its *size*, $\|\mathfrak{m}\| = \int_x \int_y \mathfrak{m}(x, y)$. The *matching ratio* of W, denoted by $\mathsf{match}(W)$, is defined as the supremum of the sizes of all matchings in W.

We write $\mathrm{MATCH}(W) \subseteq L^1(\Omega^2)$ for the set of all matchings in W. It is straightforward to check that this set is convex (like the set of fractional matchings in a finite graph) and closed (if we consider the norm topology on $L^1(\Omega^2)$). But — unlike the finite case — it need not be compact. To see this consider the graphon $U : [0, 1]^2 \to [0, 1]$ defined as $U(x, y) = 1$ for $x + y \leq 1$ and $U(x, y) = 0$ for $x + y > 1$. This example was first given in [4] in a somewhat different context. For ε positive, consider a matching $\mathfrak{m}_\varepsilon$ defined to be $1/(2\varepsilon)$ on a stripe of width ε along the diagonal $x + y = 1$ and zero otherwise. This is shown on Figure 1. It is clear that the matchings $\mathfrak{m}_\varepsilon$ do not contain any convergent subsequence, as we let $\varepsilon \to 0_+$. Considering the weak topology on the space $L^1(\Omega^2)$ (that is the topology generated by the

[3]The current choice for these functions being not-necessarily symmetric is adopted from [4]. There, this choice was dictated not by matching, but rather by creating a general concept of F-tilings even for graphs which are not vertex-transitive.

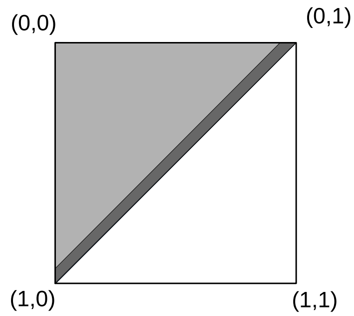

FIGURE 1. The graphon U discussed in the text. A matching $\mathfrak{m}_\varepsilon$ shown in dark gray. The support of U is shown in light gray.

dual space $L^\infty(\Omega^2)$) does not help as the same counterexample easily shows. Therefore considering the set $\mathrm{MATCH}(W)$ as a subset of the second dual of $L^1(\Omega^2)$ equipped with its weak* topology seems to be the only reasonable way to have a natural compactification of $\mathrm{MATCH}(W)$. However, we did not need go that far.

We can now proceed with the definition of fractional vertex covers of a graphon. First, recall that a function $c : V(G) \to [0,1]$ is a *fractional vertex cover* of a finite graph G if we have $c(x) + c(y) \geq 1$ for each $xy \in E(G)$. Thus, the graphon counterpart is as follows.

Definition 4. *Suppose that $W : \Omega^2 \to [0,1]$ is a graphon. We say that a function $\mathfrak{c} \in L^\infty(\Omega)$ is a* fractional vertex cover *of W if $0 \leq \mathfrak{c} \leq 1$ almost everywhere and the set*

$$\operatorname{supp} W \setminus \{(x,y) : \mathfrak{c}(x) + \mathfrak{c}(y) \geq 1\}$$

has measure 0.

A fractional vertex cover is called an *integral vertex cover* if its values are from the set $\{0,1\}$ almost everywhere.

Given a fractional vertex cover $\mathfrak{c}$ of a graphon W we define its *size*, $\|\mathfrak{c}\| = \int_x \mathfrak{c}(x)$. The fractional cover number, $\mathsf{fcov}(W)$ is the infimum of sizes of all fractional vertex covers of W.

We write $\mathrm{FCOV}(W) \subseteq L^\infty(\Omega)$ for the set of all fractional vertex covers of W. It is straightforward to check that this set is convex. Further, as was first shown in [4, Theorem 3.14], it is also compact in the space $L^\infty(\Omega)$ equipped with the weak* topology.

3. Extreme points of fractional vertex cover polytons

3.1. **Digest of properties of fractional vertex cover polytopes.** Let G be a finite graph. Then the set $\mathrm{FCOV}(G) \subseteq [0,1]^{V(G)}$ is a polytope. It is a fundamental fact in combinatorial optimization that all the vertices of this polytope are half-integral (i.e., in the form $\{0, \frac{1}{2}, 1\}^{V(G)}$), [7, Theorem 30.2]. Furthermore, all its vertices are integral if and only if G is bipartite, [7, Theorem 18.3].

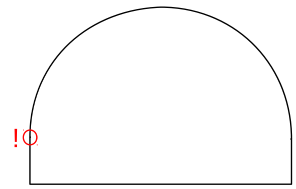

FIGURE 2. By joining a half-circle and a rectangle in $\mathbb{R}^2$ we get an extreme point that is not exposed.

3.2. **Graphon counterparts.** Suppose that $\mathcal{L}$ is a vector space, and suppose that $X \subseteq \mathcal{L}$ is a convex set. Recall that a point $x \in X$ is called an *extreme point of* X if the only pair $x', x'' \in X$ for which $x = \frac{1}{2}(x' + x'')$ is the pair $x' = x$, $x'' = x$. We shall write $\mathcal{E}(X)$ to denote the set of all extreme points of X.

When $\mathcal{L}$ is finite-dimensional and X is a polytope in $\mathcal{L}$ then the extreme points of X are exactly its vertices. The importance of the notion of extreme points comes from the Krein–Milman theorem which states that in a locally convex topological vector space, each compact convex set equals to the closed convex hull of its extreme points.

Thus, the graphon counterparts to the results described in Section 3.1 will be expressed in terms of $\mathcal{E}(\mathrm{FCOV}(W))$. Let us now state these counterparts.

Theorem 5. *Suppose that $W : \Omega^2 \to [0, 1]$ is a bipartite graphon. Then all the extreme points of $\mathrm{FCOV}(W)$ are integral.*

Theorem 6. *Suppose that $W : \Omega^2 \to [0, 1]$ is a graphon. Then all the extreme points of $\mathrm{FCOV}(W)$ are half-integral, i.e. with values from the set $\{0, \frac{1}{2}, 1\}$ almost everywhere.*

Theorem 7. *Suppose that $W : \Omega^2 \to [0, 1]$ is a graphon. If all the extreme points of $\mathrm{FCOV}(W)$ are integral then W is bipartite.*

The notion of extreme points is not the only generalization of vertices of a polytope. Another basic notion from convex analysis is that of exposed points. Its stronger variant, the notion of strongly exposed points, can be used for a characterization of the Radon–Nikodym property of Banach spaces which is an extensively studied topic. A point x in a convex set X is *exposed* if there exists a continuous linear functional for which x attains its strict maximum on X.

It is easy to see that every exposed point is extreme. The converse does not hold; a well-known counterexample in $\mathbb{R}^2$ is shown in Figure 2. We see, however, that every extreme point of the fractional vertex cover polyton of a bipartite graphon is exposed. Indeed, let $\phi \in \mathcal{E}(\mathrm{FCOV}(W))$ for some bipartite graphon $W : \Omega^2 \to [0, 1]$. Then Theorem 5 tells us that the sets $A = \phi^{-1}(0)$ and $B = \phi^{-1}(1)$ partition Ω. It is now clear that the linear functional f,

$$f(\psi) := \int_B \psi(x)\mathrm{d}x - \int_A \psi(x)\mathrm{d}x$$

is strictly maximized at ϕ on $\mathrm{FCOV}(W)$. We leave it as an open question whether every extreme point of the fractional vertex cover polyton is also exposed even for non-bipartite graphons.

3.3. **Proof of Theorem 5.** For the proof of Theorem 5 we shall need the following easy fact.

Fact 8. *Suppose that $x, y \in [0,1]$ are two reals which satisfy $x + y \geq 1$. Define $\partial(x) = \min(x, 1-x)$ and $\partial(y) = \min(y, 1-y)$. Then the numbers $x^+ = x + \partial(x)$ and $y^- = y - \partial(y)$ satisfy $x^+, y^- \in [0,1]$ and $x^+ + y^- \geq 1$.*

Proof. The fact that $x^+, y^- \in [0,1]$ is obvious. To prove that $x^+ + y^- \geq 1$, we distinguish three cases. First, suppose that $0 \leq x \leq \frac{1}{2} \leq y \leq 1$. Then $\partial(x) = x$, $\partial(y) = 1-y$, and consequently, $x^+ + y^- = 2x + 2y - 1 \geq 1$. Second, suppose that $0 \leq y \leq \frac{1}{2} \leq x \leq 1$. Then $\partial(x) = 1-x$, $\partial(y) = y$, and consequently, $x^+ + y^- = 1$. Third, suppose that $\frac{1}{2} \leq x, y \leq 1$. Then $\partial(x) = 1-x$, $\partial(y) = 1-y$, and consequently, $x^+ + y^- = 2y \geq 1$. $\square$

Proof of Theorem 5. Let $\Omega = \Omega_A \sqcup \Omega_B$ be a partition into two sets of positive measure such that W is zero almost everywhere on $(\Omega_A \times \Omega_A) \cup (\Omega_B \times \Omega_B)$. Suppose that $\mathfrak{c} \in \mathrm{FCOV}(W)$ is not integral. Using the notation from Fact 8, define two functions $\mathfrak{c}', \mathfrak{c}'' : \Omega_A \sqcup \Omega_B \to \mathbb{R}$ by $\mathfrak{c}'(a) = \mathfrak{c}(a)^+$, $\mathfrak{c}'(b) = \mathfrak{c}(b)^-$, $\mathfrak{c}''(a) = \mathfrak{c}(a)^-$, $\mathfrak{c}''(b) = \mathfrak{c}(b)^+$, for each $a \in \Omega_A$ and $b \in \Omega_B$. By Fact 8, we have that $\mathfrak{c}', \mathfrak{c}'' \in \mathrm{FCOV}(W)$. Further $\mathfrak{c} = \frac{1}{2}(\mathfrak{c}' + \mathfrak{c}'')$. As $\mathfrak{c}$ is not integral, we have that $\mathfrak{c}$ is distinct from $\mathfrak{c}'$ and $\mathfrak{c}''$. We conclude that $\mathfrak{c}$ is not an extreme point of $\mathrm{FCOV}(W)$. $\square$

3.4. **Proof of Theorem 6.** The proof of Theorem 6 is very similar to that of Theorem 5. We first state the counterpart of Fact 8 we need to this end. We omit the proof as it is almost the same as that of Fact 8.

Fact 9. *Suppose that $x, y \in [0,1]$ are two reals which satisfy $x + y \geq 1$. Define $\partial^\bullet(x) = \min(x, 1-x, |\frac{1}{2} - x|)$ and $\partial^\bullet(y) = \min(y, 1-y, |\frac{1}{2} - y|)$. Then the numbers $x^{+\bullet} = x + \partial^\bullet(x)$ and $y^{-\bullet} = y - \partial^\bullet(y)$ satisfy $x^{+\bullet}, y^{-\bullet} \in [0,1]$ and $x^{+\bullet} + y^{-\bullet} \geq 1$.*

Proof of Theorem 6. Suppose that $\mathfrak{c} \in \mathrm{FCOV}(W)$ is not half-integral. Consider the sets $\Omega_A = \{x \in \Omega : 0 \leq \mathfrak{c}(x) \leq \frac{1}{2}\}$ and $\Omega_B = \{x \in \Omega : \frac{1}{2} < \mathfrak{c}(x) \leq 1\}$. Using the notation from Fact 9, define two functions $\mathfrak{c}', \mathfrak{c}'' : \Omega_A \sqcup \Omega_B \to \mathbb{R}$ by $\mathfrak{c}'(a) = \mathfrak{c}(a)^{+\bullet}$, $\mathfrak{c}'(b) = \mathfrak{c}(b)^{-\bullet}$, $\mathfrak{c}''(a) = \mathfrak{c}(a)^{-\bullet}$, $\mathfrak{c}''(b) = \mathfrak{c}(b)^{+\bullet}$, for each $a \in \Omega_A$ and $b \in \Omega_B$. By Fact 9, we have that $\mathfrak{c}', \mathfrak{c}'' \in \mathrm{FCOV}(W)$. Further $\mathfrak{c} = \frac{1}{2}(\mathfrak{c}' + \mathfrak{c}'')$. As $\mathfrak{c}$ is not half-integral, we have that $\mathfrak{c}$ is distinct from $\mathfrak{c}'$ and $\mathfrak{c}''$. We conclude that $\mathfrak{c}$ is not an extreme point of $\mathrm{FCOV}(W)$. $\square$

3.5. **Proof of Theorem 7.** Lemmas 10 and 11 are key for proving Theorem 7. These lemmas (and the generalization of Lemma 10 given in Proposition 21) may be of independent interest.

Lemma 10. *Suppose that $W : \Omega^2 \to [0,1]$ is a graphon. Then W is bipartite if and only if for every odd integer $k \geq 3$ it holds $t(C_k, W) = 0$.*

Proof. Suppose first that there is an odd integer $k \geq 3$ such that $t(C_k, W) > 0$. Let $\Omega = \Omega_0 \sqcup \Omega_1$ be an arbitrary decomposition of Ω into two disjoint measurable subsets. Then there exists $(i_j)_{j=1}^k \in \{0,1\}^k$ such that

$$\int_{x_1 \in \Omega_{i_1}} \int_{x_2 \in \Omega_{i_2}} \cdots \int_{x_k \in \Omega_{i_k}} W(x_1, x_2) W(x_2, x_3) \dots W(x_k, x_1) > 0 \ .$$

As k is odd, there is $j \in \{1, \dots, k\}$ such that $i_j = i_{j+1}$ (here we use the cyclic indexing, i.e. $k + 1 = 1$). By Fubini's theorem

$$\int_{x_j \in \Omega_{i_j}} \int_{x_{j+1} \in \Omega_{i_{j+1}}} W(x_j, x_{j+1}) > 0 \ ,$$

in other words $\int_{\Omega_i^2} W > 0$ for some $i \in \{0, 1\}$. As the decomposition $\Omega = \Omega_0 \sqcup \Omega_1$ was chosen arbitrarily, this proves that W is not bipartite.

Now suppose that $t(C_k, W) = 0$ for every odd integer $k \geq 3$. By transfinite induction we define a transfinite sequence $\{(A_\alpha, B_\alpha) \colon \alpha \leq \gamma\}$ (for some countable ordinal γ) consisting of pairs of measurable subsets of Ω such that

(i) $A_\alpha \subseteq A_\beta$ and $B_\alpha \subseteq B_\beta$, $\alpha \leq \beta \leq \gamma$,
(ii) $\nu(A_\alpha \cup B_\alpha) < \nu(A_\beta \cup B_\beta) \leq \nu(A_\gamma \cup B_\gamma) = 1$, $\alpha < \beta \leq \gamma$,
(iii) $W \restriction_{A_\alpha^2} = 0$ a.e. and $W \restriction_{B_\alpha^2} = 0$ a.e., $\alpha \leq \gamma$,
(iv) $W \restriction_{(A_\alpha \cup B_\alpha) \times (\Omega \setminus (A_\alpha \cup B_\alpha))} = 0$ a.e., $\alpha \leq \gamma$,
(v) $\nu(A_\alpha \cap B_\alpha) = 0$, $\alpha \leq \gamma$.

Once we are done with the construction, the bipartiteness of W immediately follows by the equation $\nu(A_\gamma \cup B_\gamma) = 1$ together with (iii) and (v).

We start the construction by setting $A_0 = B_0 = \emptyset$. Now suppose that we have already constructed $\{(A_\alpha, B_\alpha) \colon \alpha < \alpha_0\}$ for some countable ordinal α_0 such that conditions (i)-(v) hold. If α_0 is a limit ordinal then we set $A_{\alpha_0} = \bigcup_{\alpha < \alpha_0} A_\alpha$ and $B_{\alpha_0} = \bigcup_{\alpha < \alpha_0} B_\alpha$. This clearly works. Otherwise, $\alpha_0 = \alpha + 1$ for some ordinal $\alpha < \alpha_0$. If $\nu(A_\alpha \cup B_\alpha) = 1$ then the construction is finished (with $\gamma = \alpha$). So suppose that $\nu(A_\alpha \cup B_\alpha) < 1$ and denote $\Omega' = \Omega \setminus (A_\alpha \cup B_\alpha)$. If $W \restriction_{\Omega' \times \Omega'} = 0$ a.e. then it suffices (by (iv)) to set $A_{\alpha_0} = A_\alpha \cup \Omega'$ and $B_{\alpha_0} = B_\alpha$. Otherwise there is $x_0 \in \Omega'$ such that $\nu\left(\{y \in \Omega' \colon W(x_0, y) > 0\}\right) > 0$. By Fubini's theorem, we may also assume that for every odd integer $k \geq 3$ it holds

$$\int_{\Omega^{k-1}} W(x_0, x_1) W(x_1, x_2) \dots W(x_{k-1}, x_0) = 0 \ . \tag{2}$$

We set

$$D_0 = \{y \in \Omega' \colon W(x_0, y) > 0\} \quad , \text{ and}$$

$$D_\ell = \left\{ y \in \Omega' \colon \int_{\Omega^\ell} W(x_0, x_1) W(x_1, x_2) \ldots W(x_\ell, y) > 0 \right\}, \text{ for } \ell \geq 1 \ .$$

Then we set

$$A = \bigcup_{\ell \text{ even}} D_\ell \quad \text{and} \quad B = \bigcup_{\ell \text{ odd}} D_\ell \ .$$

Finally, we define $A_{\alpha_0} = A_\alpha \cup A$ and $B_{\alpha_0} = B_\alpha \cup B$. The conditions (i) and (ii) are clearly satisfied, so let us verify only (iii), (iv) and (v).

As for (iii), suppose for a contradiction that $W \restriction_{A^2_{\alpha_0}}$ is positive on a set of positive measure. By the induction hypothesis, the same is true for $W \restriction_{A^2}$. So there are even integers $\ell_1, \ell_2 \geq 0$ such that $W \restriction_{D_{\ell_1} \times D_{\ell_2}}$ is positive on a set of positive measure. But then a simple application of Fubini's theorem leads to a contradiction with (2) for $k = \ell_1 + \ell_2 + 3$. So we have $W \restriction_{A^2_{\alpha_0}} = 0$ a.e. Similarly, we get $W \restriction_{B^2_{\alpha_0}} = 0$ a.e. This proves (iii).

As for (iv), suppose for a contradiction that $W \restriction_{(A_{\alpha_0} \cup B_{\alpha_0}) \times (\Omega \setminus (A_{\alpha_0} \cup B_{\alpha_0}))}$ is positive on a set of positive measure. By the induction hypothesis, the same is true for $W \restriction_{(A \cup B) \times (\Omega \setminus (A_{\alpha_0} \cup B_{\alpha_0}))} = W \restriction_{(A \cup B) \times (\Omega' \setminus (A \cup B))}$. By Fubini's theorem, there is $z \in \Omega' \setminus (A \cup B)$ such that

$$\nu\left(\{y \in A \cup B \colon W(y, z) > 0\}\right) > 0 \ .$$

So there is an integer $\ell \geq 0$ such that

$$\nu\left(\{y \in D_\ell \colon W(y, z) > 0\}\right) > 0 \ .$$

By Fubini's theorem, we easily conclude that $z \in C_{\ell+1}$. But this contradicts the fact that $z \notin A \cup B$.

As for (v), suppose for a contradiction that $\nu(A_{\alpha_0} \cap B_{\alpha_0}) > 0$. By the induction hypothesis, we also have $\nu(A \cap B) > 0$. So there are an even integer ℓ_1 and an odd integer ℓ_2 such that $\nu(D_{\ell_1} \cap D_{\ell_2}) > 0$. But then an application of Fubini's theorem leads to a contradiction with (2) for $k = \ell_1 + \ell_2 + 2$.

To finish the proof, it suffices to observe that for some countable ordinal γ we get $\nu(A_\gamma \cup B_\gamma) = 1$, and then the construction stops. □

Lemma 10 is a graphon counterpart to the well-known fact that a graph is bipartite if and only if it does not contain odd-cycles.

Lemma 11. *Suppose that $W : \Omega^2 \to [0,1]$ is a graphon. If W is not bipartite then there exists an odd integer $k \geq 3$ with the following property. For each $\varepsilon > 0$ there exist pairwise disjoint sets $A_1, \ldots, A_k \subseteq \Omega$ of the same positive measure α, such that for each $h \in [k]$, W is positive everywhere on $A_h \times A_{h+1}$ except a set of measure at most $\varepsilon\alpha^2$. Here, we use cyclic indexing, $A_{k+1} = A_1$.*

Proof. Suppose that W is not bipartite. By Lemma 10 there is an odd integer $k \geq 3$ such that

$$t := \int_{\Omega^k} W(x_1, x_2)W(x_2, x_3) \ldots W(x_k, x_1) > 0 \,. \tag{3}$$

We find a natural number n such that

$$n > \frac{k^2}{t} \,. \tag{4}$$

We fix a decomposition $\Omega = \bigcup_{i=1}^{n} \Omega_i$ of Ω into pairwise disjoint sets of the same measure $\frac{1}{n}$. We also set

$$\begin{aligned} D = \big\{(x_1, \ldots, x_k) \in \Omega^k \colon &\text{there are } i, j \in \{1, \ldots, k\} \text{ and } \ell \in \{1, \ldots, n\} \\ &\text{such that } i \neq j \text{ and } x_i, x_j \in \Omega_\ell\big\} \,. \end{aligned} \tag{5}$$

Then we have

$$\nu^{\oplus k}(D) \leq \sum_{\substack{i,j=1,\ldots,k \\ i \neq j}} \frac{1}{n} \leq \frac{k^2}{n} \overset{(4)}{<} t \,,$$

and so

$$\int_D W(x_1, x_2)W(x_2, x_3) \ldots W(x_k, x_1) \leq \nu^{\oplus k}(D) < t \,. \tag{6}$$

By (3) and (6), we get

$$\int_{\Omega^k \setminus D} W(x_1, x_2)W(x_2, x_3) \ldots W(x_k, x_1) > 0 \,.$$

By this and (5) there are pairwise distinct integers $\ell_1, \ldots, \ell_k \in \{1, \ldots, n\}$ such that

$$\int_{\Omega_{\ell_1}} \int_{\Omega_{\ell_2}} \cdots \int_{\Omega_{\ell_k}} W(x_1, x_2)W(x_2, x_3) \ldots W(x_k, x_1) > 0 \,, \tag{7}$$

and so the set

$$E := \{(x_1, x_2, \ldots, x_k) \in \Omega_{\ell_1} \times \Omega_{\ell_2} \times \ldots \times \Omega_{\ell_k} \colon W(x_1, x_2)W(x_2, x_3) \ldots W(x_k, x_1) > 0\} \tag{8}$$

is of positive measure.

Now let us fix $\varepsilon > 0$, and let $\delta > 0$ be such that

$$\frac{\nu^{\oplus k}(E) - \delta}{\nu^{\oplus k}(E) + \delta} \geq 1 - \frac{\varepsilon}{2} \,. \tag{9}$$

Recall that the σ-algebra of all measurable subsets of $\Omega_{\ell_1} \times \ldots \times \Omega_{\ell_k}$ is generated by the algebra consisting of all finite unions of measurable rectangles. Thus there is a finite union $S = \bigcup_{i=1}^{m} R_i$ of measurable rectangles $R_1, \ldots, R_m$ in $\Omega_{\ell_1} \times \ldots \times \Omega_{\ell_k}$ such that

$$\nu^{\oplus k}(E \triangle S) = \nu^{\oplus k}(E \setminus S) + \nu^{\oplus k}(S \setminus E) \leq \delta \,.$$

Without loss of generality, we may assume that the measurable rectangles $R_1, \dots, R_m$ are pairwise disjoint. Then we have

$$\frac{\nu^{\oplus k}(E \cap S)}{\nu^{\oplus k}(S)} \geq \frac{\nu^{\oplus k}(E) - \nu^{\oplus k}(E \setminus S)}{\nu^{\oplus k}(E) + \nu^{\oplus k}(S \setminus E)} \geq \frac{\nu^{\oplus k}(E) - \delta}{\nu^{\oplus k}(E) + \delta} \overset{(9)}{\geq} 1 - \frac{\varepsilon}{2} \,. \tag{10}$$

Now the left-hand side of (10) can be expressed as

$$\frac{\nu^{\oplus k}(E \cap S)}{\nu^{\oplus k}(S)} = \sum_{i=1}^{m} \frac{\nu^{\oplus k}(R_i)}{\nu^{\oplus k}(S)} \cdot \frac{\nu^{\oplus k}(E \cap R_i)}{\nu^{\oplus k}(R_i)} \,,$$

i.e. as a convex combination of $\frac{\nu^{\oplus k}(E \cap R_i)}{\nu^{\oplus k}(R_i)}$, $i = 1, \dots, m$. Therefore by (10), there is an index $i_0 \in \{1, \dots, m\}$ such that

$$\frac{\nu^{\oplus k}(E \cap R_{i_0})}{\nu^{\oplus k}(R_{i_0})} \geq 1 - \frac{\varepsilon}{2} \,. \tag{11}$$

Let R_{i_0} be of the form $R_{i_0} = B_1 \times \ldots \times B_k$. Find a natural number p such that

$$p \geq \frac{2k}{\varepsilon \nu^{\oplus k}(R_{i_0})} \,. \tag{12}$$

For every $i = 1, \dots, k$, we fix a finite decomposition $B_i = B_i^0 \cup \bigcup_{j=1}^{q_i} B_i^j$ of B_i into pairwise disjoint sets, such that $\nu(B_i^0) \leq \frac{1}{p}$ and $\nu(B_i^j) = \frac{1}{p}$ for $j = 1, \dots, q_i$. Then we clearly have

$$\nu^{\oplus k}\left(R_{i_0} \setminus \prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right) \leq \frac{k}{p} \,, \tag{13}$$

and so

$$\frac{\nu^{\oplus k}\left(E \cap \prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right)} \overset{(13)}{\geq} \frac{\nu^{\oplus k}(E \cap R_{i_0}) - \frac{k}{p}}{\nu^{\oplus k}(R_{i_0})} \overset{(11)}{\geq} 1 - \frac{\varepsilon}{2} - \frac{k}{p\nu^{\oplus k}(R_{i_0})} \overset{(12)}{\geq} 1 - \varepsilon \,. \tag{14}$$

The left-hand side of (14) can be expressed as the following convex combination:

$$\frac{\nu^{\oplus k}\left(E \cap \prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right)} = \sum_{j_1=1}^{q_1} \cdots \sum_{j_k=1}^{q_k} \frac{\nu^{\oplus k}\left(\prod_{i=1}^{k} B_i^{j_i}\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} \bigcup_{j=1}^{q_i} B_i^j\right)} \cdot \frac{\nu^{\oplus k}\left(E \cap \prod_{i=1}^{k} B_i^{j_i}\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} B_i^{j_i}\right)} \,.$$

Therefore by (14), there are $j_i \in \{1, \dots, q_i\}$, $i = 1, \dots, k$, such that

$$\frac{\nu^{\oplus k}\left(E \cap \prod_{i=1}^{k} B_i^{j_i}\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} B_i^{j_i}\right)} \geq 1 - \varepsilon \,,$$

or equivalently

$$\frac{\nu^{\oplus k}\left(\prod_{i=1}^{k} B_i^{j_i} \setminus E\right)}{\nu^{\oplus k}\left(\prod_{i=1}^{k} B_i^{j_i}\right)} \le \varepsilon \ . \tag{15}$$

We set $A_i = B_i^{j_i}$ for $i = 1, \dots, k$. Then $A_1, \dots, A_k$ are pairwise disjoint (as $A_i \subseteq B_i \subseteq \Omega_{\ell_i}$ for every i), and each of these sets has the same measure $\alpha = \frac{1}{p}$. Let us fix $h \in [k]$ and suppose towards a contradiction that

$$\nu^{\oplus 2}\left(\{(x_h, x_{h+1}) \in A_h \times A_{h+1} \colon W(x_h, x_{h+1}) = 0\}\right) > \frac{\varepsilon}{p^2} \ .$$

Then we clearly have

$$\nu^{\oplus k}\left(\{(x_1, x_2, \dots, x_k) \in A_1 \times A_2 \times \dots \times A_k \colon W(x_1, x_2)W(x_2, x_3)\dots W(x_k, x_1) = 0\}\right) > \frac{\varepsilon}{p^k} \ .$$

But this is a contradiction with (8) and (15). □

Proof of Theorem 7. We shall prove the counterpositive. Suppose that W is not bipartite. Let $\mathrm{COV}(W)$ be the closure (in the weak* topology) of the convex hull of all integral vertex covers of W. Clearly, we have $\mathrm{COV}(W) \subseteq \mathrm{FCOV}(W)$, and each integral vertex cover of $\mathrm{FCOV}(W)$ is contained in $\mathrm{COV}(W)$. Below, we shall show that

$$\mathrm{FCOV}(W) \setminus \mathrm{COV}(W) \neq \emptyset \ . \tag{16}$$

The Krein–Milman Theorem then tells us that $\mathcal{E}\big(\mathrm{FCOV}(W)\big) \setminus \mathrm{COV}(W) \neq \emptyset$. It will thus follow that there exists a non-integral fractional vertex cover in $\mathcal{E}\big(\mathrm{FCOV}(W)\big)$, as was needed to show.

Take $\mathfrak{c} : \Omega \to [0,1]$ to be constant $\frac{1}{2}$. Clearly, $\mathfrak{c} \in \mathrm{FCOV}(W)$. In order to show (16), it suffices to prove that $\mathfrak{c} \notin \mathrm{COV}(W)$. Let k be the odd integer given by Lemma 11. Let $\varepsilon = \frac{1}{32k^2}$, and let the sets $A_1, \dots, A_k$ of measure $\alpha > 0$ be given by Lemma 11.

In order to prove that $\mathfrak{c}$ is not in the weak* closure of the convex hull of integral vertex covers, consider an arbitrary ℓ-tuple of integral vertex covers $\mathfrak{c}_1, \dots, \mathfrak{c}_\ell$ of W, and numbers $\gamma_1, \dots, \gamma_\ell \geq 0$ with $\sum \gamma_i = 1$.

Consider an arbitrary $i \in [\ell]$. We say that $\mathfrak{c}_i$ *marks the set* A_h (where $h \in [k]$) if $\mathfrak{c}_i$ restricted to A_h attains the value 0 on a set of measure at most $\frac{\alpha}{4k}$. Put equivalently, $\mathfrak{c}_i$ marks the set A_h if $\mathfrak{c}_i$ restricted to A_h attains the value 1 on a set of measure at least $\frac{\alpha(4k-1)}{4k}$.

Claim 1. *For each $i \in [\ell]$, the vertex cover $\mathfrak{c}_i$ marks at least $\frac{k+1}{2}$ many of the sets $A_1, \dots, A_k$.*

Proof of Claim 1. Suppose that this is not the case. Recall that k is odd. We can find an index $h \in [k]$ that that A_h and A_{h+1} are not marked (again, using the cyclic notation $A_{k+1} = A_k$). Therefore, the $\mathfrak{c}_i$-preimages $B_h \subseteq A_h$ and $B_{h+1} \subseteq A_{h+1}$ of 0 have both measures more than $\frac{\alpha}{4k}$. It follows from Lemma 11 and the way we set ε that W is positive on a set of positive measure on $B_h \times B_{h+1}$. This contradicts the fact that $\mathfrak{c}_i$ is a vertex cover. □

Let us write $A = \bigcup_h A_h$. It follows from Claim 1 that

$$\int_A \mathfrak{c}_i \geq \frac{k+1}{2} \cdot \frac{\alpha(4k-1)}{4k} \geq \alpha \left(\frac{k}{2} + \frac{1}{6} \right) . \tag{17}$$

By convexity, we can replace $\int_A \mathfrak{c}_i$ by $\int_A \left(\sum_i \gamma_i \mathfrak{c}_i\right)$ in (17).

We now have

$$\begin{aligned} \int_A \left(\sum_i \gamma_i \mathfrak{c}_i(x) - \mathfrak{c}(x) \right) \mathrm{d}x &= \int_A \left(\sum_i \gamma_i \mathfrak{c}_i(x) \right) \mathrm{d}x - \int_A \mathfrak{c}(x) \,\mathrm{d}x \\ &\overset{(17)}{\geq} \alpha \left(\frac{k}{2} + \frac{1}{6} \right) - k\alpha \cdot \frac{1}{2} = \frac{\alpha}{6} . \end{aligned} \tag{18}$$

Since neither the set A nor the bound on the right-hand side of (18) depend on the choice of the number ℓ, the vertex covers $\mathfrak{c}_i$, and the constants γ_i, we get that $\mathfrak{c}$ is not in the weak* closure of convex combinations of integral vertex covers, as was needed. □

3.6. **An application: the Erdős–Gallai Theorem.** In this section, we prove a graphon counterpart to the following classical result of Erdős and Gallai, [2].

Theorem 12 (Erdős–Gallai, 1959)**.** *Suppose that n and ℓ are positive integers that satisfy $\ell \leq n/2$. Then any n-vertex graph with more than* $\max\{(\ell-1)(n-\ell+1) + \binom{\ell-1}{2}, \binom{2\ell-1}{2}\}$ *edges contains a matching with at least ℓ edges.*

The bound in Theorem 12 is optimal. Indeed, when $\ell \leq 0.4(n+1)$, the *extremal graph* (denoted by $\mathrm{ExG}(n,\ell)$) is the complete graph $K_{\ell-1,n-\ell+1}$ together with the complete graph inserted into the $(\ell-1)$-part. When $\ell \geq 0.4(n+1)$, the extremal graph is the complete graph of order $2\ell-1$ with $n+1-2\ell$ isolated vertices padded.

Motivated by this, for $e \in [0, \frac{1}{2}]$ we define a graphons Ψ_e and Φ_e as a graphon as follows. We partition $\Omega = B_1 \sqcup B_2$ so that $\nu(B_1) = 1 - \sqrt{1-2e}$ and $\nu(B_2) = \sqrt{1-2e}$. We define Ψ_e to be constant 0 on $C_2 \times C_2$ and 1 elsewhere. We partition $\Omega = C_1 \sqcup C_2$ so that $\nu(C_1) = (2e)^2$ and $\nu(C_2) = 1 - (2e)^2$. We define Φ_e to be constant 1 on $C_1 \times C_1$ and 0 elsewhere. These definition uniquely determine Ψ_e and Φ_e, up to isomorphism. Thus, our graphon version of the Erdős–Gallai theorem reads as follows.

Theorem 13. *Suppose that $W : \Omega^2 \to [0,1]$ is a graphon. Let $e = \frac{1}{2}\int_{x,y} W(x,y)$. Then* $\mathsf{fcov}(W) \geq \min\{\sqrt{\frac{1}{2}e}, 1 - \sqrt{1-2e}\}$. *(The maximum is attained by the former term for $e \leq 0.32$ and by the latter term for $e \geq 0.32$).*

Furthermore, we have an equality if and only if W is isomorphic to Ψ_e (if $e \leq 0.32$) or to Φ_e (if $e \geq 0.32$).

This version of the Erdős–Gallai Theorem implies an asymptotic version of the finite statement. Furthermore, it provides a corresponding stability statement.

Theorem 14. *For every $\varepsilon > 0$ there exists numbers $n_0 \in \mathbb{N}$ and $\delta > 0$ such that the following holds. If G is a graph on $n > n_0$ vertices with more than* $\max\{(\ell-1)(n-\ell+1)+\binom{\ell-1}{2}, \binom{2\ell-1}{2}\}$ *edges, then G contains a matching with at least $\ell - \varepsilon n$ edges.*

Furthermore, G contains a matching with at least $\ell + \delta n$ edges, unless G is εn^2-close to the graph $\mathrm{ExG}(n, \ell)$ *as above in the edit distance.*

The way of deriving Theorem 14 from Theorem 13 is standard, and we refer the reader to [3] where this was done in detail in the context of a tiling theorem of Komlós, [5], which is a statement of a similar flavor.

Let us emphasize that the original proof of Theorem 12 is simple and elementary (and the corresponding stability statement would not be difficult to prove with the same approach either). While our proof is not long, it makes use of the heavy machinery of graph limits, and in particular the results from [4] and from Section 3.2. However, we think that our proof offers an interesting alternative point of view on the problem.

For the proof of Theorem 13 we shall need the following fact.

Fact 15. *Suppose that $D \in [0, \frac{1}{2}]$ is fixed. Then the maximum of the function $g(a, b) = \frac{a^2}{2} + b - \frac{b^2}{2}$ on the set $\{(a, b) : 0 \le a, b, \frac{a}{2} + b = D\}$ is attained for $a = 0, b = D$ if $D \le 0.4$ and $a = 2D, b = 0$ if $D \ge 0.4$.*

Proof. We transform this into an optimization problem in one variable by considering the function $h(b) = g(2(D-b), b)$. The function h is quadratic with limit plus infinity at $-\infty$ and at $+\infty$. Thus, the maximum of h on the interval $[0, D]$ will be either at $b = 0$ or at $b = D$. We have $h(0) = 2D^2$ and $h(D) = D - \frac{D^2}{2}$. A quick calculation gives that the latter is bigger for $D < 0.4$ while the the latter is bigger for $D > 0.4$. □

Proof of Theorem 13. By Theorem 6 we can fix a half-integral vertex cover of size $\mathsf{fcov}(W)$. Then we get a partition $\Omega = A_0 \sqcup A_{1/2} \sqcup A_1$ given by the preimages of 0, 1/2 and 1. Let us write $\alpha_0 = \nu(A_0)$, $\alpha_{1/2} = \nu(A_{1/2})$, and $\alpha_1 = \nu(A_1)$. We have $\mathsf{fcov}(W) = \frac{1}{2}\nu(A_{1/2}) + \nu(A_1)$. Note that $W_{\restriction A_0 \times (A_0 \cup A_{1/2})} = 0$. Therefore,

$$\begin{aligned} e \le \frac{\alpha_{1/2}^2}{2} + \alpha_1(\frac{1}{2}\alpha_1 + \alpha_{1/2} + \alpha_0) = \frac{\alpha_{1/2}^2}{2} + \alpha_1 - \frac{\alpha_1^2}{2} \\ \le \max\left\{2\mathsf{fcov}(W)^2, \mathsf{fcov}(W) - \frac{\mathsf{fcov}(W)^2}{2}\right\}, \end{aligned} \tag{19}$$

where the last part follows from Fact 15. Pedestrian calculations show that this is equivalent to the assertion of the theorem.

We now look at the furthermore part. If $\mathsf{fcov}(W) = \max\{e - \frac{e^2}{2}, 2e^2\}$, then the first inequality in (19) is at equality. This means that W must be 1 on $A_{1/2} \times A_{1/2}$ and on $A_1 \times \Omega$. Furthermore, Fact 15 tells us that for the second inequality in (19) to be at equality, we must have $\alpha_{1/2} = 0$ or $\alpha_1 = 1$. We conclude that W is isomorphic to Ψ_e or to Φ_e. □

4. Convergence of polytons

4.1. **Fractional vertex cover polytons of a convergent graphon sequence.** Suppose that a sequence of graphons $(W_n)_n$ converges to a graphon W. We want to relate the polytons $\mathrm{FCOV}(W_n)$ to the polyton $\mathrm{FCOV}(W)$. First, observe that the polytons $\mathrm{FCOV}(W_n)$ do not converge to $\mathrm{FCOV}(W)$ in any reasonable sense in general. Indeed, for example, take W_n to be a representation of a sample of the Erdős–Rényi random graph $\mathbb{G}(2n, 1/\log n)$. It is well-known that almost surely almost all these graphs contain a perfect matching. Thus, $\mathrm{FCOV}(W_n)$ contain only fractional vertex covers of size $\frac{1}{2}$ and more. On the other hand, almost surely, the zero graphon $W = 0$ is the limit of $(W_n)_n$, and so $\mathrm{FCOV}(W)$ consists of all $[0, 1]$-valued measurable functions on Ω.

However, Theorem 16 below shows that $\mathrm{FCOV}(W)$ asymptotically contains the polytons $\mathrm{FCOV}(W_n)$. This theorem is a special case of [4, Theorem 3.14].

Theorem 16. *Suppose that $(W_n)_n$ is a sequence of graphons on Ω that converges to a graphon $W : \Omega^2 \to [0, 1]$ in the cut-norm. Suppose that $\mathfrak{c}_n \in \mathrm{FCOV}(W_n)$. Then any accumulation point of the sequence $(\mathfrak{c}_n)_n$ in the weak* topology lies in* $\mathrm{FCOV}(W)$.

4.2. **Matching polytons of a convergent graphon sequence.** The main new result of this section concerns convergence properties of the matching polytons. This result is dual to Theorem 16: if W_n converges to W then "$\mathrm{MATCH}(W_n)$ asymptotically contain $\mathrm{MATCH}(W)$".

Theorem 17. *Suppose that $W : \Omega^2 \to [0, 1]$ is a graphon on a probability space Ω, and let $\mathfrak{m} \in \mathrm{MATCH}(W)$ be fixed. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $U : \Omega^2 \to [0, 1]$ is a graphon with $\|U - W\|_\square < \delta$ then there is $\mathfrak{m}_U \in \mathrm{MATCH}(U)$ such that $\|\mathfrak{m}_U - \mathfrak{m}\|_\square < \varepsilon$.*

Since the cut-norm topology is stronger than the weak* topology, we get the following corollary.

Corollary 18. *Suppose that $(W_n)_n$ is a sequence of graphons on a probability space Ω that converges to a graphon $W : \Omega^2 \to [0, 1]$ in the cut-norm. Suppose that $\mathfrak{m} \in \mathrm{MATCH}(W)$. Then there exists a sequence $\mathfrak{m}_n \in \mathrm{MATCH}(W_n)$ such that $(\mathfrak{m}_n)_n$ converges to $\mathfrak{m}$ in the cut-norm. In particular, the sequence $(\mathfrak{m}_n)_n$ converges to $\mathfrak{m}$ in the weak* topology.*

In the proof of Theorem 17, we will need the following technical lemma.

Lemma 19. *Let (Ω, ν) be a probability space. Then for every pair $(F_1, F_2) \in L^1(\Omega^2) \times L^1(\Omega^2)$ and every $\varepsilon > 0$ there exists a partition of Ω into finitely many pairwise disjoint subsets $\Omega_1, \dots, \Omega_k$ (for a suitable natural number k), each of measure $\frac{1}{k}$, such that*

$$\sum_{i,j=1}^{k} \int_{\Omega_i \times \Omega_j} |F_1 - F_1^{ij}| < \varepsilon \qquad \text{and} \qquad \sum_{i,j=1}^{k} \int_{\Omega_i \times \Omega_j} |F_2 - F_2^{ij}| < \varepsilon \;, \tag{20}$$

where

$$F_1^{ij} = \frac{1}{\nu(\Omega_i)\nu(\Omega_j)} \int\limits_{\Omega_i \times \Omega_j} F_1 \qquad \textit{and} \qquad F_2^{ij} = \frac{1}{\nu(\Omega_i)\nu(\Omega_j)} \int\limits_{\Omega_i \times \Omega_j} F_2 \;, \qquad i,j = 1,\dots,n \;.$$

Proof. Let $\mathcal{L}$ be the family of all pairs $(F_1, F_2) \in L^1(\Omega^2) \times L^1(\Omega^2)$ such that for every $\varepsilon > 0$ there exist a partition of Ω into finitely many pairwise disjoint subsets $B_1, \dots, B_m$ (for a suitable natural number m), and real numbers D_1^{pq}, D_2^{pq}, $p, q = 1, \dots, m$, such that

$$\sum_{p,q=1}^{m} \int\limits_{B_p \times B_q} |F_1 - D_1^{pq}| < \varepsilon \qquad \text{and} \sum_{p,q=1}^{m} \int\limits_{B_p \times B_q} |F_2 - D_2^{pq}| < \varepsilon \;.$$

It is easy to verify that $\mathcal{L}$ is a closed subspace of $L^1(\Omega^2) \times L^1(\Omega^2)$ containing all pairs consisting of characteristic functions of measurable rectangles. Therefore it holds $\mathcal{L} = L^1(\Omega^2) \times L^1(\Omega^2)$.

Now let us fix $(F_1, F_2) \in L^1(\Omega^2) \times L^1(\Omega^2)$ and $\varepsilon > 0$. As $(F_1, F_2) \in \mathcal{L}$, we can find a partition of Ω into finitely many pairwise disjoint subsets $B_1, \dots, B_m$, and real numbers D_1^{pq}, D_2^{pq}, $p, q = 1, \dots, m$, such that

$$\sum_{p,q=1}^{m} \int\limits_{B_p \times B_q} |F_1 - D_1^{pq}| < \frac{1}{4}\varepsilon \qquad \text{and} \qquad \sum_{p,q=1}^{m} \int\limits_{B_p \times B_q} |F_1 - D_2^{pq}| < \frac{1}{4}\varepsilon \;. \tag{21}$$

By the absolute continuity of the Lebesgue integral there is $\delta > 0$ such that $\int\limits_E |F_1| < \frac{1}{4}\varepsilon$ and $\int\limits_E |F_2| < \frac{1}{4}\varepsilon$, whenever $E \subseteq \Omega \times \Omega$ is such that $\nu^{\oplus 2}(E) < \delta$. We fix a natural number k such that $\frac{m}{k} < \frac{1}{2}\delta$. For every $p = 1, \dots, m$, we find a decomposition $B_p = B_p^0 \sqcup B_p^1 \sqcup \dots \sqcup B_p^{n_p}$ of B_p into finitely many pairwise disjoint subsets such that $\nu(B_p^r) = \frac{1}{k}$ for $r = 1, \dots, n_p$, and $\nu(B_p^0) \leq \frac{1}{k}$. We set $F = \bigcup\limits_{p=1}^{m} B_p^0$. Then $\nu(F)$ is smaller that $\frac{1}{2}\delta$, and so

$$\int\limits_{(F\times\Omega)\cup(\Omega\times F)} |F_1| < \frac{1}{4}\varepsilon \qquad \text{and} \qquad \int\limits_{(F\times\Omega)\cup(\Omega\times F)} |F_2| < \frac{1}{4}\varepsilon \;. \tag{22}$$

Moreover, $\nu(F)$ is a multiple of $\frac{1}{k}$, and so it can be decomposed into finitely many disjoint subsets $\tilde{B}_1, \dots, \tilde{B}_l$ (for a suitable natural number l), each of measure $\frac{1}{k}$. Let $\Omega_1, \dots \Omega_k$ be some enumeration of the sets $\tilde{B}_1 \dots, \tilde{B}_l$ and B_p^r, $r = 1 \dots, n_p$, $p = 1, \dots, m$. For every $i, j = 1, \dots, k$, we define real numbers C_1^{ij} and C_2^{ij} as follows. If $\Omega_i \times \Omega_j = B_p^r \times B_q^s$ for some $p, q \in \{1, \dots, m\}$, $r \in \{1, \dots, n_p\}$ and $s \in \{1, \dots, n_q\}$ then we set $C_1^{ij} = D_1^{pq}$ and $C_2^{ij} = D_2^{pq}$.

Otherwise, we set $C_1^{ij} = C_2^{ij} = 0$. Let us fix $t \in \{0,1\}$. Then we have

$$\begin{aligned}
&\sum_{i,j=1}^{k} \int_{\Omega_i\times\Omega_j} |F_t - C_t^{ij}| \\
&= \sum_{p,q=1}^{m} \sum_{\substack{r=1,\ldots,n_p\\ s=1,\ldots,n_q}} \int_{B_p^r\times B_q^s} |F_t - D_t^{pq}| + \int_{(F\times\Omega)\cup(\Omega\times F)} |F_t| \qquad (23)\\
&\overset{(21),(22)}{<} \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon \ .
\end{aligned}$$

Claim 2. *For every $i, j = 1, \ldots, k$, we have that*

$$\int_{\Omega_i\times\Omega_j} |F_t - F_t^{ij}| \le 2 \int_{\Omega_i\times\Omega_j} |F_t - C_t^{ij}| \ .$$

Proof. Let us fix $i, j \in \{1, \ldots, k\}$. It holds

$$\int_{\Omega_i\times\Omega_j} |F_t - F_t^{ij}| = \int_{\substack{(x,y)\in\Omega_i\times\Omega_j\\ F_t(x,y)<F_t^{ij}}} (F_t^{ij} - F_t) + \int_{\substack{(x,y)\in\Omega_i\times\Omega_j\\ F_t(x,y)>F_t^{ij}}} (F_t - F_t^{ij}) \ , \qquad (24)$$

and the two integrals on the right hand side of (24) equals each other (by the definition of F_t^{ij}). Therefore it is enough to show that one of these integrals is less or equal to $\int_{\Omega_i\times\Omega_j} |F_t - C_t^{ij}|$. Assume for example that $C_t^{ij} \ge F_t^{ij}$ (the complementary case is similar). Then we have

$$\int_{\substack{(x,y)\in\Omega_i\times\Omega_j\\ F_t(x,y)<F_t^{ij}}} (F_t^{ij} - F_t) \le \int_{\substack{(x,y)\in\Omega_i\times\Omega_j\\ F_t(x,y)<C_t^{ij}}} (C_t^{ij} - F_t) \le \int_{\Omega_i\times\Omega_j} |F_t - C_t^{ij}| \ ,$$

as we wanted. □

Inequality (23) combined with Claim 2 gives us (20). □

4.3. **Proof of Theorem 17.** Let $\varepsilon > 0$ be fixed. By basic properties of L^1-functions there is $M > 0$ (which we fix now) such that if we define

$$\tilde{\mathfrak{m}}(x,y) = \min\{\mathfrak{m}(x,y), M\}$$

then we have

$$\|\tilde{\mathfrak{m}} - \mathfrak{m}\|_{L^1(\Omega^2)} < \frac{1}{2}\varepsilon \ . \qquad (25)$$

Moreover, it is obvious that such defined function $\tilde{\mathfrak{m}}$ is still a matching in the graphon W. We fix $\tilde{\varepsilon} > 0$ such that

$$3\tilde{\varepsilon} + 6\sqrt{\tilde{\varepsilon}}M + 2\tilde{\varepsilon}^{\frac{3}{2}} < \frac{1}{2}\varepsilon \ . \qquad (26)$$

Claim 3. *There is $r>0$ such that whenever $\Theta\subseteq\Omega^2$ is of positive measure such that $\frac{1}{\nu^{\oplus 2}(\Theta)}\int_\Theta W<r$ then*

$$\nu^{\oplus 2}\left(\mathrm{supp}\,(W)\cap\Theta\right)<\frac{1}{2M}\tilde{\varepsilon}\;.$$

Proof. By basic properties of measurable functions, there is $s>0$ such that

$$\nu^{\oplus 2}\{(x,y)\in\Omega^2\colon 0<W(x,y)<s\}<\frac{1}{4M}\tilde{\varepsilon}\;. \tag{27}$$

We will prove that $r=\frac{1}{4M}\tilde{\varepsilon}s$ works. Suppose for a contradiction that there is $\Theta\subseteq\Omega^2$ of positive measure with $\frac{1}{\nu^{\oplus 2}(\Theta)}\int_\Theta W<r$ such that

$$\nu^{\oplus 2}\left(\mathrm{supp}\,(W)\cap\Theta\right)\geq\frac{1}{2M}\tilde{\varepsilon}\;. \tag{28}$$

Then we have

$$\begin{aligned} r &> \frac{1}{\nu^{\oplus 2}(\Theta)}\int_\Theta W\geq\frac{1}{\nu^{\oplus 2}(\Theta)}\nu^{\oplus 2}\left(\{(x,y)\in\Theta\colon W(x,y)\geq s\}\right)\cdot s\\ &\overset{(27)}{>}\frac{1}{\nu^{\oplus 2}(\Theta)}\left(\nu^{\oplus 2}\left(\mathrm{supp}\,(W)\cap\Theta\right)-\frac{1}{4M}\tilde{\varepsilon}\right)s\overset{(28)}{\geq}\frac{1}{\nu^{\oplus 2}(\Theta)}\cdot\frac{1}{4M}\tilde{\varepsilon}s\geq\frac{1}{4M}\tilde{\varepsilon}s\;, \end{aligned}$$

which is the desired contradiction with the definition of r. □

Now we fix $r>0$ from Claim 3, and we set

$$\eta=\frac{1}{2}\tilde{\varepsilon}\cdot\frac{1}{1+2M+2\frac{M}{r}}\;. \tag{29}$$

By Lemma 19 there is a natural number k and a partition of Ω into pairwise disjoint subsets $\Omega_1,\ldots,\Omega_k$, each of measure $\frac{1}{k}$, such that

$$\sum_{i,j=1}^{k}\int_{\Omega_i\times\Omega_j}|W-W^{ij}|<\eta^2\qquad\text{and}\qquad\sum_{i,j=1}^{k}\int_{\Omega_i\times\Omega_j}|\tilde{\mathfrak{m}}-m^{ij}|<\eta^2\;, \tag{30}$$

where

$$W^{ij}=\frac{1}{\nu(\Omega_i)\nu(\Omega_j)}\int_{\Omega_i\times\Omega_j}W\qquad\text{and}\qquad m^{ij}=\frac{1}{\nu(\Omega_i)\nu(\Omega_j)}\int_{\Omega_i\times\Omega_j}\tilde{\mathfrak{m}}\;,\qquad i,j=1\ldots,n\;.$$

The first inequality from (30) easily implies that for all but at most ηk^2 pairs (i,j) we have

$$\int_{\Omega_i\times\Omega_j}\left|W-W^{ij}\right|\leq\frac{1}{k^2}\eta\;. \tag{31}$$

Similarly, the second inequality from (30) implies that for all but at most ηk^2 pairs (i,j) we have

$$\int_{\Omega_i\times\Omega_j}\left|\mathfrak{m}-m^{ij}\right|\leq\frac{1}{k^2}\eta\;. \tag{32}$$

We now set

$$\delta = \frac{1}{k^2}\eta \ , \tag{33}$$

and we will show that this choice of δ works. So let $U\colon \Omega^2 \to [0,1]$ be a graphon such that $\|U - W\|_\square < \delta$. Let $A \subseteq [k] \times [k]$ denote the set of all those pairs (i,j) for which either (31) or (32) fails. We have that $|A| \le 2\eta k^2$. We define

$$t(x,y) = \begin{cases} \frac{m^{ij}}{W^{ij}} \cdot U(x,y) & \text{if } (i,j) \notin A \text{ and } W^{ij} \ge r, \\ 0 & \text{otherwise,} \end{cases} \qquad (x,y) \in \Omega_i \times \Omega_j, \qquad i,j = 1,\ldots,k \ .$$

Claim 4. *We have that $\|t - \tilde{\mathfrak{m}}\|_\square \le \tilde{\varepsilon}$.*

Proof. We need to show that for every measurable sets $S, T \subseteq \Omega$ it holds $\left|\int_{S\times T} (t - \tilde{\mathfrak{m}})\right| \le \tilde{\varepsilon}$. So let us fix the sets $S, T \subseteq \Omega$. Let Θ_A denote the union of all the sets $\Omega_i \times \Omega_j$ for which $(i,j) \in A$. Similarly, let Θ_B denote the union of all the sets $\Omega_i \times \Omega_j$ for which $(i,j) \notin A$ and $W^{ij} < r$, and let Θ_C denote the union of all the sets $\Omega_i \times \Omega_j$ for which $(i,j) \notin A$ and $W^{ij} \ge r$.

The bulk of the work is in proving the following three subclaims.

Subclaim 1. We have

$$\left|\int_{(S\times T)\cap\Theta_A} (t - \tilde{\mathfrak{m}})\right| \le 2\eta M \ .$$

Subclaim 2. We have

$$\left|\int_{(S\times T)\cap\Theta_B} (t - \tilde{\mathfrak{m}})\right| < \frac{1}{2}\tilde{\varepsilon} \ .$$

Subclaim 3. We have

$$\left|\int_{(S\times T)\cap\Theta_C} (t - \tilde{\mathfrak{m}})\right| < \eta\left(1 + 2\frac{M}{r}\right) \ .$$

Indeed, Subclaims 1–3 complete the proof of Claim 4 as then

$$\left|\int_{S\times T} (t - \tilde{\mathfrak{m}})\right| < \eta\left(1 + 2M + 2\frac{M}{r}\right) + \frac{1}{2}\tilde{\varepsilon} \overset{(29)}{\le} \tilde{\varepsilon} \ .$$

Proof of Subclaim 1. Recall that by the definition of the set A, it holds $\nu^{\oplus 2}(\Theta_A) \le 2\eta$, and so we have

$$\left|\int_{(S\times T)\cap\Theta_A} (t - \tilde{\mathfrak{m}})\right| = \int_{(S\times T)\cap\Theta_A} \tilde{\mathfrak{m}} \le \int_{\Theta_A} \tilde{\mathfrak{m}} \le \nu^{\oplus 2}(\Theta_A) M \le 2\eta M \ .$$

$\square$

Proof of Subclaim 2. If $\nu^{\oplus 2}(\Theta_B) = 0$ then trivially

$$\left| \int\limits_{(S\times T)\cap\Theta_B} (t - \tilde{\mathfrak{m}}) \right| = 0 \ .$$

So suppose that Θ_B is of positive measure. Note that then it clearly holds $\frac{1}{\nu^{\oplus 2}(\Theta_B)} \int\limits_{\Theta_B} W < r$, and so we have

$$\begin{aligned} \left| \int\limits_{(S\times T)\cap\Theta_B} (t - \tilde{\mathfrak{m}}) \right| &= \int\limits_{(S\times T)\cap\Theta_B} \tilde{\mathfrak{m}} \leq \nu^{\oplus 2}(\mathrm{supp}\,(\tilde{\mathfrak{m}}) \cap \Theta_B) \cdot M \\ &\leq \nu^{\oplus 2}(\mathrm{supp}\,(W) \cap \Theta_B) \cdot M \overset{\text{Claim 3}}{<} \frac{1}{2}\tilde{\varepsilon} \ . \end{aligned}$$

□

Proof of Sublclaim 3. It is enough to show that whenever a pair $(i,j) \notin A$ is such that $W^{ij} \geq r$ then

$$\left| \int\limits_{(S\times T)\cap(\Omega_i\times\Omega_j)} (t - \tilde{\mathfrak{m}}) \right| < \frac{1}{k^2}\eta \left(1 + 2\frac{M}{r}\right) \ .$$

So let us fix such a pair (i,j). Then we have

$$\begin{aligned}
&\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}(t-\tilde{\mathfrak{m}})\right|\\
&\le\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}\left(\frac{m^{ij}}{W^{ij}}U-m^{ij}\right)\right|+\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}\left(m^{ij}-\tilde{\mathfrak{m}}\right)\right|\\
&\le\frac{m^{ij}}{W^{ij}}\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}\left(U-W^{ij}\right)\right|+\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}|m^{ij}-\tilde{\mathfrak{m}}|\\
&\le\frac{M}{r}\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}\left((U-W)+(W-W^{ij})\right)\right|+\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}|m^{ij}-\tilde{\mathfrak{m}}|\\
&\le\frac{M}{r}\left|\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}(U-W)\right|+\frac{M}{r}\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}|W-W^{ij}|+\int_{(S\times T)\cap(\Omega_i\times\Omega_j)}|m^{ij}-\tilde{\mathfrak{m}}|\\
\boxed{(i,j)\notin A}\quad&\le\frac{M}{r}\left|\int_{(S\cap\Omega_i)\times(T\cap\Omega_j)}(U-W)\right|+\frac{M}{r}\cdot\frac{1}{k^2}\eta+\frac{1}{k^2}\eta\\
\boxed{\text{by (33)}}\quad&<\frac{M}{r}\cdot\frac{1}{k^2}\eta+\frac{M}{r}\cdot\frac{1}{k^2}\eta+\frac{1}{k^2}\eta\\
&=\frac{1}{k^2}\eta\left(1+2\frac{M}{r}\right)\,.
\end{aligned}$$

□

□

Let B_1 denote the set of all those $x\in\Omega$ for which $\int_{y\in\Omega}t(x,y)>\int_{y\in\Omega}\tilde{\mathfrak{m}}(x,y)+\sqrt{\tilde{\varepsilon}}$. Similarly, let B_2 denote the set of all those $x\in\Omega$ for which $\int_{y\in\Omega}t(y,x)>\int_{y\in\Omega}\tilde{\mathfrak{m}}(y,x)+\sqrt{\tilde{\varepsilon}}$. Then we have $\nu(B_1)<\sqrt{\tilde{\varepsilon}}$, as otherwise

$$\tilde{\varepsilon}\overset{\text{Claim 4}}{\ge}\|t-\tilde{\mathfrak{m}}\|_\square\ge\int_{B_1\times\Omega}(t-\tilde{\mathfrak{m}})>\nu(B_1)\sqrt{\tilde{\varepsilon}}\ge\tilde{\varepsilon}\,,$$

which is not possible. In the same way, we conclude that $\nu(B_2)<\sqrt{\tilde{\varepsilon}}$. Now we are ready to define $\mathfrak{m}_U$ by setting

$$\mathfrak{m}_U(x,y)=\begin{cases}0 & \text{if } x\in B_1\cup B_2 \text{ or } y\in B_1\cup B_2,\\ \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}t(x,y) & \text{otherwise,}\end{cases}\qquad(x,y)\in\Omega^2\,. \tag{34}$$

Claim 5. *We have that* $\|\mathfrak{m}_U - t\|_{L^1(\Omega^2)} < 6\sqrt{\tilde{\varepsilon}}M + 2\tilde{\varepsilon} + 2\tilde{\varepsilon}^{\frac{3}{2}}$.

Proof. We set $B = ((B_1 \cup B_2) \times \Omega) \cup (\Omega \times (B_1 \cup B_2))$. Then we have

$$\begin{aligned}
\|\mathfrak{m}_U - t\|_{L^1(\Omega^2)} &= \int_B |\mathfrak{m}_U - t| + \int_{\Omega^2 \setminus B} |\mathfrak{m}_U - t| = \int_B t + \left(1 - \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}\right) \int_{\Omega^2\setminus B} t \\
&= \int_{(B_1\cup B_2)\times\Omega} t + \int_{\Omega\times(B_1\cup B_2)} t + \left(1 - \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}\right) \int_{(\Omega\setminus(B_1\cup B_2))^2} t \\
\overset{\boxed{\text{Claim 4}}}{\leq} & \int_{(B_1\cup B_2)\times\Omega} \tilde{\mathfrak{m}} + \tilde{\varepsilon} + \int_{\Omega\times(B_1\cup B_2)} \tilde{\mathfrak{m}} + \tilde{\varepsilon} + \left(1 - \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}\right) \left(\int_{(\Omega\setminus(B_1\cup B_2))^2} \tilde{\mathfrak{m}} + \tilde{\varepsilon} \right) \\
&\leq \nu^{\oplus 2}((B_1 \cup B_2) \times \Omega) \cdot M + \nu^{\oplus 2}(\Omega \times (B_1 \cup B_2)) \cdot M + 2\tilde{\varepsilon} + \frac{2\sqrt{\tilde{\varepsilon}}}{1+2\sqrt{\tilde{\varepsilon}}}(M + \tilde{\varepsilon}) \\
&< 4\sqrt{\tilde{\varepsilon}}M + 2\tilde{\varepsilon} + 2\sqrt{\tilde{\varepsilon}}(M + \tilde{\varepsilon}) = 6\sqrt{\tilde{\varepsilon}}M + 2\tilde{\varepsilon} + 2\tilde{\varepsilon}^{\frac{3}{2}} \ .
\end{aligned}$$

□

Now we have

$$\begin{aligned}
\|\mathfrak{m}_U - \mathfrak{m}\|_\square &\leq \|\mathfrak{m}_U - t\|_\square + \|t - \tilde{\mathfrak{m}}\|_\square + \|\tilde{\mathfrak{m}} - \mathfrak{m}\|_\square \\
&\leq \|\mathfrak{m}_U - t\|_{L^1(\Omega^2)} + \|t - \tilde{\mathfrak{m}}\|_\square + \|\tilde{\mathfrak{m}} - \mathfrak{m}\|_{L^1(\Omega^2)} \\
\boxed{\text{by (25), Claims 4 and 5}} \quad &< \frac{1}{2}\varepsilon + 3\tilde{\varepsilon} + 6\sqrt{\tilde{\varepsilon}}M + 2\tilde{\varepsilon}^{\frac{3}{2}} \\
\boxed{(26)} \quad &< \varepsilon \ .
\end{aligned}$$

So it remains to show that $\mathfrak{m}_U$ is a matching in the graphon U.

The fact that $\mathfrak{m}_U$ is a nonnegative function from $L^1(\Omega^2)$ is obvious, and we also have $\mathrm{supp}\,(\mathfrak{m}_U) \subseteq \mathrm{supp}\,(t) \subseteq \mathrm{supp}\,(U)$. So we only need to show that for almost every $x \in \Omega$ it holds

$$\int_{y\in\Omega} \mathfrak{m}_U(x,y) + \int_{y\in\Omega} \mathfrak{m}_U(y,x) \leq 1 \ . \tag{35}$$

This is trivially satisfied for every $x \in B_1 \cup B_2$ as then the left-hand side of (35) equals 0. So let us fix $x \in \Omega \setminus (B_1 \cup B_2)$. We may assume that

$$\int_{y\in\Omega} \tilde{\mathfrak{m}}(x,y) + \int_{y\in\Omega} \tilde{\mathfrak{m}}(y,x) \leq 1 \ , \tag{36}$$

as $\tilde{\mathfrak{m}}$ is a matching (in the graphon W). Then it holds

$$\int\limits_{y\in\Omega} \mathfrak{m}_U(x,y) + \int\limits_{y\in\Omega} \mathfrak{m}_U(y,x) \overset{(34)}{\leq} \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}\left(\int\limits_{y\in\Omega} t(x,y) + \int\limits_{y\in\Omega} t(y,x)\right)$$

$$\boxed{x\notin B_1\cup B_2} \quad \leq \frac{1}{1+2\sqrt{\tilde{\varepsilon}}}\left(\int\limits_{y\in\Omega} \tilde{\mathfrak{m}}(x,y) + \int\limits_{y\in\Omega} \tilde{\mathfrak{m}}(y,x) + 2\sqrt{\tilde{\varepsilon}}\right) \overset{(36)}{\leq} 1\ ,$$

which completes the proof of Theorem 17.

5. Concluding remarks

5.1. **Bipartiteness from the matching polyton.** Theorems 5 and 7 characterize bipartiteness of a graphon in terms of its fractional vertex cover polyton. For finite graphs there is another characterization in terms of the matching polytope: a graph is bipartite if and only if MATCH(G) is integral. Recall that there seems to be no counterpart to the concept of integrality of a graphon matching (c.f. Remark 3). So, we leave it as an important question to provide a characterization of bipartiteness in terms of MATCH(W).

5.2. **Perfect matching polyton.** Many variants of the above polytopes are considered in combinatorial optimization. As an example, let us mention the *perfect matching polytope* PerfMATCH(G) and the *fractional perfect matching polytope* FPerfMATCH(G) of a graph G. The corresponding graphon polyton is[4]

$$\mathrm{PerfMATCH}(W) = \{\mathfrak{m} \in \mathrm{MATCH}(W) : \|\mathfrak{m}\| = 1\}\ .$$

It might be interesting to study this, and similar polytons. That said, let us emphasize that many basic results, like Edmonds' perfect matching polytope theorem, seem not to have a graphon counterpart as they concern integrality-related properties of the polytope.

5.3. **Generalizing the results to F-tilings.** Results in Section 3 are specific to matchings — even in the finite setting. Even though we have not worked out details, we believe that our second main result, Theorem 17, extends to general F-tilings as introduced in [4] (and so does its proof).

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[4]again, we cannot distinguish between the integral and fractional version

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Appendix A. Characterization of partite graphons

In Lemma 10 we proved by transfinite induction that a graphon is bipartite if and only if it has zero density of each odd cycle. In the finite world there is a more general but equally trivial statement: a graph is r-partite if and only if it has zero density of each graph of chromatic number $r+1$ (or higher). In Proposition 21 we give a graphon version of this statement. Its proof stems from discussions with András Máthé.

We shall first need the following lemma, which can be found as [3, Lemma 2.2] (though it appears to be folklore).

Lemma 20. *Let $W : \Omega^2 \to [0,1]$ be a graphon. Suppose that $(A_n)_{n=1}^{\infty}$ is a sequence of sets in Ω with the property that*

$$\int_{(x,y)\in A_n\times A_n} W(x,y) \overset{n\to\infty}{\longrightarrow} 0 \ .$$

Suppose that the indicator functions of the sets A_n converge weak to a function $f : \Omega \to [0,1]$. Then* $\operatorname{supp} f$ *is an independent set in W.*

Proposition 21. *Suppose that $k \in \mathbb{N}$ and $W : \Omega^2 \to [0,1]$ is a graphon which has zero density of each finite graph of chromatic number at least $k+1$. Then W is k-partite.*

Proof. Let $(G_n)_n$ be samples of the inhomogeneous random graphs $\mathbb{G}(n, W)$. It is well-known that the graphs G_n converge to W in the cut-distance almost surely, see e.g., [6, Lemma 10.16]. We can therefore map the vertices $\{1, 2, \ldots, n\}$ of G_n to sets $\Omega_n^{(1)}, \Omega_n^{(2)}, \ldots, \Omega_n^{(n)}$ which partition Ω into sets of measure $\frac{1}{n}$ each, in a way that this partition witnesses ε_n-closeness of G_n to W in the cut-distance (where $\varepsilon_n \to 0_+$). In particular, whenever $O \subseteq V(G_n)$ is an independent set, we have

$$\int_{x\in\bigcup_{v\in O}\Omega_n^{(v)}} \int_{y\in\bigcup_{v\in O}\Omega_n^{(v)}} W(x,y) < \varepsilon_n \ .$$

Also, observe that almost surely, each graph G_n has zero density of each finite graph of chromatic number at least $k+1$. In particular G_n is k-colorable. Let us fix a partition $\Omega = V_n^{(1)} \sqcup V_n^{(2)} \sqcup \ldots \sqcup V_n^{(k)}$ according to one fixed k-coloring of G_n.

Consider now a weak* accumulation point $(f_1, f_2, \ldots, f_k)$ of the sequence of k-tuples of functions

$$\left(\mathbf{1}_{V_n^{(1)}}, \mathbf{1}_{V_n^{(2)}}, \ldots, \mathbf{1}_{V_n^{(k)}}\right)_n .$$

(Such an accumulation point exists by the sequential Banach–Alaoglu theorem.) We have $f_1 + f_2 + \ldots + f_k = 1$ almost everywhere on Ω. Consequently we can find measurable sets $A_i \subseteq \operatorname{supp} f_i$ so that $\Omega = A_1 \sqcup A_2 \sqcup \ldots \sqcup A_k$. Lemma 20 tells us that $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_k$ is a k-coloring of W. $\square$

Let us note that in [1] a result in a similar direction was proven.

Theorem 22 ([1])**.** *Suppose that W is a graphon, and H is a finite graph with the property that $t(H, W) = 0$. Then W is countably partite.*

Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, Praha. The Institute of Mathematics is supported by RVO:67985840.

E-mail address: `dolezal@math.cas.cz`

E-mail address: `honzahladky@gmail.com`