A Diffuse Interface Model of a Two-Phase Flow with Thermal Fluctuations

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Two phase flow with temperature fluctuations

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho, c)\mathrm{d}t$$
$$= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t - \mathrm{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2 \right) \mathrm{d}t$$

$$\mathrm{d}(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u})\mathrm{d}t = \Delta_x \mu \mathrm{d}t + \boxed{\varrho \sigma(c) \mathrm{d}W}$$

Constitutive relations

$$E_{\mathrm{free}} = \int_{\mathcal{D}} \varrho f(\varrho, c) + \frac{1}{2}|\nabla_x c|^2 \, \mathrm{d}x$$

$$\varrho \mu = \varrho \frac{\partial f(\varrho, c)}{\partial c} - \Delta_x c, \ \ p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \nu_{\mathrm{shear}} \left(\nabla_x \mathbf{u}^t + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \nu_{\mathrm{bulk}} \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Stochastic forcing

Stochastic basis

$$\left\{\Omega,\ \mathfrak{F},\ \{\mathfrak{F}_t\}_{t\geq 0},\ \mathcal{P}\right\}$$

Random driving force in the Cahn–Hilliard system

$$\sigma(c)\mathrm{d}W \equiv \sum_{k=1}^{\infty} \alpha_k \sigma_k(c)\mathrm{d}W_k,\ \text{where } \sum_{k=1}^{\infty} \alpha_k^2 < \infty,$$

$$\sigma_k \in W^{2,\infty}(R),\ \|\sigma_k\|_{W^{2,\infty}} \leq c \text{ uniformly in } k = 1, 2, \ldots$$

Initial and boundary conditions

Periodic boundary conditions

$$\mathcal{T}^N = \left\{[-\pi,\pi]|_{\{-\pi,\pi\}}\right\}^N, \ N = 1,2,3.$$

(Random) initial data

$$\varrho(0,\cdot) = \varrho_0, \ (\varrho \mathbf{u})(0,\cdot) = (\varrho \mathbf{u})_0, \ c(0,\cdot) = c_0$$

Total mass

$$\varrho_0 > 0 \text{ in } \mathcal{T}^N, \ \int_{\mathcal{T}^N} \varrho_0 \, \mathrm{d}x = M > 0, \ \mathcal{P}\text{-a.s.}$$

Weak martingale solution

Probability basis, forcing term

$$\left\{ \Omega,\ \mathfrak{F},\ \{\mathfrak{F}_t\}_{t\geq 0},\ \mathcal{P} \right\},\ \{W_k\}_{k=1}^{\infty}$$

Field equations

$$\left[\int_{\mathcal{T}^N} \varrho\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathcal{T}^N} \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$\left[\int_{\mathcal{T}^N} \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathcal{T}^N} \left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho, c)\mathrm{div}_x\boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_{\mathcal{T}^N} \left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I} \right) : \nabla_x\boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$\left[\int_{\mathcal{T}^N} \varrho c\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\mathcal{T}^N} \left[\varrho c\partial_t\varphi + \varrho c\mathbf{u}\cdot\nabla_x\varphi - \nabla_x\mu\cdot\nabla_x\varphi \right] \ \mathrm{d}x + \int_0^\tau \left(\int_{\mathcal{T}^N} \varrho\sigma(c)\varphi \ \mathrm{d}x \right) \ \mathrm{d}W$$

Dissipative martingale solutions

Energy inequality

$$\left[\psi \int_{\mathcal{T}^N} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho f(\varrho, c) + \frac{1}{2}|\nabla_x c|^2\right] \, \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \psi \int_{\mathcal{T}^N} \left[\mathbb{S}(c, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + |\nabla_x \mu|^2\right] \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \partial_t \psi \int_{\mathcal{T}^N} \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho f(\varrho, c) + \frac{1}{2}|\nabla_x c|^2\right] \, \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \psi \int_{\mathcal{T}^N} \frac{1}{2} \sum_{k=1}^{\infty} \alpha_k^2 |\nabla_x \sigma_k(c)|^2 \, \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \psi \int_{\mathcal{T}^N} \frac{1}{2} \varrho \frac{\partial^2 f(\varrho, c)}{\partial c^2} \sum_{k=1}^{\infty} \alpha_k^2 |\sigma_k(c)|^2 \, \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \psi \left(\int_{\mathcal{T}^N} \varrho \mu \sigma(c) \, \mathrm{d}x\right) \mathrm{d}W$$

The main existence result

Hypotheses

$$f(\varrho, c) = f^e(\varrho) + f^m(\varrho, c) + f^c(c),$$

$$\text{where } f^e = a\varrho^{\gamma-1},\ a > 0,\ \gamma > 3$$

$$f^c \in C^3(R),\ f^c(0) = 0,\ |\partial_c^j f^c| \lesssim 1,\ j = 2, 3$$

$$\partial_c f^c(c) \approx c \text{ for } |c| >> 1 \text{ large},$$

$$f^m(\varrho, c) = \log(\varrho) H(c),\ H \in C^3(R),\ |\partial_c^j H| \lesssim 1,\ j = 1, 2, 3$$

Conclusion

There exists a dissipative martingale solution in $[0, T]$ with a given initial law

Existing result

Cahn–Hilliard equation driven by stochastic forcing

- **Basic (additive noise):** Goudenege [2009]; Debussche and Goudenege [2011]
- **More recent:** Gal, Medjo [2014]; Goudenege, Manca [2015]; Scarpa [2017]

Compressible model of a two–phase flow

- **Modelling:** Anderson, McFadden, Wheeler [1998]; Lowengrub, Truskinowski [1998]
- **Analysis:** Abels, E.F. [2008]

Approximate scheme

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho[\mathbf{u}]_R)\mathrm{d}t = \varepsilon\Delta_x\varrho\mathrm{d}t,\ \ \varrho(0,\cdot) = \varrho_0$$

$$\begin{aligned}
&\mathrm{d}\Pi_m(\varrho\mathbf{u}) + \mathrm{div}_x\Pi_m(\varrho\mathbf{u}\otimes[\mathbf{u}]_R)\mathrm{d}t + \chi\left(\|\mathbf{u}\|_{H_m} - R\right)\nabla_x\Pi_m\left[p(\varrho,c) + \sqrt{\varepsilon}\varrho^{\alpha}\right]\mathrm{d}t \\
&\quad = \varepsilon\Delta_x\Pi_m(\varrho\mathbf{u})\mathrm{d}t + \mathrm{div}_x\Pi_m\mathbb{S}(\nabla_x\mathbf{u})\mathrm{d}t \\
&\quad - \chi\left(\|\mathbf{u}\|_{H_m} - R\right)\mathrm{div}_x\Pi_m\left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\right)\mathrm{d}t, \\
&\quad \Pi_m(\varrho\mathbf{u})(0,\cdot) = \Pi_m(\varrho_0\mathbf{u}_0)
\end{aligned}$$

$$\mathrm{d}c + [\mathbf{u}]_R\cdot\nabla_x c\mathrm{d}t = \frac{1}{\varrho}\Delta_x\mu\mathrm{d}t + \sigma(c)\mathrm{d}W,\ \ \mu = \frac{\partial f(\varrho,c)}{\partial c} - \frac{1}{\varrho}\Delta_x c,\ \ c(0,\cdot) = c_0$$

Itô – Nisio approach

$$c \mapsto \varrho[c],\ \ \mathbf{u}[c]$$

Solve a stochastic ODE (Galerkin approximation) with memory terms

Stochastic compactness

Main ingredients in the existence proof

- Changing probability setting at any approximate level
- Working in weak topologies (non–Polish spaces), Jakubowski version of Skorokhod representation theorem
- Lack of continuity in time, random distributions instead of random processes in the spirit of Breit, E.F., Hofmanová [2018]