Compressible fluid flows driven by stochastic forcing

Eduard Feireisl
based on joint work with D.Breit (Edinburgh), M.Hofmanová (Bielefeld)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague
Technische Universität Berlin

International workshop on Hyperbolic and Kinetic Problems, Taipei, 9 July - 12 July, 2018

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Driven Navier-Stokes/Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Stochastic forcing

$$\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(x, \varrho, \mathbf{u})\mathrm{d}\beta_k$$

Iconic examples

$$\mathbf{G}_k = \mathbf{f}_k(x), \ \mathbf{G}_k = \mathbf{u} d_k(x) - \text{“stochastic damping”}$$

Initial and boundary conditions

(Random) initial data

$$\varrho(0, \cdot) = \varrho_0, \ (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0$$

Spatial domain

$$Q \subset R^N, \text{ or "flat" torus } Q = \mathcal{T}^N = \left([0,1]|_{\{0,1\}} \right)^N, \ N = (1), 2, 3$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial Q} = 0 \text{ impermeability}$$

$$\mathbf{u} \times \mathbf{n}|_{\partial Q} = 0 \text{ no-slip}$$

$$[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial Q} = 0 \text{ complete slip}$$

Weak (PDE) formulation

Field equations

$$\left[\int_Q \varrho\phi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \varrho\mathbf{u}\cdot\nabla_x\phi \,\mathrm{d}x\mathrm{d}t,$$

$$\left[\int_Q \varrho\mathbf{u}\cdot\phi \,\mathrm{d}x\right]_{t=0}^{t=\tau} - \int_0^\tau \int_Q \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\phi + p(\varrho)\mathrm{div}_x\phi \,\mathrm{d}x\mathrm{d}t$$

$$= -\int_0^\tau \int_Q \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\phi \,\mathrm{d}x\mathrm{d}t + \boxed{\int_0^\tau \left(\int_Q \varrho\mathbf{G}\cdot\phi \,\mathrm{d}x\right)\mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_Q \varrho\mathbf{G}\cdot\phi \,\mathrm{d}x\right)\mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_Q \varrho\mathbf{G}_k\cdot\phi \,\mathrm{d}x\right)\mathrm{d}\beta_k$$

Admissibility

Energy inequality

$$- \int_0^T \partial_t \psi \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \, \mathrm{d}x \mathrm{d}t + \int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \mathrm{d}t$$

$$\leq \psi(0) \int_Q \left[\frac{|(\varrho \mathbf{u})_0|^2}{2 \varrho_0} + P(\varrho_0) \right] \, \mathrm{d}x$$

$$+ \frac{1}{2} \int_0^T \psi \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{G}_k(x, \varrho, \mathbf{u})|^2 \, \mathrm{d}x \right) \mathrm{d}t + \int_0^T \psi \mathrm{d}M_E$$

$$\psi \geq 0, \ \psi(T) = 0, \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Strong vs. martingale solutions

Strong solutions

- the functions ϱ, $\mathbf{u}$ are differentiable a.s., the equations are satisfied in the classical sense
- the probability space uniquely determined

Martingale solutions

- solutions defined on a different, typically, the standard probability space
- the white noise as well as the initial data coincide with the originals in law

Main difficulties

Finite–dimensional approximation

Vacuum zone, random variables $\varrho\mathbf{u}$ and $\mathbf{u}$

A priori bounds

Energy *a priori* bounds only in expectations

Stochastic compactness method

Skorokhod–Prokhorov theorem (works on Polish spaces), weak topology is not Polish

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Global existence for the Navier–Stokes system [Breit, Hofmanová [2015]

The Navier–Stokes system admits global–in–time martingale solutions for

$$p(\varrho) \approx \varrho^{\gamma},\ \gamma > \frac{N}{2}$$

Relative energy inequality

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_Q \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right] \,\mathrm{d}x$$

Relative energy inequality

$$- \int_0^T \partial_t \psi \ \mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \ \mathrm{d}t$$

$$+ \int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \ \middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \mathrm{d}t$$

Test functions

$$\mathrm{d}r = D_t^d r \ \mathrm{d}t + \mathbb{D}_t^s r \ \mathrm{d}W, \ \ \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \ \mathrm{d}t + \mathbb{D}_t^s \mathbf{U} \ \mathrm{d}W$$

Remainder

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_Q \varrho \Big(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_Q \big((r - \varrho) P''(r) D_t^d r + \nabla_x P'(r) (r\mathbf{U} - \varrho \mathbf{u}) \big) \ \mathrm{d}x$$

$$- \int_Q \mathrm{div}_x \mathbf{U} (p(\varrho) - p(r)) \ \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_Q \varrho \Big| \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - [\mathbb{D}_t^s \mathbf{U}]_k \Big|^2 \ \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_Q \varrho P'''(r) |[\mathbb{D}_t^s r]_k|^2 \ \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_Q p''(r) |[\mathbb{D}_t^s r]_k|^2 \ \mathrm{d}x$$

$$+ \int_Q \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x$$

Weak–strong uniqueness

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.
A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists

Uniqueness in law.
If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Stationary solutions to the Navier–Stokes system

Basic hypotheses

- $$|\mathbf{G}_k| + |\nabla \mathbf{G}_k| \approx \alpha_k, \ \sum_{k>0} \alpha_k^2 < \infty$$
- $$p(\varrho) \approx \varrho^\gamma, \ \gamma > \frac{N}{2}$$
- complete slip/no slip boundary conditions

Stationary solutions [Breit, EF, Hofmanová, Maslowski] [2017]

For a given (deterministic) mass

$$M = \int_Q \varrho \,\mathrm{d}x > 0$$

the Navier–Stokes system admits a stationary martingale solution.

Method of the proof

Finite–dimensional approximation

Use the Krylov–Bogolyubov theory on the approximate system

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta_x \varrho + M\left(\int_Q \varrho \,\mathrm{d}x\right)$$

+ Galerkin approximation for the momentum equation

Uniform bounds

Uniform bounds based on deterministic estimates + Itô's chain rule

Stochastic compactness method

Skorokhod–Prokhorov theorem (works on Polish spaces), here we have weak topology

Complete system – more physics?

Complete system

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho, \vartheta)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Internal energy balance

$$\mathrm{d}\varrho e(\varrho, \vartheta) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u})\mathrm{d}t + \mathrm{div}_x \mathbf{q} \mathrm{d}t = \mathbb{S}(\nabla_x \mathbf{u}) : \mathbf{u}\mathrm{d}t - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} \mathrm{d}t$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Gibbs' relation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Weak (PDE) solutions to the Euler system

Infinitely many weak (PDE) solutions, Breit, EF, Hofmanová [2017]

Let $T > 0$ and the initial data

$$\varrho_0 \in C^3(Q),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(Q)$$

be given.
There exists a sequence of *strictly positive* stopping times

$$\tau_M > 0,\ \tau_M \to \infty$$

a.s. such that the initial–value problem for the *compressible Euler system* possesses infinitely many solutions defined in $(0, T \wedge \tau_M)$. Solutions are adapted to the filtration associated to the Wiener process W.

Semi-deterministic approach - additive noise

"Additive noise" problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k$$

$$\varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k = \varrho \mathbf{G} \mathrm{d} W$$

Additive noise, Step I

Step I

$$\partial_t(\varrho\mathbf{u}-\varrho\mathbf{G}W)+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)=-\partial_t\varrho\mathbf{G}W=\mathrm{div}_x(\varrho\mathbf{u})\mathbf{G}W$$

Transformed system I

$$\mathbf{w}=\varrho\mathbf{u}-\varrho\mathbf{G}W$$

$$\partial_t\varrho+\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)=0$$

$$\partial_t\mathbf{w}+\mathrm{div}_x\left(\frac{(\mathbf{w}+\varrho\mathbf{G}W)\otimes(\mathbf{w}+\varrho\mathbf{G}W)}{\varrho}\right)+\nabla_x p(\varrho)$$

$$=\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)\mathbf{G}W$$

Additive noise, Step II

Step II

$$\mathbf{w} = \mathbf{v} + \mathbf{V} + \nabla_x \Phi,\ \mathrm{div}_x \mathbf{v} = 0, \int_Q \mathbf{v}\ \mathrm{d}x = 0,\ \mathbf{V} = \mathbf{V}(t)$$

Transformed system II

$$\mathbf{w} = \varrho \mathbf{u} - \varrho \mathbf{G} W$$

$$\partial_t \varrho + \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W)}{\varrho} \right)$$

$$+ \nabla_x p(\varrho) + \nabla_x \partial_t \Phi = \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step III

Step III

Fix Φ, ϱ, $\mathbf{V}$ so that

$$\varrho(0,\cdot)=\varrho_0,\ \mathbf{V}(0)=\frac{1}{|\Omega|}\int_Q \mathbf{u}_0\ \mathrm{d}x,\ \nabla_x\Phi(0,\cdot)=\mathbf{H}^{\perp}[\mathbf{u}_0]$$

$$\partial_t\varrho+\mathrm{div}_x(\nabla_x\Phi+\varrho\mathbf{G}W)=0$$

$$\partial_t\mathbf{V}=\frac{1}{|\Omega|}\mathrm{div}_x(\nabla_x\Phi+\varrho\mathbf{G}W)\mathbf{G}W$$

$$\mathrm{div}_x\left(\nabla_x\mathbf{M}+\nabla_x\mathbf{M}^{\perp}-\frac{2}{N}\mathrm{div}_x\mathbf{M}\right)$$

$$=\mathrm{div}_x(\nabla_x\Phi+\varrho\mathbf{G}W)\mathbf{G}W-\partial_t\mathbf{V}$$

Additive noise, Step IV

Step IV

Fix $\mathbf{h}$, $\mathbb{H}$ so that

$$\mathbf{h} = \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W, \ \mathbb{H} = \nabla_x \mathbf{M} + \nabla_x^t \mathbf{M} - \frac{2}{N} \mathrm{div}_x \mathbf{M} \mathbb{I} \in R^{N \times N}_{0,\mathrm{sym}}$$

Tranformed system III

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{H} + p(\varrho)\mathbb{I} + \partial_t \Phi \mathbb{I} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] - \frac{1}{|\Omega|} \int_Q \mathbf{u}_0 \ \mathrm{d}x$$

Additive noise, Step V

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e = \Lambda - \frac{N}{2}\left(p(\varrho) + \partial_t \Phi\right), \ \Lambda = \Lambda(t)$$

Abstract Euler system

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} - \mathbb{H} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C([0, T] \times \Omega; R^N), \ \mathbb{F} \in C([0, T] \times \Omega; R^{N \times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M} \right] < e$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M}\right] < e$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} + \mathbb{M}$$

Augmenting oscillations

Oscillatory lemma

If

$$\varrho, e, \mathbf{h} \in C(Q; R^N), \varrho, e > 0,\ \mathbb{H} \in C(Q; R^{N\times N}_{\rm sym,0})$$

$$\frac{N}{2}\lambda_{\rm max}\left[\frac{\mathbf{h}\otimes\mathbf{h}}{\varrho} - \mathbb{H}\right] < e \text{ in } Q,$$

then there exist

$$\mathbf{w}_n \in C^\infty_c(Q; R^N),\ \mathbb{G}_n \in C^\infty_c(Q; R^{N\times N}_{\rm sym,0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + {\rm div}_x \mathbb{G}_n = 0,\ {\rm div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\rm max}\left[\frac{(\mathbf{h}+\mathbf{w}_n)\otimes(\mathbf{h}+\mathbf{w}_n)}{\varrho} - (\mathbb{H}+\mathbb{G}_n)\right] < e$$

$$\mathbf{w}_n \rightharpoonup 0,\ \liminf_{n\to\infty}\int_Q \frac{|\mathbf{w}_n|^2}{\varrho}\ {\rm d}x{\rm d}t \geq \Lambda(\max_\Omega e)\int_Q\left(e - \frac{1}{2}\frac{|\mathbf{h}|^2}{\varrho}\right)^2\ {\rm d}x{\rm d}t$$

Basic ideas of proof [DeLellis and Székelyhidi]

Basic result

Unit cube and constant coefficients ϱ, e, $\mathbf{h}$, $\mathbb{H}$

Scaling

Localizing the basic result to "small" cubes by means of scaling arguments

Approximation

Replacing all continuous functions by their means on any of the "small" cubes

Difficulties in the stochastic world

Adaptiveness

All quantities must be adapted to the filtration associated to the Wiener process W

Geometric setting

Continuous functions approximated in a similar way as in the definition of Itô's integral
Admissible directions for oscillations selected by the Kuratowski, Ryll–Nardzewski theorem

Space–time localization

Stopping the Wiener process by its Hölder norm