

On the asymptotic behavior of stochastically driven compressible fluid flows

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Driven Navier-Stokes system

Field equations

$$d\rho + \operatorname{div}_x(\rho \mathbf{u})dt = 0$$

$$d(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u})dt + \nabla_x p(\rho)dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})dt + \rho \mathbf{G}(x, \rho, \mathbf{u})dW$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Stochastic forcing

$$\rho \mathbf{G}(x, \rho, \mathbf{u})dW = \sum_{k=1}^{\infty} \rho \mathbf{G}_k(x, \rho, \mathbf{u})dW_k$$

Iconic examples

$$\mathbf{G}_k = \mathbf{f}_k(x), \quad \mathbf{G}_k = \mathbf{u} d_k(x) - \text{“stochastic damping”}$$

Additive vs. multiplicative noise

Additive noise

$$\varrho \sum_{k=1}^{\infty} b_k(x) dW_k$$

$$\begin{aligned} d(\varrho \mathbf{u}) \cdots &= \varrho \sum_{k=1}^{\infty} b_k(x) dW_k = d \left(\varrho \sum_{k=1}^{\infty} b_k(x) dW_k \right) - d\varrho \sum_{k=1}^{\infty} b_k(x) W_k \\ &= d \left(\varrho \sum_{k=1}^{\infty} b_k(x) dW_k \right) + \operatorname{div}_x(\varrho \mathbf{u}) \sum_{k=1}^{\infty} b_k(x) W_k \end{aligned}$$

General multiplicative noise

$$\varrho \mathbf{G}(x, \varrho, \mathbf{u}) dW = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(x, \varrho, \mathbf{u}) dW_k$$

Initial and boundary conditions

(Random) initial data

$$\varrho(0, \cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0$$

Spatial domain

$Q \subset \mathbb{R}^N$, or “flat” torus $Q = \mathcal{T}^N = ([0, 1]_{\{0,1\}})^N$, $N = (1), 2, 3$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial Q} = 0 \text{ impermeability}$$

$$\mathbf{u} \times \mathbf{n}|_{\partial Q} = 0 \text{ no-slip}$$

$$[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial Q} = 0 \text{ complete slip}$$

Weak (PDE) formulation

Field equations

$$\begin{aligned} & \left[\int_Q \varrho \phi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \varrho \mathbf{u} \cdot \nabla_x \phi \, dx dt, \\ & \left[\int_Q \varrho \mathbf{u} \cdot \phi \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_Q \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \phi + p(\varrho) \operatorname{div}_x \phi \, dx dt \\ & = - \int_0^\tau \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \phi \, dx dt + \boxed{\int_0^\tau \left(\int_Q \varrho \mathbf{G} \cdot \phi \, dx \right) dW} \end{aligned}$$

$\phi = \phi(x)$ – a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_Q \varrho \mathbf{G} \cdot \phi \, dx \right) dW = \sum_{k=1}^{\infty} \int_0^\tau \left(\int_Q \varrho \mathbf{G}_k \cdot \phi \, dx \right) d\beta_k$$

Admissibility

Energy inequality

$$\begin{aligned} & - \int_0^T \partial_t \psi \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] dx dt + \int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx dt \\ & \leq \psi(0) \int_Q \left[\frac{|(\varrho \mathbf{u})_0|^2}{2\varrho_0} + P(\varrho_0) \right] dx \\ & + \frac{1}{2} \int_0^T \psi \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{G}_k(x, \varrho, \mathbf{u})|^2 dx \right) dt + \int_0^T \psi dM_E \\ & \psi \geq 0, \quad \psi(T) = 0, \quad P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} dz \end{aligned}$$

Strong vs. martingale solutions

Strong solutions

- the functions ϱ , \mathbf{u} are differentiable a.s., the equations are satisfied in the classical sense
- the probability space uniquely determined

Martingale solutions

- solutions defined on a different, typically, the standard probability space
- the white noise as well as the initial data coincide with the originals

in law

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Global existence for the Navier–Stokes system [Breit, Hofmanová [2015]]

The Navier–Stokes system admits global-in-time martingale solutions for

$$p(\varrho) \approx \varrho^\gamma, \quad \gamma > \frac{N}{2}$$

Weak–strong uniqueness

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.

A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists

Uniqueness in law.

If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Stochastic stationary solutions

Complete trajectories

$$\varrho, \mathbf{u} \in L^2_{\text{loc}}(0, T; L^\gamma \times W^{1,2}(Q; R^N))$$

Stationarity

The law of the solution is time shift invariant

$$\text{probability } \{(\varrho, \mathbf{u}) \in B\} = \text{probability } \{(\varrho(t + \tau), \mathbf{u}(t + \tau)) \in B\}$$

for any $\tau \geq 0$ and any

$$B - \text{ a Borel subset of } L^2_{\text{loc}}(0, T; L^\gamma \times W^{1,2}(Q; R^N))$$

Stationary solutions to the Navier–Stokes system

Basic hypotheses



$$|\mathbf{G}_k| + |\nabla \mathbf{G}_k| \approx \alpha_k, \quad \sum_{k>0} \alpha_k^2 < \infty$$



$$p(\varrho) \approx \varrho^\gamma, \quad \gamma > \frac{N}{2}$$

- complete slip/no slip boundary conditions

Stationary solutions [Breit, EF, Hofmanová, Maslowski] [2017]

For a given (deterministic) mass

$$M = \int_Q \varrho \, dx > 0$$

the Navier–Stokes system admits a stationary martingale solution.

Method of the proof

Finite-dimensional approximation

Use the Krylov–Bogolyubov theory on the approximate system

$$d\rho + \operatorname{div}_x(\rho \mathbf{u}) dt = \varepsilon \Delta_x \rho dt + M \left(\int_Q \rho dx \right) dt$$

+ Galerkin approximation for the momentum equation

Uniform bounds

Uniform bounds based on deterministic estimates + Itô's chain rule

Stochastic compactness method

Skorokhod–Prokhorov theorem (works on Polish spaces), here we have weak topology

Complete system – more physics?

Complete system

$$d\rho + \operatorname{div}_x(\rho \mathbf{u}) dt = 0$$

$$d(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) dt + \nabla_x p(\rho, \vartheta) dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) dt + \boxed{\rho \mathbf{G}(x, \rho, \mathbf{u}) dW}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Internal energy balance

$$d\rho e(\rho, \vartheta) + \operatorname{div}_x(\rho e(\rho, \vartheta) \mathbf{u}) dt + \operatorname{div}_x \mathbf{q} dt = \mathbb{S}(\nabla_x \mathbf{u}) : \mathbf{u} dt - p(\rho, \vartheta) \operatorname{div}_x \mathbf{u} dt$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Gibbs' relation

$$\vartheta Ds(\rho, \vartheta) = De(\rho, \vartheta) + p(\rho, \vartheta) D \left(\frac{1}{\rho} \right)$$