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ABSTRACT. We introduce new techniques allowing one to construct diagonals of bounded Hilbert space operators and operator tuples under "Blaschke-type" assumptions. This provides a new framework for a number of results in the literature and identifies, often large, subsets in the set of diagonals of *arbitrary* bounded operators (and their tuples). Moreover, our approach leads to substantial generalizations of the results due to Bourin, Herrero and Stout having assumptions of a similar nature.

1. INTRODUCTION

Let T be a bounded linear operator on a separable Hilbert space H. If $(e_k)_{k=1}^N$ is an orthonormal basis in H, $1 \le N \le \infty$, then T admits a matrix representation $M_T = (\langle Te_i, e_j \rangle)_{i,j=1}^N$. If $N < \infty$ then M_T is a finite matrix, and it is an essential part of the matrix theory to relate the properties of T and M_T. In particular, it is of substantial interest to express the structure of T in terms of the elements of M_T. Even in this toy setting a number of natural questions, e.g. on the interplay between the spectrum of T and the diagonal of M_T appeared to be rather complicated, see e.g. [7] and [48] for a pertinent discussion.

For an infinite-dimensional space H the relations between T and M_T become even more involved and depend on very advanced methods and techniques stemming from various domains of analysis. A nice illustration of the interplay between T and M_T is provided by the famous Kadison-Singer problem, its "matrix" reformulation and its solution by techniques originating from matrix theory. We refer e.g. to [41] for a nice account.

It is well-known that for good enough (usually Schatten class) T the diagonal of M_T carries a spectral information about T, and for trace-class T its trace is just the sum of eigenvalues by Lidskii's theorem. Note that in this case the trace-class property of T and the value of its trace do not change

under a change of basis. However, the situation becomes more complicated when T is far from being compact. One way to circumvent the problem is to vary the orthonormal basis $(e_k)_{k=1}^\infty$ and to consider the whole set $\mathcal{D}(T) := \{\langle Te_k, e_k \rangle_{k=1}^\infty\} \subset \ell_\infty(\mathbb{N})$ of diagonals of T.

A lot of research has been done to understand the structure of $\mathcal{D}(T)$ for general T. However, it still remains mysterious, even for very particular classes of operators. The aim of this paper is to introduce and study a Blaschke-type condition leading to a description of "large" subsets of $\mathcal{D}(T)$ in a priori terms. This is done in a very general set-up of bounded linear operators T and their tuples.

To put our consideration into a proper framework, let us first review major achievements made so far in the study of diagonals of operators on infinite-dimensional spaces. First, note that (as mentioned in [37]) the study of diagonals can be understood in at least two senses:

(a) as the study of $\mathcal{D}(T)$ for a class of operators T,

and

(b) as the study of $\mathcal{D}(T)$ for a fixed operator T.

Most of the relevant research has been concentrated on the first, easier problem, but there have been several papers addressing the second problem as well. Clearly the entries constituting diagonals of T belong to the numerical range $W(T)$ of T, and thus the numerical range and its subsets appear to be very natural candidates for a characterization of at least a part of $\mathcal{D}(T)$. However, the importance of numerical ranges for the study of diagonals was underlined only in [24], and to some extent in [47] and [18].

Motivated by a notorious problem from the theory of commutators, P. Fan studied in [18] a problem of existence of zero diagonals in $\mathcal{D}(T)$. Using the properties of the essential numerical range $W_e(T)$ of T, he proved that a bounded linear operator T on H admits a zero-diagonal, that is $(0) \in \mathcal{D}(T)$, if and only if there exists an orthonormal basis $(e_k)_{k=1}^\infty$ such that $s_n := \sum_{k=1}^n \langle Te_k, e_k \rangle, n \in \mathbb{N}$, possess a subsequence $(s_{n_m})_{m=1}^\infty$ satisfying $s_{n_m} \to 0, m \to \infty$. A number of related results in [18] suggested that "the diagonal of an operator carries more information about the operator than its relatively small size (compare to the "fat" matrix representation of the operator) may suggest." This line of research was continued e.g. in [20] and [19]. By different methods, it has been shown recently in [37] that an infinite-rank idempotent admits a zero-diagonal if and only if it is not a Hilbert-Schmidt perturbation of a (self-adjoint) projection.

An systematic study of the set $\mathcal{D}(T)$ has been attempted by Herrero in [24]. Addressing the challenging problem (a) above, Herrero showed that if $\{d_n\}_{n=1}^\infty$ belongs to the interior $\operatorname{Int} W_e(T)$ of $W_e(T)$ and $\{d_n\}_{n=1}^\infty$ has a limit point again in $\operatorname{Int} W_e(T)$, then $\{d_n\}_{n=1}^\infty \in \mathcal{D}(T)$. Otherwise, as shown in [24], there exists a compact operator K such that $\{d_n\}_{n=1}^\infty \in \mathcal{D}(T + K)$. Finally, if $\operatorname{dist}\{d_n, W_e(T)\} \to 0$ as $n \to \infty$ then there exists $\{d_n'\}_{n=1}^\infty$ such that $\{d_n'\}_{n=1}^\infty \in \mathcal{D}(T)$ and $|d_n - d_n'| \to 0$ as $n \to \infty$. The results by Herrero motivated the research in this paper to a large extent.

In particular, the importance of Blaschke-type conditions was suggested by analysis of diagonals for the unilateral shift in [24]. See Section 4 for more on that. As remarked in [24], the numerical range results allow one to deduce easily Fong's theorem from [22] saying that every bounded sequence $(d_n)_{n=1}^\infty$ admits a nilpotent operator (of index 2) whose diagonal is $(d_n)_{n=1}^\infty$. Moreover, in the same way, one can derive a similar result from [37] replacing nilpotents by idempotents. See Remark 4.4 for more on that.

The papers by Fan and Herrero were preceded by a deep article by Stout [47], where the diagonals of T appeared in a natural way in the study of Schur algebras and where $\mathcal{D}(T)$ was also related to $W_e(T)$. Recall that $0 \in W_e(T)$ if (and only if) there exists $(d_n)_{n=1}^\infty \in \mathcal{D}(T)$ such that $(d_n)_{n=1}^\infty \in c_0(\mathbb{N})$. At the beginning of 1970s, Anderson proved that $0 \in W_e(T)$ is in fact equivalent to the existence of a p-summable sequence in $\mathcal{D}(T)$ for every $p > 1$. Extending Anderson's result, Stout discovered that $0 \in W_e(T)$ yields a stronger property: for any $(\alpha_n)_{n=1}^\infty \notin \ell_1$ there is $(d_n)_{n=1}^\infty \in \mathcal{D}(T)$ such that $|d_n| \leq \alpha_n$ for all n. These results were crucial in the study of matrix and Schur norms for Hilbert space operators in [23]. In particular, by relating the size of entries of M_T to $W_e(T)$, the infima of maximum entry norm and Schur norms of M_T over all choices of bases in H were proved in [23] to be precisely dist $\{0, \sigma_{\text{ess}}(T)\}$, where $\sigma_{\text{ess}}(T)$ stands for the essential spectrum of T.

Comparatively recently, a relevant study of $\mathcal{D}(T)$ in a very general context of *operator-valued* diagonals has been realized in [11] by J.-C. Bourin. He proved that if $W_e(T)$ contains the open unit disc $\mathbb{D}$, then for every sequence $\{C_n\}_{n \geq 1}$ of operators acting possibly on different Hilbert spaces and satisfying $\sup_{n \geq 1} \|C_n\| < 1$, there exists a total sequence of nonzero mutually orthogonal projections $\{P_n\}_{n \geq 1}$ on H such that the compression $P_n T|_{\text{Ran } P_n}$ of T is unitarily equivalent to C_n for all $n \geq 1$. The operators $\bigoplus_{n \geq 1} P_n T P_n$, called "pinchings" of T in [11], can be considered as operator counterparts of elements from $\mathcal{D}(T)$, and they essentially coincide with those elements when C_n act on one-dimensional spaces. Note that pinching results appeared to be useful in the study of C^*-algebras, see e.g. [12] for more on that.

A rather general but unfortunately only approximate result on diagonals has been obtained by Neumann in [46] where, in particular, the ℓ^∞-closure of $\mathcal{D}(T)$ for (bounded) selfadjoint T was identified with a certain convex set. More precisely, let T be a bounded selfadjoint operator on H, and let $t_- = \inf \sigma_{\text{ess}}(T)$ and $t_+ = \sup \sigma_{\text{ess}}(T)$. Then $T = T_- + T_0 + T_+$, where T_+ and T_- are compact selfadjoint operators with positive and negative spectra respectively. Denoting $\mathbf{t}_+$ ($\mathbf{t}_-$) the diagonal of T_+ (T_-) with respect to some orthonormal basis, it was shown in [46] that the closure of $\mathcal{D}(T)$ in $\ell^\infty(\mathbb{N})$ can be described as

$$\overline{\text{conv } S\mathbf{t}_-} + [t_-, t_+]^{\mathbb{N}} + \overline{\text{conv } S\mathbf{t}_+},$$

where S is a group of permutations (bijections) of $\mathbb{N}$. As shown in [46], this result yields, in particular, a description of closed convex invariant subsets for selfadjoint operators. The results of a similar and more general nature have appeared recently in [32].

A different perspective has been opened since Kadison's striking results on diagonals of selfadjoint projections, addressing the problem (a). Generalizing the classical Schur-Horn theorem for matrices, Kadison proved in [27, 28] that a sequence $(d_n)_{n=1}^\infty$ is a diagonal of some *self-adjoint* projection if and only if it takes values in $[0, 1]$ and if the sums $a := \sum_{d_j < 1/2} d_j$ and $b := \sum_{d_j \geq 1/2} (1 - d_j)$ satisfy either $a + b = \infty$ or $a + b < \infty$ and $a - b \in \mathbb{Z}$. Illustrating the sharpness of Kadison's result and drawbacks of Neumann's approximate description, we note that the Neumann's theorem would produce all positive sequences with elements between 0 and 1. The Kadison integer condition was recognized as the Fredholm index obstruction in a subsequent paper by Arveson [6], where Kadison's dichotomy was studied for normal operators with finite spectrum. (Note also [30], where the integer was identified with so-called essential codimension of a pair of projections.) More precisely, Arveson considered a more general task of describing the diagonals for normal operators $N(X)$ with the essential spectrum $\sigma_e(T) = \sigma(T) = X$ and with X being the set of vertices of a convex polygon $P \subset \mathbb{C}$. He defined the set $\mathrm{Lim}^1(X)$ of "critical" sequences $d = (d_n)_n \in l^\infty(\mathbb{N})$ whose limit points belong to X and moreover such that the elements of d converge rapidly to these limits in the sense that $\sum_{n=1}^\infty \mathrm{dist}(d_n, X) < \infty$, that is $(d_n)_{n=1}^\infty$ satisfies the analogue of the classical Blaschke condition. He showed that there is a discrete group Γ_X depending only on the arithmetic properties of X, and a surjective mapping $d \mapsto s(d) \in \Gamma_X$ of the set of all such sequences d such that if $s(d) \neq 0$, then d is not the diagonal of any operator in $N(X)$. Moreover, it appeared that this is the only obstruction in the case of two-point sets, but also that there are other obstructions in the case of three-point sets. The case of three and more points set was settled very recently in [38] and we refer to this paper for very interesting details. We also refer to [5] containing an illuminating discussion of Kadison's theorem.

The research started by Kadison and Arveson gave rise to an intensive activity around diagonals of operators. In particular, the set $\mathcal{D}(T)$ was characterized for several classes of T (bearing certain resemblance to the selfadjoint situation) and moreover several results (e.g. Neumann's l^∞-closure result discussed above) were extended to the setting of von Neumann algebras and tuples of elements, sometimes obtaining new statements in the case of operators as well, see e.g. [4], [32], [40], and [31] and the references therein. Without mentioning all the substantial contributions, we give only a few samples related to the above discussion. Pairs of null real sequences realized as sequences of eigenvalues and diagonals of positive compact operators were characterized in [29]. A description of diagonals for selfadjoint operators T with finite spectrum was recently given in [16] (see also [10]).

Note that in this case the description of $\mathcal{D}(T)$ though explicit becomes rather technical and has an involved formulation. Very recently, $\mathcal{D}(T)$ for a class of unitary operators T was described in [26]. In particular, it was shown in [26] that a complex-valued sequence $(d_n)_{n=1}^{\infty}$ is a diagonal of some unitary operator on H if and only if $\sup_{n \geq 1} |d_n| \leq 1$ and

$$(1.1) \qquad 2(1 - \inf_{n \geq 1} |d_n|) \leq \sum_{n=1}^{\infty} (1 - |d_n|).$$

One should also mention the applications to frame theory where the diagonals arise e.g. a sequence of frame norms. For this direction of research one may consult e.g. [3], [13] and [15]. For several other related results, we refer to a recent survey [50].

In this paper we study the diagonals of bounded operators from the perspective of numerical ranges and spectrum. Being inspired by Herrero's work [24] and by Arveson's considerations in [6], we assume that the diagonal belongs to the interior of $W_e(T)$ and we introduce *a Blaschke-type* condition

$$\sum_{n=1}^{\infty} \operatorname{dist} \{d_n, \partial W_e(T)\} = \infty$$

on the size of diagonal $(d_n)_{n=1}^{\infty}$ near the boundary of $W_e(T)$. (The terminology originates from an *opposite* condition from complex analysis concerning zeros of bounded analytic functions in the unit disc. Rather then writing "non-Blaschke" here and in the sequel we have decided to name the condition above "Blaschke-type".) Note that one can easily recognize "Blaschke's" component in (1.1), however the condition is more subtle.

Given $T \in B(H)$, this set-up helps us to suggest a general and new method for constructing a big part of diagonals that works in a variety of different settings, including operator tuples and operator-valued diagonals. Moreover, for commuting tuples it allowed us to use spectral properties of T for constructing diagonals for power tuples. Thus, apart from substantially generalizing the results by Herrero and Bourin, we propose an approach that unifies and extends existing results and does not depend on specific properties of T (e.g. as being self-adjoint or unitary). We would like to stress that, addressing problem (a) above, we work with fixed operators rather than operator classes thus dealing with a more demanding and involved task. On the other hand, the drawback of our framework is that working with a fixed operator we are still far from giving a full description of $\mathcal{D}(T)$ for a fixed T, and results similar to Kadison's ones are out of reach. Although we suspect a characterization of $\mathcal{D}(T)$ is hardly possible in such a generality, see also Section 4 for a discussion of necessary conditions for diagonals and some classes of T when Blaschke-type assumptions help to obtain exhaustive characterizations.

Our first two main results given below provide Blaschke-type conditions for a sequence from the essential numerical range of a tuple $\mathcal{T}$ to be a

diagonal of $\mathcal{T}$. To treat degenerate cases, we deal with the notion of relative interior first and the result for an interior in $\mathbb{C}^n$ arises as a corollary.

Theorem 1.1. *Let $\mathcal{S} = (S_1, \ldots, S_s) \in B(H)^s$ be an s-tuple of selfadjoint operators, and let $(\lambda_k)_{k=1}^\infty \subset \mathrm{Int}\,_{\mathbb{R}^s} W_\mathrm{e}(\mathcal{S})$ satisfy*

$$\text{(1.2)} \qquad \sum_{k=1}^\infty \mathrm{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_\mathrm{e}(\mathcal{S})\} = \infty.$$

Then $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{S})$.

Theorem 1.1 has a counterpart, Corollary 4.2, for $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. In this case, assuming that $M \subset \mathbb{C}^n$ is the smallest real hyperplane containing $W(\mathcal{T})$, we conclude that any $(\lambda_k)_{k=1}^\infty \subset \mathrm{Int}\,_M W_e(\mathcal{T})$ satisfying

$$\sum_{k=1}^\infty \mathrm{dist}\,\{\lambda_k, M \setminus W_e(\mathcal{T})\} = \infty,$$

belongs to $\mathcal{D}(\mathcal{T})$.

Using Theorem 1.1 in the context of power tuples $(T, \ldots, T^n)$ for $T \in B(H)$, and thus being able to invoke the notion of spectrum, we show that for every $(\lambda_k)_{k=1}^\infty \subset \mathrm{Int}\,\widehat{\sigma}(T)$ satisfying $\sum_{k=1}^\infty \mathrm{dist}\,^n\{\lambda_k, \partial\widehat{\sigma}(T)\} = \infty$ the sequence $(\lambda_k, \lambda_k^2, \ldots, \lambda_k^n)$ belongs to $\mathcal{D}(T, \ldots, T^n)$. We are not aware of similar statements in the literature, although some related statements can be found in [43].

Theorem 1.2. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. For every $(\lambda_k)_{k=1}^\infty \subset W_\mathrm{e}(\mathcal{T})$ and every $(\alpha_k)_{k=1}^\infty \notin \ell_1$ there exists an orthonormal basis $(u_k)_{k=1}^\infty$ in H such that*

$$\text{(1.3)} \qquad \big\|\langle \mathcal{T} u_k, u_k \rangle - \lambda_k\big\| \leq |\alpha_k|$$

for all $k \in \mathbb{N}$.

Theorem 1.2 yields the existence of diagonals for compact perturbations of tuples that satisfy weakened Blaschke-type conditions. In particular, if $p > 1$ and $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}^n$ satisfy $\sum_{k=1}^\infty \mathrm{dist}\,^p\{\lambda_k, W_\mathrm{e}(\mathcal{T})\} < \infty$, then there exists an n-tuple of operators $\mathcal{K} = (K_1, \ldots, K_n)$ with $K_j, 1 \leq j \leq n$, from the Schatten class S_p such that $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{T} + \mathcal{K})$. Our results on compact perturbations lead to several characterizations of the subset $\mathcal{D}_{\mathrm{const}}(T)$ of $\mathcal{D}(T)$ consisting of constant diagonals. The problem of understanding the structure of $\mathcal{D}_{\mathrm{const}}(T)$ has been raised in [11], although there are other related results in the literature. The most interesting question on convexity of $\mathcal{D}_{\mathrm{const}}(T)$ remains still unanswered.

It is curious that the same type of technique yields the results in the context of operator diagonals thus extending Bourin's results from [11] mentioned above to the setting of tuples and replacing his uniform contractivity condition on operator diagonal by a more general assumption of Blaschke's type. In particular, the following statement holds.

Theorem 1.3. *Let $T \in B(H)$ with $W_e(T) \supset \overline{\mathbb{D}}$. Let $L_k, k \in \mathbb{N}$, be separable Hilbert spaces (finite or infinite-dimensional) and let $C_k \in B(L_k)$ be contractions satisfying $\sum_{k=1}^{\infty}(1 - \|C_k\|) = \infty$. Then there exist projections $P_{K_k}, k \in \mathbb{N}$, onto mutually orthogonal subspaces $K_k \subset H$ such that $\bigoplus_{k=1}^{\infty} K_k = H$ and $P_{K_k} T P_{K_k}$ is unitarily equivalent to C_k for all $k \in \mathbb{N}$.*

(Note that since $W_e(T)$ is compact, in the formulation above, one may replace the closure of $\mathbb{D}$ by $\mathbb{D}$ itself , and the choice of $\overline{\overline{\mathbb{D}}}$ is just a matter of taste.)

The methods used in the proof of Theorem 1.3 appeared to be quite fruitful in the study of power tuples. Under natural spectral assumptions, they allowed us also to construct operator diagonals of power tuples $(T, \ldots, T^n)$ consisting for power tuples of contractions $(C_k, \ldots, C_k^n)$ where $(C_k)_{k=1}^{\infty}$ satisfy an analogue for tuples of Blaschke's condition in Theorem 1.3. More precisely, assuming that $\widehat{\sigma}(T) \supset \overline{\mathbb{D}}$, $n \in \mathbb{N}$, and $L_k, k \in \mathbb{N}$, are separable Hilbert spaces, we prove that for any contractions $C_k \in B(L_k), k \in \mathbb{N}$, satisfying $\sum_{k=1}^{\infty}(1 - \|C_k\|)^n = \infty$ one can find mutually orthogonal subspaces $K_k \subset H$ such that $H = \bigoplus_{k=1}^{\infty} K_k$ and $P_{K_k}(T, \ldots, T^n)P_{K_k}$ is unitarily equivalent to $(C_k, \ldots, C_k^n)$ for all $k \in \mathbb{N}$.

The result has a flavor of (finite) power dilations. Note however that its proof relies on a classical unitary power dilation for a Hilbert space contraction.

2. Notation

It will be convenient to fix some notations in a separate section. In particular, we let H be an infinite-dimensional complex separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and $B(H)$ the space of all bounded linear operators on H. For a bounded linear operator T we denote by $\sigma(T)$ its spectrum, by $W(T)$ its numerical range, and by $N(T)$ its kernel.

In the following we consider an n-tuple $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Note that we do not in general assume that the operators T_j commute. For $x, y \in H$ we write shortly

$$\langle \mathcal{T}x, y \rangle = (\langle T_1 x, y \rangle, \ldots, \langle T_n x, y \rangle) \in \mathbb{C}^n \quad \text{and} \quad \mathcal{T}x = (T_1 x, \ldots, T_n x) \in H^n.$$

Similarly for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we write $\mathcal{T} - \lambda = (T_1 - \lambda_1, \ldots, T - \lambda_n)$ and

$$\|\lambda\| = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

If $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ and $R, S \in B(H)$ then we define

$$(2.1) \qquad R\mathcal{T}S := (RT_1 S, \ldots, RT_n S).$$

For a a subspace M of a Hilbert space H we denote by P_M the orthogonal projection onto M and by J_M the natural embedding $J_M : M \to H$, defined by $Jx = x, \ x \in M$. For an operator $T \in B(H)$, we denote by $T_M : M \to M$ the compression of T to M. Note that $T_M = J_M^* T J_M$.

If $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ and $\mathcal{S} = (S_1, \ldots, S_n) \in B(H')^n$ are n-tuples of operators, then we say that $\mathcal{T}$ and $\mathcal{S}$ are unitarily equivalent and write $\mathcal{T} \overset{u}{\sim} \mathcal{S}$ if there exists a unitary operator $U : H \to H'$ such that $T_j = US_jU^{-1}$ for every $j = 1, \ldots, n$.

For $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ we denote by $\mathcal{D}(\mathcal{T}) \subset \ell_\infty(\mathbb{N}, \mathbb{C}^n)$ its set of diagonals. In other words,

$$\mathcal{D}(\mathcal{T}) = \{(\langle T_1e_n, e_n\rangle, \ldots, \langle Te_n, e_n\rangle)_{n=1}^\infty\}$$

when $(e_n)_{n=1}^\infty$ varies through the set of all orthonormal bases of H. It will be important to distinguish a subset of $\mathcal{D}(\mathcal{T})$ consisting of "constant" diagonals. Define

$$\mathcal{D}_{\mathrm{const}}(\mathcal{T}) = \{\lambda \in \mathbb{C}^n : (\lambda, \lambda, \ldots) \in \mathcal{D}(\mathcal{T})\}.$$

For a closed set $K \subset \mathbb{C}^n$ we denote by ∂K the topological boundary of K, by $\mathrm{conv}\, K$ the convex hull of K, and by $\widehat{K}$ the polynomial hull of K. Recall that if $K \subset \mathbb{C}$ then $\widehat{K}$ is the union of K with all bounded components of the complement $\mathbb{C} \setminus K$. For $K \subset \mathbb{C}^n$ denote by $\mathrm{Int}\, K$ the interior of K. If $K \subset M \subset \mathbb{C}^n$ then denote by $\mathrm{Int}_M K$ the relative interior with respect to $M \subset \mathbb{C}^n$ with the induced topology.

Finally, we let $\mathbb{T}$ stand for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\mathbb{D}$ for the unit disc $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\mathbb{R}_+ = [0, \infty)$. Finally, for $\rho > 0$, write $\mathbb{T}_\rho = \{z \in \mathbb{C} : |z| = \rho\}$.

3. Preliminaries

We start with recalling certain basic notions and facts from the spectral theory of operator tuples on Hilbert spaces. They can be found e.g. in [45, Chapters 2-3].

Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ be an n-tuple of commuting operators. Recall that its joint (Harte) spectrum $\sigma(\mathcal{T})$ can be defined as the complement of the set of those $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ for which

$$\sum_{j=1}^n L_j(T_j - \lambda_j) = \sum_{j=1}^n (T_j - \lambda_j)R_j = I$$

for some $L_j, R_j, 1 \leq j \leq n$, from the algebra $B(H)$. It is well-known that $\sigma(\mathcal{T})$ is a non-empty compact subset of $\mathbb{C}^n$. One can define the joint essential spectrum $\sigma_e(\mathcal{T})$ as the (Harte) spectrum of the n-tuple $(T_1 + \mathcal{K}(H), \ldots, T_n + \mathcal{K}(H))$ in the Calkin algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of all compact operators on H. This definition of $\sigma_e(\mathcal{T})$ is rather implicit. Thus it is helpful to consider the the joint essential approximate point spectrum $\sigma_{\pi e}(\mathcal{T}) \subset \sigma_e(\mathcal{T})$ defined as the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that

$$\inf_{x \in M, \|x\|=1} \sum_{j=1}^n \|(T_j - \lambda_j)x\| = 0$$

for every subspace $M \subset H$ of finite codimension. The set $\sigma_{\pi e}(\mathcal{T})$ is quite a big part of $\sigma_e(\mathcal{T})$ so that the polynomial convex hulls $\widehat{\sigma}_e(\mathcal{T})$ and $\widehat{\sigma}_{\pi e}(\mathcal{T})$ coincide. Note that if $n = 1$ then $\sigma_e(T_1) = \{\lambda_1 \in \mathbb{C} : T_1 - \lambda_1 \text{ is not Fredholm}\}$, and for $T \in B(H)$ and $\mathcal{T} = (T, T^2, \ldots, T^n) \in B(H)^n$, one has $\sigma(\mathcal{T}) = \{(\lambda, \ldots, \lambda^n) : \lambda \in \sigma(T)\}$ and $\sigma_e(\mathcal{T}) = \{(\lambda, \ldots, \lambda^n) : \lambda \in \sigma_e(T)\}$, and the same property holds for $\sigma_{\pi e}(T)$. Denote by $\sigma_p(\mathcal{T})$ the point spectrum of $\mathcal{T}$, i.e., the set of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $\bigcap_{j=1}^n N(T_j - \lambda_j) \neq \{0\}$. If $x \in \bigcap_{j=1}^n N(T_j - \lambda_j)$ then we will write $\mathcal{T}x = \lambda x$.

It is often useful to relate $\sigma(\mathcal{T})$ to a larger and more easily accessible set $W(\mathcal{T}) \subset \mathbb{C}^n$ called the joint numerical range of $\mathcal{T}$ and defined as

$$W(\mathcal{T}) = \big\{(\langle T_1 x, x\rangle, \ldots, \langle T_n x, x\rangle) : x \in H, \|x\| = 1\big\}.$$

The set $W(\mathcal{T})$ can be identified with a subset of $\mathbb{R}^{2n}$ if one identifies the n-tuple $\mathcal{T}$ with the $(2n)$-tuple $(\operatorname{Re} T_1, \operatorname{Im} T_1, \ldots, \operatorname{Re} T_n, \operatorname{Im} T_n)$ of self-adjoint operators. Unfortunately, if $n > 1$, then $W(\mathcal{T})$ is not in general convex, see e.g. [33].

As in the spectral theory, there is also a notion of the joint essential numerical range $W_e(\mathcal{T})$ associated to $\mathcal{T}$. For $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ we define $W_e(\mathcal{T})$ as the set of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k)_{k=1}^\infty \subset H$ with

$$\lim_{k \to \infty} \langle T_j x_k, x_k\rangle = \lambda_j, \qquad j = 1, \ldots, n.$$

It is instructive to note that in the definition above one may choose $(x_k)_{k=1}^\infty \subset H$ to be an orthonormal basis of H. For $n = 1$ the proof of the latter fact can be found in [11] and [47], for the general case see [33, Theorem 2.1]. It is easy to see that $\lambda \in W_e(\mathcal{T})$ if and only if for every $\delta > 0$ and every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ such that $\|\langle \mathcal{T}x, x\rangle - \lambda\| < \delta$, see e.g. [44, Proposition 5.7] for the proof. The latter property was a basis of many of many inductive constructions in [43] and [44], and it will also be crucial in this paper.

Alternatively, $W_e(\mathcal{T})$ can be described as

$$W_e(\mathcal{T}) := \bigcap \overline{W(T_1 + K_1, \ldots, T_n + K_n)}$$

where the intersection is taken over all n-tuples $K_1, \ldots, K_n$ of compact operators on H. Moreover, by [42, Corollary 14], we can always find a tuple $(K_1^0, \ldots, K_n^0) \in K(H)^n$ such that

$$(3.1) \qquad W_e(\mathcal{T}) = \overline{W(T_1 + K_1^0, \ldots, T_n + K_n^0)}.$$

Recall that $W_e(\mathcal{T})$ is a nonempty, compact and, in contrast to $W(\mathcal{T})$, *convex* subset of $\overline{W(\mathcal{T})}$, see [8] or [33].

As a straightforward consequence of the definitions above, if the n-tuple $\mathcal{T} \in B(H)^n$ is commuting then $\sigma_{\pi e}(\mathcal{T}) \subset W_e(\mathcal{T})$. Since the polynomial hulls of $\sigma_{\pi e}(\mathcal{T})$ and $\sigma_e(\mathcal{T})$ coincide [45, Corollary 19.16], their convex hulls

coincide as well, and the convexity of $W_e(\mathcal{T})$ yields

(3.2) $\operatorname{conv} \sigma_e(\mathcal{T}) \subset W_e(\mathcal{T}).$

For a comprehensive account of essential numerical ranges one may consult [33]. See also [44] for a discussion of other numerical ranges closely related to the notion of essential numerical range.

There are several other numerical ranges useful in applications. In particular, the so-called infinite numerical range $W_\infty(\mathcal{T})$ will be relevant in the sequel. Recall that if $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$ then $W_\infty(\mathcal{T})$ can be defined by

$$W_\infty(\mathcal{T}) := \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : PT_jP = \lambda_j P, \quad j = 1, \ldots, n \right\}$$

for some infinite rank projection P. By [44], the essential numerical range $W_e(\mathcal{T})$ of $\mathcal{T}$ can be described in terms of $W_\infty(\mathcal{T})$ as

$$W_e(\mathcal{T}) = \bigcup_{\mathcal{K} \in \mathcal{K}(H)^n} W_\infty(\mathcal{T} + \mathcal{K}).$$

The infinite numerical range of a tuple is closely related to its essential numerical range as the following statement shows, see [43, Corollary 4.2] and [44, Corolary 4.3].

Theorem 3.1. *For any $\mathcal{T} \in B(H)^n$,*

(3.3) $\operatorname{Int}(W_e(\mathcal{T})) \subset W_\infty(\mathcal{T}).$

Moreover, if the tuple $\mathcal{T}$ is commuting then

(3.4) $\operatorname{Int} \operatorname{conv} \sigma(\mathcal{T}) \subset W(\mathcal{T}).$

Thus $W_\infty(\mathcal{T})$ is large if $W_e(\mathcal{T})$ is large. Note in passing that in general, by [44],

(3.5) $\operatorname{Int} \operatorname{conv} \left(W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T}) \right) \subset W(\mathcal{T}).$

The importance of $W_\infty(\mathcal{T})$ can be illustrated by the next result crucial for our subsequent arguments, For $S \subset \mathbb{C}^n$ denote by $\mathcal{M}(S)$ the set of all n-tuples of operators $\mathcal{A} = (A_1, \ldots, A_n) \in B(H)^n$ such that there exist an orthonormal basis $(x_k)_{k=1}^\infty$ in H and a sequence $(\lambda_k)_{k=1}^\infty \subset S$ satisfying $\mathcal{A}x_k = \lambda_k x_k$ for each k. The theorem below identifies compressions of a tuple $\mathcal{T}$ with a tuple of diagonal operators $\mathcal{A}$ whose diagonals belong to the infinite numerical range of $\mathcal{T}$, its proof can be found in [44].

Proposition 3.2. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Let $\mathcal{A} \in \operatorname{conv} \mathcal{M}(W_\infty(\mathcal{T}))$. Then there exists a subspace $L \subset H$ such that the compression $\mathcal{T}_L$ is unitarily equivalent to $\mathcal{A}$.*

Tuples $\mathcal{T}_n = (T, T^2, \ldots, T^n)$ consisting of powers of a single operator $T \in B(H)$ are of special interest since they allow one to reveal the structure of an operator T by looking at its powers, thus sometimes uncovering new effects (see e.g. [43]). The following statement from [43] describes big subsets of $W(T, T^2, \ldots, T^n)$ in terms of the *spectrum* of T. In this paper, it will be vital for constructing (operator) diagonals for tuples of operator powers.

Theorem 3.3. *Let $T \in B(H)$ and let λ belong to the interior of the polynomial hull $\hat{\sigma}(T)$ of $\sigma(T)$. Then*

$$(\lambda, \lambda^2, \ldots, \lambda^n) \in \operatorname{Int} W_e(T, T^2, \ldots, T^n) \subset W_\infty(T, T^2, \ldots, T^n).$$

for all $n \in \mathbb{N}$.

Thus, in particular, $\operatorname{Int} \hat{\sigma}(T) \subset \operatorname{Int} W_e(T)$, and $\overline{\mathbb{D}} \subset \operatorname{Int} \hat{\sigma}(T)$ implies that $\overline{\mathbb{D}} \subset \operatorname{Int} W_e(T)$. So the spectral assumption in Theorem 6.3 are stronger than the numerical range assumption in Theorem 1.3.

4. Diagonals: Blaschke-type condition

Let $T \in B(H)$. Recall from the introduction that according to [24], any sequence $(\lambda_k)_{k=1}^\infty \subset \operatorname{Int} W_e(T)$ with an accumulation point inside $\operatorname{Int} W_e(T)$ can be realized as a diagonal of T, i.e., there exists an orthonormal basis $(u_k)_{k=1}^\infty$ in H such that $\langle T u_k, u_k \rangle = \lambda_k$ for all k.

Below we prove a similar result under a much weaker, Blaschke-type assumption:

$$\sum_{k=1}^\infty \operatorname{dist}\{\lambda_k, \partial W_e(T)\} = \infty.$$

Note that if $W_e(T) = \overline{\mathbb{D}}$ then this assumption reduces to the condition $\sum_{k=1}^\infty (1 - |\lambda_k|) = \infty$, opposite to the classical Blaschke's one. Moreover, our technique allows us to obtain the result in a more demanding setting of operator tuples, i.e., we construct given common diagonals for n-tuples of operators $\mathcal{T}$ with respect to a common orthonormal basis.

As mentioned in the previous section, the (essential) numerical range of an n-tuple $(T_1, \ldots, T_n) \in B(H)^n$ can be identified with the (essential) numerical range of the $(2n)$-tuple $(\operatorname{Re} T_1, \operatorname{Im} T_1, \ldots, \operatorname{Re} T_n, \operatorname{Im} T_n)$ of selfadjoint operators, considered as a subset of $\mathbb{R}^{2n}$. It will be convenient to formulate the next results for tuples of self-adjoint operators.

Our arguments rely essentially on the following result describing big subsets of the numerical range of a tuple in terms of its essential numerical range.

Proposition 4.1. *Let $\mathcal{S} = (S_1, \ldots, S_s) \in B(H)^s$ be an s-tuple of selfadjoint operators. Then*

$$\operatorname{Int}_{\mathbb{R}^s} W_e(\mathcal{S}) \subset W_\infty(\mathcal{S}).$$

Proof. The proof of the proposition is exactly the same as the proof of [44, Corolary 4.3], where it was proved that $\operatorname{Int} W_e(\mathcal{T}) \subset W_\infty(\mathcal{T})$ for any n-tuple $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$, the interior being considered in $\mathbb{C}^n$. Thus, we omit the arguments. $\square$

The proof of Theorem 1.1, the main result of this section, introduces a technique which will crucial for the whole of paper.

Proof of Theorem 1.1 Let $(y_m)_{m=1}^\infty$ be a sequence of unit vectors in H such that $\bigvee_m y_m = H$. Using (1.2), we can find mutually disjoint sets $A_m, m \in \mathbb{N}$, such that $\bigcup_{m=1}^\infty A_m = \mathbb{N}$ and

$$\sum_{k \in A_m} \text{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_e(\mathcal{S})\} = \infty$$

for all $m \in \mathbb{N}$.

It is sufficient to construct an orthonormal sequence $(u_k)_{k=1}^\infty$ in H such that $\langle \mathcal{S} u_k, u_k \rangle = \lambda_k$ for all $k \in \mathbb{N}$ and

$$\ln \text{dist}^2 \Big\{ y_m, \bigvee_{k \le N} u_k \Big\} \le -\frac{\sum_{k \le N, k \in A_m} \text{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_e(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}$$

for all $N, m \in \mathbb{N}$. Indeed, since

$$\sum_{k \in A_m} \text{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_e(\mathcal{S})\} = \infty$$

for all $m \in \mathbb{N}$, we will then have

$$\lim_{N \to \infty} \text{dist}\,\Big\{ y_m, \bigvee_{k \le N} u_k \Big\} = 0,$$

so $y_m \in \bigvee_{k \in \mathbb{N}} u_k$ for all m, and so $(u_k)_{k=1}^\infty$ will be an orthonormal basis satisfying $\langle \mathcal{S} u_k, u_k \rangle = \lambda_k$ for all $k \in \mathbb{N}$.

The sequence $(u_k)_{k=1}^\infty$ will be constructed inductively. Let $N \ge 1$ and suppose that $u_1, \ldots, u_{N-1}$ is an orthonormal set satisfying $\langle \mathcal{S} u_k, u_k \rangle = \lambda_k$ for all $k \le N - 1$ and

$$\ln \text{dist}^2 \Big\{ y_m, \bigvee_{k \le N-1} u_k \Big\} \le -\frac{\sum_{k \le N-1, k \in A_m} \text{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_e(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}$$

for all $m \in \mathbb{N}$.

Write $M_{N-1} = \bigvee_{k \le N-1} u_k$, and let $m \in \mathbb{N}$ be such that $N \in A_m$. We are looking for a unit vector $u_N \in M_{N-1}^\perp$ such that $\langle \mathcal{S} u_N, u_N \rangle = \lambda_N$ and

$$\ln \text{dist}^2 \Big\{ y_m, \bigvee_{k \le N} u_k \Big\} \le -\frac{\sum_{k \le N, k \in A_m} \text{dist}\,\{\lambda_k, \mathbb{R}^s \setminus W_e(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}.$$

Recall that by Proposition 4.1, one has $\lambda_N \in \text{Int}\, W_e(\mathcal{S}) \subset W_\infty(\mathcal{S})$, hence if $y_m \in M_{N-1}$ then it suffices to take any unit vector $u_N \in M_{N-1}^\perp$ satisfying $\langle \mathcal{S} u_N, u_N \rangle = \lambda_N$.

Suppose that $y_m \notin M_{N-1}$. Then

(4.1) $$y_m = a + tb$$

with

(4.2) $a \in M_{N-1}, \qquad b \perp M_{N-1}, \|b\| = 1 \quad \text{and} \quad t = \text{dist}\,\{y_m, M_{N-1}\} > 0,$

and so
$$\ln t^2 \leq -\frac{\sum_{k \leq N-1, k \in A_m} \operatorname{dist}\{\lambda_k, \mathbb{R}^s \setminus W_{\mathrm{e}}(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}.$$

If $\langle \mathcal{S}b, b \rangle = \lambda_N$ then set $u_N := b$. If $\langle \mathcal{S}b, b \rangle \neq \lambda_N$, then let
$$\rho = \big\|\langle \mathcal{S}b, b \rangle - \lambda_N\big\| \qquad \text{and} \qquad \delta = \frac{1}{2}\operatorname{dist}\{\lambda_N, \mathbb{R}^s \setminus W_{\mathrm{e}}(\mathcal{S})\}.$$

There exists $\mu \in \operatorname{Int} W_{\mathrm{e}}(\mathcal{S}) \subset W_\infty(\mathcal{S})$ such that $\|\mu - \lambda_N\| = \delta$ and
$$\frac{\langle \mathcal{S}b, b \rangle - \lambda_N}{\rho} = \frac{\lambda_N - \mu}{\delta}.$$

Choose $x \in H$, $\|x\| = 1$ such that
$$x \perp M_{N-1}, b, S_1 b, S_1^* b, \ldots, S_s b, S_s^* b \qquad \text{and} \qquad \langle \mathcal{S}x, x \rangle = \mu,$$

and set
$$u_N := \sqrt{\frac{\rho}{\rho + \delta}}\, x + \sqrt{\frac{\delta}{\rho + \delta}}\, b.$$

We have $\|u_N\| = 1$ and $u_N \perp M_{N-1}$, and moreover
$$\begin{aligned} \langle \mathcal{S}u_N, u_N \rangle &= \frac{\rho}{\rho + \delta}\, \mu + \frac{\delta}{\rho + \delta}\, \langle \mathcal{S}b, b \rangle \\ &= \frac{\rho}{\rho + \delta}\, (\mu - \lambda_N) + \frac{\delta}{\rho + \delta}\, (\langle \mathcal{S}b, b \rangle - \lambda_N) + \lambda_N \\ &= \lambda_N. \end{aligned}$$

It remains to estimate the distance $\operatorname{dist}\{y_m, M_N\}$. In view of (4.1) and (4.2), taking into account that $\langle b, u_N \rangle = \sqrt{\frac{\delta}{\rho + \delta}}$, we have

$$\operatorname{dist}^2\{y_m, M_N\} = \operatorname{dist}^2\{tb, M_N\} = t^2 \cdot \operatorname{dist}^2\Big\{b, \bigvee_{k=1}^{N} u_k\Big\} = t^2\Big(1 - \frac{\delta}{\rho + \delta}\Big)$$

and
$$\begin{aligned} \ln \frac{\operatorname{dist}^2\{y_m, M_N\}}{\operatorname{dist}^2\{y_m, M_{N-1}\}} &= \ln\Big(1 - \frac{\delta}{\rho + \delta}\Big) \leq -\frac{\delta}{\rho + \delta} \\ &\leq -\frac{\operatorname{dist}\{\lambda_N, \mathbb{R}^s \setminus W_{\mathrm{e}}(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}. \end{aligned}$$

Hence
$$\begin{aligned} \ln \operatorname{dist}^2\{y_m, M_N\} &\leq \ln \operatorname{dist}^2\{y_m, M_{N-1}\} - \frac{\operatorname{dist}\{\lambda_N, \mathbb{R}^s \setminus W_{\mathrm{e}}(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}} \\ &\leq -\sum_{k \leq N, k \in A_m} \frac{\operatorname{dist}\{\lambda_k, \mathbb{R}^s \setminus W_{\mathrm{e}}(\mathcal{S})\}}{4 \max\{\|S_1\|, \ldots, \|S_s\|\}}. \end{aligned}$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 4.2. *Let $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. Let $M \subset \mathbb{C}^n$ be the smallest (real) hyperplane containing $W(\mathcal{T})$. Let $(\lambda_k)_{k=1}^\infty \subset \operatorname{Int}_M W_e(\mathcal{T})$ satisfy*

$$\sum_{k=1}^\infty \operatorname{dist}\{\lambda_k, M \setminus W_e(\mathcal{T})\} = \infty.$$

Then $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{T})$.

Proof. Again it is sufficient to consider tuples of selfadjoint operators. So assume that $\mathcal{S} = (S_1, \ldots, S_s) \in B(H)^s$ is an s-tuple of selfadjoint operators. Let M be the hyperplane generated by $W(\mathcal{S})$ and let $(\lambda_k)_{k=1}^\infty \subset \operatorname{Int}_M W_e(\mathcal{S})$ satisfy $\sum_{k=1}^\infty \operatorname{dist}\{\lambda_k, M \setminus W_e(\mathcal{S})\} = \infty$.

We prove the statement by induction on s. If $s = 1$, then either $M = \mathbb{R}$ and the statement follows from the previous theorem, or M is a single point, $M = \{t\}$, so that $S_1 = tI$ and the statement is clear.

Suppose that the statement is true for $s \geq 1$. Let $\mathcal{S} = (S_1, \ldots, S_{s+1})$ be an $(s+1)$-tuple of selfadjoint operators. Let M be the hyperplane generated by $W(\mathcal{S})$. If $M = \mathbb{R}^{s+1}$ then the statement follows from the previous theorem.

Suppose that $M \neq \mathbb{R}^{s+1}$. Then there exist $\alpha_0, \alpha_1, \ldots, \alpha_{s+1} \in \mathbb{R}$ such that $(\alpha_1, \ldots, \alpha_{s+1}) \neq (0, \ldots, 0)$ and for all $m = (m_1, \ldots, m_{s+1}) \in M$,

$$\alpha_0 + \sum_{j=1}^{s+1} \alpha_j m_j = 0.$$

Without loss of generality we may assume that $\alpha_{s+1} \neq 0$. Then

$$W\Big(\frac{\alpha_0}{\alpha_{s+1}}I + \sum_{j=1}^{s+1} \frac{\alpha_j}{\alpha_{s+1}}S_j\Big) = \{0\},$$

and so

(4.3) $$S_{s+1} = -\frac{\alpha_0}{\alpha_{s+1}}I - \sum_{j=1}^s \frac{\alpha_j}{\alpha_{s+1}}S_j.$$

Let $\mathcal{S}' = (S_1, \ldots, S_s)$ and let $P : \mathbb{R}^{s+1} \to \mathbb{R}^s$ be the natural projection onto the first s coordinates. For $k \in \mathbb{N}$ let $\lambda_k = (\lambda_{k,1}, \ldots, \lambda_{k,s+1})$. By (4.3), we have

$$\lambda_{k,s+1} = -\frac{\alpha_0}{\alpha_{s+1}} - \sum_{j=1}^s \frac{\alpha_j}{\alpha_{s+1}}\lambda_{k,j}$$

for all $k \in \mathbb{N}$. So it is sufficient to show that $(P\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{S}')$.

The smallest hyperplane containing $W(\mathcal{S}')$ is $P(M)$ and it is easy to see that $\sum_{k=1}^\infty \operatorname{dist}\{P\lambda_k, P(M) \setminus W_e(\mathcal{S}')\} = \infty$. By the induction assumption this implies that $(P\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{S}')$. $\quad\square$

Corollary 4.3. *Let* $\mathcal{T} = (T_1, \ldots, T_n) \in B(H)^n$. *Let* $(\lambda_k)_{k=1}^\infty \subset \operatorname{Int} W_e(\mathcal{T})$ *satisfy*

$$(4.4) \qquad \sum_{k=1}^\infty \operatorname{dist} \{\lambda_k, \mathbb{C} \setminus W_e(\mathcal{T})\} = \infty.$$

Then $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{T})$.

Remark 4.4. The corollary above, and even weaker Herrero's result mentioned in the introduction, allows one to deduce directly Fong's theorem from [23] saying that any bounded sequence in $\mathbb{C}$ can be realized as a diagonal of square zero nilpotent N. As noted in [24, p. 864], it is enough to observe that $W_e(N)$ can contain an arbitrary disc (with center at zero) for an appropriate N. Similarly, using [34, Corollary 2.6] (and Theorem 1.1 quoted there and proved in [49]), one can prove that any bounded sequence from $\mathbb{C}$ can be realized as a diagonal of idempotent thus recovering [37, Theorem 3.7]. In fact, using [34, Corollary 2.6], similar results can be proved for more general quadratic operators, but we refrain ourselves from giving straightforward details.

A natural question is how fast a sequence should approach the boundary of the essential numerical range to be realizable as a diagonal, and whether the assumption (1.2) is optimal. It appears that (1.2) cannot in general be improved as a simple Herrero's example from [24, p. 862-863] shows. Namely, it was proved in [24] that if T is the unilateral shift, then $W_e(T) = \overline{\mathbb{D}}$, and a sequence $(\lambda_k)_{k=1}^\infty \subset \mathbb{D}$ is a diagonal of T if and only if $\sum_{k=1}^\infty (1 - |\lambda_k|) = \infty$.

The same proof works in a more general setting, see also [26, Lemma 4.1] for a similar argument.

Proposition 4.5. *Let* $T \in B(H)$ *be such that* $\|T\| \leq 1$ *and* $W_e(T) = \overline{\mathbb{D}}$. *Suppose that* T *is not a Fredholm operator of index* 0. *Let* $(\lambda_k)_{k=1}^\infty \subset \mathbb{D}$. *Then*

$$(\lambda_k)_{k=1}^\infty \in \mathcal{D}(T) \iff \sum_{k=1}^\infty (1 - |\lambda_k|) = \infty.$$

Unfortunately, Blaschke-type conditions (1.2) and (4.4) are only sufficient for a sequence $(\lambda_k)_{k=1}^\infty$ to be in $\mathcal{D}(\mathcal{T})$ for an n-tuple $\mathcal{T} \in B(H)^n$. On the other hand, we can also formulate a necessary condition as well. Let us start with a single selfadjoint operator.

Proposition 4.6. *Let* $T \in B(H)$, $T \geq 0$. *Let* $(\lambda_k)_{k=1}^\infty \subset \operatorname{Int}_{\mathbb{R}} W_e(T)$ *and* $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(T)$. *Then* $\sum_{k=1}^\infty \lambda_k = \infty$.

Proof. Suppose that $(\lambda_k)_{k=1}^\infty \in \operatorname{Int}_{\mathbb{R}} W_e(T)$, $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(T)$ and $\sum_{k=1}^\infty \lambda_k < \infty$. Then there exists an orthonormal basis $(u_k)_{k=1}^\infty$ in H such that $\langle Tu_k, u_k \rangle = \lambda_k$ for all $k \in \mathbb{N}$. We have

$$\sum_{k=1}^\infty \|T^{1/2} u_k\|^2 = \sum_{k=1}^\infty \lambda_k < \infty.$$

So $T^{1/2}$ is a Hilbert-Schmidt operator. Thus T is compact, $W_e(T) = \{0\}$ and $\text{Int}\, W_e(T) = \emptyset$, a contradiction. $\qquad\square$

Corollary 4.7. *Let $\mathcal{S} = (S_1, \ldots, S_s) \in B(H)^s$ be an s-tuple of selfadjoint operators, let $(\lambda_k)_{k=1}^{\infty} \subset \text{Int}_{\mathbb{R}^s}\, W_e(\mathcal{S})$, $(\lambda_k) \in \mathcal{D}(\mathcal{S})$, and $\lambda_k = (\lambda_{k,1}, \ldots, \lambda_{k,s})$ for all $k \in \mathbb{N}$. Let $\alpha_0, \ldots, \alpha_s$ be real numbers, $(\alpha_1, \ldots, \alpha_s) \neq (0, \ldots, 0)$, $V = \alpha_0 I + \sum_{j=1}^{s} \alpha_j S_j$ and $a = \inf\{t : t \in w(V)\}$. Then*

$$\sum_{k=1}^{\infty} \left(\alpha_0 - a + \sum_{k=1}^{s} \alpha_j \lambda_{k,j} \right) = \infty.$$

Proof. We have

$$\left(\alpha_0 - a + \sum_{j=1}^{s} \alpha_j \lambda_{k,j} \right) \in \mathcal{D}\left((\alpha_0 - a)I + \sum_{j=1}^{s} \alpha_j S_j \right),$$

$$\left(\alpha_0 - a + \sum_{j=1}^{s} \alpha_j \lambda_{k,j} \right) \in \text{Int}_{\mathbb{R}}\, W_e(V - a)$$

for all $k \in \mathbb{N}$, and $(\alpha_0 - a)I + \sum_{j=1}^{s} \alpha_j S_j \geq 0$. So the statement follows from the previous proposition. $\qquad\square$

In general, despite the Blaschke-type conditions identify a subset of $\mathcal{D}(\mathcal{T})$ they are far from being characterizations of the whole set $\mathcal{D}(\mathcal{T})$ even if $\text{Int}\, W_e(\mathcal{T})$ is large. Examples of selfadjoint projections (Kadison's theorem) and of normal operators with finite spectrum (Arveson's obstruction theorem) can serve as simple illustrations of this fact.

While numerical ranges are useful for dealing with operator diagonals (as the theorem above confirms), it is also of interest to relate the spectral structure of an operator to its set of diagonals, thus showing an unexpected link between both. This task appears to be more demanding: while diagonals "live" in the numerical range, their relation to spectrum is far from being obvious. On this way, we prove a result describing a part of diagonals for powers of operator by means of the polynomial hull of its spectrum.

We start with several auxiliary statements.

Lemma 4.8. *Let $\rho > 0$ and $n \in \mathbb{N}$. Then there exists $b_n > 0$ with the following property: If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{C}^n$ then there are $s \in \mathbb{N}$, $\mu_1, \ldots, \mu_s \in \mathbb{T}_\rho$ and $\alpha_1, \ldots, \alpha_s \geq 0$ such that*

$$\sum_{j=1}^{s} \alpha_j \leq b_n \|\varepsilon\| \qquad and \qquad \sum_{j=1}^{s} \alpha_j \mu_j^k = \varepsilon_k, \qquad 1 \leq k \leq n.$$

More precisely, one can choose

$$b_n = \begin{cases} (2^n - 1)\rho^{-n}, & \rho \leq 1, \\ 2^n - 1, & \rho \geq 1. \end{cases}$$

Proof. We prove the lemma by induction on n. For $n = 1$ set $b_1 = \rho^{-1}$. If $\varepsilon_1 \in \mathbb{C}$, $\varepsilon_1 = |\varepsilon_1| \cdot e^{2\pi i \varphi}$ for some $\varphi \in [0, 1)$, then take $\lambda_1 = \rho e^{2\pi i \varphi}$ and $\alpha_1 = \frac{|\varepsilon_1|}{\rho}$. We have

$$\alpha_1 \lambda_1 = |\varepsilon_1| \cdot e^{2\pi i \varphi} = \varepsilon_1 \qquad \text{and} \qquad \alpha_1 = b_1 |\varepsilon_1|,$$

hence the lemma clearly holds for $n = 1$ and $b_1 = \rho^{-1}$.

Suppose that the lemma is true for some integer $n - 1 \geq 1$, and prove it for n. Set

$$b_n = 2b_{n-1} + \rho^{-n}.$$

By the induction assumption, there exist $l \in \mathbb{N}$, $z_1, \ldots, z_l \in \mathbb{T}_\rho$ and $\alpha_1, \ldots, \alpha_l \geq 0$ such that

$$\sum_{j=1}^{l} \alpha_j \leq b_{n-1} \|\varepsilon\|$$

and

$$\sum_{j=1}^{l} \alpha_j z_j^k = \varepsilon_k, \qquad k = 1, \ldots, n - 1.$$

Let

$$\tilde{\varepsilon}_n = \varepsilon_n - \sum_{j=1}^{l} \alpha_j z_j^n.$$

Then

$$|\tilde{\varepsilon}_n| \leq |\varepsilon_n| + \rho^n \sum_{j=1}^{l} \alpha_j \leq \|\varepsilon\| + \rho^n b_{n-1} \|\varepsilon\|.$$

Write $\tilde{\varepsilon}_n = |\tilde{\varepsilon}_n| \cdot e^{2\pi i \varphi}$ for some $\varphi \in [0, 1)$ and set

$$\xi_j = \rho e^{2\pi i (\varphi + \frac{j}{n})} \qquad \text{and} \qquad \beta_j = \frac{|\tilde{\varepsilon}_n|}{n \rho^n}, \qquad 1 \leq j \leq n.$$

If $1 \leq k \leq n - 1$ then

$$\sum_{j=1}^{n} \beta_j \xi_j^k = \frac{|\tilde{\varepsilon}_n|}{n \rho^n} \rho^k e^{2\pi i k \varphi} \sum_{j=1}^{n} e^{2\pi i j k / n} = 0.$$

Similarly,

$$\sum_{j=1}^{n} \beta_j \xi_j^n = \frac{|\tilde{\varepsilon}_n|}{n \rho^n} \rho^n e^{2\pi i \varphi} \cdot n = \tilde{\varepsilon}_n.$$

Thus for every k, $1 \leq k \leq n - 1$, we have

$$\sum_{j=1}^{l} \alpha_j z_j^k + \sum_{j=1}^{n} \beta_j \xi_j^k = \varepsilon_k$$

and

$$\sum_{j=1}^{l} \alpha_j z_j^n + \sum_{j=1}^{n} \beta_j \xi_j^n = \varepsilon_n - \tilde{\varepsilon}_n + \tilde{\varepsilon}_n = \varepsilon_n.$$

Finally,

$$\sum_{j=1}^{l} \alpha_j + \sum_{j=1}^{n} \beta_j \leq b_{n-1}\|\varepsilon\| + \frac{|\tilde{\varepsilon}_n|}{\rho^n} \leq \|\varepsilon\|\big(2b_{n-1} + \rho^{-n}\big) = b_n\|\varepsilon\|.$$

For $\rho \leq 1$ one can prove easily by induction that $b_n \leq (2^n - 1)\rho^{-n}$. Similarly, for $\rho \geq 1$ one has $b_n \leq 2^n - 1$. $\qquad\qquad\square$

The next result is of independent interest. Given $T \in B(H)$, it provides a lower bound for the distance from $(\lambda, \ldots, \lambda^n), \lambda \in \operatorname{Int} \widehat{\sigma}(T)$, to the boundary of $W_{\mathrm{e}}(T, \ldots, T^n)$, and thus relates the spectrum to Blaschke-type conditions.

Proposition 4.9. *Let* $T \in B(H)$, $n \in \mathbb{N}$ *and let* $\mathcal{T}_n = (T, T^2, \ldots, T^n)$. *If* $\lambda \in \operatorname{Int} \widehat{\sigma}(T)$ *and* $\operatorname{dist} \{\lambda, \partial\widehat{\sigma}(T)\} \leq 1$, *then*

$$\operatorname{dist} \big\{(\lambda, \lambda^2, \ldots, \lambda^n), \partial W_{\mathrm{e}}(\mathcal{T}_n)\big\} \geq 2^{-n}\operatorname{dist} \big\{\lambda, \partial\widehat{\sigma}(T)\big\}^n.$$

Proof. Without loss of generality we may assume that $\lambda = 0$. Indeed, consider the operator $\widetilde{T} = T - \lambda$ and the n-tuple $\widetilde{\mathcal{T}} = \big((T-\lambda), (T-\lambda)^2, \ldots, (T-\lambda)^n\big)$. Then

$$\operatorname{dist} \{\lambda, \partial\widehat{\sigma}(T)\} = \operatorname{dist} \{0, \partial\widehat{\sigma}(\widetilde{T})\}$$

and

$$\operatorname{dist} \big\{(\lambda, \lambda^2, \ldots, \lambda^n), \partial W_{\mathrm{e}}(\mathcal{T}_n)\big\} = \operatorname{dist} \big\{(0, \ldots, 0), \partial W_{\mathrm{e}}(\widetilde{\mathcal{T}}_n)\big\}.$$

So suppose that $\lambda = 0 \in \operatorname{Int} \widehat{\sigma}(T)$ and $d := \operatorname{dist} \{0, \partial\widehat{\sigma}(T)\} \leq 1$. Fix $\rho \in (0, d)$, and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{C}^n$ be such that $\|\varepsilon\| \leq 2^{-n} \cdot \operatorname{dist} \{0, \partial\widehat{\sigma}(T)\}$.

By Lemma 4.8, there exist $s \in \mathbb{N}$, $\mu_1, \ldots, \mu_s \in \mathbb{T}_\rho$ and $\alpha_1, \ldots, \alpha_s \geq 0$ such that

$$\sum_{j=1}^{s} \alpha_j \leq 2^n \rho^{-n}\|\varepsilon\|$$

and

$$\sum_{j=1}^{s} \alpha_j \mu_j^k = \varepsilon_k, \qquad 1 \leq k \leq n.$$

By Theorem 3.3, $(\mu_j, \mu_j^2, \ldots, \mu_j^n) \in W_{\mathrm{e}}(\mathcal{T}_n)$ for each $j \leq s$. So

$$\Big(\sum_{j=1}^{s} \alpha_j\Big)^{-1}(\varepsilon_1, \ldots, \varepsilon_n) \in \operatorname{conv} \big\{(\mu_j, \mu_j^2, \ldots, \mu_j^n) : 1 \leq j \leq s\big\} \subset W_{\mathrm{e}}(\mathcal{T}_n).$$

Moreover, $(0, \ldots, 0) \in W_{\mathrm{e}}(\mathcal{T}_n)$. So $(\varepsilon_1, \ldots, \varepsilon_n) \in W_{\mathrm{e}}(\mathcal{T}_n)$ for each $\varepsilon \in \mathbb{C}^n$ with $\|\varepsilon\| \leq \frac{\rho^n}{2^n}$. Since $\rho < d$ was arbitrary,

$$\operatorname{dist} \{(0, \ldots, 0), \partial W_{\mathrm{e}}(\mathcal{T}_n)\} \geq \frac{d^n}{2^n}.$$

$\qquad\qquad\square$

Now we are ready to link spectral properties of operator tuples to their sets of diagonals.

Corollary 4.10. *Let $T \in B(H)$, $n \in \mathbb{N}$, $(\lambda_k)_{k=1}^{\infty} \subset \operatorname{Int} \widehat{\sigma}(T)$. Suppose that $\sum_{k=1}^{\infty} \operatorname{dist}^n\{\lambda_k, \partial\widehat{\sigma}(T)\} = \infty$. Then there exists an orthonormal basis (u_k) in H such that*

$$\langle T^j u_k, u_k \rangle = \lambda_k^j, \qquad k \in \mathbb{N}, 1 \le j \le n.$$

Proof. Let $\mathcal{T}_n = (T, T^2, \ldots, T^n)$. By Proposition 4.9,

$$\sum_{k=1}^{\infty} \operatorname{dist}\left\{(\lambda_k, \lambda_k^2, \ldots, \lambda_k^n), \partial W_{\mathrm{e}}(\mathcal{T}_n)\right\} = \infty$$

and the statement follows from Corollary 4.3. $\qquad\square$

5. Diagonals: compact perturbations

Let $T \in B(H)$ and $0 \in W_{\mathrm{e}}(T)$. In [47, Theorem 2.3], Q. Stout showed that for each sequence $(\alpha_k)_{k=1}^{\infty} \notin \ell_1$ there exists an orthonormal basis $(u_k)_{k=1}^{\infty}$ in H such that the corresponding diagonal $\langle T u_k, u_k \rangle_{k=1}^{\infty}$ of T satisfies

$$|\langle T u_k, u_k \rangle| \le |\alpha_k|$$

for all $k \in \mathbb{N}$. In particular, for each $p > 1$ it is possible to construct a diagonal of T satisfying $\sum_{k=1}^{\infty} |\langle T u_k, u_k \rangle|^p < \infty$. This is an older result due to Anderson [1], see also [23, Theorem 4.1].

By the techniques of this paper, we improve Stout's result in Theorem 1.2 in two directions. First, we show that any sequence $(\lambda_k)_{k=1}^{\infty} \subset W_{\mathrm{e}}(T)$ can be approximated by a diagonal in a sense of (1.3) (Stout's statement treats just zero sequences) and, second, we are able to obtain the result for n-tuples of operators, rather than for a single operator. Note that Theorem 1.2 generalizes also the corresponding Herrero's result from [24].

Proof of Theorem 1.2 We argue as in the proof of Theorem 1.1, though the technical details deviate at several points.

Let $(y_m)_{m=1}^{\infty}$ be a sequence of unit vectors in H such that $\bigvee_m y_m = H$. We can find mutually disjoint sets $A_m, m \in \mathbb{N}$, such that $\bigcup_{m=1}^{\infty} A_m = \mathbb{N}$ and

$$\sum_{k \in A_m} |\alpha_k| = \infty$$

for all $m \in \mathbb{N}$.

Again it is sufficient to construct an orthonormal sequence $(u_k)_{k=1}^{\infty}$ in H such that $\|\langle \mathcal{T} u_k, u_k \rangle - \lambda_k\| \le |\alpha_k|$ for all $k \in \mathbb{N}$ and

$$\ln \operatorname{dist}^2\left\{y_m, \bigvee_{k \le N} u_k\right\} \le -\frac{\sum_{k \le N, k \in A_m} |\alpha_k|}{4 \max\{\|T_1\|, \ldots, \|T_n\|\}}$$

for all $N, m \in \mathbb{N}$.

The sequence $(u_k)_{k=1}^{\infty}$ will be constructed inductively. Let $N \ge 1$ and suppose that $u_1, \ldots, u_{N-1}$ is an orthonormal sequence satisfying $\|\langle \mathcal{T} u_k, u_k \rangle -$

$\lambda_k\| \le |\alpha_k|$ for all $k \le N - 1$ and

$$\ln \operatorname{dist}{}^2\Big\{y_m, \bigvee_{k \le N-1} u_k\Big\} \le -\frac{\sum_{k \le N-1, k \in A_m} |\alpha_k|}{4 \max\{\|T_1\|, \ldots, \|T_n\|\}}$$

for all $m \in \mathbb{N}$. Write $M_{N-1} = \bigvee_{k \le N-1} u_k$ and let $N \in A_m$ for some $m \in \mathbb{N}$.

Decompose y_m as $y_m = a + tb$, where $a \in M_{N-1}$, $b \perp M_{N-1}$, $\|b\| = 1$ and $t = \operatorname{dist}\{y_m, M_{N-1}\}$.

Choose a unit vector $v \in M_{N-1}^{\perp}$ such that $v \perp b, T_1, b, T_1^* b, \ldots, T_n b, T_n^* b$,

$$\big\|\langle \mathcal{T} v, v\rangle - \lambda_N\big\| \le \frac{|\alpha_N|}{2}$$

and set

$$u_N = cb + \sqrt{1 - c^2}\, v,$$

where

$$c = \min\Big\{1, \Big(\frac{|\alpha_N|}{4 \max\{\|T_1\|, \ldots, \|T_n\|\}}\Big)^{1/2}\Big\}.$$

Clearly $\|u_N\| = 1$ and $u_N \perp u_1, \ldots, u_{N-1}$.

We have

$$\begin{aligned}
\big\|\langle \mathcal{T} u_N, u_N\rangle - \lambda_N\big\| &\le \big\|\langle \mathcal{T} u_N, u_N\rangle - \langle \mathcal{T} v, v\rangle\big\| + \big\|\langle \mathcal{T} v, v\rangle - \lambda_N\big\| \\
&\le c^2 \|\langle \mathcal{T} b, b\rangle\| + c^2 \|\langle \mathcal{T} v, v\rangle\| + \frac{|\alpha_N|}{2} \\
&\le 2c^2 \max\{\|T_1\|, \ldots, \|T_n\|\} + \frac{|\alpha_N|}{2} \\
&\le |\alpha_N|.
\end{aligned}$$

Furthermore,

$$\operatorname{dist}{}^2\Big\{y_m, \bigvee_{k \le N} u_k\Big\} \le t^2 \operatorname{dist}{}^2\Big\{b, \bigvee u_N\Big\} = t^2(1 - c^2).$$

Thus

$$\begin{aligned}
\ln \operatorname{dist}{}^2\Big\{y_m, \bigvee_{k \le N} u_k\Big\} &\le \ln t^2 + \ln(1 - c^2) \le \ln t^2 - c^2 \\
&\le \ln t^2 - \frac{|\alpha_N|}{4 \max\{\|T_1\|, \ldots, \|T_n\}} \\
&\le -\frac{\sum_{k \le N, k \in A_m} |\alpha_k|}{4 \max\{\|T_1\|, \ldots, \|T_n\|\}}.
\end{aligned}$$

$\square$

Now, by means of the approximation Theorem 1.2, we can describe the set of diagonals $\mathcal{D}(\mathcal{T})$ up to p-Schatten class perturbations of $\mathcal{T}$. In this more general setting, we are able to construct the diagonals satisfying a generalized Blaschke-type condition resembling in a sense the definition of

S_p-classes. Moreover, the diagonals of perturbations may not necessarily belong to $W_e(\mathcal{T})$ but should only approximate $W_e(\mathcal{T})$ good enough, where the rate of approximation is determined by the Schatten class of perturbations.

Corollary 5.1. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Let $p > 1$. Let $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}^n$ satisfy*

$$(5.1) \qquad \sum_{k=1}^\infty \operatorname{dist}^p \{\lambda_k, W_e(\mathcal{T})\} < \infty.$$

Then there exists an n-tuple of operators $\mathcal{K} = (K_1, \dots, K_n)$ with K_j from the Schatten class S_p, $1 \le j \le n$, such that $(\lambda_k)_{k=1}^\infty \in \mathcal{D}(\mathcal{T} + \mathcal{K})$.

Proof. Let $p > 1$ be fixed and let $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}^n$ satisfy (5.1).

Find a sequence $(\lambda_k')_{k=1}^\infty \subset W_e(\mathcal{T})$ such that $\sum_{k=1}^\infty \|\lambda_k' - \lambda_k\|^p < \infty$. By Theorem 1.2, there exists an orthonormal basis $(u_k)_{k=1}^\infty$ in H such that

$$\|\langle \mathcal{T} u_k, u_k \rangle - \lambda_k'\| \le k^{-1}$$

for all k. Define $\mathcal{K} = (K_1, \dots, K_n) \in B(H)^n$ as

$$\mathcal{K} u_k = (\lambda_k - \langle \mathcal{T} u_k, u_k \rangle) u_k$$

for all k. Then

$$\langle (\mathcal{T} + \mathcal{K}) u_k, u_k \rangle = \lambda_k, \qquad k \in \mathbb{N},$$

and

$$\sum_{k=1}^\infty \|\lambda_k - \langle \mathcal{T} u_k, u_k \rangle\|^p \le \sum_{k=1}^\infty \Big(\|\lambda_k - \lambda_k'\| + \|\lambda_k' - \langle \mathcal{T} u_k, u_k \rangle\| \Big)^p < \infty.$$

So the operators $K_1, \dots, K_n$ belong to S_p. $\qquad\square$

We finish this section with a discussion of a subset $\mathcal{D}_{\text{const}}(\mathcal{T}) = \{\lambda \in \mathbb{C}^n : (\lambda, \lambda, \dots) \in \mathcal{D}(\mathcal{T})\}$ of diagonals of $T \in B(H)$ that seems to be crucial. Recall that operators possessing a zero diagonal appeared relevant in several areas of operator theory (e.g. the study of commutators) and attracted a substantial attention much before the foundational works of Kadison and Arveson on the set of (all) diagonals. Thus it is natural to try understand the whole set of constant diagonals for a fixed T and to relate its structure to the structure of $\sigma(T)$ and $W(T)$. While this task seems to be more accessible than a characterization of $\mathcal{T}$, we are far from a complete answer even for very simple operators T.

Observe that by Theorem 1.1,

$$(5.2) \qquad \operatorname{Int} W_e(\mathcal{T}) \subset \mathcal{D}_{\text{const}}(\mathcal{T}) \subset W_e(\mathcal{T}) \cap W(\mathcal{T}).$$

Since the interior of a convex set is convex, both $W_e(\mathcal{T})$ and $\operatorname{Int} W_e(\mathcal{T})$ are convex sets. However, the question whether $\mathcal{D}_{\text{const}}(\mathcal{T})$ is convex is still open, even if $n = 1$. This problem has been posed explicitly in [11, p. 213]. It is instructive to note that, even if $n = 1$, the diagonal set $\mathcal{D}_{\text{const}}(T)$ is not a subset of $W_\infty(T)$, and even the inclusion $\mathcal{D}_{\text{const}}(T) \subset W_2(T)$ may not hold, where $W_2(\mathcal{T})$ stands for the set of all $\lambda \in \mathbb{C}$ such that there exists

a two-dimensional subspace $L \subset H$ satisfying $P_L T P_L = \lambda P_L$. The next example makes this precise.

Example 5.2. Let $T = \mathrm{diag}\left(-1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right)$. By Fan's result [18, Theorem 1] mentioned in the introduction (see also [39, Appendix] for corrections), $0 \in \mathcal{D}_{\mathrm{const}}(T)$. However, $0 \notin W_2(T)$. Indeed, suppose on the contrary that $0 \in W_2(T)$. So there exists a two-dimensional subspace M such that $P_M T P_M = 0$. In particular there exists $x = (0, x_1, x_2, \dots) \in H$ such that $\|x\| = 1$ and $\langle Tx, x \rangle = 0$. However, $\langle Tx, x \rangle = \sum_{j=1}^{\infty} 2^{-j} |x_j|^2 > 0$, a contradiction.

While, we are not able to answer this question about the convexity of $\mathcal{D}_{\mathrm{const}}(\mathcal{T})$ either, we give several statements clarifying the structure of the set of constant diagonals operators. First, we describe the orbit of $\mathcal{D}_{\mathrm{const}}(\mathcal{T})$ under compact perturbations.

Proposition 5.3. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$\bigcup_{\mathcal{K} \in K(H)^n} \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K}) = W_{\mathrm{e}}(\mathcal{T}).$$

Proof. For each $\mathcal{K} \in K(H)^n$ we have

$$\mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K}) \subset W_{\mathrm{e}}(\mathcal{T} + \mathcal{K}) = W_{\mathrm{e}}(\mathcal{T}).$$

On the other hand, if $\lambda \in W_{\mathrm{e}}(\mathcal{T})$ then, by Corollary 5.1, there exists $\mathcal{K}$ (even an n-tuple of Schatten class operators) such that $\lambda \in \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K})$. $\square$

Similarly, we are able to locate a subset of $\mathcal{D}_{\mathrm{const}}(\mathcal{T})$ invariant under compact perturbations.

Proposition 5.4. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$\bigcap_{\mathcal{K} \in K(H)^n} W(\mathcal{T} + \mathcal{K}) = \bigcap_{\mathcal{K} \in K(H)^n} \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K}) = \mathrm{Int}\, W_{\mathrm{e}}(\mathcal{T}).$$

Proof. Taking into account (5.2), for each $\mathcal{K} \in K(H)^n$,

$$\mathrm{Int}\, W_{\mathrm{e}}(\mathcal{T}) = \mathrm{Int}\, W_{\mathrm{e}}(\mathcal{T} + \mathcal{K}) \subset \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K}) \subset W(\mathcal{T} + \mathcal{K}).$$

Conversely, let $\lambda \in W_{\mathrm{e}}(\mathcal{T}) \setminus \mathrm{Int}\, W_{\mathrm{e}}(\mathcal{T})$. Without loss of generality, by using (3.1) and considering a suitable compact perturbation, we may assume that $W_{\mathrm{e}}(\mathcal{T}) = \overline{W(\mathcal{T})}$. We may also assume, by a suitable translation, that $\lambda = 0$. Now let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, $0 \in W_{\mathrm{e}}(\mathcal{T}) = \overline{W(\mathcal{T})}$, $0 \notin \mathrm{Int}\, W_{\mathrm{e}}(\mathcal{T})$. The numerical range of the n-tuple $\mathcal{T}$ can be identified with the numerical range of the $(2n)$-tuple $(\mathrm{Re}\, T_1, \mathrm{Im}\, T_1, \dots, \mathrm{Re}\, T_n, \mathrm{Im}\, T_n)$ of selfadjoint operators. By a suitable rotation in $\mathbb{R}^{2n} \sim \mathbb{C}^n$ we may assume that $\langle (\mathrm{Re}\, T_1) x, x \rangle \geq 0$ for all $x \in H$ and $0 \in \overline{W(\mathcal{T})}$. Let $K_1 = \mathrm{diag}\left(1, 2^{-1}, 2^{-2}, \dots\right)$ (in any orthonormal basis). Clearly K_1 is a compact

operator on H and $\langle (\operatorname{Re} T_1 + K_1)x, x \rangle > 0$ for all $x \neq 0$. So $0 \notin W(\operatorname{Re} T_1 + K_1)$ and then $0 \notin \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K})$ for the tuple $\mathcal{K} = (K_1, 0, \ldots, 0)$. Thus,

$$\bigcap_{\mathcal{K} \in K(H)^n} \mathcal{D}_{\mathrm{const}}(\mathcal{T} + \mathcal{K}) \subset \bigcap_{\mathcal{K} \in K(H)^n} W(\mathcal{T} + \mathcal{K}) \subset \operatorname{Int} W_{\mathrm{e}}(\mathcal{T}).$$

$\square$

Note that in several specific cases $\mathcal{D}_{\mathrm{const}}(T)$ allows an explicit description. For instance, by [11], if $T \in B(H)$ then $W(T)$ is relatively open (i.e., it is a single point, an open segment or an open set in $\mathbb{C}$) if and only if $W(T) = \mathcal{D}_{\mathrm{const}}(T)$. The class of operators with open $W(T)$ is substantial. For instance, it includes weighted periodic shifts and a number of Toeplitz operators.

6. Operator-valued diagonals: Blaschke-type condition

Let $T \in B(H)$ satisfy $W_{\mathrm{e}}(T) \supset \overline{\mathbb{D}}$. Let $(C_k)_{k=1}^\infty$ be a sequence of contractions on Hilbert spaces H_k (possibly different from H) such that $\sup_k \|C_k\| < 1$. By [11, Theorem 2.1], the operator T has a "pinching" $\bigoplus_{k=1}^\infty C_k$, i.e., there exist mutually orthogonal subspaces $K_k \subset H$ such that $H = \bigoplus_{k=1}^\infty K_k$ and $P_{K_k} T P_{K_k} \overset{u}{\sim} C_k$ for all k.

In Theorem 1.3, which can be considered as an operator-valued version of Theorem 1.1, we show that the assumption $\sup_{k \geq 1} \|C_k\| < 1$ can be replaced by a much weaker Blaschke-type condition $\sum_{k=1}^\infty (1 - \|C_k\|) = \infty$ resembling a similar condition (1.2) above. Clearly, the operator-valued version of Theorem 1.1 is more involved and its proof requires new arguments. However, the scheme of the proof is similar to the one used in Section 4.

First, we will need an auxiliary lemma.

Lemma 6.1. *Let $T \in B(H)$ with $W_{\mathrm{e}}(T) \supset \overline{\mathbb{D}}$. Then there exist mutually orthogonal subspaces $H_k \subset H, k \in \mathbb{N}$, such that*

$$W_{\mathrm{e}}(P_{H_k} T P_{H_k}) \supset \overline{\mathbb{D}}, \qquad k \in \mathbb{N}.$$

Proof. Let $f : \mathbb{N} \to \mathbb{N}^3$ be a bijection, and write $f(k) = (k_1, k_2, k_3)$. Let $(w_s)_{s=1}^\infty \subset \mathbb{T}$ be a sequence dense in $\mathbb{T}$.

Inductively we construct a sequence of mutually orthogonal unit vectors $(x_k)_{k=1}^\infty \in H$ such that

$$\left| \langle T x_k, x_k \rangle - w_{k_1} \right| < \frac{1}{k_2}.$$

Now suppose that $(x_k)_{k=1}^\infty \subset H$ is constructed in this way. For $m \in \mathbb{N}$ define

$$H_m = \bigvee \{x_k : k_3 = m\},$$

and note that the subspaces $H_m, m \in \mathbb{N}$ are mutually orthogonal. Let $s, m \in \mathbb{N}$ be fixed. Then $\{u_j = x_{f^{-1}(s,j,m)} : j \geq 1\}$ form an orthonormal

sequence in H_m such that

$$\left|\langle Tu_j, u_j\rangle - w_s\right| < \frac{1}{j}$$

for all $j \in \mathbb{N}$. Thus $w_s \in W_{\mathrm{e}}(P_{H_m}TP_{H_m})$ and, since this holds for all $s, m \in \mathbb{N}$, we have $W_{\mathrm{e}}(P_{H_m}TP_{H_m}) \supset \overline{\mathbb{D}}$ for all $m \in \mathbb{N}$. $\qquad\square$

So everything is prepared for the proof of Theorem 1.3 and we give it below.

Proof of Theorem 1.3 Let $H_m, m \in \mathbb{N}$, be the subspaces given by Lemma 6.1, and let $(y_m)_{m=1}^{\infty}$ be a sequence of unit vectors in H such that $\bigvee_m y_m = H$. Using the assumption, we can find mutually disjoint sets $A_m, m \in \mathbb{N}$, such that $\bigcup_{m=1}^{\infty} A_m = \mathbb{N}$ and

$$\sum_{k \in A_m} (1 - \|C_k\|) = \infty$$

for all $m \in \mathbb{N}$. Next we construct mutually orthogonal subspaces $K_k, k \in \mathbb{N}$, such that $\dim K_k/(H_k \cap K_k) \leq 1$, $P_{K_k}TP_{K_k} \overset{u}{\sim} C_k$ and

$$(6.1) \qquad \ln \operatorname{dist}^2\left\{y_m, \bigvee_{k \leq N} K_k\right\} \leq - \sum_{k \leq N, k \in A_m} \frac{1 - \|C_k\|}{16\|T\|}$$

for all $m, N \in \mathbb{N}$. Note that as a consequence $\bigoplus_{k=1}^{\infty} K_k = H$. Indeed, for each $m \in \mathbb{N}$ we have

$$\lim_{N \to \infty} \ln \operatorname{dist}^2\left\{y_m, \bigvee_{k \leq N} K_k\right\} \leq - \lim_{N \to \infty} \sum_{k \leq N, k \in A_m} \frac{1 - \|C_k\|}{16\|T\|} = -\infty.$$

So $\operatorname{dist}\left\{y_m, \bigvee_{k \in \mathbb{N}} K_k\right\} = 0$ and $y_m \in \bigvee_{k \in \mathbb{N}} K_k$. Hence $H = \bigoplus_{k=1}^{\infty} K_k$.

As above, the sequence of mutually orthogonal subspaces $(K_k)_{k=1}^{\infty}$ satisfying (6.1) will be constructed inductively.

Let $N \in \mathbb{N}$ and suppose that the subspaces $K_k, k \leq N-1$, have already been constructed. Then $N \in A_m$ for some $m \in \mathbb{N}$. Write $y_m = a + tb$ where

$$a \in \bigvee_{k \leq N-1} K_k, \quad b \perp \bigvee_{k \leq N-1} K_k, \quad \|b\| = 1$$

and $t = \operatorname{dist}\left\{y_m, \bigvee_{k \leq N-1} K_k\right\} \leq 1$.

Setting $H'_N = H_N \cap \{b, Tb, T^*b\}^{\perp} \cap \bigcap_{k=1}^{N-1} K_k^{\perp}$, observe that $W_{\mathrm{e}}(P_{H'_N}TP_{H'_N}) \supset \overline{\mathbb{D}}$.

Now fix any unit vector $x \in L_N$, and let $P \in B(L_N)$ be the orthogonal projection onto the one-dimensional subspace generated by x. If

$$\rho =: \sqrt{\frac{1 - \|C_N\|}{16\|T\|}}.$$

then $\rho \leq \frac{1}{4}$ and $\sqrt{1-\rho^2} \geq \frac{1}{2}$. Consider

$$
\begin{aligned}
C'_N =&(I-P)C_N(I-P) + \frac{1}{\sqrt{1-\rho^2}}(I-P)C_N P \\
&+ \frac{1}{\sqrt{1-\rho^2}}PC_N(I-P) + \frac{PC_N P - \rho^2\langle Tb,b\rangle P}{1-\rho^2} \\
=&C_N + \Big(\frac{1}{\sqrt{1-\rho^2}} - 1\Big)(I-P)C_N P + \Big(\frac{1}{\sqrt{1-\rho^2}} - 1\Big)PC_N(I-P) \\
&+ \Big(\frac{1}{1-\rho^2} - 1\Big)PC_N P - \frac{\rho^2}{1-\rho^2}\langle Tb,b\rangle P,
\end{aligned}
$$

where I denotes the identity operator on L_N. We have

$$
\begin{aligned}
\|C'_N\| \leq& \|C_N\| + 2\Big(\frac{1}{\sqrt{1-\rho^2}} - 1\Big) + \Big(\frac{1}{1-\rho^2} - 1\Big) + \frac{\rho^2}{1-\rho^2}\|T\| \\
\leq& \|C_N\| + 2\frac{1-\sqrt{1-\rho^2}}{\sqrt{1-\rho^2}} + \frac{\rho^2}{1-\rho^2} + \frac{\rho^2}{1-\rho^2}\|T\| \\
\leq& \|C_N\| + 4\big(1-\sqrt{1-\rho^2}\big) + \frac{2\|T\|\rho^2}{1-\rho^2} \\
\leq& \|C_N\| + 4\rho^2 + 4\rho^2\|T\| \\
\leq& \|C_N\| + 8\rho^2\|T\| \\
\leq& \|C_N\| + \frac{1-\|C_N\|}{2} < 1.
\end{aligned}
$$

So, by for example [11, Theorem 2.1], there exists a subspace $K'_N \subset H'_N$ such that $P_{K'_N}TP_{K'_N} \overset{u}{\sim} C'_N$. Thus there exists a unitary operator $U : L_N \to K'_N$ such that

$$
U^{-1}P_{K'_N}TP_{K'_N}U = C'_N.
$$

Set

$$
v = \sqrt{1-\rho^2}Ux + \rho b.
$$

Since $Ux \in K'_N \subset H'_N \perp b$, we have $\|v\| = 1$ and $v \perp U(I-P)L_N$. Let

$$
K_N = U(I-P)L_N \vee \{v\}
$$

and let $V : L_N \to K_N$ be defined by

$$
V \restriction_{(I-P)L_N} := U \restriction_{(I-P)L_N} \qquad \text{and} \qquad Vx := v.
$$

We have $K'_N \subset H'_N \subset H_N$, and so $\dim K_N/(K_N \cap H_N) \leq 1$. Clearly V is a unitary operator, and $K_N \perp \bigvee_{k \leq N-1} K_k$. For $z = u + sx$, where

$u \in (I - P)L_N$ and $s \in \mathbb{C}$, we have

$$
\begin{aligned}
\langle V^{-1} P_{K_N} T P_{K_N} V z, z \rangle & \\
&= \langle V^{-1} T V z, z \rangle = \langle T(Uu + sv), Uu + sv \rangle \\
&= \langle T(Uu + s\sqrt{1 - \rho^2} Ux + s\rho b), Uu + s\sqrt{1 - \rho^2} Ux + s\rho b \rangle \\
&= \langle T(Uu + s\sqrt{1 - \rho^2} Ux), Uu + s\sqrt{1 - \rho^2} Ux \rangle + s^2 \rho^2 \langle Tb, b \rangle \\
&= \langle C_N'(u + s\sqrt{1 - \rho^2} x), u + s\sqrt{1 - \rho^2} x \rangle + s^2 \rho^2 \langle Tb, b \rangle \\
&= \langle (I - P) C_N' (I - P) z, z \rangle + \sqrt{1 - \rho^2} \langle (I - P) C_N' P z, z \rangle \\
&\quad + \sqrt{1 - \rho^2} \langle P C_N' (I - P) z, z \rangle + (1 - \rho^2) \langle P C_N' P z, z \rangle + \rho^2 \langle Tb, b \rangle \langle P z, z \rangle \\
&= \langle C_N z, z \rangle.
\end{aligned}
$$

Hence $P_{K_N} T P_{K_N} \overset{u}{\sim} C_N$.

Moreover, since $\langle b, v \rangle = \rho$ and

$$
\operatorname{dist}^2 \left\{ b, \bigvee v \right\} = 1 - \rho^2 = 1 - \frac{1 - \|C_N\|}{16 \|T\|},
$$

we infer that

$$
\ln \frac{\operatorname{dist}^2 \left\{ y_m, \bigvee_{k \le N} K_k \right\}}{\operatorname{dist}^2 \left\{ y_m, \bigvee_{k \le N-1} K_k \right\}} = \ln \left(1 - \frac{1 - \|C_N\|}{16 w(T)} \right) \le -\frac{1 - \|C_N\|}{16 \|T\|}
$$

and

$$
\ln \operatorname{dist}^2 \left\{ y_m, \bigvee_{k \le N} K_k \right\} \le -\sum_{k \le N, k \in A_m} \frac{1 - \|C_k\|}{16 \|T\|}.
$$

This finishes the proof. $\qquad \square$

Replacing the numerical range condition $W_e(T) \supset \overline{\mathbb{D}}$ in Theorem 1.3 by the stronger spectral assumption $\widehat{\sigma}(T) \supset \overline{\mathbb{D}}$, we can put Theorem 1.3 in a more general and demanding context of tuples of powers of T. To this aim we will need the next lemma.

Lemma 6.2. *Let $T \in B(H)$ satisfy $\widehat{\sigma}(T) \supset \overline{\mathbb{D}}$. Let $n \in \mathbb{N}$, let L be a separable Hilbert space, and let $C, A_1, \dots, A_n \in B(L)$ be such that $\|C\| < 1$ and $\|A_j\| \le \frac{(1 - \|C\|)^n}{n 2^{2n+4}}, j = 1, \dots, n$. Then there exists a subspace $K \subset H$ such that*

$$
P_k(T, T^2, \dots, T^n) P_k \overset{u}{\sim} (C + A_1, C^2 + A_2, \dots, C^n + A_n).
$$

Proof. Set

$$
c := \|C\|, \quad d := \frac{(1 - c)^n}{n 2^{2n+4}}, \quad c' := c + \frac{c(1 - c)^n}{2^{n+1}}, \quad \text{and} \quad d' := \frac{1}{2^{n+1}}.
$$

We prove first that

(6.2) $$ \frac{c}{c'} + 2n \frac{d}{d'} < 1. $$

Since $\frac{1}{1+a} < 1 - \frac{a}{2}$ for all $a \in [0, 1)$, we have

$$\frac{1}{1 + \frac{(1-c)^n}{2^{n+1}}} + \frac{(1-c)^n}{2^{n+2}} < 1,$$

which is equivalent to (6.2).

Choose

$$\varepsilon' := \frac{1}{4n}\left(1 - \frac{c}{c'} - 2n\frac{d}{d'}\right),$$

and $\varepsilon < \frac{\varepsilon'}{n2^n}$.

Let $M \supset L$ and let $U \in B(M)$ be the minimal unitary dilation of the contraction $\frac{C}{c}$. Extend the operators $A_1, \ldots, A_n$ to $\tilde{A}_1, \ldots, \tilde{A}_n \in B(M)$ by defining $\tilde{A}_j \restriction_L = A_j \restriction_L$ and $\tilde{A}_j \restriction_{M \ominus L} = 0$.

By the Weyl-von Neumann's theorem, cU can be written as $cU = D + K$ where $D \in B(M)$ is a diagonal operator with entries of moduli c and $K \in B(M)$ is a compact operator, $\|K\| \le \varepsilon$. For $j = 1, \ldots, n$ let $K_j = (cU)^j - D^j$. We have $K_1 = K$ and

$$K_j = \sum_{s=0}^{j-1}(cU)^s(cU - D)D^{j-s-1}.$$

So K_j is a compact operator for each $j = 1, \ldots, n$ and $\|K_j\| \le n\varepsilon$.

Similarly, operators $\operatorname{Re}\tilde{A}_j$ and $\operatorname{Im}\tilde{A}_j$ can be written as $\operatorname{Re}\tilde{A} = D'_j + K'_j$ and $\operatorname{Im}\tilde{A}_j = D''_j + K''_j$, where D'_j and D''_j are diagonal operators with entries of moduli not exceeding d, and K'_j and K''_j are compact selfadjoint operators with $\|K'_j\| \le \varepsilon$ and $\|K''_j\| \le \varepsilon$. We have

$$\left(cU + \tilde{A}_1, \ldots, (cU)^n + \tilde{A}_n\right)$$
$$= \frac{c}{c'}\left(\frac{c'}{c}\left(D, \ldots, D^n\right)\right)$$
$$+ \sum_{j=1}^{n}\frac{d}{d'}\left(\frac{d'}{d}\left(\underbrace{0, \ldots, 0}_{j-1}, D'_j + iD''_j, 0, \ldots, 0\right)\right)$$
$$+ \sum_{j=1}^{n}\varepsilon'\left(\frac{1}{\varepsilon'}\left(\underbrace{0, \ldots, 0}_{j-1}, \operatorname{Re}K_j + i\operatorname{Im}K_j + K'_j + iK''_j, 0, \ldots, 0\right)\right).$$

Write for short $\mathcal{T} = (T, \ldots, T^n)$. If $z \in \mathbb{C}$, $|z| = c$, then

$$\left\|\frac{c'}{c}(z, z^2, \ldots, c^n) - (z, z^2, \ldots, c^n)\right\| = \left(\frac{c'}{c} - 1\right) \cdot c = c' - c = \frac{(1-c)^n c}{2^{n+1}},$$

hence $\frac{c'}{c}\left(D, \ldots, D^n\right)$ is a jointly diagonal n-tuple with all entries in $\operatorname{Int}W_e(\mathcal{T}) \subset W_\infty(\mathcal{T})$. So, by Proposition 4.9, $\frac{c'}{c}\left(D, \ldots, D^n\right) \in \mathcal{M}(W_\infty(\mathcal{T}))$.

Similarly,

$$\frac{d'}{d}\left(\underbrace{0, \ldots, 0}_{j-1}, D'_j, 0, \ldots, 0\right)$$

is a diagonal tuple such that the operator norm of each of its entry does not exceed $d' = \frac{1}{2^{n+1}}$. Moreover, by Proposition 4.9,

$$\frac{d'}{d}\big(\underbrace{0,\dots,0}_{j-1}, D'_j, 0, \dots, 0\big) \in \mathcal{M}(W_\infty(\mathcal{T})).$$

In the same way

$$\frac{d'}{d}\big(\underbrace{0,\dots,0}_{j-1}, D''_j, 0, \dots, 0\big) \in \mathcal{M}(W_\infty(\mathcal{T})).$$

Finally, $\operatorname{Re} K_j, \operatorname{Im} K_j, K'_j, K''_j, 1 \le j \le n$, are compact selfadjoint operators with entries of moduli $\le n\varepsilon$. Thus the tuples

$$\frac{1}{\varepsilon'}\big(\underbrace{0,\dots,0}_{j-1}, \operatorname{Re} K_k, 0, \dots, 0\big), \qquad \frac{1}{\varepsilon'}\big(\underbrace{0,\dots,0}_{j-1}, i\operatorname{Im} K_j + K'_j, 0, \dots, 0\big),$$

$$\frac{1}{\varepsilon'}\big(\underbrace{0,\dots,0}_{j-1}, K'_j, 0, \dots, 0\big) \qquad \text{and} \qquad \frac{1}{\varepsilon'}\big(\underbrace{0,\dots,0}_{j-1}, iK''_j, 0, \dots, 0\big)$$

are diagonal, and operator norms of their entries are not larger than $\frac{n\varepsilon}{\varepsilon'} < \frac{1}{2^n}$. Hence the tuples belong to $\mathcal{M}(W_\infty(\mathcal{T}))$.

Finally, by construction, $\big(cU + \tilde{A}_1, \dots, (cU)^n + \tilde{A}_n\big)$ is a convex combination of n-tuples belonging to $\mathcal{M}(W_\infty(\mathcal{T}))$. So, Proposition 3.2 implies that there exists a subspace $K' \subset H$ such that

$$\big(cU + \tilde{A}_1, \dots, (cU)^n + \tilde{A}_n\big) \overset{u}{\sim} P_{K'}(T, \dots, T^n)P_{K'}.$$

Since

$$P_L\big(cU + \tilde{A}_1, \dots, (cU)^n + \tilde{A}_n\big)P_L = (C + A_1, \dots, C^n + A_n),$$

this implies the statement of the theorem. $\qquad\qquad\square$

For a sequence of Hilbert space contractions $(C_k)_{k=1}^\infty$ with norms not approaching 1 too fast, the following statement yields pinchings $(C_k, \dots, C_k^n)$ for a tuple $(T, \dots, T^n), T \in B(H)$, if the spectrum of T is sufficiently large. Its assumptions are close to be optimal even if $n = 1$, see [11].

Theorem 6.3. *Let $T \in B(H)$, $\hat{\sigma}(T) \supset \overline{\mathbb{D}}$, $n \in \mathbb{N}$. Let $L_k, k \in \mathbb{N}$, be separable Hilbert spaces, and let $C_k \in B(L_k), k \in \mathbb{N}$, be such that $\sum_{k=1}^\infty (1 - \|C_k\|)^n = \infty$. Then there are mutually orthogonal subspaces $K_k, k \in \mathbb{N}$, of H such that*

$$H = \bigoplus_{k=1}^\infty K_k \qquad \text{and} \qquad P_{K_k}(T, \dots, T^n)P_{K_k} \overset{u}{\sim} (C_k, \dots, C_k^n)$$

for all $k \in \mathbb{N}$.

Proof. The proof is analogous to that of Theorem 1.3, so we present it only briefly.

As in the proof of Theorem 1.3, one can find mutually orthogonal subspaces $H_m \subset H, m \in \mathbb{N}$, such that

$$W_{\mathrm{e}}\big(P_{H_m}(T,\ldots,T^n)P_{H_m}\big) = W_{\mathrm{e}}(T,\ldots,T^n).$$

Let $(y_m)_{m=1}^{\infty}$ be a sequence of unit vectors in H such that $\bigvee_m y_m = H$. We can find a sequence of mutually disjoint sets $(A_m)_{m\in\mathbb{N}}$ such that $\bigcup_{m=1}^{\infty} A_m = \mathbb{N}$ and

$$\sum_{k\in A_m} (1 - \|C_k\|)^n = \infty$$

for all $m \in \mathbb{N}$. We construct mutually orthogonal subspaces $K_k \subset H, k \in \mathbb{N}$, satisfying $P_{K_k}(T,\ldots,T^n)P_{K_k} \overset{u}{\sim} (C_k,\ldots,C_k^n)$, $\dim K_k/(H_k \cap K_k) \le 1$ and

$$\ln \mathrm{dist}^2\Big\{y_m, \bigvee_{k\le N} K_k\Big\} \le - \sum_{k\le N, k\in A_m} \frac{(1-\|C_k\|)^n}{n2^{2n+7}\|T\|^n}$$

for all $m, n \in \mathbb{N}$.

Let $N \in \mathbb{N}$ and suppose that the subspaces $K_k, k \le N-1$, have already been constructed. Let $N \in A_m$. Let $y_m = a + tb$ where

$$a \in \bigvee_{k\le N-1} K_k, \quad b \perp \bigvee_{k\le N-1} K_k, \quad \|b\| = 1$$

and $t = \mathrm{dist}\Big\{y_m, \bigvee_{k\le N-1} K_k\Big\} \le 1$.

Define

$$H'_N = H_N \cap \{b, T^j b, T^{*j} b : j = 1,\ldots,n\}^{\perp} \cap \bigcap_{k=1}^{N-1} K_k^{\perp},$$

and note that $W_{\mathrm{e}}(P_{H'_N} T P_{H'_N}) \supset \overline{\mathbb{D}}$.

Fix any unit vector $x \in L_N$ and let $P \in B(L_N)$ be the orthogonal projection onto the one-dimensional subspace generated by x. If

$$\rho = \sqrt{\frac{(1-\|C_k\|)^n}{n2^{2n+7}\|T\|^n}}$$

then $\rho \le \frac{1}{4}$ and $\sqrt{1-\rho^2} \ge \frac{1}{2}$. For each $j = 1,\ldots,n$ consider

$$C'_{N,j} = (I-P)C_N^j(I-P) + \frac{1}{\sqrt{1-\rho^2}}(I-P)C_N^j P$$

$$+ \frac{1}{\sqrt{1-\rho^2}}PC_N^j(I-P) + \frac{PC_N^j P - \rho^2\langle T^j b, b\rangle P}{1-\rho^2},$$

where I denotes the identity operator on L_N. We have

$$\|C'_{N,j} - C_N^j\| \le 2\Big(\frac{1}{\sqrt{1-\rho^2}} - 1\Big) + \Big(\frac{1}{1-\rho^2} - 1\Big) + \frac{\rho^2}{1-\rho^2}\|T\|^j$$

$$\le 8\rho^2\|T\|^j.$$

By Lemma 6.2, there exists a subspace $K'_N \subset H'_N$ such that
$$P_{K'_N}(T, \dots, T^n)P_{K'_N} \overset{u}{\sim} (C'_{N,1}, \dots, C'_{N,n}),$$
i.e., there exists a unitary operator $U : L_N \to K'_N$ such that
$$U^{-1}P_{K'_N}(T, \dots, T^n)P_{K'_N}U = (C'_{N,1}, \dots, C'_{N,n}).$$
Set
$$v = \sqrt{1 - \rho^2}Ux + \rho b.$$
Since $Ux \in K'_N \subset H'_N$ and $H'_N \perp b$, we have $\|v\| = 1$ and $v \perp U(I - P)L_N$. Let $K_N = U(I - P)L_N \vee \{v\}$ and note that $K_N \perp \bigvee_{k \leq N-1} K_k$. Define a unitary operator $V : L_N \to K_N$ by
$$V \upharpoonright_{(I-P)L_N} := U \upharpoonright_{(I-P)L_N} \qquad \text{and} \qquad Vx := v.$$
For $z = u + sx$, $u \in (I - P)L_N$, $s \in \mathbb{C}$ and $1 \leq j \leq n$ we have
$$\left\langle V^{-1}P_{K_N}T^j P_{K_N}Vz, z\right\rangle$$
$$= \left\langle T^j(Uu + sv), Uu + sv\right\rangle$$
$$= \left\langle T^j(Uu + s\sqrt{1 - \rho^2}Ux), Uu + s\sqrt{1 - \rho^2}Ux\right\rangle + s^2\rho^2\langle T^j b, b\rangle$$
$$= \left\langle (I - P)C'_{N,j}(I - P)z, z\right\rangle + \sqrt{1 - \rho^2}\left\langle (I - P)C'_{N,j}Pz, z\right\rangle$$
$$+ \sqrt{1 - \rho^2}\left\langle PC'_{N,j}(I - P)z, z\right\rangle$$
$$+ (1 - \rho^2)\langle PC'_{N,j}Pz, z\rangle + \rho^2\langle T^j b, b\rangle\langle Pz, z\rangle$$
$$= \langle C_N^j z, z\rangle.$$
Hence
$$P_{K_N}(T, \dots, T^n)P_{K_N} \overset{u}{\sim} (C_N, \dots, C_N^n).$$
Moreover, we have $\langle b, v\rangle = \rho$ and
$$\text{dist}^2\left\{b, \bigvee v\right\} = 1 - \rho^2 = 1 - \frac{(1 - \|C_N\|)^n}{n2^{2n+7}\|T\|^n}.$$
Thus
$$\ln \frac{\text{dist}^2\left\{y_m, \bigvee_{k \leq N} K_k\right\}}{\text{dist}^2\left\{y_m, \bigvee_{k \leq N-1} K_k\right\}} = \ln\left(1 - \frac{(1 - \|C_N\|)^n}{n2^{2n+7}\|T\|^n}\right) \leq -\frac{(1 - \|C_N\|)^n}{n2^{2n+7}\|T\|^n}.$$
As in Theorem 1.3, this finishes the proof. $\qquad \square$

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