



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Sign patterns of  $J$ -orthogonal matrices**

*Frank J. Hall*

*Zhongshan Li*

*Caroline T. Parnass*

*Miroslav Rozložník*

Preprint No. 46-2017

PRAHA 2017



# Sign patterns of $J$ -orthogonal matrices

Frank J. Hall, Zhongshan Li, Caroline T. Parnass  
Department of Mathematics and Statistics  
Georgia State University, Atlanta, GA 30303, USA  
fhall@gsu.edu, zli@gsu.edu, cparnass1@student.gsu.edu

Miroslav Rozložník \*  
Institute of Mathematics, Czech Academy of Sciences  
Žitná 25, 115 67 Praha 1, Czech Republic  
miro@math.cas.cz

July 26, 2017

## Abstract

This paper builds upon the results in the article “G-matrices,  $J$ -orthogonal matrices, and their sign patterns”, Czechoslovak Math. J. 66 (2016), 653–670, by Hall and Rozložník. A number of further general results on the sign patterns of the  $J$ -orthogonal matrices are proved. Properties of block diagonal matrices and their sign patterns are examined. It is shown that all  $4 \times 4$  full sign patterns allow  $J$ -orthogonality. Important tools in this analysis are Theorem 2.2 on the exchange operator and Theorem 3.2 on the characterization of  $J$ -orthogonal matrices in the paper “ $J$ -orthogonal matrices: properties and generation”, SIAM Review 45 (3) (2003), 504–519, by Higham. As a result, it follows that for  $n \leq 4$  all  $n \times n$  full sign patterns allow a  $J$ -orthogonal matrix as well as a G-matrix. In addition, the  $3 \times 3$  sign patterns of the  $J$ -orthogonal matrices which have zero entries are characterized.

*AMS Subj. Class.:* 15A80; 15A15; 15A23

*Keywords:* G-matrix;  $J$ -orthogonal matrix; Sign pattern matrix; Sign patterns that allow  $J$ -orthogonality.

## 1 Introduction

Following [6], we say that a real matrix  $A$  is a  $G$ -matrix if  $A$  is nonsingular and there exist nonsingular diagonal matrices  $D_1$  and  $D_2$  such that

$$A^{-T} = D_1 A D_2, \quad (1)$$

where  $A^{-T}$  denotes the transpose of the inverse of  $A$ . Denote by  $J$  a diagonal (signature) matrix, each of whose diagonal entries is  $+1$  or  $-1$ . As in [10], a nonsingular real matrix  $Q$  is called  $J$ -orthogonal if

$$Q^T J Q = J, \quad (2)$$

or equivalently, if

$$Q^{-T} = J Q J. \quad (3)$$

---

\*This research is supported by the Czech Science Foundation under the project GA17-12925S.

Of course, every orthogonal matrix  $Q$  is a  $J$ -orthogonal matrix, where  $J$  is the identity matrix. And clearly, from (3), every  $J$ -orthogonal matrix is a G-matrix. On the other hand, as shown in [9], a G-matrix can always be transformed to a  $J$ -orthogonal matrix.

**Definition 1.1.** We say that two real matrices  $A$  and  $B$  are *positive-diagonally equivalent* if there are diagonal matrices  $D_1$  and  $D_2$  with all diagonal entries positive such that  $B = D_1AD_2$ .

**Theorem 1.2.** [9, Theorem 2.6] *A matrix  $A$  is a G-matrix if and only if  $A$  is positive-diagonally equivalent to a column permutation of a  $J$ -orthogonal matrix.*

Some easily proved properties of  $J$ -orthogonal matrices are as follows.

**Theorem 1.3.** (i) *For a fixed signature matrix  $J$ , the set of all  $J$ -orthogonal matrices is a multiplicative group, which is also closed under the operations of transposition, negation, and multiplication on either side by any signature matrix of the same order.*

(ii) *The direct sum of square diagonal blocks  $A_{11}, \dots, A_{kk}$  is a  $J$ -orthogonal matrix if and only if each diagonal block  $A_{ii}$  is a  $J_i$ -orthogonal matrix, where  $J_i$  is the corresponding diagonal block of  $J$ .*

(iii) *The Kronecker product of  $J_i$ -orthogonal matrices is a  $J$ -orthogonal matrix with  $J$  equal to the Kronecker product of the  $J_i$ 's.*

(iv) *If  $Q$  is  $J$ -orthogonal and  $P$  is a permutation of the same order, then  $P^TQP$  is  $J_1$ -orthogonal with  $J_1 = P^TJP$ .*

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the sign of entries in the matrix. An  $m \times n$  matrix whose entries are from the set  $\{+, -, 0\}$  is called a *sign pattern matrix* (or a *sign pattern*, or a *pattern*). A sign pattern is said to be *full* if it does not have any 0 entry. For a real matrix  $B$ ,  $\text{sgn}(B)$  is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of  $B$  by  $+$  (respectively,  $-$ ,  $0$ ). For a sign pattern matrix  $A$ , the *sign pattern class* of  $A$  is defined by

$$Q(A) = \{B : \text{sgn}(B) = A\}.$$

A sign pattern matrix  $P$  is called a *permutation sign pattern* (*generalized permutation sign pattern*) if exactly one entry in each row and column is equal to  $+$  ( $+$  or  $-$ ) and all the other entries are 0. A *permutation similarity* of the  $n \times n$  sign pattern  $A$  has the form  $P^TAP$ , where  $P$  is an  $n \times n$  permutation matrix. A *signature pattern* is a diagonal sign pattern matrix each of whose diagonal entries is  $+$  or  $-$ . A sign pattern  $B$  is *signature equivalent* to the sign pattern  $A$  provided  $B = S_1AS_2$ , where  $S_1$  and  $S_2$  are signature patterns. A *signature similarity* of the  $n \times n$  sign pattern  $A$  has the form  $SAS$ , where  $S$  is an  $n \times n$  signature pattern.

Suppose  $P$  is a property referring to a real matrix. A sign pattern  $A$  is said to *require*  $P$  if every matrix in  $Q(A)$  has property  $P$ ;  $A$  is said to *allow*  $P$  if some real matrix in  $Q(A)$  has property  $P$ .

A square sign pattern  $A$  is *sign singular* if every matrix  $B \in Q(A)$  is singular. It is well-known that an  $n \times n$  sign pattern matrix  $A$  is sign singular if and only if  $A$  has no “composite cycle” of length  $n$ . The reader is referred to [3] or [8] for more information on sign pattern matrices.

Of course, when  $J = I_n$ , a  $J$ -orthogonal matrix is an orthogonal matrix. Let  $\mathcal{PO}_n$  denote the set of  $n \times n$  sign patterns that allow an orthogonal matrix. A more general question than characterizing  $\mathcal{PO}_n$  is the following: what are the sign patterns which allow a  $J$ -orthogonal matrix? Specifically, it is of interest to find sign patterns which allow a  $J$ -orthogonal matrix, but do not allow an orthogonal matrix. We shall let  $\mathcal{J}_n$  denote the set of all sign patterns of the  $n \times n$   $J$ -orthogonal matrices (for various possible  $J$ ), that is,

the class of  $n \times n$  sign patterns that allow a  $J$ -orthogonal matrix. Clearly, if  $A \in \mathcal{J}_n$ , then  $A$  cannot be sign singular. As in [6], we let  $\mathcal{G}_n$  denote the class of all  $n \times n$  sign pattern matrices  $A$  that allow a  $G$ -matrix.

As already mentioned in [9], from Theorem 1.2 we immediately have the following connection with  $G$ -matrices.

**Theorem 1.4.** [9, Theorem 4.3] *The sign patterns in  $\mathcal{G}_n$  are exactly the column permutations of the sign patterns in  $\mathcal{J}_n$ .*

In particular, if the sign pattern  $A$  allows a  $J$ -orthogonal matrix, then  $A$  allows a  $G$ -matrix.

Now, the all  $+$  (also, all  $-$ )  $n \times n$  sign pattern is the sign pattern of a nonsingular Cauchy matrix, which is a  $G$ -matrix, see [5]. Thus:

**Theorem 1.5.** [9, Theorem 4.4] *The all  $+$  (also, all  $-$ )  $n \times n$  sign pattern allows a  $J$ -orthogonal matrix (but of course not an orthogonal matrix, unless  $n = 1$ ).*

The following straightforward result was also mentioned in [9].

**Lemma 1.6.** [9, Lemma 6.3] *The set  $\mathcal{J}_n$  is closed under the following operations:*

- i) negation;
- ii) transposition;
- iii) permutation similarity;
- iv) signature equivalence.

The use of these operations yields “equivalent” sign patterns, and this will be used subsequently.

Theorem 1.4 may be paraphrased as follows:  $\mathcal{G}_n = \mathcal{J}_n \mathcal{P}_n$ , where  $\mathcal{P}_n$  is the set of all  $n \times n$  permutation sign patterns. Observe that  $\mathcal{G}_n^T = \mathcal{G}_n$ ,  $\mathcal{J}_n^T = \mathcal{J}_n$ , and  $\mathcal{P}_n^T = \mathcal{P}_n$ . By taking the transpose of each element in the sets in the equation  $\mathcal{G}_n = \mathcal{J}_n \mathcal{P}_n$ , we get  $\mathcal{G}_n = \mathcal{P}_n \mathcal{J}_n$ , which is the content of the next theorem.

**Theorem 1.7.** *The set of all  $n \times n$  sign patterns that allow a  $G$ -matrix is the same as the set of all row permutations of the  $n \times n$  sign patterns allowing  $J$ -orthogonality.*

In fact, we can generalize this result as follows:

**Theorem 1.8.** *The set of all  $n \times n$  sign patterns that allow a  $G$ -matrix is the same as the set of all permutation equivalences of the  $n \times n$  sign patterns allowing  $J$ -orthogonality.*

*Proof.* From Theorem 1.4, we have  $\mathcal{G}_n = \mathcal{J}_n \mathcal{P}_n$ . Thus to complete the proof, it suffices to show that  $\mathcal{J}_n \mathcal{P}_n = \mathcal{P}_n \mathcal{J}_n \mathcal{P}_n$ . Since the identity permutation sign pattern is in  $\mathcal{P}_n$ , obviously  $\mathcal{J}_n \mathcal{P}_n \subseteq \mathcal{P}_n \mathcal{J}_n \mathcal{P}_n$ . To show the reverse inclusion, let  $P_1 Q_1 P_2 \in \mathcal{P}_n \mathcal{J}_n \mathcal{P}_n$ , where  $P_1, P_2$  are permutation sign patterns and  $Q_1$  allows  $J$ -orthogonality. Then  $P_1 Q_1 P_1^T$  allows  $J$ -orthogonality and hence  $P_1 Q_1 P_2 = (P_1 Q_1 P_1^T)(P_1 P_2) \in \mathcal{J}_n \mathcal{P}_n$ .  $\square$

Let  $A$  be an  $n \times n$  sign pattern matrix. From [9], the very important fundamental *sign potentially  $J$ -orthogonal* (SPJO) conditions are that there exists a  $(+, -)$  signature pattern  $J$  such that

$$A^T J A \xleftrightarrow{c} J \tag{4}$$

and

$$A J A^T \xleftrightarrow{c} J, \tag{5}$$

where  $\xleftrightarrow{c}$  denotes (generalized) sign pattern compatibility.

These are necessary conditions for  $A \in \mathcal{J}_n$ . If these conditions do not hold, then  $A \notin \mathcal{J}_n$ . When  $J = I$ , we get the normal SPO conditions for orthogonal matrices, see for example [4]. The SPJO conditions are not sufficient for an  $n \times n$  sign pattern matrix to allow  $J$ -orthogonality, as illustrated in [9].

Observe that  $A^T J A$  and  $A J A^T$  are symmetric generalized sign pattern matrices. So, to verify the SPJO conditions we need only to find a  $J$  which fulfills the upper-triangular part of the compatible conditions. Let  $J = \text{diag}(\omega_1, \dots, \omega_n)$ . Note that (4) and (5) may be restated as

$$\sum_{k=1}^n \omega_k a_{ki} a_{kj} \xleftrightarrow{c} \delta_{ij} \omega_j \quad \text{for all } i, j \quad (6)$$

and

$$\sum_{k=1}^n \omega_k a_{ik} a_{jk} \xleftrightarrow{c} \delta_{ij} \omega_j \quad \text{for all } i, j. \quad (7)$$

(With an  $n \times n$   $(+, -)$  sign pattern  $A$ , for  $i = j$ , (6) and (7) automatically hold for any  $J$ .)

In [9], the following important result was proved.

**Theorem 1.9.** [9, Theorem 6.11] *For all  $n \geq 1$ , each  $n \times n$  full sign pattern  $A$  satisfies the SPJO conditions.*

If we allow zero entries, then Theorem 1.9 may fail. For example, an  $n \times n$  sign pattern  $A$  with a zero column does not satisfy  $A^T J A \xleftrightarrow{c} J$  and an  $n \times n$  sign pattern  $A$  with a zero row does not satisfy  $A J A^T \xleftrightarrow{c} J$ , for any signature pattern  $J$ .

A number of other general results on the sign patterns are also proved in Section 2 and used in subsequent sections. The  $3 \times 3$  sign patterns of the  $J$ -orthogonal matrices which have zero entries are characterized in Section 3. In Section 4 it is shown that all  $4 \times 4$  full sign patterns allow  $J$ -orthogonality; important tools in this analysis are Theorem 2.2 on the exchange operator and Theorem 3.2 on the characterization of  $J$ -orthogonal matrices in the paper [10] by Nick Higham. As a result, it then follows that for  $n \leq 4$  all  $n \times n$  full sign patterns allow a  $J$ -orthogonal matrix as well as a G-matrix. It is also shown that if an  $n \times n$  full sign pattern  $A$  allows a  $J$ -orthogonal matrix, then  $A$  allows a rational  $J$ -orthogonal matrix with the same signature matrix.

## 2 Block diagonal matrices and their sign patterns

The following structural result of G-matrices was established in [9]. For the notion of fully indecomposable matrices, we refer the reader to [2].

**Theorem 2.1.** [9, Theorem 2.1] *Let  $A$  be a nonsingular real matrix in block upper triangular form*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

*where all the diagonal blocks are square. Then  $A$  is a G-matrix if and only if each  $A_{ii}$  ( $i = 1, \dots, m$ ) is a G-matrix and all the strictly upper triangular blocks  $A_{ij}$  are equal to 0. Furthermore, if  $A$  is a G-matrix that has a row (or a column) with no 0 entry, then  $A$  is fully indecomposable.*

**Example 2.2.** Let

$$A = \begin{bmatrix} 0 & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & 0 \end{bmatrix}.$$

Notice that  $A$  is permutationally equivalent to the sign pattern

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix},$$

which by Theorem 2.1 does not allow a  $G$ -matrix. Hence,  $A$  does not allow a  $J$ -orthogonal matrix. The same holds for the similar  $n \times n$  sign pattern.

More generally, by Theorem 2.1 and Theorem 1.8, we have the following.

**Theorem 2.3.** *Let  $A$  be an  $n \times n$  sign pattern matrix, and  $P$  and  $Q$  be permutation patterns such that  $PAQ$  has the block upper triangular form*

$$PAQ = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. If  $A \in \mathcal{J}_n$ , then  $PAQ \in \mathcal{G}_n$ , each  $A_{ii}$  ( $i = 1, \dots, m$ ) allows a  $G$ -matrix, and all the strictly upper triangular blocks  $A_{ij}$  are equal to 0. If  $PAQ \notin \mathcal{G}_n$ , then  $A \notin \mathcal{J}_n$ .

We note that when the sign pattern  $A$  in Theorem 2.3 is not sign singular, such a  $PAQ$  block upper triangular form where specifically the square diagonal blocks are fully indecomposable, is always possible [2, Theorem 4.2.6].

Of specific interest is the following.

**Theorem 2.4.** *If  $A$  is an  $n \times n$  sign pattern matrix with exactly  $n + 1$  nonzero entries, then  $A \notin \mathcal{J}_n$ .*

*Proof.* If  $A$  has no composite cycle of length  $n$ , then of course,  $A \notin \mathcal{J}_n$ . If  $A$  has a composite cycle of length  $n$ , then for some permutation sign pattern  $P$ ,  $AP$  has no zero diagonal entries and exactly one nonzero off-diagonal entry. By Theorem 2.1,  $AP \notin \mathcal{G}_n$ . Hence, by Theorem 2.3,  $A \notin \mathcal{J}_n$ .  $\square$

We can also apply Theorem 1.3 to sign patterns.

**Theorem 2.5.** *Let the  $n \times n$  sign pattern matrix  $A$  be the direct sum*

$$A = \begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. Then  $A$  allows a  $J$ -orthogonal matrix if and only if each  $A_{ii}$  ( $i = 1, \dots, m$ ) allows a  $J$ -orthogonal matrix.

**Remark 2.6.** The Kronecker product of sign patterns which allow a  $J$ -orthogonal matrix also allows a  $J$ -orthogonal matrix. For a fixed signature matrix  $J$ , a product of  $J$ -orthogonal matrices can produce a different sign pattern allowing a  $J$ -orthogonal matrix for the same  $J$ .

Observe that any generalized permutation pattern allows  $J$ -orthogonality with  $J = I$ , since if  $B$  is a generalized permutation matrix, then  $B^T I B = B^T B = I$ . Hence, we have another result to be subsequently used in this paper.

**Theorem 2.7.** *If  $A$  is an  $n \times n$  generalized permutation sign pattern, then  $A \in \mathcal{J}_n$ .*

The following can be of general use.

**Theorem 2.8.** *Suppose that  $B$  is an  $n \times n$  real nonsingular matrix and  $B$  is both  $J_1$ -orthogonal and  $J_2$ -orthogonal, where  $J_1 = \text{diag}(I_{p_1}, -I_{q_1})$ ,  $J_2 = \text{diag}(I_{p_2}, -I_{q_2})$ , and  $J_1 \neq J_2$ . Then we have that*

$$B = \begin{bmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix},$$

where the partitioning of  $B$  results from the partitioning of the matrix  $J_2 J_1$ .

*Proof.* Since the matrix  $B$  is  $J_1$ -orthogonal, from  $B^T J_1 B = J_1$  we have that  $J_1 B = B^{-T} J_1$ . Similarly, from  $B^T J_2 B = J_2$  we have  $B^{-T} J_2 = J_2 B$ . The previous two identities give  $(J_2 J_1) B = B (J_2 J_1)$ , where  $J_2 J_1 = \text{diag}(I_{\min(p_1, p_2)}, -I_{\max(p_1, p_2) - \min(p_1, p_2)}, I_{\min(q_1, q_2)})$ . By partitioning  $B$  in the same way as  $J_2 J_1$ , we have

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}.$$

Thus, it follows from  $(J_2 J_1) B = B (J_2 J_1)$  that  $B_{12} = B_{21} = B_{23} = B_{32} = 0$ . □

**Remark 2.9.** It is clear that there exists a permutation matrix  $P$  such that  $\tilde{J} = P (J_2 J_1) P^T = \text{diag}(I_p, -I_q)$ , where  $p = \min(p_1, p_2) + \min(q_1, q_2)$  and  $q = \max(p_1, p_2) - \min(p_1, p_2)$ . Using this permutation matrix  $P$  the matrix  $B$  can be transformed by permutation similarity into the block diagonal matrix

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{13} & 0 \\ B_{31} & B_{33} & 0 \\ 0 & 0 & B_{22} \end{bmatrix}$$

that is  $\tilde{J}$ -orthogonal satisfying  $\tilde{B}^T \tilde{J} \tilde{B} = \tilde{J}$ .

**Corollary 2.10.** *If  $A$  is an  $n \times n$  full sign pattern matrix, then there does not exist  $B \in Q(A)$  such that is both  $J_1$ -orthogonal and  $J_2$ -orthogonal, where  $J_1 \neq \pm J_2$ .*

If  $A$  is say an  $n \times n$  signature pattern, then clearly  $A$  is a sign pattern in  $\mathcal{J}_n$  which allows all the  $2^n$  signature matrices  $J$ . A given full sign pattern matrix  $A$  also may allow  $J$ -orthogonal matrices corresponding to more than one distinct (none are negatives of another) signature matrices  $J$ .



**Example 2.11.** Given the sign pattern

$$A = \begin{bmatrix} + & - & - \\ + & + & - \\ + & - & - \end{bmatrix},$$

there are two possible choices for  $J$  that satisfy the SPJO conditions, namely  $J_1 = \text{diag}(1, 1, -1)$  and  $J_2 = \text{diag}(1, -1, -1)$ .

For  $d > \frac{1}{\sqrt{2}}$ , the real matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ d & \frac{1}{2d} & -\frac{2d^2-1}{2d} \\ d & -\frac{1}{2d} & -\frac{2d^2+1}{2d} \end{bmatrix} \in Q(A)$$

is  $J_1$ -orthogonal. For example, if  $d = 1$ , then

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} \in Q(A)$$

satisfies  $B^T J_1 B = J_1$ .

On the other hand, for  $j > \frac{1}{\sqrt{2}}$ , the real matrix

$$\begin{bmatrix} \frac{2j^2+1}{2j} & -\frac{2j^2-1}{2j} & -1 \\ \frac{1}{2j} & \frac{1}{2j} & -1 \\ j & -j & -1 \end{bmatrix} \in Q(A)$$

is  $J_2$ -orthogonal. For example, if  $j = 1$ , then

$$B = \begin{bmatrix} 3/2 & -1/2 & -1 \\ 1/2 & 1/2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \in Q(A)$$

satisfies  $B^T J_2 B = J_2$ .

Notice that in the above example the signature matrices are equivalent (although not all the resulting  $J$ -orthogonal matrices are equivalent). In the following we exhibit non-equivalent signature matrices.

**Example 2.12.** Consider the  $4 \times 4$  all  $+$  sign pattern. This pattern is  $J$ -orthogonal with the two non-equivalent  $J$  matrices  $J_1 = \text{diag}(1, 1, 1, -1)$  and  $J_2 = \text{diag}(1, 1, -1, -1)$ .

$$A_1 = \frac{1}{3} \begin{pmatrix} 4 & 1 & 1 & 3 \\ 1 & 4 & 1 & 3 \\ 1 & 1 & 4 & 3 \\ 3 & 3 & 3 & 6 \end{pmatrix} \text{ is } J_1\text{-orthogonal,}$$

$$A_2 = \frac{1}{3} \begin{pmatrix} 4 & 1 & 2 & 2 \\ 1 & 4 & 2 & 2 \\ 2 & 2 & 4 & 1 \\ 2 & 2 & 1 & 4 \end{pmatrix} \text{ is } J_2\text{-orthogonal.}$$

An interesting question is the following: Is it true that whenever a square full sign pattern  $A$  and a signature pattern  $J$  satisfy the SPJO conditions, then  $A$  allows  $J$ -orthogonality (for this particular  $J$ )? That the answer is no is seen in [4, Example 3.8] where the  $6 \times 6$  pattern is SPO but is not in  $\mathcal{PO}_6$ . However, specifying the  $*$  entries as  $+$ , this pattern is in  $\mathcal{J}_6$ , as shown as follows.

**Example 2.13.** Let

$$A = \begin{pmatrix} + & + & + & + & + & + \\ + & + & + & + & + & - \\ + & + & + & + & - & + \\ + & + & + & + & - & - \\ + & + & - & - & + & + \\ + & - & + & - & + & + \end{pmatrix}$$

and let  $J = \text{diag}(1, -1, -1, -1, -1, -1)$ . Then we produced the following decimal approximation of a matrix  $B \in Q(A)$  such that  $B^T J B = J$  to within four decimal places:

$$B = \begin{pmatrix} 1.8457 & 0.1748 & 1.2301 & 0.5382 & 0.0023 & 0.7572 \\ 0.4467 & 0.4877 & 0.5807 & 0.5934 & 0.3467 & -0.3900 \\ 1.2188 & 0.1332 & 0.7813 & 0.7961 & -0.0450 & 1.1053 \\ 0.1207 & 0.4068 & 0.1700 & 0.0456 & -0.8983 & -0.1055 \\ 0.0121 & 0.7684 & -0.0923 & -0.4680 & 0.2659 & 0.3339 \\ 0.8408 & -0.1379 & 1.2361 & -0.2876 & 0.0113 & 0.2776 \end{pmatrix}$$

### 3 Characterization of sign patterns in $\mathcal{J}_3$ with 0 entries

We want to identify all those  $3 \times 3$  sign patterns with 0 entries which allow  $J$ -orthogonality. (The  $2 \times 2$  case should be clear.) To organize our argument, we consider sign patterns with varying numbers of zero entries.

Note that all  $3 \times 3$  full sign patterns allow  $J$ -orthogonality [9].

**Sign patterns with 9, 8, or 7 zero entries.** Any  $3 \times 3$  sign pattern with only 2, 1 or 0 nonzero entries cannot contain a composite cycle of length 3; thus any such pattern is sign singular and hence cannot allow  $J$ -orthogonality, since if  $B$  is  $J$ -orthogonal, then  $B$  is nonsingular.

**Sign patterns with 6 zero entries.** Note that a  $3 \times 3$  sign pattern with exactly 3 nonzero entries must not be sign singular in order to allow  $J$ -orthogonality, so we only consider such sign patterns which have a composite cycle of length 3, namely, the  $3 \times 3$  generalized permutation patterns. By Theorem 2.7, these patterns allow  $J$ -orthogonality. Thus, the sign patterns in  $\mathcal{J}_3$  with exactly 6 zero entries are precisely the  $3 \times 3$  generalized permutation patterns.

**Sign patterns with 5 zero entries.** That no  $3 \times 3$  sign pattern with exactly five zero entries allows a  $J$ -orthogonal matrix simply follows from Theorem 2.4.

**Sign patterns with four zero entries.** In order to determine the sign patterns with four zero entries that allow  $J$ -orthogonality, we can systematically consider the number of zero entries on the main diagonal.

Let  $\star$  denote a  $+$  or  $-$  entry. Note that if we require all nonzero entries on the main diagonal, then up to equivalence, there are three patterns to consider. Two of these patterns

$$\begin{bmatrix} \star & \star & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}, \begin{bmatrix} \star & \star & 0 \\ 0 & \star & \star \\ 0 & 0 & \star \end{bmatrix}$$

do not satisfy the SPJO conditions for any  $J$ , while it can be seen that

$$\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

does allow  $J$ -orthogonality.

Now suppose there is one zero entry on the main diagonal. Then we may permute it to the  $(1, 1)$  position. By systematic inspection it can be seen that no pattern of this form allows  $J$ -orthogonality.

Now if there are two zero entries on the main diagonal, then up to equivalence, there is one pattern of this form that allows  $J$ -orthogonality:

$$\begin{bmatrix} \star & 0 & \star \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix}.$$

Finally, with three zero entries on the main diagonal, there is no pattern that allows  $J$ -orthogonality.

**Sign patterns with three or two zero entries.** We can conduct a similar investigation of the sign patterns by systematically inspecting the possibilities. Once again the SPJO conditions come into play. In this way, we find that there is no  $3 \times 3$  sign pattern with exactly three or two zero entries that allows  $J$ -orthogonality.

**Sign patterns with one zero entry.** In this case, we first eliminate from consideration all those sign patterns which are sign potentially orthogonal, since for  $n = 3$ , every SPO pattern allows orthogonality [1].

So suppose  $A$  is a  $3 \times 3$  non-SPO pattern with exactly one zero entry. If the zero is on the main diagonal, we permute it to the  $(3, 3)$  position. Suppose first that the inner product of the first two columns is not 0 or  $\neq$ . Since they are nonzero, these columns are either the same or negative of each other. So we can multiply on the left and right by suitable signature patterns so that all the entries in the first two columns are  $+$ . We can also multiply the third column by  $-$  if necessary to obtain the form

$$A = \begin{bmatrix} + & + & + \\ + & + & \star \\ + & + & 0 \end{bmatrix},$$

leaving two possible patterns up to equivalence. Note that if  $\star = -$ , then  $A$  does not satisfy the SPJO conditions for any  $J$ . On the other hand, if  $\star = +$ , then we can obtain a  $J$ -orthogonal matrix of this form; for example

$$\begin{bmatrix} 2 & 1 & \sqrt{2} \\ \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \sqrt{3} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \end{bmatrix}$$

allows  $J$ -orthogonality with  $J = \text{diag}(1, -1, -1)$ .

Similarly, if the first and third columns are not SPO, then by signature equivalence we can obtain the form

$$\begin{bmatrix} + & + & + \\ + & \star & + \\ + & \star & 0 \end{bmatrix},$$

while if the second and third columns are not SPO, we obtain

$$\begin{bmatrix} + & + & + \\ \star & + & + \\ \star & + & 0 \end{bmatrix}.$$

Upon inspection we find that no matrix of the above forms (except for all the  $\star$  equal to  $+$ , as described above) allows  $J$ -orthogonality.

Now suppose that the zero entry is off the main diagonal, and without loss of generality, permute the zero to the  $(2, 3)$  position. Then similar to the above discussion, we obtain three possible forms:

$$\begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & \star \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & \star & 0 \\ + & \star & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ \star & + & 0 \\ \star & + & + \end{bmatrix}.$$

Of these possible patterns, four allow  $J$ -orthogonality. They are listed below along with examples of  $J$ -orthogonal matrices with those sign patterns:

$$A = \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \sqrt{3} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ 2 & 1 & 2 \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & - \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{\sqrt{3}}{2} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 2 & 2 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \sqrt{5} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} + & + & + \\ - & + & 0 \\ + & + & + \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \in Q(A) \text{ is } J\text{-orthogonal with } J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

These are all of the non-SPO  $3 \times 3$  sign patterns, up to equivalence, with exactly one zero entry which allow  $J$ -orthogonality.

We have thus proved the following result.

**Theorem 3.1.** *Up to equivalence, the sign patterns in  $\mathcal{J}_3$  with at least one zero entry are*

$$\begin{bmatrix} + & + & 0 \\ + & \star & 0 \\ 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ + & 0 & \star \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & + \end{bmatrix}, \\ \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & + & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & + & 0 \\ + & + & + \end{bmatrix},$$

as well as the  $3 \times 3$  generalized permutation sign patterns and the  $3 \times 3$  SPO sign patterns with one zero entry, where  $\star$  denotes a  $+$  or  $-$  entry.

## 4 The $4 \times 4$ full sign pattern case

An initial investigation of the question of whether the full  $n \times n$  sign patterns always allow a  $J$ -orthogonal matrix was begun in [9], and for  $n \leq 3$  it was shown to be true.

**Remark 4.1.** It was observed in [4] that for  $n \leq 4$ , the SPO patterns are the same as the sign patterns in  $\mathcal{PO}_n$ , and that this is also the case for full sign patterns of order 5, see [1] and [13]. So, regarding the above question with  $n \leq 5$ , we need only to consider non-SPO patterns.

We establish that every  $4 \times 4$  full sign pattern matrix allows  $J$ -orthogonality. As observed above, for  $n \leq 4$ , the SPO patterns are the same as the patterns in  $\mathcal{PO}_n$ . Therefore, since every orthogonal matrix is also  $J$ -orthogonal, we need only consider those patterns which are not sign potentially orthogonal. Without loss of generality, we can suppose each pattern is not sign potentially column orthogonal, since  $\mathcal{J}_4$  is closed under transposition.

Note that a given full sign pattern can be multiplied on the left and right by signature patterns so that it has the form

$$\begin{bmatrix} + & + & + & + \\ + & & & \\ + & & & \\ + & & & \end{bmatrix}.$$

Moreover, since we are considering sign patterns which are not sign potentially column orthogonal and which have no zero entries, this means that two columns must be the same. Thus we can use permutation similarity and signature equivalence to reduce to the case

$$\begin{bmatrix} + & + & + & + \\ + & + & & \\ + & + & & \\ + & + & & \end{bmatrix}, \tag{8}$$

which leaves 64 possible sign patterns.

We can reduce the number of cases by noting that the cases

$$\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & & \\ + & + & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} + & + & + & + \\ + & + & - & + \\ + & + & & \\ + & + & & \end{bmatrix}$$

are equivalent, since we can switch the third and fourth columns, and simultaneously switching the third and fourth rows.

Using (8) as our template, there are now three possibilities to consider. **We first consider the case that the (2, 3) and (2, 4) entries are both +, for which there are 16 subcases.** Four of these are symmetric staircase patterns and therefore in  $\mathcal{J}_4$ , [9, Theorem 6.2]. A further 2 patterns are permutationally similar to symmetric staircase patterns, so these too are in  $\mathcal{J}_4$ .

Now consider the non-symmetric staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & - & - \end{bmatrix}.$$

If we construct  $J_1$  and  $J_2$  as in [9, Remark 5.7] we see that  $J_2 = PJ_1P^T$  where  $P = [e_1, e_2, e_4, e_3]$ ; and now similar to [9, Example 5.9],  $AP = A$ . This leads us to conclude that  $A \in \mathcal{J}_4$ . The transpose of  $A$  is also in  $\mathcal{J}_4$ . Additionally,

$$B = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & - \\ + & + & + & + \end{bmatrix}$$

is permutationally similar to  $A$ , so  $B, B^T \in \mathcal{J}_4$ .

Now let

$$B = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}$$

which is in  $\mathcal{J}_4$  as in [9, Example 5.9]. Let  $S = \text{diag}(+, +, +, -)$  and  $P = [e_3, e_2, e_1, e_4]$ . Then

$$C = P^T B S P = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4; \quad C^T \in \mathcal{J}_4.$$

This leaves 3 patterns up to equivalence which we still must show are in  $\mathcal{J}_4$ :

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \quad \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & - & - \end{bmatrix}.$$

(The only other case not mentioned is the transpose of the third pattern above.)

**Next, we again use (8) as our template and now consider the case that the (2, 3) and (3, 4) entries are both  $-$ .** In this case, there is one staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix}$$

for which, following the example in [9, Remark 5.7], we see that  $AP = A$ . So  $A \in \mathcal{J}_4$ . Note from  $A$  we can also multiply the third and fourth columns by  $-$  and permute the first and second lines to obtain

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4. \tag{9}$$

Two more matrices in this case can be obtained as follows:

We begin with the staircase pattern

$$A = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & - & - \end{bmatrix}.$$

If we compute  $P$  as in [9, Remark 5.7], then we find that  $AP \neq A$ . But in fact,

$$AP = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & + & + \\ + & - & - & + \end{bmatrix}.$$

So this pattern is in  $\mathcal{J}_4$ . Now we can obtain the pattern

$$B = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}$$

from  $AP$  by permutation similarity, so  $B \in \mathcal{J}_4$ . If  $P = [e_1, e_2, e_4, e_3]$ , then  $P^T B P$  is another  $-,-$  pattern in  $\mathcal{J}_4$ .

Similarly, if we begin with the staircase pattern  $A = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix}$ , we find that

$$AP = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & + \\ + & - & - & + \end{bmatrix} \in \mathcal{J}_4 \quad (10)$$

and by permutation similarity  $B = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & + & - \end{bmatrix} \in \mathcal{J}_4$ . If  $P = [e_1, e_2, e_4, e_3]$ , then  $P^T B P$  is another  $-,-$  pattern in  $\mathcal{J}_4$ .

We can obtain another pattern by letting  $A$  be the pattern in equation (10), and letting  $S = \text{diag}(+, -, -, +)$  and  $P = [e_2, e_4, e_3, e_1]$ . Then  $B = P^T A S P = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & - & + \end{bmatrix} \in \mathcal{J}_4$ , and permuting the third and fourth lines of  $B$  yields another  $-,-$  pattern in  $\mathcal{J}_4$ .

There are 5 more  $-,-$  patterns up to equivalence still to be determined. They are, for our reference

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & + \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & + & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & + & - \\ + & + & - & + \end{bmatrix}.$$

**The final case to consider is that in our template (8), the (2, 3) entry is + and the (2, 4) entry is -.**

The staircase pattern  $A = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{bmatrix}$  is contained in [9, Example 5.9]. If we take  $S = \text{diag}(+, +, +, -)$

and  $P = [e_2, e_1, e_3, e_4]$ , then  $P^T A S P = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \in \mathcal{J}_4$ .

We can obtain two more patterns from (9) above by taking the transpose and performing permutation similarity to obtain the patterns

$$\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & + & - \end{bmatrix}.$$



The other patterns in this case are equivalent either to previous unresolved patterns or one of the 5 unresolved patterns below:

$$\begin{aligned} & \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & - & - \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix} \\ & \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & - & + \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & - & - \end{bmatrix}. \end{aligned}$$

In summary, to this point, there remain **11 unresolved patterns, up to equivalence**:

$$\begin{aligned} A_1 &= \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, A_2 = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, A_3 = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & - & + \end{bmatrix}, A_4 = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \\ A_5 &= \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & - \\ + & + & - & + \end{bmatrix}, A_6 = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & + \\ + & + & - & - \end{bmatrix}, A_7 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & - \\ + & + & + & + \end{bmatrix}, A_8 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \\ A_9 &= \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & - \\ + & + & - & - \end{bmatrix}, A_{10} = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & + \\ + & + & - & - \end{bmatrix}, A_{11} = \begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & + \\ + & + & - & - \end{bmatrix}. \end{aligned}$$

To settle most of these remaining sign patterns, we use the following result contained in [10]. As stated in [10], this decomposition was first derived in [7]; it is also mentioned in [10] that in a preliminary version of [12] (which was published later) the authors treat this decomposition in more depth.

**Theorem 4.2.** [10, Theorem 3.2] *We define*

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = n.$$

Assume also that  $p \leq q$ . Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

be  $J$ -orthogonal with  $B_{11} \in \mathbb{R}^{p,p}$ ,  $B_{12} \in \mathbb{R}^{p,q}$ ,  $B_{21} \in \mathbb{R}^{q,p}$  and  $B_{22} \in \mathbb{R}^{q,q}$ . Then there are orthogonal matrices  $U_1, V_1 \in \mathbb{R}^{p,p}$  and  $U_2, V_2 \in \mathbb{R}^{q,q}$  such that

$$\begin{bmatrix} U_1^T & 0 \\ 0 & U_2^T \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} C & -S & 0 \\ -S & C & 0 \\ 0 & 0 & I_{q-p} \end{bmatrix}, \quad (11)$$

where  $C = \text{diag}(c_i)$ ,  $S = \text{diag}(s_i)$  and  $C^2 - S^2 = I_p$  ( $c_i > s_i \geq 0$ ). Any matrix  $B$  satisfying (11) is  $J$ -orthogonal.

**Remark 4.3.** In the case  $n = 4$  and  $J = \text{diag}(1, 1, -1, -1)$ , every  $J$ -orthogonal matrix  $B$  has a factorization of the form

$$\begin{aligned} B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} C & -S \\ -S & C \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & V_2^T \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & -U_2 \end{bmatrix} \begin{bmatrix} C & S \\ S & C \end{bmatrix} \begin{bmatrix} V_1^T & 0 \\ 0 & -V_2^T \end{bmatrix}. \end{aligned}$$

For  $J = \text{diag}(1, 1, -1, -1)$ , with suitable choices of  $2 \times 2$  orthogonal matrices  $U_1, U_2$  and  $V_1, V_2$ , we can generate  $4 \times 4$   $J$ -orthogonal matrices with some prescribed sign patterns. Note that some sign patterns are quite difficult to achieve by a product of two  $2 \times 2$  orthogonal matrices and a diagonal matrix. For a fixed pair  $V_1, V_2$  the two block rows of the matrix  $B$  can be interpreted as two orthogonal transformations of four vectors in the plane. The sign pattern will allow a  $J$ -orthogonal matrix only if there exists an orthogonal transformation mapping the four vectors with the sign pattern of the first block row to the four vectors with the sign pattern of the second block row. This is clearly not always possible.

**Remark 4.4.** In the case  $n = 4$  and  $J = \text{diag}(1, -1, -1, -1)$ , every  $J$ -orthogonal matrix  $B$  has a factorization of the form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} c_1 u_1 v_1 & -s_1 u_1 (V_2 e_1)^T \\ -s_1 v_1 U_2 e_1 & U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T \end{bmatrix}, \quad (12)$$

where  $u_1, v_1 \in \mathbb{R}$ ,  $U_2, V_2 \in \mathbb{R}^{3,3}$  are orthogonal and  $e_1 = [1 \ 0 \ 0]^T \in \mathbb{R}^3$ .

It was noted that a given  $4 \times 4$  full sign pattern can be multiplied on the left and right by signature patterns so that it has the form

$$\begin{bmatrix} + & + & + & + \\ + & & & \\ + & & & \\ + & & & \end{bmatrix}. \quad (13)$$

For  $J = \text{diag}(1, -1, -1, -1)$  this sign pattern essentially leads to the condition  $u_1 v_1 = 1$  due to  $c_1 > 1$ . Taking  $u_1 = -1$  and  $v_1 = -1$  we get to the conditions that both  $U_2 e_1$  and  $V_2 e_1$  should have the sign pattern equal to  $(+ \ + \ +)^T$ . So, given the orthogonal matrices  $U_2, V_2 \in \mathbb{R}^{3,3}$  such that  $\text{sgn}(U_2 e_1) = \text{sgn}(V_2 e_1) = (+ \ + \ +)^T$ , then there exists a  $J$ -orthogonal matrix of the form (12) with the sign pattern (13). The sign pattern of the lower right diagonal block is given by the sign pattern of the matrix

$$U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T = U_2 V_2^T + (c_1 - 1) U_2 e_1 e_1^T V_2^T.$$

Note that the sign pattern of  $U_2 e_1 e_1^T V_2^T$  is the  $3 \times 3$  matrix of all  $+$ . In addition, for sufficiently small  $c_1 - 1$ , the sign pattern of  $U_2 \begin{bmatrix} c_1 & 0 \\ 0 & I_2 \end{bmatrix} V_2^T$  becomes equal to the sign pattern of the  $3 \times 3$  orthogonal matrix  $U_2 V_2^T$ . This is the way we can generate  $3 \times 3$   $J$ -orthogonal matrices with some prescribed sign patterns of the form (13).

We can handle  $A_3$  by the approach mentioned in Remark 4.4. Let us take the orthogonal  $3 \times 3$  matrices  $U_2$  and  $V_2$  as

$$U_2 = \frac{1}{3} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} V_2; \quad V_2 = \frac{1}{7} \begin{bmatrix} 6 & -3 & 2 \\ 3 & 2 & -6 \\ 2 & 6 & 3 \end{bmatrix}.$$

Then the matrix  $U_2 V_2^T$  has exactly the same sign pattern as the lower right block of the pattern  $A_3$  and it can be also verified that the first column of the matrix  $U_2$  has all positive entries. Then, as can be checked, the matrix

$$B = \begin{bmatrix} 2 & \frac{6}{7}\sqrt{3} & \frac{3}{7}\sqrt{3} & \frac{2}{7}\sqrt{3} \\ \frac{11}{21}\sqrt{3} & \frac{164}{147} & \frac{82}{147} & -\frac{76}{147} \\ \frac{16}{21}\sqrt{3} & \frac{145}{147} & \frac{146}{147} & \frac{130}{147} \\ \frac{8}{21}\sqrt{3} & \frac{146}{147} & -\frac{74}{147} & \frac{65}{147} \end{bmatrix}$$

is  $J$ -orthogonal with respect to  $J = \text{diag}(1, -1, -1, -1)$ . Eight other patterns from the list of unresolved patterns can also be handled by this approach. The key is that the lower right block allows a  $3 \times 3$  orthogonal matrix.

$A_7$  is equivalent to the sign pattern

$$\begin{bmatrix} - & + & - & + \\ + & + & + & + \\ + & + & + & + \\ + & - & + & - \end{bmatrix}.$$

This latter sign pattern can be handled by the approach mentioned in Remark 4.3. We choose

$$U_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad -U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_1^T = -V_2^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}.$$

Then, as can be checked, the matrix

$$B = \begin{bmatrix} -\sqrt{2} & \sqrt{2} & -\frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 2 & 2 \\ 2 & 2 & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

is  $J$ -orthogonal with respect to  $J = \text{diag}(1, 1, -1, -1)$ .

Thus  $A_9$  is the only remaining unresolved  $4 \times 4$  full sign pattern.

In order to state another very elegant and useful structural characterization of  $J$ -orthogonal matrices, we need the notion of the exchange operator. Let  $p$  and  $n$  be positive integers with  $p \leq n$ . Let  $B$  be an  $n \times n$  matrix partitioned as  $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$  such that  $B_{11}$  is  $p \times p$  and is nonsingular. The exchange operator applied to  $B$  with respect to the above partition yields

$$\text{exc}(B) = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12} \\ B_{21}B_{11}^{-1} & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{bmatrix}.$$

The following theorem found in [11, Theorem 2.1] and [10, Theorem 2.2] characterizes the close connections between orthogonal matrices and  $J$ -orthogonal matrices.

**Theorem 4.5.** *Let  $p$  and  $n$  be positive integers with  $p \leq n$ . Let  $B$  be an  $n \times n$  real matrix partitioned as  $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$  such that  $B_{11}$  is  $p \times p$ . Let  $J = \text{diag}(I_p, -I_{n-p})$ . If  $B$  is  $J$ -orthogonal, then the leading  $p \times p$  principal submatrix of  $B$  is nonsingular and  $\text{exc}(B)$  is orthogonal. Conversely, if  $B$  is orthogonal and  $B_{11}$  is nonsingular, then  $\text{exc}(B)$  is  $J$ -orthogonal.*

Therefore, every  $J$ -orthogonal matrix can be constructed from a suitable orthogonal matrix using the exchange operator and permutation similarity. This approach can be used to show that a given full sign pattern allows  $J$ -orthogonality for a particular  $J$ . This process can be done for hundreds of thousands of “random” rational orthogonal matrices using MATLAB.

It turns out for every  $4 \times 4$  full sign pattern  $A$  that satisfies the SPJO conditions for a specific signature pattern  $J$ , we can generate a  $J$ -orthogonal matrix in  $Q(A)$ . In particular, note that  $A_9$  satisfies the SPJO conditions with the signature pattern  $J_1 = \text{diag}(+, -, +, -)$ . With the help of MATLAB running the preceding procedure, for  $J = \text{diag}(1, 1, -1, -1)$ , we obtain the following  $J$ -orthogonal matrix

$$B = \frac{1}{12} \begin{bmatrix} 8 & 18 & 12 & 10 \\ 26 & -9 & 18 & -17 \\ 20 & 6 & 24 & -2 \\ 14 & -15 & 6 & -23 \end{bmatrix},$$

which satisfies  $P^T B P = A_9$ , where  $P = [e_1, e_3, e_2, e_4]$ . It follows that  $A_9$  allows a  $J_1$ -orthogonal matrix with  $J_1 = P^T J P = \text{diag}(1, -1, 1, -1)$ , and hence,  $A_9 \in \mathcal{J}_4$ .

We now reach the following conclusion.

**Theorem 4.6.** *Every  $4 \times 4$  full sign pattern allows a  $J$ -orthogonal matrix.*

Combined with known results on full sign patterns of orders at most 3, we get the following result.

**Corollary 4.7.** *For  $n \leq 4$ , every  $n \times n$  full sign pattern allows a  $J$ -orthogonal matrix.*

In view of Theorem 1.4, we also have

**Corollary 4.8.** *For  $n \leq 4$ , every  $n \times n$  full sign pattern allows a  $G$ -matrix.*

Thus, we have the following nice result.

**Corollary 4.9.** *For every  $n \times n$  full sign pattern  $A$  with  $n \leq 4$ ,  $A \in \mathcal{G}_n$  iff  $A \in \mathcal{J}_n$ .*

Suppose a full  $n \times n$  sign pattern  $A$  allows a  $J$ -orthogonal matrix  $B \in Q(A)$ . Without loss of generality, we may assume that all the positive entries of  $J$  occur at the leading diagonal entries. By Theorem 4.5,  $\text{exc}(B)$  is an orthogonal matrix. Observe that  $\text{exc}(\text{exc}(B))=B$ . Write  $\text{exc}(B)$  as a product of real Householder matrices  $H_{v_1}, \dots, H_{v_k}$  (where  $k \leq n$ ). Replace each  $v_i$  with a rational approximation  $\tilde{v}_i$ . Since matrix multiplication and exchange operator are continuous, we see that when the rational approximations  $\tilde{v}_i$  are sufficiently close to  $v_i$ ,  $\tilde{B} = \text{exc}(H_{\tilde{v}_1} \cdots H_{\tilde{v}_k})$  is a rational  $J$ -orthogonal matrix in  $Q(A)$ . Thus we have shown the following interesting result.

**Theorem 4.10.** *Let  $A$  be a full  $n \times n$  sign pattern. If  $A$  allows a  $J$ -orthogonal matrix, then  $A$  allows a rational  $J$ -orthogonal matrix with the same signature matrix  $J$ . In particular, if  $A$  allows orthogonality, then  $A$  allows a rational orthogonal matrix.*

As a consequence, if the  $n \times n$  full sign pattern  $A$  does not allow a rational  $J$ -orthogonal matrix for any signature matrix  $J$ , then  $A$  does not allow a real  $J$ -orthogonal matrix.

## 5 Concluding remarks

The question of whether every  $n \times n$  full sign pattern allows a  $J$ -orthogonal matrix is still open. It seems to be a complicated and impressive problem. Even for  $n = 5$  the number of cases is daunting. Some other techniques will need to be developed.

## References

- [1] L. B. Beasley and D. J. Scully, Linear operators which preserve combinatorial orthogonality, *Linear Algebra Appl.* 201 (1994), 171–180.
- [2] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991.
- [3] R. A. Brualdi and B. L. Shader, *Matrices of Sign-Solvable Linear Systems*, Cambridge University Press, Cambridge, 1995.
- [4] C. A. Eschenbach, F. J. Hall, D. L. Harrell, and Z. Li, When does the inverse have the same sign pattern as the transpose?, *Czechoslovak Math. J.* 49 (1999), 255–275.
- [5] M. Fiedler, Notes on Hilbert and Cauchy matrices, *Linear Algebra Appl.* 432 (2010), 351–356.
- [6] M. Fiedler and F. J. Hall, G-matrices, *Linear Algebra Appl.* 436 (2012), 731–742.
- [7] E. J. Grimme, D. C. Sorensen, and P. Van Dooren, Model reduction of state space systems via an implicitly restarted Lanczos method, *Numerical Algorithms*, 12(1-2) (1996), 1–31.
- [8] F. J. Hall and Z. Li, Sign pattern matrices, Chapt. 42 in L. Hogben (Ed.), *Handbook of Linear Algebra*, 2nd ed., Chapman and Hall/CRC Press, Boca Raton, 2013.
- [9] F. J. Hall and M. Rozložnik, G-matrices,  $J$ -orthogonal matrices, and their sign patterns, *Czechoslovak Math. J.* 66 (2016), 653–670.
- [10] N. J. Higham,  $J$ -orthogonal matrices: properties and generation, *SIAM Review* 45 (3) (2003), 504–519.
- [11] M. Stewart and G.W. Stewart, On hyperbolic triangularization: stability and pivoting, *SIAM J. Matrix Anal. Appl.* 19 (1998), no. 4, 847–860.
- [12] M. Stewart and P. Van Dooren, On the factorization of hyperbolic and unitary transformations into rotations, *SIAM J. Matrix Anal. Appl.* 27 (2006), no. 3, 876–890.
- [13] C. Waters, Sign patterns that allow orthogonality, *Linear Algebra Appl.* 235 (1996), 1–13.