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The polyconvexity for functions of several closed differential forms, with applications to electro-magneto-elastostatics

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The Polyconvexity for Functions of Several Closed Differential Forms, with Applications to Electro-Magneto-Elastostatics ⋆

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Abstract This note deals with integral functionals of the form

$$I(\omega_1, \dots, \omega_m) = \int_\Omega f(\omega_1, \dots, \omega_m)\, dx$$

where $\omega_1, \dots, \omega_m$ are closed differential forms of different degrees on a bounded open set $\Omega \subset \mathbb{R}^n$. The convexity properties for the integrand f are introduced, called the mult. ext. quasiconvexity, mult. ext. one convexity and mult. ext. polyconvexity,⋆⋆ treated as particular cases of the $\mathscr{A}$-quasiconvexity theory corresponding to the constraints

$$d\omega_1 = \dots = d\omega_m = 0 \quad (d = \text{the exterior derivative}). \tag{$*$}$$

It is shown that any mult. ext. quasiaffine function is a linear combination of expressions of the form $\omega_1^{r_1} \wedge \dots \wedge \omega_m^{r_m}$, where the powers are understood in the sense of the exterior multiplication and where $r_1, \dots, r_m$ range all nonnegative integers for which $\omega_1^{r_1} \wedge \dots \wedge \omega_m^{r_m}$ does not vanish. As a consequence, a function $f = f(\omega_1, \dots, \omega_m)$ is mult. ext. polyconvex if and only if it can be written as

$$f(\omega_1, \dots, \omega_m) = \Phi(\dots, \omega_1^{r_1} \wedge \dots \wedge \omega_m^{r_m}, \dots)$$

where $r_1, \dots, r_m$ range the above-mentioned set of integers and Φ is a convex function. Under this notion of mult. ext. polyconvexity, an existence theorem for the minimum energy state is proved. The polyconvexity in the classical calculus of variations is shown to be a particular case of the present approach. Our main motivation work was, however, the polyconvexity for electro-magneto-elastic interactions in continuous bodies, where the constraints $(*)$ come from the combination of Maxwell's equations with the compatibility of deformations. It will be shown that the mult. ext. polyconvexity takes the form determined by an involved direct calculation in an earlier paper of the author [32].

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⋆⋆ I extrapolate the terminology of [4], which introduces the ext. quasiconvexity, ext. one convexity and ext. polyconvexity in the case of a single differential form.

Contents

1 Introduction

This note deals with the integral functionals of the form

$$I(\omega_1, \dots, \omega_m) = \int_\Omega f(\omega_1, \dots, \omega_m)\, dx \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\omega_1, \dots, \omega_m$ are closed differential forms on Ω, i.e., forms satisfying the differential constrains

$$d\omega_1 = \dots = d\omega_m = 0 \qquad (d = \text{the exterior derivative}). \tag{1.2}$$

The degrees $s(1), \dots, s(m)$ of the forms $\omega_1, \dots, \omega_m$ are generally different from each other, with $1 \le s(i) \le n$. Accordingly, f is a continuous integrand defined on the product

$$\Delta_s := \wedge_{s(1)} \times \cdots \times \wedge_{s(m)}$$

of the spaces $\wedge_s$ of s-vectors on $\mathbb{R}^n$. The reader is referred to Sections 9 and 10 for the terminology and notation for the exterior algebra and analysis employed here. Here we only note that we identify the spaces $\wedge_s$ with their duals $\wedge^s$; thus we do not distinguish the s-vectors from s-covectors and use these terms interchangeably.

Our main interest is in the convexity properties for integrand f, called below the mult. ext. quasiconvexity, mult. ext. one convexity and mult. ext. polyconvexity. These notions are particular cases the $\mathscr{A}$-quasiconvexity [18, 24–25, 34, 9] corresponding to the differential constraints (1.2).

An integrand $f : \Delta_s \to \mathbb{R} \cup \{\infty\}$ is said to be mult. ext. quasiconvex if

$$\int_Q f(\omega_1 + \psi_1(x), \dots, \omega_m + \psi_m(x))\, dx \ge f(\omega_1, \dots, \omega_m) \tag{1.3}$$

for each constant multivectors $\omega_1, \dots, \omega_m$, each m-tuple of differential forms ψ_i on $\mathbb{R}^n$ that are periodic with respect to $Q = (0, 1)^n$ and satisfy

$$d\psi_i = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q \psi_i(x)\, dx = 0, \quad i = 1, \dots, m.$$

The integrand f is said to be mult. ext. quasiaffine if (1.3) holds with the equality sign for all ω_i and ψ_i occurring there. The integrand f is said to be mult. ext. polyconvex if it can be expressed as a convex function of a finite number of mult. ext. quasiaffine functions. It then follows from Jensens's inequality that mult. ext. polyconvexity implies mult. ext. quasiconvexity.

The main result of this note is the determination of all mult. ext. quasiaffine and mult. ext. polyconvex functions. It turns out (see Theorem 2.8, below) that any mult. ext. quasiaffine function is a linear combination, with constant coefficients, of the products of integer exterior powers of ω_i, i.e., linear combinations of the system of functions*

$$\mathfrak{F}_s = \{\omega_1^{r1} \wedge \cdots \wedge \omega_m^{rm} : (r_1, \ldots, r_m) \in \mathfrak{R}_s\}, \tag{1.4}$$

where

$$\omega^r = \underbrace{\omega \wedge \cdots \wedge \omega}_{r \text{ times}}$$

and the collection $\mathfrak{R}_s$ of exponents is given by**

$$\mathfrak{R}_s := \{(r_1, \ldots, r_m) \in \mathbb{N}_0^m : \sum_{i=1}^{m} r_i s(i) \le n \text{ and } r_i \le 1 \text{ if } s(i) \text{ is odd}\}. \tag{1.5}$$

Thus if f is a mult. ext. quasiaffine function then for each $(r_1, \ldots, r_m) \in \mathfrak{R}_s$ there exists a multivector of appropriate degree $\alpha_{r1\ldots rm}$ such that

$$f(\omega_1, \ldots, \omega_m) = \sum_{(r_1, \ldots, r_m) \in \mathfrak{R}_s} \alpha_{r1\ldots rm} \cdot \omega_1^{r1} \wedge \cdots \wedge \omega_m^{rm} \tag{1.6}$$

for each $(\omega_1, \ldots, \omega_m) \in \Delta_s$. The mult. ext. polyconvex functions are convex functions of the elements of the list $\mathfrak{F}_s$ (Theorem 2.9), i.e.,

$$f(\omega_1, \ldots, \omega_m) = \Phi(\ldots, \omega_1^{r1} \wedge \cdots \wedge \omega_m^{rm}, \ldots) \tag{1.7}$$

for each $(\omega_1, \ldots, \omega_m) \in \Delta_s$ where $(r_1, \ldots, r_m)$ range the set $\mathfrak{R}_s$ and Φ is a convex function. Under this notion of mult. ext. polyconvexity, an existence theorem is proved for the minimum energy state for integrands that need not be finite everywhere on their domains (Theorem 3.1). The descriptions (1.6) and (1.7) of mult. ext. quasiaffine and mult. ext. polyconvex functions are extensions of earlier results of Bandyopadhyay, Dacorogna & Sil [4] dealing with ext. quasiaffine and ext. polyconvex functions of a *single* differential form.

The case of several differential forms has been motivated by the desire to encompass the classical calculus of variations with several unknowns, nonlinear elasticity and electro-magneto-elasticity; areas that remain outside the scope of [4]. Referring to Sections 4 and 5 for details, we now outline these motivations.

Example A: Classical calculus of variations and nonlinear elasticity This example shows that the general results of the paper yield the well-known structure of quasiaffine and polyconvex functions of the calculus of variations. Here one deals with integrals of the form

* In (1.4) and at similar places below, the symbol $\omega_1^{r_1} \wedge \cdots \wedge \omega_m^{r_m}$ stands for the function f defined on Δ_s by $f(\omega_1, \ldots, \omega_m) = \omega_1^{r_1} \wedge \cdots \wedge \omega_m^{r_m}$ for each $(\omega_1, \ldots, \omega_m) \in \Delta_s$.

** The conditions on $(\omega_1, \ldots, \omega_m)$ in (1.5) are necessary and sufficient to guarantee that the product in $\mathfrak{F}_s$ in (1.4) does not vanish identically.

$$I(u_1,\ldots,u_m)=\int_\Omega f(\nabla u_1,\ldots,\nabla u_m)\,dx \tag{1.8}$$

where $u=(u_1,\ldots,u_m)$ is an m-tuple of scalar dependent variables. Under our identifications of vectors and covectors, the gradients ∇u_i are 1-forms and thus the integral in (1.8) is a particular case of that in (1.1) with

$$\omega_i=\nabla u_i \quad \text{and} \quad s(i)=1, \quad i=1,\ldots,m. \tag{1.9}$$

The exterior derivative of a 1-form is the curl and thus the differential constraints (1.2) are satisfied by the interchangeability of the second partial derivatives. The mult. ext. quasiconvexity coincides with Morrey's quasiconvexity, and mult. ext. quasiaffinity with the quasiaffinity derived from Morrey's quasiconvexity. In view of (1.9), the exponents $\boldsymbol{r}=(r_1,\ldots,r_m)$ from the list $\mathfrak{R}_{\boldsymbol{s}}$ have all entries equal to 0 or 1, and so the list of mult. ext. quasiaffine functions reduces to

$$\mathfrak{F}_{\boldsymbol{s}}=\{\omega_{I_1}\wedge\cdots\wedge\omega_{I_h}:I\in\mathbb{I}_h^m,\ 0\le h\le q\}, \tag{1.10}$$

where

$$q:=\min\{m,n\}$$

and

$$\mathbb{I}_h^m=\{I=(I_1,\ldots,I_h)\in\mathbb{N}^h:1\le I_1<\ldots<I_h\le m\} \tag{1.11}$$

is the set of all m dimensional multiindices of order h. It will be shown in Section 4 that the list (1.10) is isomorphic to the well-known collection $\mathfrak{M}_{\boldsymbol{s}}$ of all minors of the matrix $F=\nabla u$ of all possible orders h, i.e., to the set

$$\mathfrak{M}_{\boldsymbol{s}}=\{F_{IJ}^{(h)}:h=1,\ldots,q,\ I\in\mathbb{I}_h^m,\ J\in\mathbb{I}_h^n\}, \tag{1.12}$$

where

$$F_{IJ}^{(h)}:=\det[F_{I_aJ_b}]_{1\le a,\,b\le h},$$

with F_{iA} the matrix elements of F. Thus the integrand f is mult. ext. polyconvex if

$$f(F)=\text{a convex function of the minors in (1.12)},$$

which is Ball's original notion of polyconvexity. In the nonlinear elasticity we have $m=n=3$ and the triplet of 1-forms

$$\omega_1=\nabla u_1, \quad \omega_2=\nabla u_2, \quad \omega_3=\nabla u_3 \tag{1.13}$$

from (1.9) represents the deformation gradient F. The list (1.10) of mult. ext. quasiaffine functions is

$$\mathfrak{F}_{\boldsymbol{s}}:\quad 1,\ \ \omega_i,\ \ \omega_j\wedge\omega_k,\ \ \omega_1\wedge\omega_2\wedge\omega_3,\quad 1\le i\le 3,\quad 1\le j<k\le 3 \tag{1.14}$$

consisting of the absolute, linear, bilinear and trilinear expressions, which is isomorphic, expression-by-expression, to Ball's list

$$1,\quad F,\quad \operatorname{cof}F,\quad \det F.$$

The associated polyconvexity takes the well-known form

$$f(F)=\Phi(F,\operatorname{cof}F,\det F)$$

where Φ is a convex function.

Example B: Electro-magneto-elastostatics The electro-magneto-elastic interactions have recently received much theoretical attention in view of the technological application of electro- or magneto-sensitive elastomers, smart materials whose mechanical properties change instantly by the application of an electric or magnetic fields. The total energy is the sum of the energy of the body, the energy of the vacuum electromagnetic field in the exterior of the body, and the term corresponding to the loads. Only the first term is of interest here, which in dimension $n = 3$ takes the form

$$I(u, D, B) = \int_{\Omega} f(F, D, B)\, dx \tag{1.15}$$

where $F = \nabla u \equiv (\nabla u_1, \nabla u_2, \nabla u_3)$ is the deformation gradient as above and $D = (D_1, D_2, D_3)$ and $B = (B_1, B_2, B_3)$ are the referential (lagrangean) electric displacement and magnetic induction, satisfying

$$\operatorname{div} D = 0, \quad \operatorname{div} B = 0 \quad \text{in} \quad \Omega. \tag{1.16}$$

The integral in (1.15) is of the format (1.1) with $m = 5$, where $\omega_1, \omega_2, \omega_3$ are the 1-forms representing the deformation gradient as in (1.13) and ω_4, ω_5 are 2-forms

$$\omega_4 = D_1 d\hat{x}_1 + D_2 d\hat{x}_2 + D_3 d\hat{x}_3, \quad \omega_5 = B_1 d\hat{x}_1 + B_2 d\hat{x}_2 + B_3 d\hat{x}_3,$$

where

$$d\hat{x}_1 = dx_2 \wedge dx_3, \quad d\hat{x}_2 = dx_3 \wedge dx_1, \quad d\hat{x}_3 = dx_1 \wedge dx_2.$$

The equations $d\omega_4 = d\omega_5 = 0$ are equivalent to (1.16) (see Section 5). The list of mult. ext. quasiaffine functions is obtained by combining the mechanical list (1.14) with

$$\omega_i, \quad \omega_i \wedge \omega_j, \quad i = 4,\ 5, \quad 1 \le j \le 3,$$

which leads to the isomorphic set

$$1, \quad F, \quad \operatorname{cof} F, \quad \det F, \quad D, \quad B, \quad FD, \quad FB,$$

with the unexpected cross-effect terms FD and FB, as determined by a direct calculation in [32]. The associated polyconvexity reads

$$f(F, D, B) = \Phi(F, \operatorname{cof} F, \det F, D, B, FD, FB)$$

where Φ is a convex function of the indicated variables.

This paper is organized as follows. Section 2 introduces the central convexity concepts and presents the main results of the paper for them without proofs. Section 3 presents a sample-type existence theorem for minimizers of the total energy under the mult. ext. polyconvexity; in contrast with the analogous result under mult. ext. quasiconvexity, the integrand may take infinite values. Section 4 shows that the classical calculus of variations and nonlinear elasticity may be viewed as particular cases of the present theory. Section 5 describes the mult. ext. polyconvexity for electro-magneto-elastostatics. Section 6 and 7 present the proofs of the main results of the paper, Proposition 2.3 and Theorem 2.8. Appendix A (Section 8) outlines the $\mathscr{A}$-quasiconvexity theory and Appendices B and C (Sections 9 and 10) present an axiomatic (index-free) approach to the exterior algebra and analysis.

Notation Throughout the paper, n is a positive integer, the dimension of the underlying space. We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the sets of all positive or nonnegative integers, respectively, by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, by $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ the extended real line and by $\mathbb{M}^{m \times n}$ the space of real m by n matrices. If Ω is an open subset of $\mathbb{R}^n$ and Z a finite-dimensional vector space then $C^\infty(\Omega, Z)$ denotes the set of all indefinitely differentiable Z-valued maps on Ω and by $C_0^\infty(\Omega, Z)$ the set of all indefinitely differentiable Z-valued maps on $\mathbb{R}^n$ with compact support which is contained in Ω. Further, we let $Q = (0, 1)^n$ be the unit cube and denote by $C^\infty_{\mathrm{per}}(\mathbb{R}^n, Z)$ the set of all indefinitely differentiable Q-periodic Z-valued maps on $\mathbb{R}^n$. If f is a Z-valued function on Ω, we denote by $f_{,i}$ the partial derivative of $f(x)$ with respect to x_i. If s is a positive integer then $\mathbb{P}_s$ is the set of all permutations $\pi : \{1, \dots, s\} \to \{1, \dots, s\}$; $\operatorname{sgn}(\pi)$ is the index (sign) of $\pi \in \mathbb{P}_s$.

2 Main results

The reader is referred to the appendices in Sections 9 and 10 for the notations and definitions for the exterior algebra and analysis employed here.

2.1 Definitions Let $\boldsymbol{s} = (s(1), \dots, s(m))$ be an m-tuple of integers satisfying $1 \le s(i) \le n$.

(i) We define $\Delta_{\boldsymbol{s}} := \wedge_{s(1)} \times \cdots \times \wedge_{s(m)}$ and $\Gamma_{\boldsymbol{s}} = \wedge_{s(1)-1} \times \cdots \times \wedge_{s(m)-1}$;

(ii) for any $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \Gamma_{\boldsymbol{s}}$ we define $\boldsymbol{\lambda} \wedge \eta$ and $\eta \wedge \boldsymbol{\lambda} \in \Delta_{\boldsymbol{s}}$ by

$$\boldsymbol{\lambda} \wedge \eta := (\lambda_1 \wedge \eta, \dots, \lambda_m \wedge \eta) \quad \text{and} \quad \eta \wedge \boldsymbol{\lambda} := (\eta \wedge \lambda_1 \dots, \eta \wedge \lambda_m);$$

(iii) for any $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in C^1(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$ we write

$$d\boldsymbol{\xi} = (d\xi_1, \dots, d\xi_m),$$

which is an element of $C^0(\mathbb{R}^n, \Delta_{\boldsymbol{s}})$;

(iv) we define the characteristic cone Λ by

$$\Lambda = \{\boldsymbol{\lambda} \wedge \eta : \boldsymbol{\lambda} \in \Gamma_{\boldsymbol{s}} \text{ and } \eta \in \mathbb{S}^{n-1}\}; \tag{2.1}$$

(v) an integrand of type $\boldsymbol{s}$ is any continuous function $f : \Delta_{\boldsymbol{s}} \to \bar{\mathbb{R}}$. The arguments of f are thus m tuples $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ of multivectors of degrees $s(1), \dots, s(m)$, respectively.

Specializing the general concepts of the $\mathscr{A}$-quasiconvexity theory (see Section 8, below) to the constraints (1.2), we introduce the following notions.

2.2 Definitions An integrand f of type $\boldsymbol{s}$ is said to be

(i) mult. ext. quasiconvex at $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ if

$$\int_Q f(\boldsymbol{\omega} + \boldsymbol{\psi}(x))\, dx \ge f(\boldsymbol{\omega}) \tag{2.2}$$

for every $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ and every $\boldsymbol{\psi} \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \Delta_{\boldsymbol{s}})$ such that

$$d\boldsymbol{\psi} = 0 \quad \text{on} \quad \mathbb{R}^n \quad \text{and} \quad \int_Q \boldsymbol{\psi}(x)\, dx = 0; \tag{2.3}$$

(ii) mult. ext. quasiaffine if f takes only finite values and both f and $-f$ are mult. ext. quasiconvex;

(iii) mult. ext. polyconvex if there exists a finite number of mult. ext. quasiaffine functions $f_1, \dots, f_g$ and a convex lower semicontinuous function $\Phi : \mathbb{R}^g \to \bar{\mathbb{R}}$ such that

$$f(\boldsymbol{\omega}) = \Phi(f_1(\boldsymbol{\omega}), \dots, f_g(\boldsymbol{\omega}))$$

for each $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$.

The following alternative forms of the mult. ext. quasiconvexity condition, with different classes of test functions $\boldsymbol{\psi}$, will be useful.

2.3 Proposition *For an integrand f of type $\boldsymbol{s}$ the following conditions are equivalent*:

(i) *f is mult. ext. quasiconvex*;

(ii) (2.2) *holds for every $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ and every $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n, \Delta_{\boldsymbol{s}})$ satisfying* (2.3);

(iii) *we have*

$$\int_Q f(\boldsymbol{\omega} + d\boldsymbol{\xi}(x))\, dx \ge f(\boldsymbol{\omega}) \tag{2.4}$$

for every $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ and every $\boldsymbol{\xi} \in C_{\mathrm{per}}^\infty(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$;

(iv) *we have* (2.4) *for every $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ and every $\boldsymbol{\xi} \in C_0^\infty(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$*.

The proof is given in Section 6. The equivalence (iii) $\Leftrightarrow$ (iv) is well-known in the classical calculus of variations, see, e.g., [2; Remark, p. 141] or [33]. Condition (iv) is adopted in [4] to define the ext. quasiconvexity of an integrand depending on a single differential form. All the above equivalences depend crucially on the fact that the equation $d\boldsymbol{\psi} = 0$ has a potential, i.e., it is satisfied locally if and only if $\boldsymbol{\psi} = d\boldsymbol{\xi}$ for some other form $\boldsymbol{\xi}$.

2.4 Theorem *Any mult. ext. polyconvex function is mult. ext. quasiconvex.*

This is a standard application of Jensen's inequality; see, e.g., [9; Corollary 2.5]. Besides being a sufficient condition for the mult. ext. quasiconvexity, the main motivation for the mult. ext. polyconvexity is that it allows to prove the existence of a minimizer of the variational problem without the restrictive growth conditions needed in the case of mult. ext. quasiconvexity, see Section 3.

To understand the condition of the mult. ext. polyconvexity, we need a good description of mult. ext. quasiaffine functions. These will be approached via an apriori larger class of mult. ext. one affine functions to be now introduced. The analysis will show that in the present special case the classes of mult. ext. quasiaffine functions and mult. ext. one affine functions coincide (as they do in the classical calculus of variations).

2.5 Definition An integrand f of type $\boldsymbol{s}$ is said to be

(i) mult. ext. one convex if

$$f(t\boldsymbol{\omega}_1 + (1-t)\boldsymbol{\omega}_2) \le tf(\boldsymbol{\omega}_1) + (1-t)f(\boldsymbol{\omega}_2)$$

for every $t \in (0, 1)$ and every $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \Delta_{\boldsymbol{s}}$ such that $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \in \Lambda$,

(ii) mult. ext. one affine if f takes only finite values and both f and $-f$ are mult. ext. one convex.

The descriptions of mult. ext. quasiaffine and mult. ext. polyconvex functions uses the exterior products of exterior powers to be now introduced.

2.6 Definitions

(i) If $\omega \in \wedge_s$ is on s-vector and r a nonnegative integer such that $rs \le n$, we define the exterior power $\omega^r \in \wedge_{rs}$ by

$$\omega^r = \begin{cases} 1 & \text{if } r = 0, \\ \underbrace{\omega \wedge \cdots \wedge \omega}_{r \text{ times}} & \text{if } r > 0. \end{cases}$$

If r is odd and $r \ge 2$ then $\omega^r = 0$ by a simple application of (9.1).

(ii) For a given vector of degrees $\boldsymbol{s} = (s(1), \dots, s(m))$ we define the set $\mathfrak{R}_{\boldsymbol{s}}$ of admissible exponents by

$$\mathfrak{R}_{\boldsymbol{s}} = \{\boldsymbol{r} = (r_1, \dots, r_m) \in \mathbb{N}_0^m : \sum_{i=1}^{m} r_i s(i) \le n \text{ and } r_i \le 1 \text{ if } s(i) \text{ is odd}\}. \tag{2.5}$$

(iii) If $\boldsymbol{r} = (r_1, \dots, r_m) \in \mathfrak{R}_{\boldsymbol{s}}$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in \Delta_{\boldsymbol{s}}$, we put

$$\boldsymbol{\omega}^{\boldsymbol{r}} := \bigwedge_{i=1}^{m} \omega_i^{r_i}$$

which an element of $\wedge_{\dim(\boldsymbol{r})}$, where $\dim(\boldsymbol{r}) := \sum_{i=1}^m r_i s(i)$.

(iv) We define the set of admissible powers

$$\mathfrak{F}_{\boldsymbol{s}} := \{\boldsymbol{\omega}^{\boldsymbol{r}} : \boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}}\}. \tag{2.6}$$

Recall that we interpret the elements $\boldsymbol{\omega}^{\boldsymbol{r}}$ of $\mathfrak{F}_{\boldsymbol{s}}$ as functions $f_{\boldsymbol{r}} : \Delta_{\boldsymbol{s}} \to \wedge_{\dim(\boldsymbol{r})}$ given by $f_{\boldsymbol{r}}(\boldsymbol{\omega}) = \boldsymbol{\omega}^{\boldsymbol{r}}$, $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$.

We are about to analyze mult. ext. quasiaffine functions. It will be convenient to take into account also the following condition, which will turn out to be equivalent to the mult. ext. quasiaffinity.

2.7 Definition An integrand of type $\boldsymbol{s}$ is said to be a mult. ext. null lagrangian if for any bounded open set $\Omega \subset \mathbb{R}^n$, any $\boldsymbol{\xi} \in C^\infty(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$, and any $\boldsymbol{\theta} \in C_0^\infty(\Omega, \Gamma_{\boldsymbol{s}})$ we have

$$\int_\Omega f(d\boldsymbol{\xi} + d\boldsymbol{\theta})\, dx = \int_\Omega f(d\boldsymbol{\xi})\, dx.$$

This reduces to Condition (iii) of Proposition 2.3 if $\boldsymbol{\omega} := d\boldsymbol{\xi}$ is constant and $\Omega = Q$.

2.8 Theorem *For an integrand f of type $\boldsymbol{s}$ the following three conditions are equivalent*:

(i) *f is mult. ext. quasiaffine*;

(ii) *f is mult. ext. one affine*;

(iii) *f is a linear combination, with constant coefficients, of the products from the set of admissible powers $\mathfrak{F}_{\boldsymbol{s}}$; thus for each $\boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}}$ there exists $\alpha_{\boldsymbol{r}} \in \wedge_{\dim(\boldsymbol{r})}$ such that*

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}}} \alpha_{\boldsymbol{r}} \cdot \boldsymbol{\omega}^{\boldsymbol{r}} \tag{2.7}$$

for each $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$;

(iv) *f is a mult. ext. null lagrangian.*

Thus the set $\mathfrak{F}_{\boldsymbol{s}}$ is a basis of the finite-dimensional space of mult. ext. quasiaffine functions. The proof of Theorem 2.8 is deferred to Section 7. The particular case $m = 1$ is [4; Theorem 17].

2.9 Theorem *Let $\boldsymbol{r}(1),\dots,\boldsymbol{r}(g)$ be any enumeration of the elements of $\Re_{\boldsymbol{s}}$. An integrand f of type $\boldsymbol{s}$ is mult. ext. polyconvex if and only if there exists a convex lower semicontinuous function $\Phi : \wedge_{\dim(\boldsymbol{r}(1))} \times \dots \times \wedge_{\dim(\boldsymbol{r}(g))} \to \bar{\mathbb{R}}$ such that*

$$f(\boldsymbol{\omega}) = \Phi(\boldsymbol{\omega}^{\boldsymbol{r}(1)},\dots,\boldsymbol{\omega}^{\boldsymbol{r}(g)})$$

for each $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$.

This is a direct consequence of Theorem 2.8.

3 Existence of minimizers of energy under mult. ext. polyconvexity

We consider the integral functional

$$I(\boldsymbol{\omega}) = \int_\Omega \big(f(\boldsymbol{\omega}) - \sum_{i=1}^m \mathfrak{X}_i \cdot \omega_i\big)\, dx \tag{3.1}$$

depending on the collection $\boldsymbol{\omega} = (\omega_1,\dots,\omega_m)$ of closed differential forms on an open bounded region $\Omega \subset \mathbb{R}^n$ where f is an integrand f of type $\boldsymbol{s} = (s(1),\dots,s(m))$. The prescribed functions $\mathfrak{X}_i : \Omega \to \wedge_{s(i)}$ represent external influences such as the body forces and boundary tractions in elasticity, as will be shown below.

To formulate the assumptions, we let let $p_1,\dots,p_m$ be numbers in $(1,\infty)$, put $p_i' = p_i/(p_i - 1)$ and consider the following conditions:

$\mathbf{H}_1$ f is a mult. ext. polyconvex (continuous) integrand of type $\boldsymbol{s}$;

$\mathbf{H}_2$ f satisfies

$$f(\omega_1,\dots,\omega_m) \ge c(|\omega_1|^{p_1} + \dots + |\omega_m|^{p_m} - 1)$$

for some $c > 0$ and all $(\omega_1,\dots,\omega_m) \in \Delta_{\boldsymbol{s}}$;

$\mathbf{H}_3$ the numbers $p_1,\dots,p_n$ satisfy

$$r_1/p_1 + \dots + r_m/p_m \le 1 \quad \text{for all } (r_1,\dots,r_m) \in \Re_{\boldsymbol{s}};$$

$\mathbf{H}_4$ $\mathfrak{X}_i \in L^{p_i'}(\Omega, \wedge_{s(i)})$, $i = 1,\dots,m$.

For the purpose of the treatment below, we define the domain $\mathscr{D}$ of the functional I in (3.1) to be the set of all $\boldsymbol{\omega} = (\omega_1,\dots,\omega_m) \in L^{p_1}(\Omega,\wedge_{s(1)}) \times \dots \times L^{p_m}(\Omega,\wedge_{s(m)})$ which satisfy

$$d\omega_1 = \dots = d\omega_m = 0 \quad \text{on } \Omega \tag{3.2}$$

in the weak sense, which means that

$$\int_\Omega \omega_i \cdot \operatorname{div} \psi_i \, dx = 0$$

for each $\psi_i \in C_0^\infty(\Omega, \wedge_{s(i)+1})$ and all $i = 1,\dots,m$. We refer to Definition 10.3, below, for the definition of the weak exterior derivative.

It is not assumed that the integrand f is finite. Condition $\mathbf{H}_2$ implies that f is bounded from below and thus the integral in (3.1) is well defined as a finite number or ∞. We denote by

$$\operatorname{dom} f = \{\boldsymbol{\sigma} \in \Delta_{\boldsymbol{s}} : f(\boldsymbol{\sigma}) < \infty\}$$

the effective domain of f. The assumed continuity of f (which is a part of the definition of an integrand of type $\boldsymbol{s}$) implies that $\operatorname{dom} f$ is an open subset of $\Delta_{\boldsymbol{s}}$.

The following theorem presents an existence result for a minimizer of I under Neumann's boundary conditions.

3.1 Theorem *Suppose that Hypotheses $\mathbf{H}_1 - \mathbf{H}_4$ hold. Then*
(i) *if I is not identically equal to ∞ on $\mathscr{D}$ then I has a minimizer $\boldsymbol{\omega}$ in $\mathscr{D}$, i.e., an element such that*

$$I(\boldsymbol{\omega}) \le I(\boldsymbol{\sigma})$$

for all $\boldsymbol{\sigma} \in \mathscr{D}$;
(ii) *each minimizer $\boldsymbol{\omega}$ satisfies $f(\boldsymbol{\omega}(x)) < \infty$ for almost every $x \in \Omega$;*
(iii) *if f is differentiable on* dom$f$ *and $\boldsymbol{\omega}$ is a minimizer whose range is contained in a compact subset of* dom$f$ *then we have the weak form of the Euler-Lagrange equations*

$$\operatorname{div}(\mathrm{D}_{\omega i} f - \mathfrak{X}_i) = 0 \quad \text{on} \quad \mathbb{R}^n, \quad i = 1, \dots, m \tag{3.3}$$

which means that

$$\int_\Omega d\sigma_i \cdot (\mathrm{D}_{\omega i} f - \mathfrak{X}_i)\, dx = 0 \tag{3.4}$$

for every $\sigma_i \in C_0^\infty(\mathbb{R}^n, \wedge_{s(i)-1})$.

3.2 Remarks

(i) The possible infinities of the integrand in Theorem 3.1 should be contrasted with the existence theorem under mult. ext. quasiconvexity, which would require

$$c(|\boldsymbol{\omega}|^p - 1) \le f(\boldsymbol{\omega}) \le C(|\boldsymbol{\omega}|^p - 1)$$

for some $p > 1$, $c > 0$, $C > 0$ and all $\boldsymbol{\omega}$; hence f must be finite for all $\boldsymbol{\omega}$. Only for such integrands there is a sequential lower semicontinuity theorem, see [18; Theorem 3.7].

(ii) The coercivity hypothesis $\mathbf{H}_2$ under $p_1, \dots, p_m$ as in $\mathbf{H}_3$ may be unnecessarily strong in concrete cases. For example, in the case of elasticity (see Section 4, below), the unknown $\boldsymbol{\omega}$ is a triplet 1-forms and thus $\mathbf{H}_3$ requires that the triplet p_1, p_2, p_3 satisfy

$$p_1 \ge 3, \quad p_2 \ge 3, \quad p_3 \ge 3,$$

while the existence theorems in nonlinear elasticity [1, 23] require much weaker coercivity conditions.

To prove the existence of the solution, we need the compensated compactness and the lower semicontinuity results in Theorems 3.3 and Theorem 3.5.

3.3 Theorem *Let p_i, $i = 1, \dots, m$, be a collection of numbers satisfying*

$$1 < p_i \le \infty \quad \text{and} \quad 1/p_1 + \ldots + 1/p_m \le 1$$

and let $\{\psi_{i,k}\}_{k=1}^\infty \subset L^{p_i}(\Omega, \wedge_{l_i})$, $i = 1, \dots, m$, be sequences such that with some $\psi_i \in L^{p_i}(\Omega, \wedge_{l_i})$ we have

$$\psi_{i,k} \rightharpoonup \psi_i \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^{p_i}(\Omega, \wedge_{l_i}) \tag{3.5}$$

and for each $i = 1, \dots, m$,

$$\text{the sequence } \{d\psi_{i,k}\}_{k=1}^\infty \text{ is bounded in } L^{q_i}(\Omega, \wedge_{l_i+1})$$

where $q_i \ge 1$ satisfy $np_i/(n + p_i) < q_i \le p_i$. Then

$$\bigwedge_{i=1}^m \psi_{i,k} \overset{*}{\rightharpoonup} \bigwedge_{i=1}^m \psi_i \quad \text{in} \quad \mathscr{M}(\Omega, \wedge_l) \tag{3.6}$$

where $l = l_1 + \ldots + l_m$, *and where* (3.6) *denotes the convergence in the sense of measures, i.e.,*

$$\int_\Omega \sigma \cdot \bigwedge_{i=1}^{m} \psi_{i,k}\, dx \to \int_\Omega \sigma \cdot \bigwedge_{i=1}^{m} \psi_i\, dx$$

for every continuous function $\sigma : \mathbb{R}^n \to \wedge_l$ *such that* $\sigma = 0$ *outside* Ω.

This is due to Robbin, Rogers & Temple [29; Theorem 1.1] and Iwaniec & Lutoborski [21; Theorem 5.1] (see also Remark 3.4). If $p_i = \infty$ then (3.5) should be understood to mean weak* convergence in $L^\infty(\Omega, \wedge_{l_i})$. The exterior derivatives in the above statement are understood in the weak sense as in Definition 10.3. However, in the proof below, Theorem 3.3 will be applied to sequences for which the weak exterior derivatives vanish.

3.4 Remarks

(i) Theorem 3.3 is a generalization of the div-curl lemma by Murat [24] and Tartar [34]. Their version is $m = 2, l_1 = 1, l_2 = n - 1$.

(ii) The L^∞ version of Theorem 3.3, i.e., the case $p_1 = \cdots = p_m = \infty$ much predates the celebrated work of Murat and Tartar: it is due to Whitney and dates back to 1957 [35; Chapter IX, Theorem 17A]. This, however, went unnoticed and the result was forgotten, the exception being the present author in [31]. Rephrasing slightly, we state Whitney's result as follows. If in the notation of Theorem 3.3 we have

$$\psi_{i,k} \overset{*}{\rightharpoonup} \psi_i \quad \text{as } k \to \infty \text{ in } L^\infty(\Omega, \wedge_{l_i}),$$

and for each $i = 1, \ldots, m$,

$$\text{the sequence } \{d\psi_{i,k}\}_{k=1}^\infty \text{ is bounded in } L^\infty(\Omega, \wedge_{l_i+1})$$

then

$$\bigwedge_{i=1}^{m} \psi_{i,k} \overset{*}{\rightharpoonup} \bigwedge_{i=1}^{m} \psi_i \quad \text{in } L^\infty(\Omega, \wedge_l).$$

Whitney proves the case $m = 2$; an obvious iteration of this particular case leads to the above statement.

3.5 Theorem (Reshetnyak [28], Ball & Murat [3]) *Let* $\Phi : \mathbb{R}^h \to \bar{\mathbb{R}}$ *be convex, lower semicontinuous and bounded below. If* θ, $\theta_k \in L^1(\Omega, \mathbb{R}^h)$ *satisfy*

$$\theta_k \overset{*}{\rightharpoonup} \theta \quad \text{in } \mathscr{M}(\Omega, \mathbb{R}^h)$$

then

$$\liminf_{j \to \infty} \int_\Omega \Phi(\theta_k)\, dx \ge \int_\Omega \Phi(\theta)\, dx.$$

Proof of Theorem 3.1 (i): Let $\boldsymbol{\omega}_k = (\omega_{1,k}, \ldots, \omega_{m,k}) \in \mathscr{D}$ be a minimization sequence. By Hypothesis $\mathbf{H}_2$, the sequence $\boldsymbol{\omega}_k$ is bounded in $L^{p_1}(\Omega, \wedge_{s(1)}) \times \cdots \times L^{p_m}(\Omega, \wedge_{s(m)})$. The reflexivity implies that there exists a subsequence, again denoted by $\boldsymbol{\omega}_k$, such that

$$\boldsymbol{\omega}_k \rightharpoonup \boldsymbol{\omega} \quad \text{in } L^{p_1}(\Omega, \wedge_{s(1)}) \times \cdots \times L^{p_m}(\Omega, \wedge_{s(m)}).$$

Theorem 3.3 and Hypothesis $\mathbf{H}_3$ imply that for every $\boldsymbol{r} \in \mathfrak{R}_s$ we have

$$\boldsymbol{\omega}_k^{\boldsymbol{r}} \xrightarrow{*} \boldsymbol{\omega}^{\boldsymbol{r}} \quad \text{in} \quad \mathscr{M}(\Omega, \wedge_{\dim(r)}).$$

By hypothesis $\mathbf{H}_1$, f is polyconvex and hence there exists a convex lower semicontinuous function $\Phi : \mathbb{R}^g \to \bar{\mathbb{R}}$ such that

$$f(\boldsymbol{\omega}) = \Phi(\boldsymbol{\omega}^{\boldsymbol{r}(1)}, \dots, \boldsymbol{\omega}^{\boldsymbol{r}(g)})$$

for each $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$. By Theorem 3.5,

$$\liminf_{k \to \infty} \int_\Omega \Phi(\boldsymbol{\omega}_k^{\boldsymbol{r}(1)}, \dots, \boldsymbol{\omega}_k^{\boldsymbol{r}(g)})\, dx \ge \int_\Omega \Phi(\boldsymbol{\omega}^{\boldsymbol{r}(1)}, \dots, \boldsymbol{\omega}^{\boldsymbol{r}(g)})\, dx.$$

This can be rewritten as

$$\liminf_{k \to \infty} \int_\Omega f(\boldsymbol{\omega}_k)\, dx \ge \int_\Omega f(\boldsymbol{\omega})\, dx.$$

As also

$$\lim_{k \to \infty} \int_\Omega \mathfrak{X}_i \cdot \omega_{i,k}\, dx = \int_\Omega \mathfrak{X}_i \cdot \omega_i\, dx,$$

$i = 1, \dots, m$, we have

$$\liminf_{k \to \infty} I(\boldsymbol{\omega}_k) \ge I(\boldsymbol{\omega}).$$

Since the condition (3.2) survives the limit, we see that $\boldsymbol{\omega}$ is in $\mathscr{D}$ and thus it minimzes I on $\mathscr{D}$. The proof of (i) is complete.

(ii): Follows immediately form $I(\boldsymbol{\omega}) < \infty$.

(iii): Let $\boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^n, \Delta_{\boldsymbol{s}})$. The hypothesis of (iii) implies that $\boldsymbol{\omega} + t\boldsymbol{\psi} \in \operatorname{dom} f$ for all sufficiently small $|t|$. Then

$$I(\boldsymbol{\omega} + t\boldsymbol{\psi}) \ge I(\boldsymbol{\omega}).$$

The differentiation with respect to t at $t = 0$ gives, standardly,

$$\int_\Omega \sum_{i=1}^m \psi_i \cdot (\mathrm{D}_{\omega_i} f - \mathfrak{X}_i)\, dx = 0$$

and hence

$$\int_\Omega \psi_i \cdot (\mathrm{D}_{\omega_i} f - \mathfrak{X}_i)\, dx = 0 \tag{3.7}$$

for each $i = 1, \dots, m$. If $\sigma_i \in C_0^\infty(\mathbb{R}^n, \wedge_{s(i)-1})$ then $\omega_i := d\sigma_i$ satisfies $d\omega_i = 0$ and (3.7) reduces to (3.4). □

3.6 Remark If Ω has a Lipschitz boundary and $\mathfrak{S}_i := \mathrm{D}_{\omega_i} f$ and $\mathfrak{X}_i$ are continuously differentiable on $\operatorname{cl} \Omega$ then (3.3) is equivalent to a more standard form of equilibrium equations

$$\operatorname{div} \mathfrak{S}_i + \mathfrak{b}_i = 0 \quad \text{in} \quad \Omega, \qquad \mathfrak{S}_i \llcorner \nu = \mathfrak{s}_i \quad \text{on} \quad \partial\Omega, \tag{3.8}$$

where ν is the normal to $\partial\Omega$ and

$$\mathfrak{b}_i = -\operatorname{div} \mathfrak{X}_i, \quad \mathfrak{s}_i = \mathfrak{X}_i \llcorner \nu,$$

$i = 1, \dots, m$. Drawing a mechanical analogy, $\mathfrak{S}_i$ is the stress, $\mathfrak{b}_i$ the body force, and $\mathfrak{s}_i$ the surface traction.

Proof We use the identity (10.6) to rewrite (3.4) to obtain

$$\int_\Omega \big(\operatorname{div}_0((\mathfrak{S}_i - \mathfrak{X}_i) \llcorner \sigma) - \sigma \cdot \operatorname{div}(\mathfrak{S}_i - \mathfrak{X}_i)\big)\, dx = 0;$$

then we apply the classical divergence theorem (for vector fields) to the first term

$$\int_{\partial\Omega} ((\mathfrak{S}_i - \mathfrak{X}_i) \llcorner \sigma) \cdot \nu \, dA(x) - \int_\Omega \sigma \cdot \operatorname{div}(\mathfrak{S}_i - \mathfrak{X}_i)\, dx = 0.$$

The arbitrariness of σ then provides

$$\operatorname{div}(\mathfrak{S}_i - \mathfrak{X}_i) = 0 \quad \text{in} \quad \Omega, \qquad (\mathfrak{S}_i - \mathfrak{X}_i) \llcorner \nu = 0 \quad \text{on} \quad \partial\Omega,$$

which then gives (3.8). □

4 Example A: Classical calculus of variations and nonlinear elasticity

4.1 The integrand and its variables The classical calculus of variations deals with the integral functionals of the form

$$I(u_1, \dots, u_m) = \int_\Omega f(\nabla u_1, \dots, \nabla u_m)\, dx \tag{4.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ as always and $u = (u_1, \dots, u_m)$ is an m-tuple of scalar dependent variables. The integral in (4.1) is directly in the format (1.1) with the 1-forms

$$\omega_i = \nabla u_i.$$

The forms ω_i satisfy (1.2) since for 1-forms the exterior derivative reduces to curl and we have curl $\nabla u_i = 0$ by the interchangeability of the exterior derivatives. The orders of the forms are $s(1) = \dots = s(m) = 1$, the vector of orders is $\boldsymbol{s} = (1, \dots, 1) \in \mathbb{R}^m$ and the domain of f is $\Delta_{\boldsymbol{s}} := [\wedge_1^n]^m$, where here and below we use the notation

$$\wedge_h^m := \wedge_h \mathbb{R}^m \quad \text{and} \quad \wedge_h^n := \wedge_h \mathbb{R}^n$$

for any nonnegative integer h satisfying

$$0 \le h \le q := \min\{m, n\}$$

in the notation of Section 9.

We denote the integrand collectively as $F = \nabla u$. We interpret its pointwise value $F(x)$ either as a matrix in $\mathbb{M}^{m \times n}$ or as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., as an element of $\operatorname{Lin}(\mathbb{R}^n, \mathbb{R}^m)$. Below we identify

$$\mathbb{M}^{m \times n} \simeq [\wedge_1^n]^m$$

by writing

$$F \simeq \boldsymbol{\omega} \tag{4.2}$$

for any $F = [F_{iA}] \in \mathbb{M}^{m \times n}$ and any $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in [\wedge_1^n]^m$ if

$$\omega_i = \sum_{A=1}^n F_{iA}\, e_A, \tag{4.3}$$

$i = 1, \dots, m$, where $e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n$.

It is immediate that in the present case the mult. ext. quasiconvexity coincides with Morrey's quasiconvexity, mult. ext. quasiaffinity with the quasiaffinity derived from Morrey's quasiconvexity, and mult. ext. one convexity with the rank one convexity. We now turn to discussing the relationship between the mult. ext. polyconvexity and Ball's polyconvexity [1].

4.2 Mult. ext. polyconvexity and Ball's polyconvexity Recall that a function $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ is mult. ext. polyconvex if it can be written as a convex lower semicontinuous function of the collection $\mathfrak{F}_{\boldsymbol{s}}$ of admissible powers in (2.6). On the other hand, a function $f : \mathbb{R}^{m\times n}$ is said to be polyconvex in Ball's sense if it can be written as a convex lower semicontinuous function of the set $\mathfrak{M}_{\boldsymbol{s}}$ of all minors of $F = [F_{iA}]$, given by

$$\mathfrak{M}_{\boldsymbol{s}} = \{F_{IJ}^{(h)} : h = 1, \dots, q,\ I \in \mathbb{I}_h^m,\ J \in \mathbb{I}_h^n\},$$

where $\mathbb{I}_h^m$ and $\mathbb{I}_h^n$ are the sets of m-dimensional and n-dimensional multiindices of order h (see (1.11)) and

$$F_{IJ}^{(h)} := \det [F_{I_a J_b}]_{1\le a,\, b\le h}.$$

Thus to prove the equivalence of the two notions of polyconvexity, it suffices to show that under the identification (4.2)–(4.3) we have a bijective linear correspondence between the functions in $\mathfrak{F}_{\boldsymbol{s}}$ and those in $\mathfrak{M}_{\boldsymbol{s}}$. Since $\boldsymbol{s} = (1, \dots, 1) \in \mathbb{R}^m$, the collection of admissible exponents (2.5) is

$$\mathfrak{R}_{\boldsymbol{s}} = \{\boldsymbol{r} = (r_1, \dots, r_m) : r_i \in \{0, 1\},\ \sum_{i=1}^m r_i \le n\}.$$

If, for a given $\boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}}$ we denote by h the number of the occurrences of the value 1 in the sequence $\boldsymbol{r}$ and by $I = (I_1, \dots, I_h)$ the increasing sequence of the indices $i \in \{1, \dots, m\}$ with $r_i = 1$, then $I \in \mathbb{I}_h^m$ and

$$\boldsymbol{\omega}^{\boldsymbol{r}} = \boldsymbol{\omega}_I := \omega_{I_1} \wedge \dots \wedge \omega_{I_h}. \tag{4.4}$$

for every $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in [\wedge_1^n]^m$. Since the correspondence between $\boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}}$ and $I \in \mathbb{I}_0^m \cup \ldots \cup \mathbb{I}_q^m$ is bijective, we have

$$\mathfrak{F}_{\boldsymbol{s}} = \{\omega_{I_1} \wedge \dots \wedge \omega_{I_h} : I \in \mathbb{I}_h^m,\ 0 \le h \le q\}.$$

If $\boldsymbol{\omega}$ and $F \in \mathbb{M}^{m\times n}$ are related as in (4.3), then writing $\omega_{I_a} = \sum_{A_a=1}^n F_{I_a A_a} e_{A_a}$, inserting these expressions into (4.4) and expanding the products, we obtain

$$\boldsymbol{\omega}_I = \sum_{A_1=1, \dots, A_h=1}^{n} F_{I_1 A_1} \cdots F_{I_h A_h} e_{A_1} \wedge \dots \wedge e_{A_h}. \tag{4.5}$$

We now note that $e_{A_1} \wedge \dots \wedge e_{A_h} \neq 0$ if and only if there exists a permutation $\pi \in \mathbb{P}_h$ and $J = (J_1, \dots, J_q) \in \mathbb{I}_h^n$ such that

$$(A_1, \dots, A_h) = (J_{\pi(1)}, \dots, J_{\pi(h)})$$

and if this is the case, then

$$e_{A_1} \wedge \dots \wedge e_{A_h} = \operatorname{sgn}(\pi)\, e_J.$$

Thus (4.5) reduces to

$$\boldsymbol{\omega}_I = \sum_{J\in\mathbb{I}_h^n} \sum_{\pi\in\mathbb{P}_h} \operatorname{sgn}(\pi)\, F_{I_1 J_{\pi(1)}} \cdots F_{I_h J_{\pi(h)}}\, e_J = \sum_{J\in\mathbb{I}_h^n} F_{IJ}^{(h)} e_J.$$

This formula establishes the desired bijective correspondence between $\mathfrak{F}_{\boldsymbol{s}}$ and $\mathfrak{M}_{\boldsymbol{s}}$ and hence the desired equivalence of the mult. ext. polyconvexity and polyconvexity in Ball's sense.

In view of Propositions 9.6 and 9.7 yet another (slightly more natural) description is in terms of the exterior powers $\wedge_h F$. One can say that $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ is mult. ext. polyconvex (and hence polyconvex in Ball's sense) if and only if

$$f(F) = \Phi(\wedge_1 F, \dots, \wedge_q F),$$

$F \in \mathbb{M}^{m\times n}$, where Φ is a convex lower semicontinuous function defined on

$$\mathrm{Lin}(\wedge_1^n, \wedge_1^m) \times \cdots \times \mathrm{Lin}(\wedge_q^n, \wedge_q^m).$$

5 Example B: Electro-magneto-elastostatics

The statics of electro- and magneto-sensitive elastomers has received considerable attention in recent years [6, 12–15, 7–8, 11, 20] in view of their technological applications. The main point in modeling these materials is the coupling of the nonlinear mechanical response with the electric or magnetic response. The goal of this section is to determine the mult. ext. polyconvexity corresponding to this case; the reader is referred to [32] for more details. Let $\Omega \subset \mathbb{R}^n$ where $n = 2$ or 3.

The basic electromagnetic variables are the referential (lagrangean) electric displacement D, magnetic induction B, the electric field E and the magnetic field H, defined on the entire space $\mathbb{R}^3$, satisfying the static Maxwell's equations

$$\left.\begin{aligned} \operatorname{div} D &= 0, \qquad \operatorname{div} B = 0, \\ \operatorname{curl} E &= 0, \qquad \operatorname{curl} H = 0, \end{aligned}\right\} \quad \text{in} \quad \mathbb{R}^n \tag{5.1}$$

where div and curl are the referential versions of these operators, and the equations are undersood in the weak sense. The mechanical variables are the deformation $u : \Omega \to \mathbb{R}^n$, deformation gradient $F = \nabla u$ and the referential stress $S : \Omega \to \mathbb{M}^{n\times n}$. The latter satisfies

$$\begin{aligned} \operatorname{div} S + b = 0 \qquad & \text{in} \quad \Omega, \\ Sv = s_0 \qquad & \text{on} \quad \partial\Omega, \end{aligned} \tag{5.2}$$

where b is the body force, v is the normal to $\partial\Omega$, and s_0 is the surface traction on $\partial\Omega$.

To formulate the constitutive equations, we note that many choices of independent/dependent variables are possible. We take the triplet (F, D, B) as independent variables, start from the free energy function $f : \mathbb{M}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, i.e., $f = f(F, D, B)$, and put

$$S = \mathrm{D}_F f, \quad E = \mathrm{D}_D f, \quad H = \mathrm{D}_B f.$$

The variational formulation seeks the equilibrium states as minimizers of the total energy; then the divergence equations $(5.1)_1$ enter into the definition of admissible variations (competitors) in the minimum energy principle. On the other hand, the curl equations $(5.1)_2$ and the mechanical equilibrium (5.2) will follow as the Euler-Lagrange equations for the minimizer.

The energy of the body is given by

$$I(u, D, B) = \int_\Omega f(F, D, B)\, dx; \tag{5.3}$$

the total energy then consists of this term plus the energy of the vacuum electromagnetic field in the exterior of Ω and the term describing the loads. The integral in (5.3) falls within the format (1.1) under the identifications which we now describe separately for $n = 3$ and 2.

5.1 Dimension three Here $m = 5$ and the forms $\omega_1, \ldots, \omega_5$ are as follows:

$$\omega_i = F_{i1} dx_1 + F_{i2} dx_2 + F_{i3} dx_3, \quad 1 \le i \le 3,$$
$$\omega_4 = D_1 d\hat{x}_1 + D_2 d\hat{x}_2 + D_3 d\hat{x}_3, \quad \omega_5 = B_1 d\hat{x}_1 + B_2 d\hat{x}_2 + B_3 d\hat{x}_3, \tag{5.4}$$

where

$$d\hat{x}_i = \frac{1}{2} \sum_{i,j=1}^{3} \varepsilon_{ijk} dx_j \wedge dx_k$$

and F is the deformation gradient. Thus ω_1, ω_2, and ω_3 are 1-forms and ω_4 ω_5 are 2-forms. The constraints (1.2) with $1 \le i \le 3$ follow from the interchangeability of the second partial derivatives while (1.2) for $i = 4, 5$ are equivalent to $(5.1)_1$ since (10.4) gives

$$d\omega_4 = \sum_{i,j=1}^{3} D_{j,i} dx_i \wedge d\hat{x}_j = \sum_{i=1}^{3} D_{i,i} dx_1 \wedge dx_2 \wedge dx_3$$

because $dx_i \wedge d\hat{x}_j = \delta_{ij} dx_1 \wedge dx_2 \wedge dx_3$.

To determine the mult. ext. quasiaffine functions, we refer to Theorem 2.8, which reduces the question to determining the list $\mathfrak{F}_s$ in (2.6) where $\mathfrak{R}_s$ is defined in (2.5). This construction of $\mathfrak{F}_s$ can be rephrased as follows: **(a)** construct all possible multiple exterior products of the elements of the set

$$\omega_1, \quad \omega_2, \quad \omega_3, \quad \omega_4, \quad \omega_5$$

(including the products of an element with itself) which result in forms of order $\le n$. **(b)** Then eliminate all redundancies from the resulting list (as some products will occur several times or will differ by the sign or will vanish). **(c)** The result is a set which differs from $\mathfrak{F}_s$ at most by the signs of its elements. This procedure produces the following list of mult. ext. quasiaffine functions

$$1, \quad \omega_i, \quad \omega_j \wedge \omega_k, \quad \omega_1 \wedge \omega_2 \wedge \omega_3, \quad 1 \le i \le 3, \quad 1 \le j < k \le 3, \tag{5.5}$$
$$\omega_4, \quad \omega_5, \quad \omega_i \wedge \omega_4, \quad \omega_i \wedge \omega_5, \quad 1 \le i \le 3. \tag{5.6}$$

This is isomorphic to

$$1, \quad F, \quad \operatorname{cof} F, \quad \det F, \quad D, \quad B, \quad FD, \quad FB. \tag{5.7}$$

Indeed, it was shown in Section 4 that (5.5) is isomorphic to the first four members of (5.7); the first two members of (5.6) of course correspond to D and B, and for the remaining two memebrs it suffices to note that

$$\omega_i \wedge \omega_4 = (FD)_i \, dx_1 \wedge dx_2 \wedge dx_3, \quad \omega_i \wedge \omega_5 = (FB)_i \, dx_1 \wedge dx_2 \wedge dx_3.$$

The list (5.7) was determined by a direct calculation in [32]. Thus the free energy $f = f(F, D, B)$ is mult. ext. polyconvex if there exists a convex function Φ such that

$$f(F, D, B) = \Phi(F, \operatorname{cof} F, \det F, D, B, FD, FB) \tag{5.8}$$

for each $(F, D, B) \in \mathbb{M}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n$ [32; Theorem 6.5].*

* After the research presented in this section and in [32] had been completed, I became aware of the recent papers by Gil & Ortigosa [19] and Ortigosa & Gil1 [26–27]. (I thank M. Itskov for

5.2 Dimension two Then (5.3) takes the form (1.1) with $m = 4$ and with the 1-forms $\omega_1, \ldots, \omega_4$ given by

$$\omega_i = F_{i1}\,dx_1 + F_{i2}\,dx_2, \quad 1 \le i \le 2,$$
$$\omega_3 = D_1\,dx_2 - D_2\,dx_1, \quad \omega_4 = B_1\,dx_2 - B_2\,dx_1.$$

The reader will have no difficulty to check that the list $\mathfrak{F}_s$ of mult. ext. quasiaffine functions is

$$1, \quad \omega_1, \quad \omega_2, \quad \omega_3, \quad \omega_4, \quad \omega_i \wedge \omega_j, \quad 1 \le i < j \le 4,$$

which is isomorphic to

$$1, \quad F, \quad \det F, \quad D, \quad B, \quad FD, \quad FB, \quad D \times B. \tag{5.9}$$

We note that the term $D \times B$ comes from the 2-form $\omega_3 \wedge \omega_4 \in \mathfrak{F}_s$ since ω_3 and ω_4 are 1-forms. This has no analog in dimension $n = 3$ since the corresponding term $\omega_4 \wedge \omega_5$ (where ω_4, ω_5 are as in (5.4)), being a product of two 2-forms, is a 4-form in dimension 3, and hence $\omega_4 \wedge \omega_5$ vanishes. From (5.9) one finds that $f = f(F, D, B)$ is mult. ext. polyconvex if there exists a convex function Φ such that

$$f(F, D, B) = \Phi(F, \det F, D, B, FD, FB, D \times B)$$

[32; Theorem 6.5].

6 Proof of Proposition 2.3

We shall prove

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}.$$

(i) $\Rightarrow$ (ii): Immediate.

(ii) $\Rightarrow$ (iv): If $\boldsymbol{\xi} \in C_0^\infty(\mathbb{R}^n, \Gamma_s)$ then $\boldsymbol{\psi} := d\boldsymbol{\xi}$ satisfies (2.3) and hence (2.2) reduces to (2.4).

(iv) $\Rightarrow$ (iii): Assume that we have

$$\int_Q f(\boldsymbol{\omega} + d\boldsymbol{\theta}(x))\,dx \ge f(\boldsymbol{\omega}) \tag{6.1}$$

for every $\boldsymbol{\omega} \in \Delta_s$ and every $\boldsymbol{\theta} \in C_0^\infty(\mathbb{R}^n, \Gamma_s)$ and prove (2.4) for every $\boldsymbol{\xi} \in C_{\mathrm{per}}^\infty(\mathbb{R}^n, \Gamma_s)$. A scaling argument shows that (6.1) implies that for any positive integer a we have a similar condition for the cube $Q_a = (0, a)^n$, namely,

$$\int_{Q_a} f(\boldsymbol{\omega} + d\boldsymbol{\theta}(x))\,dx \ge a^n f(\boldsymbol{\omega}) \tag{6.2}$$

for all $\boldsymbol{\theta} \in C_0^\infty(Q_a, \Gamma_s)$. Let $q \in C^\infty(\mathbb{R}, \mathbb{R})$ be any function such that $q = 0$ on $(-\infty, 0]$ and $q = 1$ on $[1, \infty)$. Let $m_a : \mathbb{R}^n \to \mathbb{R}$ be defined by

drawing my attention to these papers.) In [19] and [26–27], the authors *postulate,* under the name multi-variable convexity, Condition (5.8) as a convenient way to satisfy the electro-magneto-elastic ellipticity condition. The relationship of their condition to the $\mathscr{A}$-quasiconvexity theory, the main topic of the present section, is not studied in these papers.

$$m_a(x) = q(x_1)q(a - x_1) \cdots q(x_n)q(a - x_n)$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then $m_a \in C_0^\infty(Q_a, \mathbb{R})$ and putting

$$Z_a := (1, a - 1)^n, \quad Z_a^{\mathrm{c}} := Q_a \sim (1, a - 1)^n,$$

we have

$$m_a = 1 \quad \text{on} \quad Z_a, \quad m_a = 0 \quad \text{on} \quad \mathbb{R}^n \sim Z_a, \quad |m_a| + |\nabla m_a| \le C \quad \text{on} \quad Z_a^{\mathrm{c}},$$

where C is independent of a. If $\boldsymbol{\xi} \in C_{\mathrm{per}}^\infty(\mathbb{R}^n, \Gamma_s)$ then $\boldsymbol{\theta}_a := m_a \boldsymbol{\xi} \in C_0^\infty(Q_a, \Gamma_s)$ and the formula $d\boldsymbol{\theta}_a = m_a d\boldsymbol{\xi} + \nabla m_a \wedge \boldsymbol{\xi}$ shows that

$$d\boldsymbol{\theta}_a = d\boldsymbol{\xi} \quad \text{on} \quad Z_a \quad \text{and} \quad |d\boldsymbol{\theta}_a| \le C \quad \text{on} \quad Z_a^{\mathrm{c}} \tag{6.3}$$

where the bound C is independent of a. The left-hand side of (6.2) is written as the sum of

$$\int_{Z_a} f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))\, dx \quad \text{and} \quad \int_{Z_a^{\mathrm{c}}} f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))\, dx;$$

noting that

$$\int_{Z_a} f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))\, dx = (a - 2)^n \int_Q f(\boldsymbol{\omega} + d\boldsymbol{\xi}(x))\, dx$$

by the periodicity of $\boldsymbol{\xi}$, we obtain

$$(a - 2)^n \int_Q f(\boldsymbol{\omega} + d\boldsymbol{\theta}(x))\, dx + \int_{Z_a^{\mathrm{c}}} f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))\, dx \ge a^n f(\boldsymbol{\omega}). \tag{6.4}$$

Next we note that

$$\Big| \int_{Z_a^{\mathrm{c}}} f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))\, dx \Big| \le a^{n-1} C \tag{6.5}$$

where the bound C is independent of a since the volume of the boundary layer Z_a^{c} satisfies $\mathscr{L}^n(Z_a^{\mathrm{c}}) \le n a^{n-1}$ and since $|f(\boldsymbol{\omega} + d\boldsymbol{\theta}_a(x))| \le C$ on Z_a^{c} by $(6.3)_2$. Dividing (6.4) by a^n, letting $a \to \infty$ and noting that the second term on the left-hand side tends to 0 by (6.5), we obtain (2.4).

(iii) $\Rightarrow$ (i): Suppose that (iii) holds and prove (i). Thus let $\boldsymbol{\omega} \in \Delta_s$ and let $\boldsymbol{\psi} \in C_{\mathrm{per}}^\infty(\mathbb{R}^n, \Delta_s)$ satisfy (2.3), and prove that (2.2) holds. Since every such a $\boldsymbol{\psi}$ can be approximated by a sequence of trigonometric polynomials in $C_{\mathrm{per}}^\infty(\mathbb{R}^n, \Delta_s)$ satisfying (2.3), we can assume from the start that $\boldsymbol{\psi}$ is a trigonometric polynomial, i.e.,

$$\boldsymbol{\psi}(x) = \sum_{k \in \mathbb{Z}^n} \tilde{\boldsymbol{\psi}}_k \mathrm{e}^{2\pi \mathrm{i} k \cdot x},$$

$x \in \mathbb{R}^n$, where $\tilde{\boldsymbol{\psi}}_k \in \Delta_s$ and only finitely many $\tilde{\boldsymbol{\psi}}_k$ are different from 0. The conditions (2.3) give

$$k \wedge \tilde{\boldsymbol{\psi}}_k = 0 \quad \text{for all} \quad k \in \mathbb{Z}^n \quad \text{and} \quad \tilde{\boldsymbol{\psi}}_0 = 0 \tag{6.6}$$

and we have additionally the reality conditions

$$\tilde{\boldsymbol{\psi}}_k^* = \tilde{\boldsymbol{\psi}}_{-k} \tag{6.7}$$

where now we work (tacitly) in the complexifications of the spaces, and $*$ denotes the complex conjugation. The first of (6.6) implies that for each $k \in \mathbb{Z}^n$, $k \neq 0$, there exists an $\tilde{\boldsymbol{\xi}}_k \in \Gamma_s$ such that

$$\tilde{\psi}_k = k \wedge \tilde{\xi}_k \tag{6.8}$$

and only finitely many $\tilde{\xi}_k$ are different from 0. Moreover, Condition (6.7) and the 'odd' nature of the requirement (6.8) shows that the coefficients $\tilde{\psi}_k$ can be chosen as to satisfy $\tilde{\xi}_k^* = -\tilde{\xi}_{-k}$. Thus, putting $\tilde{\xi}_0 = 0$ and defining ξ by

$$\xi(x) = (2\pi\mathrm{i})^{-1} \sum_{k \in \mathbb{Z}^n} \tilde{\xi}_k \mathrm{e}^{2\pi\mathrm{i} k \cdot x},$$

we see that ξ takes only real values and $\xi \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \Gamma_s)$. Thus (2.4) holds for this ξ and since by the construction $\psi = d\xi$, we see that (2.4) reduces to (2.2). □

7 Proof of Theorem 2.8

We prove Theorem 2.8 by establishing the following cycle of implications:

$$\left.\begin{array}{ll} \text{mult. ext. affinity} & \Rightarrow \\ \text{mult. ext. one affinity} & \Rightarrow \\ \text{the explicit form (2.7)} & \Rightarrow \\ f \text{ is a mult. ext. null lagrangian} & \Rightarrow \\ \text{mult. ext. affinity.} & \end{array}\right\} \tag{7.1}$$

The first implication is the general assertion in Theorem 8.4; one has only to verify that the general definition (8.2) of the characteristic cone reduces to (2.1) in the case of the constraints given by (1.2). But this is immediate.

We now turn to the remaining three implications in (7.1).

7.1 Lemma *Any mult. ext. one affine integrand f of type $\boldsymbol{s}$ is a polynomial of degree $\le n$ in the components of $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$; moreover, for any integer $p \ge 2$ the derivative $\mathrm{D}^p f(\boldsymbol{\omega})$ of f of order p at any point $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ satisfies*

$$\mathrm{D}^p f(\boldsymbol{\lambda}_1 \wedge \eta_{\tau(1)}, \dots, \boldsymbol{\lambda}_p \wedge \eta_{\tau(p)}) = \operatorname{sgn}(\tau)\, \mathrm{D}^p f(\boldsymbol{\lambda}_1 \wedge \eta_1, \dots, \boldsymbol{\lambda}_p \wedge \eta_p). \tag{7.2}$$

for any $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p \in \Gamma_{\boldsymbol{s}}$, $\eta_1, \dots, \eta_p \in \mathbb{R}^n$ and $\tau \in \mathbb{P}_p$. Here and below we consistently omit the argument $\boldsymbol{\omega}$ of $\mathrm{D}^p f$.

By Definition 9.4, the alternating property (7.2) is equivalent to the condition

$$\mathrm{D}^p f(\boldsymbol{\lambda}_1 \wedge \eta_1, \dots, \boldsymbol{\lambda}_p \wedge \eta_p) = 0 \tag{7.3}$$

whenever $\eta_1, \dots, \eta_p$ are linearly dependent. In the broader context of the theory of compensated compactness, the analog of (7.3) is a necessary, and under the constant rank assumption also sufficient condition for the weak continuity of a function f (see [34; Theorem 18], [25; Theorem 3.4]). Here f is a function of a general variable, not necessarily of a collection of differential forms. Alternatively, the analog of (7.3) is necessary for f to be quasiaffine in the context of higher-order variational problems (see [2; Theorem 3.4]). Nevertheless, Lemma 7.1 does not follow from any of these results, because its hypothesis is (slightly) weaker.

Proof (Cf. [30; Proof of Propositions 13.5.2 and 13.5.3].) Let f be a mult. ext. one affine integrand of type $\boldsymbol{s}$. Prove first the assertion of the lemma under the additional assumption that f is infinitely differentiable.

(ii): Differentiating the mult. ext. one affinity condition

$$f(t\boldsymbol{\omega}_1 + (1-t)\boldsymbol{\omega}_2) = tf(\boldsymbol{\omega}_1) + (1-t)f(\boldsymbol{\omega}_2)$$

with respect to t twice at $t = 0$ and using $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 = \boldsymbol{\lambda}_1 \wedge \eta_1$ for some $\boldsymbol{\lambda}_1 \in \Gamma_s$ and $\eta \in \mathbb{R}^n$, we obtain

$$\mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_1 \wedge \eta_1) = 0. \tag{7.4}$$

Next, we shall employ twice the following direct consequence of the polarization identity, which we record formally for a future convenience: If $\mathrm{B}(\cdot,\cdot)$ is a bilinear form on a vector space X then

$$\mathrm{B}(a,a) = 0 \quad \text{for all} \quad a \in \mathrm{X} \quad \Leftrightarrow \quad \mathrm{B}(a,b) + \mathrm{B}(b,a) = 0 \quad \text{for all} \quad a,\ b \in \mathrm{X}. \tag{7.5}$$

Recalling (7.4) and applying (7.5) to the bilinear form

$$\mathrm{B}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_2 \wedge \eta_1)$$

one obtains $\mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_2 \wedge \eta_1) + \mathrm{D}^2 f(\boldsymbol{\lambda}_2 \wedge \eta_1, \boldsymbol{\lambda}_1 \wedge \eta_1) = 0$ and hence

$$\mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_1 \wedge \eta_2) = 0$$

by the interchangeability of the second partial derivatives. Thus applying (7.5) to the bilinear form

$$\mathrm{B}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_2 \wedge \eta_2) = 0$$

and using the interchangeability of the second partial derivatives we obtain

$$\mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_2 \wedge \eta_2) + \mathrm{D}^2 f(\boldsymbol{\lambda}_1 \wedge \eta_2, \boldsymbol{\lambda}_2 \wedge \eta_1) = 0.$$

Differentiating the last identity $p-2$ times in the directions $\boldsymbol{\lambda}_3 \wedge \eta_3, \dots, \boldsymbol{\lambda}_r \wedge \eta_r$ we obtain

$$\begin{aligned} &\mathrm{D}^p f(\boldsymbol{\lambda}_1 \wedge \eta_1, \boldsymbol{\lambda}_2 \wedge \eta_2, \boldsymbol{\lambda}_3 \wedge \eta_3, \dots, \boldsymbol{\lambda}_r \wedge \eta_r) \\ &\qquad + \mathrm{D}^p f(\boldsymbol{\lambda}_1 \wedge \eta_2, \boldsymbol{\lambda}_2 \wedge \eta_1, \boldsymbol{\lambda}_3 \wedge \eta_3, \dots, \boldsymbol{\lambda}_r \wedge \eta_r) = 0. \end{aligned}$$

This establishes (7.2) for the special case of the permutation τ which interchanges the first two indices in $\{1, 2, \dots, p\}$. Using the invariance of $\mathrm{D}^p f$ under all permutations from $\mathbb{P}_p$, one extends (7.2) to any permutation τ which interchanges any pair of indices in $\{1, 2, \dots, p\}$. Since any permutation is a composition of these special cases, one establishes (7.2) generally.

Applying (7.2) with $p = n + 1$ and using the elementary fact that there is no nonzero alternating $n+1$-form on $\mathbb{R}^n$, we obtain

$$\mathrm{D}^{n+1} f(\boldsymbol{\lambda}_1 \wedge \eta_1, \dots, \boldsymbol{\lambda}_{n+1} \wedge \eta_{n+1}) = 0 \tag{7.6}$$

for each $\boldsymbol{\omega} \in \Delta_s$, each $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n+1} \in \Gamma_s$ and each $\eta_1, \dots, \eta_{n+1} \in \mathbb{R}^n$. Since

$$\operatorname{span}\{\boldsymbol{\lambda} \wedge \eta : \boldsymbol{\lambda} \in \Gamma_s,\ \eta \in \mathbb{R}^n\} = \Delta_s,$$

Equation (7.6) implies

$$\mathrm{D}^{n+1} f(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{n+1}) = 0$$

for any $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_{n+1} \in \Delta_s$ by the $n+1$-linearity of $\mathrm{D}^{n+1} f$. Thus f is a polynomial of degree at most n.

This proves the lemma under the additional assumption that f is infinitely differentiable. If f is merely continuous, we approximate it by the sequence f_ρ, $\rho > 0$, of mollifications of f. Clearly, the functions f_ρ are mult. ext. one affine as a consequence of the same property of f. By the already proved part, each f_ρ is a polynomial of degree $\le n$ and hence the limit f of f_ρ is again a polynomial of degree $\le n$. □

7.2 Remark Let f be a mult. ext. one affine integrand f of type $\boldsymbol{s}$. By Lemma 7.1, f is a polynomial in the variable $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ of degree $\deg(f) \le n$. Let us decompose f into the homogeneous polynomials, i.e., let us write

$$f = \sum_{p=0}^{\deg(f)} g_p \tag{7.7}$$

where each g_p is a homogeneous polynomial in the variable $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$ of degree p, i.e., one satisfying $g_p(t\boldsymbol{\omega}) = t^p g_p(\boldsymbol{\omega})$ for each $t \in \mathbb{R}$. An easy scaling argument shows that the mult. ext. one affinity of f is inherited by each of g_p separately. Thus to establish the general form of an mult. ext. one affine integrand f, we can consider only homogeneous polynomials of degrees $p = 0, \dots, n$, and to sum the results of these particular analyses at the end.

A crucial step in the succeeding analysis of mult. ext. one affine integrands is that even homogeneous degree p mult. ext. one affine integrands can be decomposed into smaller and more tractable pieces, each of which is mult. ext. one affine as well, as Equation (7.9) of the next proposition shows.

This requires the following notation. For any integer $p \in \{0, \dots, n\}$ let $\mathscr{B}(\boldsymbol{s}, p)$ be the set of all sequences $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$ of nonnegative integers satisfying $\sum_{i=1}^m \beta_i = p$. For any $\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)$ let $(\zeta(\boldsymbol{\beta}, 1), \dots, \zeta(\boldsymbol{\beta}, p))$ be a p-tuple of positive integers with the components $\zeta(\boldsymbol{\beta}, k)$, $1 \le k \le p$, given by

$$\zeta(\boldsymbol{\beta}, k) = \begin{cases} s(1) & \text{if } \ 1 \le k \le \beta_1, \\ s(2) & \text{if } \ \beta_1 + 1 \le k \le \beta_1 + \beta_2, \\ \vdots & \\ s(i) & \text{if } \ \beta_1 + \ldots + \beta_{i-1} + 1 \le k \le \beta_1 + \ldots + \beta_i, \\ \vdots & \\ s(m) & \text{if } \ \beta_1 + \ldots + \beta_{m-1} + 1 \le k \le \beta_1 + \ldots + \beta_m. \end{cases}$$

For any object c and any nonnegative integer β let $[\![\mathrm{c}]\!]_\beta$ be the β-tuple $\underbrace{(\mathrm{c}, \dots, \mathrm{c})}_{\beta \text{ times}}$ if $\beta > 0$ and $[\![\mathrm{c}]\!]_\beta = $ nothing if $\beta = 0$.

7.3 Proposition *Let f be a mult. ext. one affine integrand that is simultaneously a homogeneous polynomial of degree p. Then for each $\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)$ there exists a real-valued p-linear form $G_{\boldsymbol{\beta}} = G_{\boldsymbol{\beta}}(\sigma_1, \dots, \sigma_p)$ of p variables*

$$(\sigma_1, \dots, \sigma_p) \in \wedge_{\zeta(\boldsymbol{\beta},1)} \times \wedge_{\zeta(\boldsymbol{\beta},2)} \times \cdots \times \wedge_{\zeta(\boldsymbol{\beta},p)} \tag{7.8}$$

with the following two properties:

(i) *for each $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in \Delta_{\boldsymbol{s}}$,*

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)} G_{\boldsymbol{\beta}}([\![\omega_1]\!]_{\beta_1}, \dots, [\![\omega_m]\!]_{\beta_m}); \tag{7.9}$$

(ii) *we have*

$$G_{\boldsymbol{\beta}}(\lambda_1 \wedge \eta_{\tau(1)}, \dots, \lambda_p \wedge \eta_{\tau(p)}) = \operatorname{sgn}(\tau)\, G_{\boldsymbol{\beta}}(\lambda_1 \wedge \eta_1, \dots, \lambda_p \wedge \eta_p) \tag{7.10}$$

whenever

$$(\lambda_1, \dots, \lambda_p) \in \wedge_{\zeta(\boldsymbol{\beta},1)-1} \times \wedge_{\zeta(\boldsymbol{\beta},2)-1} \times \cdots \times \wedge_{\zeta(\boldsymbol{\beta},p)-1},$$
$$(\eta_1, \dots, \eta_p) \in [\mathbb{R}^n]^p, \quad \text{and} \quad \tau \in \mathbb{P}_p.$$

Each expression

$$G_{\boldsymbol{\beta}}([\![\omega_1]\!]_{\beta_1}, \dots, [\![\omega_m]\!]_{\beta_m}) \tag{7.11}$$

in the decomposition (7.9) contains exactly β_1 repetitions of ω_1, β_2 repetitions of ω_2, etc., with some of the ω_k's (possibly) omitted. Each block (7.11) has the alternating property (7.10), which, of course, is inherited from the same property of f, as stated in (7.2).

Proof For any $i \in \{1, \dots, m\}$ and any object c, we denote by $\langle \mathrm{c}; i\rangle$ the m-tuple

$$\langle \mathrm{c}; i\rangle = (0, \dots, 0, \mathrm{c}, 0, \dots, 0)$$

with the entry c on the i-th place. Writing $\boldsymbol{\omega} = \sum_{i=1}^m \langle \omega_i; i\rangle$ in the formula

$$f(\boldsymbol{\omega}) = \frac{\mathrm{D}^p f(\boldsymbol{\omega}, \dots, \boldsymbol{\omega})}{p!}$$

and expanding, one obtains the decomposition

$$f(\boldsymbol{\omega}) = \frac{1}{p!} \sum_{i_1, \dots, i_p = 1}^{m} \mathrm{D}^p f\big(\langle \omega_{i_1}; i_1\rangle, \dots, \langle \omega_{i_p}; i_p\rangle\big). \tag{7.12}$$

Given $\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)$, we denote by $S(\boldsymbol{\beta})$ the collection of all sequences $\gamma = (\gamma_1, \dots, \gamma_p)$ of integers between 1 and m such that γ contains the value $i \in \{1, \dots, m\}$ exactly β_i times. Then (7.12) takes the form

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)} H_{\boldsymbol{\beta}}(\boldsymbol{\omega})$$

where

$$H_{\boldsymbol{\beta}}(\boldsymbol{\omega}) = \frac{1}{p!} \sum_{\gamma \in S(\boldsymbol{\beta})} \mathrm{D}^p f\big(\langle \omega_{\gamma_1}; \gamma_1\rangle, \dots, \langle \omega_{\gamma_p}; \gamma_p\rangle\big).$$

We observe that $(\zeta(\boldsymbol{\beta}, 1), \dots, \zeta(\boldsymbol{\beta}, p))$ is a member of $S(\boldsymbol{\beta})$, in fact the unique element of $S(\boldsymbol{\beta})$ that is a nondecreasing sequence. A general element γ of $S(\boldsymbol{\beta})$ is a permutation of $(\zeta(\boldsymbol{\beta}, 1), \dots, \zeta(\boldsymbol{\beta}, p))$. Combining this with the symmetry of $\mathrm{D}^p f$ under permutations, one obtains (7.9) with

$$G_{\boldsymbol{\beta}}(\sigma_1, \dots, \sigma_p) = \mathrm{D}^p f\big(\sigma_{\zeta(\boldsymbol{\beta},1)}, \dots, \sigma_{\zeta(\boldsymbol{\beta},p)}\big)$$

for any $(\sigma_1, \dots, \sigma_p)$ as in (7.8). Then the alternating property (7.10) follows by obvious choices of arguments in (7.2). □

The functions $G_{\boldsymbol{\beta}}$ of Proposition 7.3 admit the following simple description.

7.4 Lemma *For any $\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s}, p)$ there exists $\alpha = \alpha_{\boldsymbol{\beta}} \in \wedge_d$, where $d = \sum_{i=1}^m s(i)\beta_i$, such that*

$$G_{\boldsymbol{\beta}}(\sigma_1 \dots, \sigma_p) = \alpha_{\boldsymbol{\beta}} \cdot (\sigma_1 \wedge \cdots \wedge \sigma_p) \tag{7.13}$$

for any $(\sigma_1, \dots, \sigma_p)$ as in (7.8).

Proof Let us consider the expression $G_{\boldsymbol{\beta}}(\sigma_1 \dots, \sigma_p)$ with the arguments σ_k equal to simple vectors, i.e.,

$$\sigma_k = v_{k,1} \wedge \cdots \wedge v_{k,\zeta(\boldsymbol{\beta},k)}, \quad 1 \le k \le p,$$

where $v_{k,1}, \dots, v_{k,\zeta(\boldsymbol{\beta},k)}$ are vectors in $\mathbb{R}^n$. The result is a d-linear form F on $[\mathbb{R}^n]^d$ of the variables

$$\big(v_{1,1}, \dots, v_{1,\zeta(\boldsymbol{\beta},1)}, v_{2,1}, \dots, v_{2,\zeta(\boldsymbol{\beta},2)}, \dots, v_{p,1}, \dots, v_{p,\zeta(\boldsymbol{\beta},p)}\big) \in [\mathbb{R}^n]^d. \tag{7.14}$$

Let us show that the form F is alternating, i.e.,

$$F(w_{\varepsilon(1)}, \dots, w_{\varepsilon(d)}) = \operatorname{sgn}(\varepsilon)\, F(w_1, \dots, w_d) \tag{7.15}$$

for every $w_1, \dots, w_d \in \mathbb{R}^n$ and every permutation $\varepsilon \in \mathbb{P}_d$. To prove this, it suffices to establish (7.15) only for all simple permutations, i.e., those which exchange only two indices in $\{1, \dots, d\}$. The construction of F gives that (7.15) holds if ε represents an exchange within the same block in the right-hand side of (7.14), i.e., for all ε which exchanges

$$v_{k,i} \quad \leftrightarrow \quad v_{k,j} \quad \text{where} \quad 1 \le k \le p,\ 1 \le i,j \le \zeta(\boldsymbol{\beta},k). \tag{7.16}$$

Next, to establish (7.15) for permutations exchanging

$$v_{k,i} \quad \leftrightarrow \quad v_{l,j} \quad \text{where} \quad 1 \le k \ne l \le p,\ 1 \le i \le \zeta(\boldsymbol{\beta},k) \ \ 1 \le j \le \zeta(\boldsymbol{\beta},l), \tag{7.17}$$

we invoke (7.10). Indeed, that equation shows that (7.15) holds for some particular cases of (7.17), viz., for any ε that exchanges

$$v_{k,\zeta(\boldsymbol{\beta},k)} \quad \leftrightarrow \quad v_{l,\zeta(\boldsymbol{\beta},l)} \quad \text{where} \ \ 1 \le k \ne l \le p \ \ \text{are arbitrary.}$$

Combining this with the exchanges within single blocks as in (7.16) provides (7.15) for arbitrary simple permutation, and hence also for every permutation $\varepsilon \in \mathbb{P}_d$. Then (7.15) and Proposition 9.5 (below) imply that there exists a $\alpha_{\boldsymbol{\beta}} \in \wedge_d$ such that

$$F(w_1, \dots, w_d) = \alpha_{\boldsymbol{\beta}} \cdot (w_1 \wedge \cdots \wedge w_d)$$

for every $w_1, \dots, w_d \in \mathbb{R}^n$. This establishes (7.13) when $\sigma_1, \dots, \sigma_d$ are simple multivectors; the linearity then extends (7.13) to any collection $\sigma_1, \dots, \sigma_d$. □

7.5 Lemma *An integrand f of type $\boldsymbol{s}$ is mult. ext. one affine then it is of the form described in Item* (iii) *of Theorem* 2.8.

Proof As explained in Remark 7.2, it suffices to consider the case of a homogeneous polynomial f of degree $p = 0, \dots, n$. Assuming this, we combine (7.9) with (7.13) to obtain

$$f(\boldsymbol{\omega}) = \sum_{\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s},p)} \alpha_{\boldsymbol{\beta}} \cdot \big(\omega_{\zeta(\boldsymbol{\beta},1)} \wedge \cdots \wedge \omega_{\zeta(\boldsymbol{\beta},p)}\big) \tag{7.18}$$

Referring to the remark in Definition 2.6(i), if $\boldsymbol{\beta} \in \mathscr{B}(\boldsymbol{s},p)$ is such that for two different indices k, l we have $\zeta(\boldsymbol{\beta},k) = \zeta(\boldsymbol{\beta},l)$ and $s(\zeta(\boldsymbol{\beta},k))$ is odd, then

$$\omega_{\zeta(\boldsymbol{\beta},1)} \wedge \cdots \wedge \omega_{\zeta(\boldsymbol{\beta},p)} = 0$$

for any $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$. Thus all $\boldsymbol{\beta}$ of this type can be omitted in the sum (7.18). The set of all remaining $\boldsymbol{\beta}$ coincides with the set $\mathfrak{R}_{\boldsymbol{s}}$ of admissible exponents. Observing that $\omega_{\zeta(\boldsymbol{\beta},1)} \wedge \cdots \wedge \omega_{\zeta(\boldsymbol{\beta},p)} = \boldsymbol{\omega}^{\boldsymbol{\beta}}$ we see that (7.18) takes the form

$$f(\boldsymbol{\omega}) = \sum_{\substack{\boldsymbol{r} \in \mathfrak{R}_{\boldsymbol{s}} \\ \deg(\boldsymbol{r})=p}} \alpha_{\boldsymbol{r}} \cdot \boldsymbol{\omega}^{\boldsymbol{r}}$$

where $\deg(\boldsymbol{r}) = \sum_{i=1}^{m} r_i$. This gives the desired form of homogeneous mult. ext. one affine integrand which is a homogeneous degree p polynomial. The decomposition (7.7) then leads to the general expression (2.7). □

7.6 Lemma *If an integrand f of type $\boldsymbol{s}$ has the form described in Item* (iii) *of Theorem* 2.8*, then it is a mult. ext. null lagrangian.*

Proof We have to prove that any function f of the form $f = \boldsymbol{\omega}^{\boldsymbol{r}}$, $\boldsymbol{\omega} \in \Delta_{\boldsymbol{s}}$, where $r \in \mathfrak{R}_{\boldsymbol{s}}$, satisfies

$$\int_{\Omega} f(d\boldsymbol{\xi} + d\boldsymbol{\theta})\, dx = \int_{\Omega} f(d\boldsymbol{\xi})\, dx \tag{7.19}$$

for any bounded open set $\Omega \subset \mathbb{R}^n$, any $\boldsymbol{\xi} \in C^\infty(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$, and any $\boldsymbol{\theta} \in C_0^\infty(\Omega, \Gamma_{\boldsymbol{s}})$. Note first that it suffices to establish (7.19) for regions Ω with smooth boundary. Indeed, if Ω does not necessarily have smooth boundary and $\boldsymbol{\xi} \in C^\infty(\mathbb{R}^n, \Gamma_{\boldsymbol{s}})$, and $\boldsymbol{\theta} \in C_0^\infty(\Omega, \Gamma_{\boldsymbol{s}})$, we take any bounded region $\tilde{\Omega}$ with smooth boundary such that $\Omega \subset \tilde{\Omega}$. The assumed validity of (7.19) for $\tilde{\Omega}$ then yields

$$\int_{\tilde{\Omega}} f(d\boldsymbol{\xi} + d\boldsymbol{\theta})\, dx = \int_{\tilde{\Omega}} f(d\boldsymbol{\xi})\, dx \tag{7.20}$$

and noting that on $\tilde{\Omega} \sim \Omega$ we have $\boldsymbol{\theta} = 0$, we can subtract the integral $\int_{\tilde{\Omega}\sim\Omega} f(d\boldsymbol{\xi})\, dx$ from (7.20) to obtain (7.19).

Thus assume that Ω has smooth boundary and prove (7.19). The case of a constant function f corresponding to $\boldsymbol{r} = (0, \dots, 0)$ being clear, we assume that $r_i \geq 1$ for some $i \in \{1, \dots, m\}$. Writing $\omega_i^{r_i} = \omega_i \wedge \omega_i^{r_i - 1}$, we factorize the function f according to

$$f(\boldsymbol{\omega}) = \omega_i \wedge \boldsymbol{\omega}^{\boldsymbol{q}}$$

where $\boldsymbol{q} = (r_1, \dots, r_i - 1, \dots, r_m)$. Then

$$f(d\boldsymbol{\xi} + d\boldsymbol{\theta}) = d(\xi_i + \theta_i) \wedge (d\boldsymbol{\xi} + d\boldsymbol{\theta})^{\boldsymbol{q}} = d\big((\xi_i + \theta_i) \wedge (d\boldsymbol{\xi} + d\boldsymbol{\theta})^{\boldsymbol{q}}\big)$$

and Stokes' theorem yields

$$\int_{\Omega} f(d\boldsymbol{\xi} + d\boldsymbol{\theta})\, dx = \int_{\partial\Omega} (\xi_i + \theta_i) \wedge (d\boldsymbol{\xi} + d\boldsymbol{\theta})^{\boldsymbol{q}} \wedge \nu\, dA = \int_{\partial\Omega} \xi_i \wedge (d\boldsymbol{\xi})^{\boldsymbol{q}} \wedge \nu\, dA$$

where A is the area measure and ν the normal to $\partial\Omega$ and where we have used that $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$ vanish on $\partial\Omega$. A similar application of Stokes' theorem yields

$$\int_{\Omega} f(d\boldsymbol{\xi})\, dx = \int_{\partial\Omega} \xi_i \wedge (d\boldsymbol{\xi})^{\boldsymbol{q}} \wedge \nu\, dA.$$

A comparison of the last to equations provides (7.19). □

7.7 Lemma *Any mult. ext. null lagrangian is mult. ext. quasiaffine.*

Proof It suffices to apply Definition 2.7 to $\boldsymbol{\xi}$ such that $d\boldsymbol{\xi} \equiv \boldsymbol{\omega}$ is constant, and $\Omega = Q$. This shows that f satisfies Condition (iv) in Theorem 2.8 and hence that theorem yields that f is mult. ext. quasiconvex. □

Collecting the above lemmas, we see that the cycle of implications (7.1) is complete and so Theorem 2.8 holds.

8 Appendix A: $\mathscr{A}$-quasiconvexity

The purpose of this section is to discuss the general notions of $\mathscr{A}$-quasiaffinity, $\mathscr{A}$-polyconvexity and Λ-convexity and Λ-ellipticity conditions [18, 24–25, 34, 9].

8.1 The differential operator $\mathscr{A}$ and the characteristic cone Λ The following dimensions will be needed in the subsequent discussion:

$$\left.\begin{aligned} n &= \text{the number of independent variables, } x = (x_1, \dots, x_n), \\ d &= \text{the number of dependent variables, } u = (u_1, \dots, u_d), \\ l &= \text{the number of differential constrains.} \end{aligned}\right\} \quad (8.1)$$

We shall consider the following first-order differential constraint on a map $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$:

$$\mathscr{A}v = 0$$

where

$$\mathscr{A}v = \sum_{i=1}^n A^{(i)} \frac{\partial v}{\partial x_i}$$

with $A^{(i)} \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, $i = 1, \dots, n$. For each $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ define

$$\mathbb{A}(\eta) = \sum_{i=1}^n \eta_i A^{(i)},$$

which is an element of $\mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$, and make the standing assumption that the rank of $\mathbb{A}(\eta)$ is the same for all $\eta \neq 0$. Next, we put

$$\Lambda = \{u \in \mathbb{R}^d : \mathbb{A}(\eta)u = 0 \ \text{ for some } \ \eta \in \mathbb{R}^n : \eta \neq 0\}. \quad (8.2)$$

8.2 Definition A continuous function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be

(i) $\mathscr{A}$-quasiconvex if

$$\int_Q f(u + v(x))\, dx \ge f(u) \quad (8.3)$$

for all $u \in \mathbb{R}^d$ and all $v \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, \mathbb{R}^d)$ such that $\mathscr{A}v = 0$ on $\mathbb{R}^n$ and $\int_Q v\, dx = 0$.

(ii) $\mathscr{A}$-quasiaffine if f takes only finite values and both f and $-f$ are $\mathscr{A}$-quasiconvex, i.e., if (8.3) holds with the equality sign for all u and v described in (i).

(iii) $\mathscr{A}$-polyconvex if there exists a finite number of $\mathscr{A}$-quasiaffine functions $f_1, \dots, f_g$ and a convex lower semicontinuous function $\Phi : \mathbb{R}^g \to \bar{\mathbb{R}}$ such that

$$f(u) = \Phi(f_1(u), \dots, f_g(u))$$

for each $u \in \mathbb{R}^n$.

We also introduce the following terminology to ease the formulations below.

8.3 Definition A continuous function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be

(i) Λ-convex if

$$f(tu_1 + (1-t)u_2) \le tf(u_1) + (1-t)f(u_2)$$

for every $t \in (0, 1)$ and $u_1, u_2 \in \mathbb{R}^d$ such that $u_2 - u_1 \in \Lambda$;

(ii) Λ-affine if it takes only finite values and both f and $-f$ are Λ-convex.

8.4 Theorem ([18; Proposition 3.4]) *If $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous $\mathscr{A}$-quasiconvex function then f is Λ-convex; consequently, if f is $\mathscr{A}$-quasiaffine then f is Λ-affine.*

9 Appendix B: Grassmann's algebra

This and the following sections briefly recapitulate the basic notions of exterior algebra and analysis. I follow [5; Chapter 4] and [16; Chapters One & Four] in taking a straightforwardly abstract attitude.

Throughout this section, V denotes an n-dimensional real inner product space. The inner product enables us to identify V with its dual; consequently, we do not distinguish between multivectors and covectors and between differential forms and multivector fields. Of course the theory is applied with $V = \mathbb{R}^n$ in the preceding sections.

9.1 Theorem *There exists a unique* (*up to an isomorphism*) *associative algebra $\wedge_* V$ with identity e with the following properties*:
(i) $\wedge_* V \supset V$;
(ii) *every $v \in V$ satisfies $v \wedge v = 0$* (*here $\wedge$ denotes the product in $\wedge_* V$*);
(iii) $\dim \wedge_* V = 2^n$;
(iv) $\wedge_* V$ *is generated by e and V.*

We refer to [5; Sections 4.3 & 4.4] for a proof. We call $\wedge_* V$ the Grassmann algebra over V, its elements multivectors, and $\wedge$ the exterior product. The following proposition decomposes $\wedge_* V$ into a direct sum of subspaces $\wedge_s V$ of homogeneous elements of different degrees. An alternative approach introduces the spaces $\wedge_s V$ first and then defines (often implicitly) $\wedge_*$ as the direct sum.

9.2 Proposition *If for each nonnegative integer s we put*

$$\wedge_s V = \begin{cases} \operatorname{span}\{\wedge_{i=1}^s u_i : u_1, \dots, u_s \in V\} & \text{if } s > 0, \\ \wedge_s V = \operatorname{span}\{e\} & \text{if } s = 0 \end{cases}$$

then

$$\wedge_* V = \bigoplus_{s=0}^n \wedge_s V$$

and

$$(\wedge_s V) \wedge (\wedge_t V) \subset \wedge_{s+t} V \quad \text{for all nonnegative integers } s,\ t;$$

moreover, the following commutativity-anticommutativity rules hold:

$$u \wedge v = (-1)^{st} v \wedge u, \tag{9.1}$$

$$v_{\pi(1)} \wedge \cdots \wedge v_{\pi(s)}) = \operatorname{sgn}(\pi) v_1 \wedge \cdots \wedge v_s$$

for any $u \in \wedge_s V$, $v \in \wedge_t V$, any collection $v_1, \dots, v_s \in V$ and any permutation $\pi \in \mathbb{P}_s$.

The elements of $\wedge_s V$ are called s-vectors or equivalently multivectors of degree s. It is well-known that

$$\dim \wedge_s V = \binom{n}{s}.$$

The inner product on V extends to an inner product on $\wedge_* V$ via Grassmann's formula (9.2) as follows.

9.3 Theorem

(i) *There exists a unique inner product on* $\wedge_* V$, *denoted by* $u \cdot v$ *for each* $u, v \in V$, *such that* $u \cdot v = 0$ *if* $u \in \wedge_s$ *and* $v \in \wedge_t$ *with* $s \neq t$ *and with*

$$(\bigwedge_{i=1}^{s} u_i) \cdot (\bigwedge_{i=1}^{s} v_i) = \det\left[u_i \cdot v_j\right]_{1 \le i,j \le s} \tag{9.2}$$

for every positive integer s *and every* $u_1, \dots, u_s, v_1, \dots, v_s \in V$;

(ii) *if* $u \in \wedge_s V$, $v \in \wedge_t V$ *and* $s \ge t$ *then there exists a unique* $u \llcorner v \in \wedge_{s-t}$ *such that*

$$(u \llcorner v) \cdot w = u \cdot (w \wedge v)$$

for all $w \in \wedge_{s-t} V$ *while if* $s \le t$ *then there exists a unique* $u \lrcorner v \in \wedge_{t-s}$ *such that*

$$(u \lrcorner v) \cdot w = (u \wedge w) \cdot v$$

for all $w \in \wedge_{t-s} V$.

The reader is referred to, e.g., [16; §1.7.5] for a proof.

9.4 Definition We say that $\phi : V^s \to \mathbb{R}$ is an alternating s-form on V if it has the following two properties:

(i) ϕ is s-linear, i.e., for each integer k satisfying $1 \le k \le s$ and each $v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_s \in V$ the map $v_k \mapsto \phi(v_1, \dots, v_s)$ is linear on V;

(ii) we have $\phi(v_1, \dots, v_s) = 0$ if the collection $(v_1, \dots, v_s) \in V^s$ contains at least two identical elements.

We have the following two equivalent forms of Property (ii):

(iii) for each $(v_1, \dots, v_s) \in V^s$ and each permutation $\pi \in \mathbb{P}_s$ we have

$$\phi(v_{\pi(1)}, \dots, v_{\pi(s)}) = \operatorname{sgn}(\pi)\phi(v_1, \dots, v_s);$$

(iv) if the collection $(v_1, \dots, v_s) \in V^s$ is linearly dependent then $\phi(v_1, \dots, v_s) = 0$.

The following proposition establishes the well-known relationship between multivectors and alternating forms on V.

9.5 Proposition $\phi : V^s \to \mathbb{R}$ *is an alternating* s*-form on* V *if and only if there exists a* $w \in \wedge_s V$ *such that*

$$\phi(v_1, \dots, v_s) = w \cdot (v_1 \wedge \dots \wedge v_s)$$

for each $(v_1, \dots, v_s) \in V^s$; *this establishes a bijective correspondence between alternating* s*-forms on* V *and the elements of* $\wedge_s V$.

The following result provides an invariant (coordinate-free) approach to determinants and subdeterminants of a linear transformation; see in Proposition 9.7.

9.6 Proposition *If* W *is another finite dimensional real inner product space then every* $F \in \operatorname{Lin}(V, W)$ *has a unique extension* $\wedge_* F$ *to an unital algebra homomorphism from* $\wedge_* V$ *into* $\wedge_* W$. *The map* $\wedge_* F$ *satisfies* $(\wedge_* F) \wedge_s V \subset \wedge_s W$.

See [5; Remark (3), p. 56] for proof. The restriction $\wedge_s F$ of $\wedge_* F$ to $\wedge_s V$ is called the exterior power of F of degree s. Clearly, $\wedge_s F = 0$ for all $s > q := \min\{m, n\}$. Since $\wedge_* F$ is a homomorphism, we have

$$\wedge_s F(u_1 \wedge \cdots \wedge u_s) = (Fu_1) \wedge \cdots \wedge (Fu_s)$$

for every $u_1, \ldots, u_s \in V$, which is often taken as a defining property of $\wedge_s F$. If $F \in \mathrm{Lin}(V, W)$ and $E \in \mathrm{Lin}(W, X)$ then

$$\wedge_*(EF) = (\wedge_* E)(\wedge_* F) \quad \text{and} \quad \wedge_s (EF) = (\wedge_s E)(\wedge_s F).$$

The last proposition of this section identifies the matrix elements of the exterior power $\wedge_s F$ of a transformation $F \in \mathrm{Lin}(V, W)$ with minors of order s of the matrix of F. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be arbitrarily chosen bases in V and W, respectively. If we define

$$e_J = e_{J_1} \wedge \cdots \wedge e_{J_s}, \quad f_I = f_{I_1} \wedge \cdots \wedge f_{I_s} \tag{9.3}$$

for each $J = (J_1, \ldots, J_s) \in \mathbb{I}_s^n$ and each $I \in \mathbb{I}_s^m$, then the collections e_J, $J \in \mathbb{I}_s$, and f_I, $I \in \mathbb{I}_s^m$, are bases in $\wedge_s V$ and $\wedge_s W$, respectively. The matrix elements of F and $\wedge_s F$ are defined standardly by

$$Fe_j = \sum_{i=1}^m F_{ij} f_i, \quad (\wedge_s F) e_J = \sum_{I \in \mathbb{I}_s^m} F_{IJ}^{(s)} f_I,$$

$i = 1, \ldots, n$, $J \in \mathbb{I}_s^n$.

9.7 Proposition *The matrix elements of $\wedge_s F$ are given by*

$$F_{IJ}^{(s)} := \det [F_{I_a J_b}]_{1 \le a, b \le s}, \quad I \in \mathbb{I}_s^m, \quad J \in \mathbb{I}_s^n.$$

See, e.g., [16; Subsection 1.3.4] or [22; Section XIX.2].

10 Appendix C: Exterior calculus

In this section, Ω is an open subset of $\mathbb{R}^n$. We write

$$\wedge_* := \wedge_* \mathbb{R}^n, \quad \wedge_s := \wedge_s \mathbb{R}^n, \quad \mathbb{I}_s := \mathbb{I}_s^n;$$

it will be convenient to put $\wedge_{-1} = \{0\}$. An s-form on Ω (or interchangeably an s-vector field on Ω) is any map $\omega : \Omega \to \wedge_s$. We let $e_1, \ldots, e_n$ be the canonical basis in $\mathbb{R}^n$ and use the induced basis e_I, $I \in \mathbb{I}_s$ in $\wedge_s$; see (9.3). The expansion of ω in this basis is

$$\omega = \sum_{I \in \mathbb{I}_s} \omega_I \, e_I, \tag{10.1}$$

where $\omega_I \in C^\infty(\Omega, \mathbb{R})$. A more standard notation denotes by $x_i : \mathbb{R}^n \to \mathbb{R}$ the i-th coordinate function, associating with any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the number x_i, notes that $dx_i = e_i$, and rewrites (10.1) as

$$\omega = \sum_{1 \le I_1 < \cdots < I_s \le n} \omega_{I_1 \ldots I_s} \, dx_{I_1} \wedge \cdots \wedge dx_{I_s}.$$

10.1 Theorem *For each open subset* Ω *of* $\mathbb{R}^n$ *there exist unique linear maps* d *and* div *from* $C^\infty(\Omega, \wedge_*)$ *into itself such that*

$$d(fu) = \nabla f \wedge u, \quad \operatorname{div}(fu) = \nabla f \lrcorner\, u \tag{10.2}$$

for every $u \in \wedge_*$ *and every* $f \in C^\infty(\Omega, \mathbb{R})$.

The operation d is called the exterior derivative and the operation div the (generalized) divergence [as it coincides with the usual divergence on 1-vector fields (see Item (iii) of Theorem 10.2, below)]. Alternatively, the divergence (or its multiple by a factor ± 1 that may depend on the dimension and degree) is denoted by δ and called the interior derivative or codifferential in the literature. The definition based on (10.2) is possible only because $\mathbb{R}^n$ is flat; the formulas (10.2) are meaningless on manifolds. Moreover, the definition of div makes use of the inner product on $\mathbb{R}^n$ via the operation $\lrcorner$. With an association to the formulas $\operatorname{curl} v = \nabla \times v$ and $\operatorname{div} v = \nabla \cdot v$ from the elementary vector calculus, we may write in a perfect analogy

$$d\omega = \nabla \wedge \omega, \quad \operatorname{div} \omega = \nabla \lrcorner\, \omega$$

as motivated by (10.2) and by the formulas to follow.

Proof The operations d and div defined by

$$d\omega = \sum_{i=1}^{n} e_i \wedge \omega_{,i}, \quad \operatorname{div} \omega = \sum_{i=1}^{n} e_i \lrcorner\, \omega_{,i} \tag{10.3}$$

for any $\omega \in C^\infty(\Omega, \wedge_*)$ plainly satisfy (10.2). On the other hand, expanding any $\omega \in C^\infty(\Omega, \wedge_*)$ into components in the basis e_I, $I \in \mathbb{I}_s$, $s = 1, \dots, n$, using the linearity of d and div, a multiple application of (10.2) shows that d and div must be given by the formulas (10.3). □

10.2 Theorem *The maps* d *and* div *from Theorem* 10.1 *have the following properties*:

(i) *we have*

$$d^2 := d \circ d = 0, \quad \operatorname{div}^2 := \operatorname{div} \circ \operatorname{div} = 0;$$

(ii) *for each nonnegative integer s, the map d maps* $C^\infty(\Omega, \wedge_s)$ *into* $C^\infty(\Omega, \wedge_{s+1})$ *and* div *maps* $C^\infty(\Omega, \wedge_s)$ *into* $C^\infty(\Omega, \wedge_{s-1})$;

(iii) *on* $C^\infty(\Omega, \mathbb{R})$, d *coincides with the usual gradient and on* $C^\infty(\Omega, \mathbb{R}^n)$, div *coincides with the usual divergence*;

(iv) *if* $\psi \in C^\infty(\Omega, \wedge_s)$ *and* $\omega \in C^\infty(\Omega, \wedge_t)$ *then*

$$d(\psi \wedge \omega) = d\psi \wedge \omega + (-1)^s \psi \wedge d\omega \tag{10.4}$$

and if additionally $s > t$ *then*

$$\operatorname{div}(\psi \llcorner \omega) = (\operatorname{div} \psi) \llcorner \omega + (-1)^{s-t-1} \psi \llcorner d\omega; \tag{10.5}$$

(v) *if* $\psi \in C^\infty(\Omega, \wedge_s)$ *and* $\omega \in C^\infty(\Omega, \wedge_{s-1})$ *then*

$$\operatorname{div}(\psi \llcorner \omega) = \omega \cdot \operatorname{div} \psi + \psi \cdot d\omega \tag{10.6}$$

where the divergence on the left-hand side is the usual divergence of a 1*-vector field.*

Proof We refer to, e.g., [5; Section 4.6] for the proof of Items (i)–(iii) and (10.4). Formula (10.5), which is perhaps less standard, is proved analogously; it is stated without proof in [16; p. 356] as the second member of the list of eight formulas for the boundary operator $\partial = -\operatorname{div}$.★ The proof is completed by noting that (10.6) is a particular case of (10.5). □

10.3 Definition Let $1 \le p, q \le \infty$ and $\omega \in L^p(\Omega, \wedge_s)$.

(i) We say that ω has the weak exterior derivative in L^q if there exists a $d\omega \in L^q(\Omega, \wedge_{s+1})$ such that

$$\int_\Omega \omega \cdot \operatorname{div} \psi \, dx = -\int_\Omega \psi \cdot d\omega \, dx \tag{10.7}$$

for every $\psi \in C_0^\infty(\Omega, \wedge_{s+1})$.

(ii) We say that ω has the weak interior derivative in L^q if there exists a $\operatorname{div} \omega \in L^q(\Omega, \wedge_{s-1})$ such that

$$\int_\Omega \operatorname{div} \omega \cdot \chi \, dx = -\int_\Omega \omega \cdot d\chi \, dx$$

for every $\chi \in C_0^\infty(\Omega, \wedge_{s-1})$.

If ω is continuously differentiable then the weak definitions of $d\omega$, $\operatorname{div} \omega$ coincide with the classical ones. Indeed, in the context of Item (i), one can employ Formula (10.6) to show that

$$\operatorname{div}(\psi \llcorner \omega) = \omega \cdot \operatorname{div} \psi + \psi \cdot d\omega \tag{10.8}$$

where on the left-hand side div is the classical divergence of a 1-vector field and $\operatorname{div} \psi$ and $d\omega$ are the classical derivatives given by (10.3). Since $\psi \llcorner \omega$ vanishes near the boundary of Ω, the elementary divergence theorem for n-dimensional regions in $\mathbb{R}^n$ and 1-vector fields and (10.8) give

$$0 = \int_\Omega \operatorname{div}(\psi \llcorner \omega) \, dx = \int_\Omega \omega \cdot \operatorname{div} \psi \, dx + \int_\Omega \psi \cdot d\omega \, dx$$

and hence (10.7). The same reasoning applies to $\operatorname{div} \omega$.

The same procedures as in Items (i) and (ii) lead to the definitions of d and div in the class of $\wedge_s$-valued distributions $\mathscr{D}'(\Omega, \wedge_s)$, i.e., to the space of de Rham's currents in Ω [10, 17, 16]. If $T \in \mathscr{D}'(\Omega, \wedge_s)$, we define $dT \in \mathscr{D}'(\Omega, \wedge_{s+1})$ and $\operatorname{div} T \in \mathscr{D}'(\Omega, \wedge_{s-1})$ by

$$\langle dT, \psi \rangle = -\langle T, \operatorname{div} \psi \rangle, \quad \langle \operatorname{div} T, \chi \rangle = -\langle T, d\chi \rangle$$

for each $\psi \in \mathscr{D}(\Omega, \wedge_{s+1})$ and $\chi \in \mathscr{D}(\Omega, \wedge_{s-1})$ where we use the brackets to denote the values of the distributions on test functions. The so-defined divergence operator is related to the boundary operator ∂ central to the homological integration theory by $\partial T = -\operatorname{div} T$.

★ A different sign convention is used to define $\llcorner$ so that the signs are different in [16; p. 356].

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