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The general form of the relaxation of a purely interfacial energy for structured deformations

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Abstract This paper deals with the relaxation of energies of media with structured deformations introduced by Del Pier & Owen [9–10]. Structured deformations provide a multiscale geometry that captures the contributions at the macrolevel of both smooth and non-smooth geometrical changes (disarrangements) at submacroscopic levels. The paper examines the special case of Choksi & Fonseca's energetics of structured deformations [6] in which the unrelaxed energy does not contain the bulk contribution. Thus the energy is purely interfacial, but of a general form. New formulas for the relaxed bulk and interfacial energies are proved. The bulk relaxed energy is shown to coincide with the subadditive envelope of the unrelaxed interfacial energy while the relaxed interfacial energy is the restriction of the envelope to rank 1 tensors. Moreover, it is shown that the minimizing sequence required to define the bulk energy in the relaxation scheme of Choksi & Fonseca [6] can be realized in the more restrictive class required in the relaxation scheme of Baía, Matias & Santos [3], thus establishing the equivalence of the two approaches for general purely interfacial energies. The relaxations of the specific interfacial energies of Owen & Paroni [15] and Barroso, Matias, Morandotti & Owen [4] are simple consequences of our general results.

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Keywords Structured deformations, relaxation, subadditive envelope, interfacial energy, bulk energy, functions of measures

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1 Introduction

This paper deals with the relaxation of nonclassical continua modeled as media with structured deformations introduced by Del Piero & Owen [9–10].⋆ In their original setting, a structured deformation is a triplet $(\mathscr{K}, g, G)$ of objects of the following nature. The set $\mathscr{K} \subset \mathbb{R}^3$, the crack site, is a subset of vanishing Lebesgue measure of the reference region Ω, the map $g : \Omega \sim \mathscr{K} \to \mathbb{R}^3$, the deformation map, is piecewise continuously differentiable and injective, and G is a piecewise continuous map from $\Omega \sim \mathscr{K}$ to the set of invertible second order tensors describing deformation without disarrangements.

Within this context, *simple* deformations are triples $(\mathscr{K}, g, \nabla g)$ where g is a piecewise smooth injective map with jump discontinuities describing partial or full separation of pieces of the body. In view of this, in the general case of a structured deformation $(\mathscr{K}, g, G)$, the tensor

$$M = \nabla g - G,$$

the deformation due to disarrangements, measures the departure of $(\mathscr{K}, g, G)$ from the simple deformation $(\mathscr{K}, g, \nabla g)$.

Choksi & Fonseca [6] introduced into the theory of structured deformations energy considerations and the ideas of relaxation. For further studies in one and multidimensional settings see Del Piero [7–8]. It is well-known that the existing techniques of relaxation of the calculus of variations and continuum mechanics are unable to cope with injectivity requirements. Accordingly, Choksi & Fonseca neglect the injectivity requirement; in addition, they assume weaker regularity. In their interpretation, structured deformations are pairs (g, G) where $g : \Omega \to \mathbb{R}^n$ is a special $\mathbb{R}^n$-valued map of bounded variation from the space $SBV(\Omega)$ and $G : \Omega \to \mathrm{Lin}$ is an integrable Lin-valued map from the space $L^1(\Omega)$.⋆⋆ Thus

$$SD(\Omega) := SBV(\Omega) \times L^1(\Omega)$$

is the set of all structured deformations. Structured deformations of the form $(g, \nabla g)$ with $g \in SBV(\Omega)$ are called *simple deformations* in this paper.

The relaxation starts from the energy

$$E(g) = \int_{\Omega} W(\nabla g)\, d\mathscr{v} + \int_{J(g)} \psi([\![g]\!], \nu_g)\, d\mathscr{a} \tag{1.1}$$

of a simple deformation $g \in SBV(\Omega)$. Here $\mathscr{v}$ and $\mathscr{a}$ are the Lebesgue measure and the $n-1$-dimensional Hausdorff measure in $\mathbb{R}^n$, ∇g is the absolutely continuous part of the derivative (= gradient) $\mathrm{D}g$ of g, while the singular part

$$\mathrm{D}^s g := [\![g]\!] \otimes \nu_g \mathscr{a} \llcorner J(g)$$

is a tensor-valued singular measure describing the discontinuities of g; that part is formed from the jump set $J(g) \subset \Omega$ of g, the jump $[\![g]\!]$ of g on $J(g)$, and the

⋆ The reader is referred to the proceedings [11] and to the recent survey [2] for additional references and for further developments.

⋆⋆ For brevity of notation, we omit the target spaces and write $SBV(\Omega) \equiv SBV(\Omega, \mathbb{R}^n)$ and $L^1(\Omega) \equiv L^1(\Omega, \mathrm{Lin})$. See Section 3 for more notation and detailed definitions.

normal ν_g to $J(g)$. The reader is referred to (3.1), below, for a detailed description of these objects. The material is characterized by the bulk energy density $W : \mathrm{Lin} \to \mathbb{R}$ and by the interfacial (or cohesion) energy $\psi : \mathbb{D}_n \to \mathbb{R}$, where we denote

$$\mathbb{D}_n = \mathbb{R}^n \times \mathbb{S}^{n-1}.$$

The Approximation Theorem of Del Piero & Owen [9; Theorem 5.8] says that every structured deformation is a well-defined limit of simple deformations. In the framework of Choksi & Fonseca [6] (see also [17]) this means that corresponding to each structured deformation $(g, G) \in SD(\Omega)$ there exists a sequence $(g_k, \nabla g_k) \in SD(\Omega)$ (i.e., with g_k in $SBV(\Omega)$) such that

$$\left.\begin{aligned} & g_k \to g && \text{in } L^1(\Omega), \\ & \nabla g_k \overset{*}{\rightharpoonup} G && \text{in } \mathscr{M}(\Omega, \mathrm{Lin}), \\ & \sup\{|\nabla g_k|_{L^1(\Omega)} : k = 1, \dots\} < \infty. \end{aligned}\right\} \tag{1.2}$$

The relaxed energy of a structured deformation $(g, G) \in SD(\Omega)$ is defined by

$$I(g, G) = \inf\{\liminf_{k \to \infty} E(g_k) : g_k \in SBV(\Omega) \text{ satisfies } (1.2)\}.$$

Thus, a sequence approaching the above infimum realizes the most economical way to build up the deformation (g, G) using approximations in SBV. The relaxation theorem of Choksi & Fonseca [6; Theorems 2.6 & 2.17 and Remark 3.3] says that under some assumptions on W and ψ (a particular case of which is Assumption 2.1, below), the relaxed energy admits the integral representation

$$I(g, G) = \int_\Omega H(\nabla g, G)\, d\mathscr{v} + \int_{J(g)} h([\![g]\!], \nu_g)\, d\mathscr{a} \tag{1.3}$$

where H and h are some functions determined explicitly in the cited theorems (Theorem 2.2 presents formulas for H and h for a particular case).

This paper deals with the relaxation of energy functions E for which the bulk contribution vanishes, i.e., with energy functions of the form

$$E(g) = \int_{J(g)} \psi([\![g]\!], \nu_g)\, d\mathscr{a}$$

for each $g \in SBV(\Omega)$. The main result, Theorem 2.3, below, gives explicit descriptions of the functions H and h from (1.3) and applies them to give simplified proofs of two particular cases Examples 2.5 and 2.6 given previously in [15] and [4].

2 The main result and examples

We make the following standing hypotheses about ψ.

Assumptions 2.1.
(i) The function $\psi : \mathbb{D}_n \to \mathbb{R}$ is continuous;
(ii) we have $\psi(-a, -b) = \psi(a, b)$ and

$$0 \le \psi(a, b) \le C_1 |a| \tag{2.1}$$

for every $(a,b) \in \mathbb{D}_n$ and some $C_1 > 0$;

(iii) the function $\psi(\cdot, \nu)$ is subadditive and positively homogeneous for each $\nu \in \mathbb{S}^{n-1}$.

To ease the statements of the results, we extend any function $\zeta : \mathbb{D}_n \to [0, \infty)$ to and equally denoted function $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by homogeneity with respect to the second variable, i.e., by assuming that the extended function satisfies

$$\zeta(a, tb) = t\zeta(a,b)$$

for any $t \geq 0$ and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$. This convention applies in particular to the functions ψ and h.

We need some notation to formulate the main results. Let $Q = (-1/2, 1/2)^n$ and for every $M \in \mathrm{Lin}$ let $w_M : \partial Q \to \mathbb{R}^n$ be given by

$$w_M(x) = Mx \quad \text{for every } x \in \partial Q.$$

Furthermore, if $(a,b) \in \mathbb{D}_n$, let Q_b be any cube of unit edge, of center at $0 \in \mathbb{R}^n$, and of two faces normal to b, and let $z_{a,b} : Q_b \to \mathbb{R}^n$ be the map defined by

$$z_{a,b}(x) = \frac{1}{2} a\big(\operatorname{sgn}(x \cdot b) + 1\big), \quad x \in Q_b.$$

Finally, if $u \in SBV(\Omega)$, let us put

$$\Psi(\mathrm{D}^{\mathrm{s}} u) := \int_{J(u)} \psi([\![u]\!], \nu_u)\, d\mathcal{H}.$$

The following statement is a particular case $W = 0$ of the relaxation theorem of Choksi & Fonseca [6; Theorems 2.6 & 2.17 and Remark 3.3].

Theorem 2.2. *The effective energies H and h are given by*

$$\begin{aligned} H(A,B) = \inf \Big\{ \Psi(\mathrm{D}^{\mathrm{s}} u) : u \in SBV(Q) :& \\ u = w_A \ \text{on} \ \partial Q, \ \int_Q \nabla u \, d\mathcal{V} = B \Big\}& \end{aligned} \tag{2.2}$$

for each $A, B \in \mathrm{Lin}$, and

$$\begin{aligned} h(a,b) = \inf \Big\{ \Psi(\mathrm{D}^{\mathrm{s}} u) : u \in SBV(Q_b),& \\ u = z_{a,b} \ \text{on} \ \partial Q_b, \ \nabla u = 0 \ \text{on} \ Q_b \Big\}& \end{aligned} \tag{2.3}$$

for each $(a,b) \in \mathbb{D}_n$.

The following theorem, the main result of this paper, shows that the functions W and h admit a much more explicit description in terms of a single function Φ.

Theorem 2.3. *The functions H and h in* (1.3) *are given by*

$$H(A,B) = \Phi(A - B), \quad h(a,b) = \Phi(a \otimes b) \tag{2.4}$$

for every $A, B \in \mathrm{Lin}$ and $(a,b) \in \mathbb{D}_n$, where Φ is a subadditive and positively homogeneous function on Lin *defined by each of the following equivalent Assertions* (i)–(iv); *moreover, for dyadic arguments we have an additional Assertion* (v).

(i) *Φ is the biggest subadditive function on* Lin *satisfying*

$$\Phi(a \otimes b) \le \psi(a,b) \text{ for every } (a,b) \in \mathbb{D}_n \tag{2.5}$$

i.e.,

$$\Phi(M) = \sup\big\{\Theta(M) : \Theta \text{ is subadditive on Lin and } \Theta(a \otimes b) \le \psi(a,b) \text{ for every } (a,b) \in \mathbb{D}_n\big\}; \tag{2.6}$$

(ii) *for every* $M \in \mathrm{Lin}$,★

$$\Phi(M) = \inf\Big\{\sum_{i=1}^{m} \psi(a_i,b_i) : (a_i,b_i) \in \mathbb{D}_n,\ i = 1,\dots,m,\ \sum_{i=1}^{m} a_i \otimes b_i = M\Big\}; \tag{2.7}$$

(iii) *for every* $M \in \mathrm{Lin}$,

$$\Phi(M) = \inf\big\{\Psi(\mathrm{D}^{s}u) : u \in SBV(Q),\ u = w_M \text{ on } \partial Q,\ \nabla u = 0 \text{ on } Q\big\}; \tag{2.8}$$

(iv) *for every* $M \in \mathrm{Lin}$,

$$\Phi(M) = \inf\big\{\Psi(\mathrm{D}^{s}u) : u \in SBV(Q),\ u = w_M \text{ on } \partial Q,\ \int_Q \nabla u \, d\mathcal{V} = 0\big\}; \tag{2.9}$$

(v) *for arguments of the form* $a \otimes b$, *where* $(a,b) \in \mathbb{D}_n$, *we have*

$$\Phi(a \otimes b) = \inf\big\{\Psi(\mathrm{D}^{s}u) : u \in SBV(Q_b),\ u = z_{a,b} \text{ on } \partial Q_b,\ \nabla u = 0 \text{ on } Q_b\big\}. \tag{2.10}$$

The proof of Theorem 2.3 is given in Sections 5 and 6, below.

Remarks 2.4.

(a) Since the pointwise supremum of any family of subadditive functions is subadditive (e.g., [14; Theorem 7.2.2]), Equation (2.6) really defines a subadditive function.

(b) Among the above characterizations of Φ, the closely related novel forms (i) and (ii) must be considered as the most important. The main advantage of (i) and (ii) is that they establish connexions to the wealth of results of the convexity theory. These will be employed to analyze the examples to be formulated below.

(c) In one dimension, one can orient the normals to jumps to be always the vector +1 (rather than −1) and hence the dependence of ψ on the second variable can be suppressed, $\psi = \psi(a)$, $a \in \mathbb{R}$. Assumption 2.1(iii) then says that ψ is subadditive and positively homogeneous. Thus the subadditive envelope Φ of ψ is ψ itself, and all mentions of a subadditive envelope can be avoided. This is not the case if Assumption 2.1(iii) is relaxed. Indeed, working in one dimension, Del Piero [7–8] calculated the relaxation of the energy (1.1) with the interfacial energy ψ of a general form, avoiding Assumption 2.1(iii). His main result contains the subadditive envelope of ψ also. In the light of the above discussion, this envelope which plays a different, but related

★ Throughout the paper, the letter m denotes any positive integer.

role. The relaxation of a purely interfacial energy of a more general form than that postulated in Assumptions 2.1 in arbitrary dimension will be treated in a future paper.

(d) The expressions in (iii)–(v) already occurred previously, albeit without noting that the are mutually equivalent and equivalent to (i) and (ii), except for some particular cases to be mentioned below. The formula for H in $(2.4)_1$ with Φ defined in (iv) and the formula for h in $(2.4)_2$ with Φ defined in (v) are direct consequences of the Choksi & Fonseca's expressions in (2.2) and (2.3). The formula for H with Φ given by (iii) crops up in the relaxation schemes by Baía, Matias, and Santos [3; Eq. (3.2)] and by Barroso, Matias, Morandotti and Owen [5; Theorem 3.2]. The relaxation schemes in the last two papers are designed for second-order structured deformations and hence they are not strictly comparable with that of Choksi & Fonseca described above.

(e) The infimum (iv) could be, in principle, bigger that in (iv). Nevertheless, the infima are generally the same. This has been established previously in [4] for the special choices of ψ described in the following examples, which motivated the present study.

Example 2.5 ([15; Theorem 4, particular case $L = \mathrm{I}$]). *If*

$$\psi_{|\cdot|}(a,b) = |a \cdot b| \quad \textit{and} \quad \psi_\pm(a,b) = \{a \cdot b\}_\pm$$

for every $(a,b) \in \mathbb{D}_n$, *where* $\{\cdot\}_+$ *and* $\{\cdot\}_-$ *denote the positive and negative parts of a real number, then*

$$\Phi_{|\cdot|}(M) = |\mathrm{tr}\, M| \quad \textit{and} \quad \Phi_\pm(M) = \{\mathrm{tr}\, M\}_\pm \tag{2.11}$$

for every $M \in \mathrm{Lin}$. *The effective energies* $H_{|\cdot|}$, $H_\pm$, $h_{|\cdot|}$, *and* $h_\pm$ *are determined through* $\Phi_{|\cdot|}$ *and* $\Phi_\pm$ *by* (2.4).

As shown in [15], $\{\mathrm{tr}\, M\}_+$ is a volume density of disarrangements due to submacroscopic separations, $\{\mathrm{tr}\, M\}_-$ is a volume density of disarrangements due to submacroscopic switches and interpenetrations, and $|\mathrm{tr}\, M|$ is a volume density of all three of these non-tangential disarrangements: separations, switches, and interpenetrations. The evaluation in [15] of H (equivalently, of Φ) for (2.11) is rather complicated; a recent paper by Barroso, Matias, Morandotti & Owen [4] presents some simplification and the realization of the minimizing sequence in the narrower class (iv) in Theorem 2.3 mentioned earlier. Our version of the derivation, which includes the minimizing sequence from (iv) via Theorem 2.3 also, is given in Section 7.

Example 2.6 ([4; Equation (5.3)]). *If*

$$\psi(a,b) = |a \cdot p|$$

for $(a,b) \in \mathbb{D}_n$, *where* $p \in \mathbb{R}^n$ *is a fixed vector, then*

$$\Phi(M) = |M^{\mathrm{T}} p| \tag{2.12}$$

for any $M \in \mathrm{Lin}$.

3 Notation; functions of bounded variation

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers, by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, by Lin the set of all linear transformations from $\mathbb{R}^n$ into itself, often identified with the set of $n \times n$ matrices with real elements. We use the symbols '$\cdot$' and '$|\cdot|$' to denote the scalar product and the euclidean norm on $\mathbb{R}^n$ and on Lin. The latter are defined by $A \cdot B := \operatorname{tr}(AB^{\mathrm{T}})$ and $|A| = \sqrt{A \cdot A}$ where $A^{\mathrm{T}} \in$ Lin is the transpose of A and tr denotes the trace.

A real-valued function f defined on a vector space X is said to be subadditive if $f(x+y) \le f(x) + f(y)$ for every $x, y \in \mathrm{X}$ and positively homogeneous if $f(tx) = tf(x)$ for every $t \ge 0$ and $x \in \mathrm{X}$.

If Ω is an open subset of $\mathbb{R}^n$, we denote by $L^1(\Omega)$ the space of Lin-valued integrable maps on Ω. We denote by $\mathscr{M}(\Omega, \mathrm{Lin})$ the set of all (finite) Lin-valued measures on Ω. If $\mu \in \mathscr{M}(\Omega, \mathrm{Lin})$, we denote by $\mu \llcorner B$ the restriction of μ to a Borel set $B \subset \Omega$. If $G, G_k \in L^1(\Omega)$, $k = 1, 2, \ldots$, we say that G_k converges to G in the sense of measures, and write

$$G_k \overset{*}{\rightharpoonup} G \quad \text{in} \quad \mathscr{M}(\Omega, \mathrm{Lin}),$$

if $\int_\Omega G_k \cdot H \, d\mathscr{v} \to \int_\Omega G \cdot H \, d\mathscr{v}$ for every continuous map $H : \mathbb{R}^n \to \mathrm{Lin}$ which vanishes outside Ω.

We state some basic definitions and properties of the space $BV(\Omega) = BV(\Omega, \mathbb{R}^n)$ of maps of bounded variation and of the space $SBV(\Omega) = SBV(\Omega, \mathbb{R}^n)$ special maps of bounded variation. For more details, see [1, 12, 18], and [13].

We define the set $BV(\Omega)$ as the set of all $u \in L^1(\Omega) = L^1(\Omega, \mathbb{R}^n)$ such that there exists a measure $\mathrm{D}u \in \mathscr{M}(\Omega, \mathrm{Lin})$ satisfying

$$\int_\Omega u \cdot \operatorname{div} T \, d\mathscr{v} = -\int_\Omega T \cdot d\,\mathrm{D}u$$

for each indefinitely differentiable map $T : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ which vanishes outside some compact subset of Ω. Here $\operatorname{div} T$ is an $\mathbb{R}^n$-valued map on Ω given by $(\operatorname{div} T)_i = \sum_{j=1}^n T_{ij,j}$ where the comma followed by an index j denotes the partial derivative with respect to j th variable. The measure $\mathrm{D}u$ is uniquely determined and called the weak (or generalized) derivative of u. We shall need the following form of the Gauss-Green theorem for BV: if Ω is a domain with lipschitzian boundary and $u \in BV(\Omega)$ then there exist an $\mathscr{a}$ integrable map $u^{\partial\Omega} : \partial\Omega \to \mathbb{R}^n$ such that

$$\mathrm{D}u(\Omega) \equiv \int_\Omega d\mathrm{D}u = \int_{\partial(\Omega)} u^{\partial\Omega} \otimes \nu_\Omega \, d\mathscr{a}$$

where ν_Ω is the outer normal to $\partial\Omega$. The map $u^{\partial\Omega}$ is determined to within a change on a set of $\mathscr{a}$ measure 0 and is called the trace of u.

We define the set $SBV(\Omega)$ as the set of all $u \in BV(\Omega)$ for which $\mathrm{D}u$ has the form

$$\mathrm{D}u = \nabla u \, \mathscr{v} \llcorner \Omega + [\![u]\!] \otimes \nu_u \mathscr{a} \llcorner J(u) \tag{3.1}$$

where ∇u, the absolutely continuous part of $\mathrm{D}u$, is a map in $L^1(\Omega)$ and the term

$$\mathrm{D}^{\mathrm{s}} u := [\![u]\!] \otimes \nu_u \mathscr{a} \llcorner J(u)$$

on the right-hand side of (3.1) is called the jump (or singular) part of $\mathrm{D}u$. The objects $J(u) \subset \Omega$, $[\![u]\!] : J(u) \to \mathbb{R}^n$ and $\nu_u : J(u) \to \mathbb{S}^{n-1}$ are called the jump set of

u, the jump of u and the normal to $J(u)$, respectively. Here $J(u)$ is the set of all $x \in \Omega$ for which there exist $\nu_u(x) \in \mathbb{S}^{n-1}$ and $u^{\pm}(x) \in \mathbb{R}^n$ such that we have the approximate limits

$$u^{\pm}(x) = \operatorname*{ap\,lim}_{\substack{y \to x \\ y \in \mathbf{H}^{\pm}(x, \nu_u(x))}} u(x),$$

where $\mathbf{H}^{\pm}(x, \nu_u(x)) = \{y \in \mathbb{R}^n : \pm(y - x) \cdot \nu_u(x) > 0\}$. For a given $x \in \Omega$, either the triplet $(\nu_u, u^+, u^-) = (\nu_u(x), u^+(x), u^-(x))$ does not exist or it is uniquely determined to within the change $(\nu_u, u^+, u^-) \mapsto (-\nu_u, u^-, u^+)$. With one of these choices, one puts $[\![u]\!] = u^+ - u^-$ and notes that $[\![u]\!] \otimes \nu_u$ is unique.

Finally, we denote by $\langle r \rangle$ the integral part of $r \in \mathbb{R}$. Clearly,

$$r - 1 \le \langle r \rangle \le r \quad \text{and} \quad 0 \le r - \langle r \rangle \le 1. \tag{3.2}$$

Writing $r = kt$, where $t \in \mathbb{R}$ and $k > 0$, and dividing by k we obtain

$$0 \le t - \langle kt \rangle / k \le 1/k \tag{3.3}$$

and hence

$$\langle kt \rangle / k \to t \quad \text{as} \quad k \to \infty \tag{3.4}$$

uniformly in $t \in \mathbb{R}$.

4 Preliminary results

We put

$$\mathscr{A}(M) := \{u \in SBV(Q) : u = w_M \text{ on } \partial Q,\ \nabla u = 0 \text{ on } Q\},$$
$$\mathscr{B}(M) := \{u \in SBV(Q) : u = w_M \text{ on } \partial Q,\ \int_Q \nabla u \, d\mathscr{v} = 0\},$$

for any $M \in \text{Lin}$. We start with the following preliminary results.

Proposition 4.1. *If $A, B \in \text{Lin}$ and $u \in \mathscr{B}(A)$, $v \in \mathscr{B}(B)$, then $u + v \in \mathscr{B}(A + B)$ and*

$$\Psi(\mathrm{D}^s u + \mathrm{D}^s v) \le \Psi(\mathrm{D}^s u) + \Psi(\mathrm{D}^s v); \tag{4.1}$$

if $\mathscr{a}(J(u) \cap J(v)) = 0$, then we have the equality sign in (4.1).

Proof We have

$$J(u + v) = K_u \cup K_v \cup L \tag{4.2}$$

where

$$L = J(u) \cap J(v), \quad K_u = J(u) \sim K, \quad K_v = J(v) \sim K.$$

Next, we observe that on L we have $\nu_u(x) = \pm \nu_v(x)$ for $\mathscr{a}$-almost every $x \in L$; since we have a freedom in the choice of the sign of ν_v, we assume $\nu_u(x) = \nu_v(x)$ and denote $\mu = \nu_u$ on L. Then

$$[u + v] \otimes \nu_{u+v} = \begin{cases} [u] \otimes \nu_u & \text{on } K_u, \\ [v] \otimes \nu_v & \text{on } K_v, \\ ([u] + [v]) \otimes \mu & \text{on } L. \end{cases} \tag{4.3}$$

By the subadditivity of ψ we have

$$\psi([u]+[v],\mu) \le \psi([u],\mu)+\psi([v],\mu) = \psi([u],\nu_u)+\psi([v],\nu_v)$$

and hence (4.3) provides

$$\psi([u+v],\nu_{u+v}) \begin{cases} = \psi([u],\nu_u) & \text{on } K_u, \\ = \psi([v],\nu_v) & \text{on } K_v, \\ \le \psi([u],\nu_u)+\psi([v],\nu_v) & \text{on } L. \end{cases}$$

Integrating over $J(u+v)$ and using (4.2) we obtain

$$\begin{aligned} \Psi(\mathrm{D}^{\mathrm{s}}u+\mathrm{D}^{\mathrm{s}}v) &= \int\limits_{J(u+v)} \psi([u+v],\nu_{u+v})\,d\mathscr{H} \\ &\le \int\limits_{K_u} \psi([u],\nu_u)\,d\mathscr{H} + \int\limits_{K_v} \psi([v],\nu_v)\,d\mathscr{H} \\ &\quad + \int\limits_{L} \psi([u],\nu_u)\,d\mathscr{H} + \int\limits_{L} \psi([v],\nu_v)\,d\mathscr{H} \\ &= \Psi(\mathrm{D}^{\mathrm{s}}u)+\Psi(\mathrm{D}^{\mathrm{s}}v), \end{aligned}$$

which completes the proof of (4.1). □

Remark 4.2. If the interfacial energy density ψ has the special form

$$\psi(a,b) = \Lambda(a\otimes b) \tag{4.4}$$

where $\Lambda : \mathrm{Lin} \to [0,\infty)$ is a subadditive and positively homogeneous function then $\Psi(\mathrm{D}^{\mathrm{s}}u)$ is given by

$$\Psi(\mathrm{D}^{\mathrm{s}}u) = \Lambda(\mathrm{D}^{\mathrm{s}}u)$$

where $\mathrm{D}^{\mathrm{s}}u := [\![u]\!]\otimes\nu_u\mathscr{H} \llcorner J(u)$ is the singular part of the derivative $\mathrm{D}u$ of u and

$$\Lambda(\mathrm{D}^{\mathrm{s}}u) := \int\limits_{J(u)} \Lambda([\![u]\!]\otimes\nu_u)\,d\mathscr{H}$$

is an instance of Reshetnyak's [16] functional $\mu \mapsto \Lambda(\mu)$ of a measure $\mu \in \mathscr{M}(Q,\mathrm{Lin})$; see, e.g., [1; Equation (2.29)]. The subadditivity and positive homogeneity of of Φ (asserted in Proposition 4.1) is then an instance of the general result [1; Proposition 2.37] asserting the same properties of the functional $\mu \mapsto \Lambda(\mu)$. Indeed, if $M_i \in \mathrm{Lin}$ and $u_i \in \mathscr{A}(M_i)$, $i=1,2$, then $u_1+u_2 \in \mathscr{A}(M_1+M_2)$ and therefore

$$\Phi(M_1+M_2) \le \Lambda\big(\mathrm{D}^{\mathrm{s}}(u_1+u_2)\big) = \Lambda(\mathrm{D}^{\mathrm{s}}u_1+\mathrm{D}^{\mathrm{s}}u_2) \le \Lambda(\mathrm{D}^{\mathrm{s}}u_1)+\Lambda(\mathrm{D}^{\mathrm{s}}u_2);$$

taking the infimum over all $u_1 \in \mathscr{A}(M_1)$, $u_2 \in \mathscr{A}(M_2)$ gives

$$\Phi(M_1+M_2) \le \Phi(M_1)+\Phi(M_2).$$

The positive homogeneity of follows similarly. We note that the interfacial energies in Examples 2.5 and 2.6 have the form (4.4), but this is not the case generally.

The following elementary result records some formulas to be employed below.

Remark 4.3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with lipschitzian boundary. A countable family Ω_α, $\alpha \in \mathbb{N}$, of pairwise disjoint subsets of Ω with lipschitzian boundary is said to be a partition of Ω if one can write $\Omega = \bigcup_{\alpha=1}^{\infty} \Omega_\alpha$ to within a set of null Lebesgue measure. Let us agree to say that $\varphi \in L^1(\Omega, \mathbb{R})$ is piecewise constant if there exists a partition Ω_α such that φ is constant on each Ω_α. If v_α is the outer normal to Ω_α and if a_α is the value of φ on Ω_α, then $\varphi \in BV(\Omega, \mathbb{R})$ if and only if

$$\sum \Big\{ \int_{\partial\Omega_\alpha \cap \partial\Omega_\beta} |a_\alpha - a_\beta| \, d\mathcal{a} : (\alpha, \beta) \in \mathcal{I} \Big\} < \infty \tag{4.5}$$

where

$$\mathcal{I} = \{(\alpha, \beta) \in \mathbb{N}^2 : \alpha < \beta, \mathcal{a}(\partial\Omega_\alpha \cap \partial\Omega_\beta) > 0\}.$$

If this is the case, we have the formulas

$$J(\varphi) = \bigcup \{\partial\Omega_\alpha \cap \partial\Omega_\beta : (\alpha, \beta) \in \mathcal{I}\},$$

$$[\![\varphi]\!] v_\varphi = (a_\alpha - a_\beta) v_\beta \text{ on } \partial\Omega_\alpha \cap \partial\Omega_\beta \text{ for any } (\alpha, \beta) \in \mathcal{I},$$

$$\mathrm{D}\varphi = [\![\varphi]\!] v_\varphi \mathcal{a} \llcorner J(\varphi) \tag{4.6}$$

to within changes on sets of null $\mathcal{a}$ measure. The total variation (mass) $\mathrm{M}(\mathrm{D}\varphi)$ of $\mathrm{D}\varphi$ is equal to the sum in (4.5).

Proof Assume that (4.5) holds and prove that $\varphi \in BV(\Omega, \mathbb{R})$ and that the three formulas above hold. We note that if (4.5) holds then $\mu := [\![\varphi]\!] v_\varphi \mathcal{a} \llcorner J(\varphi)$ is a ("finite") measure in $\mathcal{M}(\Omega, \mathbb{R}^n)$. Let us prove that μ is the weak derivative of φ, which will also prove $\varphi \in BV(\Omega, \mathbb{R})$. Thus we have pro rove that

$$\int_\Omega \varphi \nabla f \, d\mathcal{v} = - \int_{J(\varphi)} f [\![\varphi]\!] \, d\mathcal{a} \tag{4.7}$$

for every class infinity function f with support in Ω. The application of the Gauss-Green theorem to each of the sets Ω_α provides

$$\int_{\Omega_\alpha} \varphi \nabla f \, d\mathcal{v} \equiv a_\alpha \int_{\Omega_\alpha} \nabla f \, d\mathcal{v} = a_\alpha \int_{\partial\Omega_\alpha} f v_\alpha \, d\mathcal{a}.$$

Summing these equations over all α and using that $v_\alpha = -v_\beta$ one obtains (4.7) and hence we have $\varphi \in BV(\Omega, \mathbb{R})$, (4.6) and all the remaining assertions of the remark. The converse implication is proved by reversing the above arguments. □

5 The function Φ

The goal of this section is to prove that the functions defined in Items (i), (ii), (iii), and (iv) of Theorem 2.3 coincide. We denote these functions by Φ_1, Φ_2, Φ_3, and Φ_4, respectively, and prove that they are the same by establishing the following cycle of relations:

$$\Phi_1 \geq \Phi_2 \geq \Phi_3 \geq \Phi_4 = \Phi_1.$$

Proposition 5.1. $\Phi_1 \geq \Phi_2$.

Proof It is easy to show that Φ_2 is a subadditive function. Thus the definition of Φ_1 gives the assertion. □

The proof of the following lemma contains a construction of the central minimizing sequence $u_k \in \mathscr{A}(M)$ for Theorem 2.3(iii). This sequence will be defined as the superposition of (a slight modification of) the sequence of step deformations s_k, $k = 1, \ldots$, defined on Q by

$$s_k(x) = k^{-1} a \langle kx \cdot b \rangle,$$

$x \in Q$. Clearly, $\nabla s_k = 0$ and in view of (3.4),

$$s_k(x) \to a(x \cdot b) \text{ on } Q$$

as $k \to \infty$. Thus s_k satisfies the boundary condition $s_k = w_{a \otimes b}$ on ∂Q in the asymptotic sense; however, the definition of $\mathscr{A}(a \otimes b)$ requires the exact form of that boundary condition. For this reason, we have to slightly modify s_k near the boundary ∂Q without violating the equation $\nabla s_k = 0$.

Lemma 5.2. *If* $M \in \mathrm{Lin}$ *and* $(a_i, b_i) \in \mathbb{D}_n$, $i = 1, \ldots, m$, *satisfy*

$$M = \sum_{i=1}^{m} a_i \otimes b_i$$

then there exists a sequence $u_k \in \mathscr{A}(M)$, $k = 1, \ldots$, *such that*

$$\limsup_{k \to \infty} \Psi(\mathrm{D}^{\mathrm{s}} u_k) \leq \sum_{i=1}^{m} \psi(a_i, b_i). \tag{5.1}$$

We refer to Remark 5.3 for a mild condition on the sequence (a_i, b_i) that guarantees that the lim sup in (5.1) strengthens to lim and the inequality sign to the equality sign.

Proof We shall first construct the sequence u_k for the particular case when $M = a \otimes b$ is a dyad and then superimpose the sequences corresponding to the dyads $a_i \otimes b_i$, $i = 1, \ldots, m$, to obtain the general case. Thus let $(a, b) \in \mathbb{D}_n$ and construct a sequence $u_k \in \mathscr{A}(a \otimes b)$, $k = 1, \ldots$, such that

$$\lim_{k \to \infty} \Psi(\mathrm{D}^{\mathrm{s}} u_k) = \psi(a, b). \tag{5.2}$$

Introduce the sets

$$C_k = (1 - k^{-2})Q, \quad L_l = (1 - (l+1)^{-2})Q \sim (1 - l^{-2})Q,$$

$k, l \in \mathbb{N}$ and observe that

$$Q = C_k \cup \bigcup_{l=k}^{\infty} L_l \tag{5.3}$$

with mutually disjoint summands for any $k \in \mathbb{N}$. Here the multiple tS of a set $S \subset \mathbb{R}^n$ by a real number t is defined by $tS = \{tx : x \in S\}$. Equation (5.3) presents a decomposition of Q into the main set C_k which is a large subset of Q for large k, while $L_k, L_{k+1}, \ldots$ present infinitely many rectangular layers filling the gap $Q \sim C_k$ and refining more and more towards the boundary of Q.

We use these sets C_k, L_k, L_{k+1}, …, to define a sequence of scalar functions $\varphi_k : Q \to \mathbb{R}$, $k = 2, \dots$, by

$$\varphi_k(x) = \begin{cases} (k-1)^{-2}\langle (k-1)^2 x \cdot b\rangle & \text{if } x \in C_k, \\ l^{-2}\langle l^2 x \cdot b\rangle & \text{if } x \in L_l \quad \text{for some } l \ge k\,. \end{cases} \tag{5.4}$$

Let us use Remark 4.3 to prove that $\varphi_k \in BV(Q, \mathbb{R})$. Clearly, φ_k is a piecewise constant function in the sense of that remark. Using $(3.2)_1$, one finds that $x \cdot b - 1 \le \varphi_k(x) \le x \cdot b$; hence $|\varphi_k|$ is bounded on Q and thus $\varphi_k \in L^1(Q, \mathbb{R})$. It remains to verify (4.5). Let us show that in the present case (4.5) reads

$$\int_{J(\varphi_k)} \big|[\![\varphi_k]\!]\big|\, d\mathcal{A} < \infty, \tag{5.5}$$

where

$$J(\varphi_k) = C_k^\circ \cup \bigcup_{l=k}^{\infty} (L_l^\circ \cup L_l^\partial) \tag{5.6}$$

is the jump set, with

$$C_k^\circ = \{x \in C_k : k^2 x \cdot b \in \mathbb{Z}\},$$
$$L_l^\circ = \{x \in L_l : l^2 x \cdot b \in \mathbb{Z}\}, \quad L_l^\partial = (1 - l^{-2})\partial Q,$$

and on $J(\varphi_k)$,

$$[\![\varphi_k]\!] \nu_{\varphi_k} = \begin{cases} (k-1)^{-2} b & \text{on } C_k^\circ, \\ l^{-2} b & \text{on } L_l^\circ \quad \text{where } l \ge k, \\ \eta_l \nu_k & \text{on } L_l^\partial \quad \text{where } l \ge k, \end{cases} \tag{5.7}$$

is the jump and normal to the jump set, with

$$\eta_l(x) = l^{-2}\langle l^2 x \cdot b\rangle - (l-1)^{-2}\langle (l-1)^2 x \cdot b\rangle$$

and with ν_k denoting the outer normal to the scaled cube $(1 - k^{-2})Q$. The formulas (5.6) and (5.7) follow from the identifications given in Remark 4.3. One has to enumerate the regions of constancy of φ_k in an arbitrary way to obtain the system of sets Ω_α, $\alpha = 1, \dots$, and use the formulas of that remark. The details are left to the reader. This establishes the equivalence of the inequalities (4.5) and (5.5). To prove that (5.5) really holds, one finds from (5.7) that

$$\int_{J(u_k)} \big|[\![\varphi_k]\!]\big|\, d\mathcal{A} = (k-1)^{-2}\mathcal{A}(C_k^\circ) + \sum_{l=k}^{\infty} l^{-2}\mathcal{A}(L_l^\circ) + \sum_{l=k}^{\infty} \int_{L_l^\partial} |\eta_l|\, d\mathcal{A}. \tag{5.8}$$

We estimate the terms $\mathcal{A}(C_k^\circ)$, $\mathcal{A}(L_l^\circ)$ and $\int_{L_l^\partial} |\eta_l(x)|\, d\mathcal{A}$ as follows. First, prove that

$$\big|\mathcal{A}(C_k^\circ) - (k-1)^2 \mathcal{V}(C_k)\big| \le 2n, \quad \big|\mathcal{A}(L_l^\circ) - l^2 \mathcal{V}(L_l)\big| \le 4n \tag{5.9}$$

and hence

$$\mathcal{A}(C_k^\circ) \le 2n + (k-1)^2 \mathcal{V}(C_k), \quad \mathcal{A}(L_l^\circ) \le 4n + l^2 \mathcal{V}(L_l). \tag{5.10}$$

Let us prove $(5.9)_2$; the proof of $(5.9)_1$ is similar. Let $\omega : L_l \to \mathbb{R}$ be defined by

$$\omega(x) = l^2 x \cdot b - \langle l^2 x \cdot b \rangle, \quad x \in L_l.$$

Then $\omega \in BV(L_l, \mathbb{R})$, $\mathrm{D}\omega = l^2 b - b\,\mathscr{a} \llcorner L_l^\circ$ and hence the Gauss-Green theorem yields

$$\mathrm{D}\omega(L_l) = l^2 \mathscr{v}(L_l) b - b\,\mathscr{a}(L_l^\circ) = \int_{\partial L_l} \omega \nu_{L_l} \, d\mathscr{a},$$

from which

$$|m\mathscr{v}(L_l) - \mathscr{a}(L_l^\circ)| \le \int_{\partial L_l} |\omega| \, d\mathscr{a}.$$

We now observe that $|\omega| \le 1$ on ∂L_l and $\partial L_l = L_{l+1}^\partial \cup L_l^\partial$. Thus

$$\int_{\partial L_l} |\omega| \, d\mathscr{a} \le \mathscr{a}(L_{l+1}^\partial) + \mathscr{a}(L_l^\partial) \le 4n$$

since, elementarily, $\mathscr{a}(L_{l+1}^\partial) \le 2n$, $\mathscr{a}(L_l^\partial) \le 2n$. Thus we have $(5.9)_2$. Next prove that

$$|\eta_l(x)| \le 2(l-1)^{-2} \quad \text{on} \quad L_l^\partial.$$

Indeed, writing

$$|\eta_l(x)| = |\big(l^{-2}\langle l^2 x \cdot b\rangle - x \cdot b\big) - \big((l-1)^{-2}\langle (l-1)^2 x \cdot b\rangle - x \cdot b\big)|,$$

using the triangle inequality and the inequality (3.3) twice, with $t = x \cdot b$ and $k = l^2$ and $k = (l-1)^2$, one obtains

$$|\eta_l(x)| \le l^{-2} + (l-1)^{-2} \le 2(l-1)^{-2}.$$

and hence

$$\int_{L_l^\partial} |\eta_l| \, d\mathscr{a} \le 2(l-1)^{-2} \mathscr{a}(L_l^\partial) \le 4n(l-1)^{-2}. \tag{5.11}$$

The estimates (5.10) and (5.11) and the formula (5.8) provide

$$\begin{aligned} \int_{J(u_k)} \big|[\![\varphi_k]\!]\big| \, d\mathscr{a} &\le 2n(k-1)^{-2} + \mathscr{v}(C_k) \\ &\quad + \sum_{l=k}^{\infty} (4nl^{-2} + \mathscr{v}(L_l)) + \sum_{l=k}^{\infty} 4n(l-1)^{-2} \\ &\le 1 + 2n(k-1)^{-2} + 8n \sum_{l=k}^{\infty} (l-1)^{-2} < \infty \end{aligned}$$

where we have used

$$\mathscr{v}(C_k) + \sum_{l=k}^{\infty} \mathscr{v}(L_l) = \mathscr{v}(Q) = 1.$$

Thus we have (5.5); hence $\varphi_k \in BV(\Omega, \mathbb{R})$ for every k and

$$\mathrm{D}\varphi_k = [\![\varphi_k]\!] \nu_{\varphi_k} \mathscr{a} \llcorner J(\varphi_k)$$

and

$$\nabla \varphi_k = 0. \tag{5.12}$$

Finally, note that the boundary trace φ_k^∂ of φ_k on ∂Q satisfies

$$\varphi_k^\partial(x) = x \cdot b \quad \text{for every} \quad x \in \partial Q. \tag{5.13}$$

While a rigorous proof of this can be given by using the essential limit of φ_k at $x \in Q$, we here only note that the definition of φ_k yields that

$$\lim_{j\to\infty} \varphi_k(x_j) = x \cdot b \tag{5.14}$$

for any $x \in \partial Q$ and any sequence $x_j \in Q$ converging to x. For this it suffices to note that in view of (5.3) one finds that x_j must belong to some L_l for some $l = l(j) \geq k$. The limit $x_j \to x$ then implies that $l(k) \to \infty$ and then the definition (5.4) and the formula (3.4) provide (5.14).

We define the sequence $u_k : Q \to \mathbb{R}^n$, $k = 2, \dots$, by

$$u_k(x) = a\varphi_k(x)$$

for every $x \in Q$. By $\varphi_k \in SBV(Q, \mathbb{R})$ and by (5.12) and (5.13) we have $u_k \in \mathscr{A}(a \otimes b)$. Further, $[\![u_k]\!] = [\![\varphi_k]\!] a \otimes \nu_{\varphi_k}$; consequently, by (5.7),

$$\psi([\![u_k]\!], \nu_{u_k}) = \begin{cases} (k-1)^{-2}\psi(a,b) & \text{on} \quad C_k^\circ, \\ l^{-2}\psi(a,b) & \text{on} \quad L_l^\circ \quad \text{for any} \quad l \geq k, \\ \psi(\eta_l a, \nu_l) & \text{on} \quad L_l^\partial \quad \text{for any} \quad l \geq k \end{cases}$$

and hence

$$\Psi(\mathrm{D}^{\mathrm{s}} u_k) = \int_{J(u_k)} \psi([\![u_k]\!], \nu_{u_k})\, d\mathscr{a} = (k-1)^{-2}\psi(a,b)\mathscr{a}(C_k^\circ) + \rho_k \tag{5.15}$$

where

$$\rho_k = \sum_{l=k}^{\infty} l^{-2}\psi(a,b)\mathscr{a}(L_l^\circ) + \sum_{l=k}^{\infty} \int_{L_l^\partial} \psi(\eta_l a, \nu_l)\, d\mathscr{a}.$$

Dividing $(5.9)_1$ by $(k-1)^2$, we obtain

$$(k-1)^{-2}\mathscr{a}(C_k^\circ) \to 1 \tag{5.16}$$

since $\mathscr{v}(C_k) \to 1$. Using (2.1), we obtain that the nonnegative number ρ_k is bounded by a (constant multiple of) the quantity

$$\begin{aligned} d_k &= \sum_{l=k}^{\infty} l^{-2}\mathscr{a}(L_l^\circ) + \sum_{l=k}^{\infty} \int_{L_l^\partial} |\eta_l|\, d\mathscr{a} \\ &\leq \sum_{l=k}^{\infty} \mathscr{v}(L_l) + 2n(k-1)^{-2} + 4n \sum_{l=k}^{\infty} (l-1)^{-2} \\ &\leq k^{-2} + 2n(k-1)^{-2} + 4n \sum_{l=k}^{\infty} (l-1)^{-2} \end{aligned}$$

and hence $\rho_k \to 0$. Equations (5.15) and (5.16) then yield (5.2).

We now complete the proof in the general case. By the preceding part of the proof, for each $i \in \{1, \dots, m\}$ there exists a sequence $u_k^i \in \mathscr{A}(a_i \otimes b_i, 0)$, $k = 1, \dots$, such that

$$\Psi(\mathrm{D}^{\mathrm{s}} u_k^i) \to \psi(a_i, b_i) \tag{5.17}$$

as $k \to \infty$. Define $u_k := \sum_{i=1}^m u_k^i$ for every k. By (4.1) we have

$$\Psi(\mathrm{D}^{\mathrm{s}} u_k) \leq \sum_{i=1}^{m} \Psi(\mathrm{D}^{\mathrm{s}} u_k^i). \tag{5.18}$$

Hence

$$\limsup_{k\to\infty} \Psi(\mathrm{D}^{\mathrm{s}} u_k) \le \lim_{k\to\infty} \sum_{i=1}^{m} \Psi(\mathrm{D}^{\mathrm{s}} u_k^i) = \sum_{i=1}^{m} \psi(a_i, b_i)$$

by (5.17). □

Remark 5.3. If the sequence (a_i, b_i) satisfies the condition

$$b_i \neq b_j \quad \text{and} \quad b_i \neq -b_j \quad \text{whenever} \quad 1 \le i < j \le m, \tag{5.19}$$

then the sequence u_k can be chosen as to satisfy, instead of the inequality (5.1), the equality

$$\lim_{k\to\infty} \Psi(\mathrm{D}^{\mathrm{s}} u_k) = \sum_{i=1}^{m} \psi(a_i, b_i).$$

Indeed, the inspection of the proof of Lemma 5.2 shows that the source of the inequality (5.1) is the subadditivity in (5.18) which cannot be replaced by the equality unless the discontinuity sets $J(u_i)$ pairwise intersect on a set of null $\mathscr{H}$-measure (see Proposition 4.1). Condition (5.19) guarantees that. However, Inequality (5.1) suffices for our purposes.

Proposition 5.4. $\Phi_2 \ge \Phi_3 \ge \Phi_4$.

Proof To prove $\Phi_2 \ge \Phi_3$, we take any sequence $(a_i, b_i) \in \mathbb{D}_n$, $i = 1, \dots, m$, such that $\sum_{i=1}^{m} a_i \otimes b_i = M$ and consider the infimum as in the definition of Φ_2 in (2.7). Hence, for the given sequence $(a_i, b_i) \in \mathbb{D}_n$, we construct a sequence of maps $u_k \in \mathscr{A}(M)$, $k = 1, \dots$, as in Lemma 5.2. Then

$$\Phi_3(M) \le \Psi(\mathrm{D}^{\mathrm{s}} u_k)$$

by the definition of Φ_3. Letting $k \to \infty$ and using (5.1), we obtain

$$\Phi_3(M) \le \sum_{i=1}^{m} \psi(a_i, b_i).$$

Taking the infimum over all sequences a_i, b_i, one obtains from the definition of Φ_2 the inequality $\Phi_3(M) \le \Phi_2(M)$. The inequality $\Phi_3 \ge \Phi_4$ is immediate. □

Proposition 5.5. $\Phi_4 = \Phi_1$.

Proof We seek to prove that Φ_4 is the biggest subadditive function satisfying $\Phi_4(a \otimes b) \le \psi(a, b)$ for any $(a, b) \in \mathbb{D}_n$. To prove the subadditivity of Φ_4, let A, $B \in$ Lin and $u \in \mathscr{B}(A)$, $v \in \mathscr{B}(B)$. Proposition 4.1 and (2.8) yield $u + v \in \mathscr{B}(A + B)$ and

$$\Phi_4(A + B) \le \Psi(\mathrm{D}^{\mathrm{s}} u + \mathrm{D}^{\mathrm{s}} v) \le \Psi(\mathrm{D}^{\mathrm{s}} u) + \Psi(\mathrm{D}^{\mathrm{s}} v).$$

Taking the infimum over all u, and v then gives the subadditivity

$$\Phi_4(A + B) \le \Phi_4(A) + \Phi_4(B).$$

Next we note that the biggest subadditive function Θ such that

$$\Theta(a \otimes b) \le \psi(a, b) \tag{5.20}$$

for any $(a, b) \in \mathbb{D}_n$ is automatically positively homogeneous; thus it suffices to prove the maximality of Φ_4 among all subadditive and positively homogeneous functions satisfying (5.20). Thus let Θ be such function and let $M \in \mathrm{Lin}$ and $u \in \mathscr{B}(M)$. Then by (5.20) and by Jensen's inequality for positively homogeneous subadditive functions

$$\begin{aligned}\Psi(\mathrm{D}^s u) &:= \int_{J(u)} \psi([\![u]\!], \nu_u)\, d\mathcal{a} \\ &\geq \int_{J(u)} \Theta([\![u]\!] \otimes \nu_u)\, d\mathcal{a} \\ &\geq \Theta\Big(\int_{J(u)} [\![u]\!] \otimes \nu_u\, d\mathcal{a}\Big).\end{aligned} \tag{5.21}$$

We now combine the boundary condition $u = w_M$ on ∂Q and the relation $\int_Q \nabla u\, d\mathcal{v} = 0$ with the Gauss-Green theorem to obtain

$$\begin{aligned}\int_{J(u)} [\![u]\!] \otimes \nu_u\, d\mathcal{a} &= \int_{J(u)} [\![u]\!] \otimes \nu_u\, d\mathcal{a} + \int_Q \nabla u\, d\mathcal{v} \\ &= \int_Q 1 d\mathrm{D}u \\ &= \int_{\partial Q} Mx \otimes \nu_Q\, d\mathcal{a} = M.\end{aligned}$$

Thus (5.21) yields

$$\Psi(\mathrm{D}^s u) \geq \Theta(H).$$

Taking the infimum over all $u \in \mathscr{B}(M)$, we obtain $\Phi_4(M) \geq \Theta(M)$. □

This proves $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4$. *We define the function* Φ *by* $\Phi = \Phi_1$.

6 Completion of the proof of Theorem 2.3

For this section, we put, for every $(a, b) \in \mathbb{D}_n$,

$$\mathscr{C}(a, b) := \{u \in SBV(Q_b) : u = z_{a,b} \text{ on } \partial Q_b,\ \nabla u = 0 \text{ on } Q_b\}$$

and denote by $\Phi_5(a, b)$ the infimum in (2.10). We then extend Φ_5 to $\mathbb{R}^n \times \mathbb{R}^n$ by homogeneity in the second variable.

Proposition 6.1. *We have* $W(A, B) = \Phi(A - B)$ *for every* $A, B \in \mathrm{Lin}$.

Proof We employ Theorem 2.2 and the definition of Φ in (2.9). Invoking (2.2), we take any $u \in SBV(Q)$ satisfying $u = w_A$ on ∂Q, and $\int_Q \nabla u\, d\mathcal{v} = B$. Then v, given by $v(x) = u(x) - Bx, x \in Q$, satisfies $v \in \mathscr{B}(A - B)$ and $\Psi(\mathrm{D}^s u) = \Psi(\mathrm{D}^s v)$. □

Lemma 6.2. *We have* $\Phi_5(a, b) \leq \Phi(a \otimes b)$ *for every* $(a, b) \in \mathbb{D}_n$.

Proof Let $(a, b) \in \mathbb{D}_n$ and let $(a_i, b_i) \in \mathbb{D}_n$, $i = 1, \dots, m$, be a sequence satisfying

$$a \otimes b = \sum_{i=1}^{m} a_i \otimes b_i. \tag{6.1}$$

Our goal is to construct a sequence $u_k \in \mathscr{C}(a, b)$, $k = 1, \dots$, such that

$$\limsup_{k \to \infty} \int_{J(u_k)} \psi([\![u_k]\!], \nu_{u_k})\, d\mathcal{a} \leq \sum_{i=1}^{m} \psi(a_i, b_i). \tag{6.2}$$

To define u_k, let

$$P = \{x \in \mathbb{R}^n : x \cdot b = 0\}$$

be the plane through the origin perpendicular to b, let Π be the projection from $\mathbb{R}^n$ onto P, let

$$F = P \cap Q_b$$

and put

$$B_k = \{x \in \mathbb{R}^n : \Pi(x) \in (1-k^{-1})F,\ 0 \le x \cdot b < k^{-1}\}$$

for any $k \in \mathbb{N}$. Define $u_k : Q_b \to \mathbb{R}^n$ by

$$u_k(x) = \begin{cases} v_k(x) & \text{if } x \in B_k, \\ z_{a,b}(x) & \text{else,} \end{cases}$$

$x \in Q_b$, where

$$v_k(x) = \sum_{i=1}^{m} k^{-1} a_i \langle k^2 x \cdot b_i \rangle \quad \text{for any} \quad x \in \mathbb{R}^n \quad \text{and} \quad k \in \mathbb{N}.$$

Employing Remark 4.2, we see that $u_k \in SBV(Q_b)$; furthermore, clearly, $u_k = z_{a,b}$ on ∂Q_b and $\nabla u_k = 0$ on Q_b; hence $u_k \in \mathscr{C}(a,b)$.

We proceed to prove (6.2). We have

$$J(u_k) = N_k \cup M_k \cup L_k \cup S_k \tag{6.3}$$

where

$$\begin{aligned} N_k &= F \sim (1-k^{-1})F, \\ M_k &= \{x \in \partial B : 0 < x \cdot b < k^{-1}\}, \\ S_k &= \{x \in \mathbb{R}^n : \Pi(x) \in (1-k^{-1})F,\ x \cdot b = k^{-1}\}, \\ L_k &= \bigcup_{i=1}^{m} L_k^i \quad \text{where} \quad L_k^i = \{x \in B_k : k^2 x \cdot b_i \in \mathbb{Z}\}. \end{aligned} \tag{6.4}$$

The jump of u_k and the normal to the jump set are

$$[\![u_k]\!](x)\nu_{u_k}(x) = \begin{cases} k^{-1} \sum_{i=1}^{m} a_i \otimes b_i\, 1_{L_k^i}(x) & \text{if } x \in L_k, \\ a \otimes b & \text{if } x \in N_k, \\ (a - v_k(x)) \otimes \nu_k & \text{if } x \in M_k, \\ (a - v_k(x)) \otimes b & \text{if } x \in S_k, \end{cases}$$

$x \in J(u_k)$, where ν_k is the outer normal to B_k and $1_{L_k^i}$ is the characteristic function of the set L_k^i. Hence the subadditivity of ψ in the first variable yields

$$\int_{L_k} \psi([\![u_k]\!], \nu_{u_k})\, d\mathscr{H} \le k^{-1} \sum_{i=1}^{m} \psi(a_i, b_i) \mathscr{H}(L_k^i);$$

consequently

$$\begin{aligned}\int_{J(u_k)} \psi(\llbracket u_k \rrbracket, \nu_{u_k})\, d\mathcal{H} \le k^{-1} \sum_{i=1}^{m} \psi(a_i, b_i)\mathcal{H}(L_k^i) \\ + \psi(a, b)\mathcal{H}(N_k) \\ + \int_{M_k} \psi(a - v_k(x)), \nu_k)\, d\mathcal{H} \\ + \int_{S_k} \psi(a - v_k(x)), b)\, d\mathcal{H}.\end{aligned} \tag{6.5}$$

Let us now analyze the terms on the right-hand side of (6.5). Using the considerations as in the proof of Lemma 5.2 (see (5.9) and (5.10)) one finds that

$$k^{-1}\mathcal{H}(L_k^i) \to 1$$

as $k \to \infty$ for every $i = 1, \dots, m$. Thus

$$k^{-1} \sum_{i=1}^{m} \psi(a_i, b_i)\mathcal{H}(L_k^i) \to \sum_{i=1}^{m} \psi(a_i, b_i). \tag{6.6}$$

Further,

$$\psi(a, b)\mathcal{H}(N_k) \to 0 \tag{6.7}$$

since, obviously,

$$\mathcal{H}(N_k) \to 0.$$

Next, note that by (6.1) and (3.3),

$$\begin{aligned}|ka(x \cdot b) - v_k(x)| &= |ka(x \cdot b) - \sum_{i=1}^{m} k^{-1} a_i \langle k^2 x \cdot b_i \rangle| \\ &= \Big|k \sum_{i=1}^{m} \big(a_i(x \cdot b_i) - k^{-2} a_i \langle k^2 x \cdot b_i \rangle\big)\Big| \\ &\le k \sum_{i=1}^{m} |a_i| \big|(x \cdot b_i) - k^{-2} \langle k^2 x \cdot b_i \rangle\big| \\ &\le k^{-1} \sum_{i=1}^{m} |a_i|.\end{aligned}$$

Then if $x \in M_k$,

$$\begin{aligned}|a - v_k(x)| &\le |a - ka(x \cdot b)| + |ka(x \cdot b) - v_k(x)| \\ &\le |a| + k|a||x \cdot b| + k^{-1} \sum_{i=1}^{m} |a_i| \\ &\le |a| + |a| + k^{-1} \sum_{i=1}^{m} |a_i|\end{aligned}$$

since $k|x \cdot b| \le 1$ on M_k. Thus $|a - v_k(x)| \le c < \infty$ for any $x \in M_k$ and any $k = 1, \dots$ A combination with (2.1) and

$$\mathcal{H}(M_k) \to 0$$

then provides

$$\int_{M_k} \psi(a - v_k(x)), \nu_k)\, d\mathcal{H} \to 0. \tag{6.8}$$

Similarly, if $x \in S_k$ then $kx \cdot b = 1$ and hence

$$|a - v_k(x)| \le |ka(x \cdot b) - v_k(x)| \le k^{-1} \sum_{i=1}^{m} |a_i| \to 0.$$

Thus (2.1) yields

$$\int_{S_k} \psi(a - v_k(x)), b)\, d\mathscr{a} \to 0 \tag{6.9}$$

since $\mathscr{a}(S_k) \le 1$ for all k. Consequently, a combination of (6.5) with (6.6)–(6.9) provides (6.2) and hence the definition of Φ_5 gives

$$\Phi_5(a, b) \le \sum_{i=1}^{m} \psi(a_i, b_i)$$

for any sequence (a_i, b_i) satisfying (6.1). Taking the infimum of the right-hand side over all such sequences and using the definition of $\Phi_2 \equiv \Phi$ we obtain the assertion. □

Lemma 6.3. *We have* $\Phi_5(a, b) \ge \Phi(a \otimes b)$ *for every* $(a, b) \in \mathbb{D}_n$.

Proof Let $u \in \mathscr{C}(a, b)$. Then, by Jensen's inequality,

$$\begin{aligned} \int_{J(u)} \psi([\![u]\!], v_u)\, d\mathscr{a} &\ge \int_{J(u)} \Phi([\![u]\!] \otimes v_u)\, d\mathscr{a} \\ &\ge \Phi\Big(\int_{J(u)} [\![u]\!] \otimes v_u\, d\mathscr{a} \Big) \\ &= \Phi(a \otimes b) \end{aligned}$$

since the boundary condition $u = z_{a,b}$ on ∂Q_b implies

$$\int_{J(u)} [\![u]\!] \otimes v_u\, d\mathscr{a} = a \otimes b.$$

That is, we have

$$\int_{J(u)} \psi([\![u]\!], v_u)\, d\mathscr{a} \ge \Phi(a \otimes b)$$

for every $u \in \mathscr{C}(a, b)$. Taking the infimum, we obtain $\Phi_5(a, b) \ge \Phi(a \otimes b)$. □

Proposition 6.4. *We have* $h(a, b) = \Phi(a \otimes b)$ *for every* $(a, b) \in \mathbb{D}_n$.

Proof This follows immediately from (2.3) and (2.10). □

This completes the proof of Theorem 2.3.

7 Derivation of the examples

***Example* 2.5.** Equation (2.11): We consider $\psi_{|\cdot|}(a, b) = |a \cdot b|$ first, and prove $(2.11)_1$. Clearly, the function $\Theta(M) = |\operatorname{tr} M|$ is a subadditive function satisfying (2.5) with $\psi = \psi_{|\cdot|}$ and hence (2.7) gives $\Phi_{|\cdot|}(M) \ge |\operatorname{tr} M|$ for any $M \in$ Lin. To prove the opposite inequality, we note that the definition (2.6) of $\Phi_{|\cdot|}$ gives

$$\psi_{|\cdot|}(a, b) = \Theta(a \otimes b) \le \Phi_{|\cdot|}(a \otimes b) \le \psi_{|\cdot|}(a, b)$$

for every $(a,b) \in \mathbb{D}_n$ and hence

$$\Phi_{|\cdot|}(a \otimes b) = |a \cdot b| \quad \text{and in particular} \quad \Phi_{|\cdot|}(a \otimes b) = 0 \ \text{ if } \ a \cdot b = 0$$

which determines $\Phi_{|\cdot|}$ on tensor products $a \otimes b$. As a consequence, if $N \in \mathrm{Lin}$ can be written as

$$N = \sum_{i=1}^{m} a_i \otimes b_i \tag{7.1}$$

where $(a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, \ldots, m$, where

$$a_i \cdot b_i = 0 \ \text{ for all } \ i = 1, \ldots, m, \tag{7.2}$$

then $\Phi_{|\cdot|}(N) = 0$ since

$$0 \le \Phi_{|\cdot|}(N) \le \sum_{i=1}^{m} \Phi_{|\cdot|}(a_i \otimes b_i) \le \sum_{i=1}^{m} \psi(a_i, b_i) = \sum_{i=1}^{m} |a_i \cdot b_i| = 0.$$

To determine $\Phi_{|\cdot|}$ on a general $M \in \mathrm{Lin}$, we write $M = A + W$ where A and W are the symmetric and skew parts of M. Let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors of A with the eigenvalues λ_i; hence $A = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i$. Then

$$M = B + N$$

where

$$B = (\operatorname{tr} M) e_1 \otimes e_1,$$

$$N = W + \sum_{i=2}^{n} \lambda_i \big(e_i \otimes e_1 - e_1 \otimes e_i - (e_1 + e_i) \otimes (e_1 - e_i) \big).$$

Since W is a linear combination of the dyads $e_i \otimes e_j$, $1 \le i \neq j \le n$, one sees that N is of the form (7.1), (7.2) and hence $\Phi_{|\cdot|}(N) = 0$; consequently

$$\Phi_{|\cdot|}(M) \le \Phi_{|\cdot|}(B) + \Phi_{|\cdot|}(N) = \Phi_{|\cdot|}(B) = \psi\big((\operatorname{tr} M) e_1, e_1\big) = |\operatorname{tr} M|.$$

Equations (2.4) complete the proof of $(2.11)_1$.

To prove the two equations in $(2.11)_2$, we employ $(2.11)_1$ and $(2.11)_2$ as follows. One has $\psi_\pm(a,b) = \frac{1}{2}\big(|a \cdot b| \pm a \cdot b\big)$ and hence if $(a_i, b_i) \in \mathbb{D}_n$ and $M \in \mathrm{Lin}$ satisfy $\sum_{i=1}^{m} a_i \otimes b_i = M$ then

$$\sum_{i=1}^{m} \psi_\pm(a_i, b_i) = \frac{1}{2}\Big(\sum_{i=1}^{m} \psi_{|\cdot|}(a_i, b_i) \pm \operatorname{tr} M \Big).$$

Taking the infimum as in (2.7) and using the above evaluation of $\Phi_{|\cdot|}$ gives

$$\Phi_\pm(M) = \frac{1}{2}\big(\Phi_{|\cdot|}(M) \pm \operatorname{tr} M\big) = \frac{1}{2}\big(|\operatorname{tr} M| \pm \operatorname{tr} M\big) = \{\operatorname{tr} M\}_\pm$$

which is $(2.11)_2$. □

***Example* 2.6.** Equation (2.12): The function $\Theta(M) = |M^{\mathrm{T}} p|$ is a subadditive function satisfying (2.5) and we obtain in the same way as in the proof of Example 2.5 that $\Phi(M) \ge |M^{\mathrm{T}} p|$ for any $M \in \mathrm{Lin}$ and

$$\Phi(a \otimes b) = |a \cdot p|, \quad \text{and in particular} \quad \Phi(a \otimes b) = 0 \ \text{ if } \ a \cdot p = 0. \tag{7.3}$$

To prove $\Phi(M) \le |M^{\mathrm{T}} p|$, we assume without loss in generality that $|p| = 1$ and let $\{p, e_2, \ldots e_n\}$ be any orthonormal basis. In view of $\mathrm{I} = p \otimes p + \sum_{i=2}^{n} e_i \otimes e_i$ we have

$$M = \mathrm{I}M = p \otimes M^{\mathrm{T}} p + \sum_{i=2}^{n} e_i \otimes M^{\mathrm{T}} e_i;$$

normalizing the second members of the dyads, we obtain

$$M = |M^{\mathrm{T}} p| p \otimes \operatorname{sgn}(M^{\mathrm{T}} p) + \sum_{i=2}^{n} |M^{\mathrm{T}} e_i| e_i \otimes \operatorname{sgn}(M^{\mathrm{T}} e_i).$$

The subadditivity of Φ provides

$$\Phi(M) \le \Phi(|M^{\mathrm{T}} p| p \otimes \operatorname{sgn}(M^{\mathrm{T}} p)) + \sum_{i=2}^{n} |\Phi(M^{\mathrm{T}} e_i| e_i \otimes \operatorname{sgn}(M^{\mathrm{T}} e_i)) = |M^{\mathrm{T}} p|$$

by (7.3). Thus $\Phi(M) \le |M^{\mathrm{T}} p|$ and the proof of (2.12) is complete. □

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