Peridynamics and Coleman & Noll's retardation theorem

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For Paolo, with a deep esteem

1 Prologue: Coleman & Noll's retardation theorem, 1960

An Approximation Theorem for Functionals, with Applications in Continuum Mechanics, [Coleman and Noll, 1960]
Viscoelasticity theory (rheology): The present value of the stress $\boldsymbol{T}(\boldsymbol{x},t)$ at $\boldsymbol{x}$ depends on the history $\boldsymbol{F}^t(\boldsymbol{x},s) = \boldsymbol{F}(\boldsymbol{x},t-s)$, $s \geq 0$ of the deformation gradient $\boldsymbol{F}$ from $-\infty$ up to t,

$$\boldsymbol{T}(\boldsymbol{x},t) = \mathfrak{T}(\boldsymbol{F}^t(\boldsymbol{x},\cdot))$$

where $\mathfrak{T}$ is generally a nonlinear functional, obeying

the hypothesis of fading memory

remote past of $\boldsymbol{F}$ has a negligible influence on the present value of $\boldsymbol{T}$.

For a functional $\mathfrak{T}$ of unspecified structure this is expressed by

- choice of the mathematical space of histories
- continuity and differentiability of $\mathfrak{T}$.

$$\Downarrow$$

for slow motions $\mathfrak{T}$ can be approximated:

$$\mathfrak{T}(\boldsymbol{F}^t(\boldsymbol{x},\cdot)) \sim \boldsymbol{f}(\boldsymbol{F}(\boldsymbol{x},t), \dot{\boldsymbol{F}}(\boldsymbol{x},t), \ddot{\boldsymbol{F}}(\boldsymbol{x},t), \dots)$$

e.g., for an incompresible fluid the first approximation

$$\boldsymbol{T} \sim -p\mathbf{1} + 2\eta\boldsymbol{D} + \lambda(\mathrm{tr}\,\boldsymbol{D})\mathbf{1} \quad \text{where} \quad \boldsymbol{D} = \frac{1}{2}(\mathrm{grad}\,\boldsymbol{v} + \mathrm{grad}\,\boldsymbol{v}^{\mathrm{T}}).$$

Spatial version: On Retardation Theorems, [Coleman, 1971]**.**

time derivatives replaced by spatial gradients

Goal of the talk:

- recapitulate Coleman's theorem
- apply it to an isotropic solid

***Remark* 1** Paolo Podio-Guidugli informed me after I had announced the title of this talk that he and Gianfranco Capriz planned a similar work, which, though, never materialized.

***Remark* 2** Victor J. Mizel draw my attention to the paper [Coleman, 1971] in 1992.

2 Introduction

Peridynamics is a nonlocal continuum theory that does not use the spatial derivatives of the displacement field
S. A. Silling in [Silling, 2000],
S. A. Silling, M. Epton, O. Weckner, J. Xu & E. Askari in [Silling et al., 2007] revised and broadened;
Predecessors I. A. Kunin in [Kunin, 1982], [Kunin, 1983] and by A. C. Eringen in [Eringen, 2002].

The equation of motion:
$\Omega \subset \mathbb{R}^n$ reference configuration
$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t)$ deformation,
$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$ body force,
ρ is the density,

$$\rho \ddot{\boldsymbol{\xi}} = \mathfrak{F}(\boldsymbol{\xi}) + \boldsymbol{b}$$

$\mathfrak{F}(\boldsymbol{\xi})(\boldsymbol{x})$ force at $\boldsymbol{x}$ exerted on $\boldsymbol{x}$ by the rest of the body.

The exact form of the operator $\mathfrak{F}$ often differs in different authors.
[Silling, 2000] and [Silling et al., 2007] proposes the following forms:

$$\mathfrak{F}(\xi)(\boldsymbol{x}) = \int_{\Omega} \boldsymbol{f}(\xi(\boldsymbol{y}) - \xi(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x})\, dV_{\boldsymbol{y}},$$

$$\mathfrak{F}(\xi)(\boldsymbol{x}) = \int_{\Omega} \big(\boldsymbol{T}(\xi(\boldsymbol{y}) - \xi(\boldsymbol{x})) - \boldsymbol{T}(\xi(\boldsymbol{x}) - \xi(\boldsymbol{y}))\big)\, dV_{\boldsymbol{y}},$$

respectively, where $\boldsymbol{f}$ and $\boldsymbol{T}$ are materially dependent functions

3 Asymptotic expansion for vanishing nonlocality

Does the theory reduce to the classical local or higher–grade continuum theory under certain circumstances? What are these "cicumstances?"

The concrete form of the functional $\mathfrak{F}$ often contains a physical parameter, called the horizon by S. A. Silling, of dimension of length

$\Rightarrow$ the limit of vanishing viscosity.

For a general, formally unspecified $\mathfrak{F}$, one has to apply

B. D. Coleman's spatial retardation.

4 Influence Functions

Assume

- $\Omega = \mathbb{R}^n$
- the force $\mathfrak{F}(\xi)$ calculated at $\boldsymbol{x} = \boldsymbol{0}$

We assume that the values of the deformation $\xi(\boldsymbol{x})$ for $|\boldsymbol{x}| \to \infty$ influence $\mathfrak{F}(\xi)$ in a negligible way. This is expressed by working in the Banach space

$$\mathscr{L}_{p,\,h} = \left\{ \xi : \mathbb{R}^n \to \mathbb{R}^n \text{ measurable, } \int_{\mathbb{R}^n} (\xi(\boldsymbol{x}) h(\boldsymbol{x}))^p \, dV_{\boldsymbol{x}} < \infty \right\}$$

with the norm

$$|\xi|_{p,\,h} = \left(\int_{\mathbb{R}^n} (\xi(\boldsymbol{x}) h(\boldsymbol{x}))^p \, dV_{\boldsymbol{x}} \right)^{1/p}$$

where $1 \leq p \leq \infty$ and $h : \mathbb{R}^n \to (0, \infty)$, an influence function. h is said to be of order $s \geq 0$: if for each $\sigma > 0$ there exists a constant M_σ such that

$$\sup\left\{\frac{h(\boldsymbol{x}/\alpha)}{\alpha^s h(\boldsymbol{x})} : \boldsymbol{x} \in \mathbb{R}^n,\ |\boldsymbol{x}| \geq \sigma\right\} \leq M_\sigma$$

for all $\alpha \in (0, 1]$.

Remark 1. *If h is an influence function of order s then*

$$h(\boldsymbol{x}) \leq c|\boldsymbol{x}|^{-s}, \quad \boldsymbol{x} \in \mathbb{R}^n.$$

The function

$$h(\boldsymbol{x}) = \frac{1}{1 + |\boldsymbol{x}|^s}$$

is an influence function of order s, while an exponential,

$$h(\boldsymbol{x}) = \mathrm{e}^{-\gamma|\boldsymbol{x}|}, \quad \gamma > 0$$

is an influence function of all orders.

5 Fields and their Taylor Approximations

Lemma 2. *Let h be an influence function of order s. Let $\xi : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$|\boldsymbol{\xi}(\boldsymbol{x})| \leq K|\boldsymbol{x}|^k$$

for all $\boldsymbol{x} \in \mathbb{R}^n$ and some $K, k \geq 0$. If

$$k < \begin{cases} s - n/p & \text{if} \quad 1 < p < \infty, \\ s & \text{if} \quad p = \infty, \end{cases}$$

then $\boldsymbol{\xi} \in \mathscr{L}_{p,h}$.

A field $\boldsymbol{\xi} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be m-times differentiable at $\boldsymbol{x} = \boldsymbol{0}$, if

$$\boldsymbol{\xi}(\boldsymbol{x}) = \sum_{k=0}^{m} \frac{1}{k!} \nabla^k \boldsymbol{\xi} \cdot \boldsymbol{x}^k + \mathrm{o}(|\boldsymbol{x}|^m)$$

where $\nabla^k \boldsymbol{\xi}$ is a tensor of order $k+1$, symmetric in the las k indices,

$$\boldsymbol{x}^k = \boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}.$$

The Taylor transformation Π_m is a linear transformation defined for all fields which are m-times differentiable at $\boldsymbol{x} = \boldsymbol{0}$:

$$\Pi_m \boldsymbol{\xi} = \sum_{k=0}^{m} \frac{1}{k!} \nabla^k \boldsymbol{\xi} \cdot \boldsymbol{x}^k.$$

Theorem 3. *If the order of the influence function* h *is* s, *then the Taylor transformation* Π_m *maps the set* $\mathscr{D}_m$ *of all* m *times differentiable functions into* $\mathscr{L}_{p,h}$ *provided that*

$$m < \begin{cases} s - n/p & \text{if} \quad 1 < p < \infty, \\ s & \text{if} \quad p = \infty. \end{cases}$$

6 Retardation

The retardation operator Γ_α with retardation factor $\alpha(0,1]$ is the linear transformation $\boldsymbol{\xi} \to \boldsymbol{\xi}_\alpha$ defined by

$$\Gamma_\alpha \boldsymbol{\xi}(\boldsymbol{x}) = \boldsymbol{\xi}_\alpha(\boldsymbol{x}) = \boldsymbol{\xi}(\alpha \boldsymbol{x}).$$

Theorem 4. Γ_α *maps* $\mathscr{L}_{p,\,h}$ *into itself. Let* h *be an influence function of order* s. *Suppose that* m *and* p *satisfy*

$$m < \begin{cases} s - n/p & \text{if} \quad 1 < p < \infty, \\ s & \text{if} \quad p = \infty. \end{cases}$$

Then

$$\lim_{\alpha \to 0} \frac{1}{\alpha^m} |\boldsymbol{\xi}_\alpha - \Pi_m \boldsymbol{\xi}_\alpha|_{p,\,h} = 0$$

or each $\boldsymbol{\xi} \in \mathscr{D}_m$.

7 Response Functions

A function $\mathfrak{F} : \mathscr{L}_{p,\,h} \to \mathbb{R}^n$ is said to be m-times Fréchet-differentiable at $\mathbf{0}$ if bounded homogeneous polynomials $\delta^k \mathfrak{F}$, of degree $k = 0, \ldots, m$, such that

$$\mathfrak{F}(\boldsymbol{\xi}) = \sum_{k=0}^{m} \frac{1}{k\,!} \delta^k \mathfrak{F}(\boldsymbol{\xi}) + \mathrm{o}(|\boldsymbol{\xi}|^m_{p,\,h})$$

for all $\boldsymbol{\xi} \in \mathscr{L}_{p,\,h}$.
If $\boldsymbol{\omega}$ is a tensor of order $k+1$, symmetric in the last k indices, we define a homogeneous monomial $\boldsymbol{\omega}^\dagger$ of of degree k of n variables $\boldsymbol{z} = (z_1, \ldots, z_n)$ by

$$\boldsymbol{\omega}^\dagger(\boldsymbol{z}) = \boldsymbol{\omega} \cdot \boldsymbol{z}^k / k\,!\,.$$

Also

$$\binom{k}{j_1, \ldots, j_s} = \frac{k\,!}{j_1\,! \cdots j_s\,!}$$

Theorem 5. *If h is an influence function of order s, $\mathfrak{F}$ is s times differentiable on $\mathscr{L}_{p,h}$ and $\boldsymbol{\xi} \in \mathscr{D}_s$ a deformation then*

$$\mathfrak{F}(\xi_\alpha) = \sum_{k=1}^{s} \frac{\alpha^k c_k(\nabla \boldsymbol{\xi}, \ldots, \nabla^s \boldsymbol{\xi})}{k!} + \mathrm{o}(\alpha^s)$$

as $\alpha \to 0$ where

$$c_k(\nabla \xi, \ldots, \nabla^s \xi) = \sum_{\substack{j_1 \in \mathbb{N}_0, \ldots, j_s \in \mathbb{N}_0 \\ j_1 + \ldots + j_s = k \\ j_1 + 2j_2 \ldots + s j_s \le s}} \binom{k}{j} \delta^k \mathfrak{F}(\underbrace{\nabla^1 \xi^\dagger, \ldots, \nabla^1 \xi^\dagger}_{j_1 - \text{times}}, \ldots, \underbrace{\nabla^s \xi^\dagger, \ldots, \nabla^s \xi^\dagger}_{j_s - \text{times}})$$

Remark 6 (Particular cases).

(i) If $s = 1$ then

$$\mathfrak{F}(\xi_\alpha) \sim \mathfrak{F}(\mathbf{0}) + c_1(\nabla \xi)$$

where

$$c_1(\nabla \xi) = \delta \mathfrak{F}(\nabla \xi^\dagger).$$

(ii) If $s = 2$ then

$$\mathfrak{F}(\xi_\alpha) \sim \mathfrak{F}(\mathbf{0}) + c_1(\nabla\xi, \nabla^2\xi) + \frac{1}{2}c_2(\nabla\xi, \nabla^2\xi)$$

where

$$c_1(\nabla\xi, \nabla^2\xi) = \delta\mathfrak{F}(\nabla\xi\dagger) + \delta\mathfrak{F}(\nabla^2\xi\dagger),$$
$$c_2(\nabla\xi, \nabla^2\xi) = \delta^2\mathfrak{F}(\nabla\xi\dagger, \nabla\xi\dagger).$$

(iii) If $s = 3$ then

$$\mathfrak{F}(\xi_\alpha) \sim \mathfrak{F}(\mathbf{0}) + c_1(\nabla\xi, \nabla^2\xi, \nabla^3\xi) + \frac{1}{2}c_2(\nabla\xi, \nabla^2\xi, \nabla^3\xi) + \frac{1}{3!}c_3(\nabla\xi, \nabla^2\xi, \nabla^3\xi)$$

where

$$c_1(\nabla\xi, \nabla^2\xi, \nabla^3\xi) = \delta\mathfrak{F}(\nabla\xi\dagger) + \delta\mathfrak{F}(\nabla^2\xi\dagger) + \delta\mathfrak{F}(\nabla^3\xi\dagger),$$
$$c_2(\nabla\xi, \nabla^2\xi, \nabla^3\xi) = \delta^2\mathfrak{F}(\nabla\xi\dagger, \nabla\xi\dagger) + 2\delta^2\mathfrak{F}(\nabla\xi\dagger, \nabla^2\xi\dagger),$$
$$c_3(\nabla\xi, \nabla^2\xi, \nabla^3\xi) = \delta^3\mathfrak{F}(\nabla\xi\dagger, \nabla\xi\dagger, \nabla\xi\dagger).$$

8 Linear isotropic peridynamic materials

Now for any $\boldsymbol{x} \in \mathbb{R}^n$, he first derivative of $\mathfrak{F}$ is a linear function of the form

$$\delta \mathfrak{F}(\boldsymbol{u})(\boldsymbol{x}) = \int_{\Omega} \boldsymbol{K}(\boldsymbol{y} - \boldsymbol{x})\big(\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})\big)\, dV_{\boldsymbol{y}}$$

where the form of the kernel $\boldsymbol{K}$ is dictated by the representation theorem of isotropic functions, i.e.,

$$\boldsymbol{K}(\boldsymbol{p}) = \psi(\boldsymbol{p})|\boldsymbol{p}|^2\mathbf{1} + \omega(\boldsymbol{p})\,\boldsymbol{p} \otimes \boldsymbol{p}$$

$\boldsymbol{p} \in \mathbb{R}^n$, where ψ and ω are radial scalar functions determined by the properties of the material. We write $\psi(r)$ and $\omega(r)$ for $\tilde{\psi}(r)$ and $\tilde{\omega}(r)$, e.g., $\int_0^\infty \psi(r)\, dr := \int_0^\infty \tilde{\psi}(r)\, dr$. No confusion can arise. Further, for any bounded function $\boldsymbol{g}$ on $\mathbb{R}^n$ with values in any normed space with the norm $|\cdot|$ we put

$$\|\boldsymbol{g}\|_\infty := \sup\{|\boldsymbol{g}(\boldsymbol{p})| : \boldsymbol{p} \in \mathbb{R}^n\} < \infty.$$

Let, finally, κ_{n-1} be the area of the unit sphere $\mathbb{S}^{n-1} := \{\boldsymbol{p} \in \mathbb{R}^n : |\boldsymbol{p}| = 1\}$ in $\mathbb{R}^n$.

Theorem 7. *Let $k \geq 1$ be an integer and let $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^n$ have bounded continuous derivatives of all orders $\leq 2k+1$. Then*

$$\left.\begin{aligned} &\mathfrak{F}(\boldsymbol{u}_\alpha) = \sum_{s=1}^{k} \alpha^{2s-2}\,\mathfrak{N}^{(s)}\,\boldsymbol{u} + \alpha^{2k}\,\mathfrak{S}_\alpha^{(k)}\,\boldsymbol{u} \quad \text{on} \quad \mathbb{R}^n \\ &\text{where } \ \|\,\mathfrak{S}_\alpha^{(k)}\,\boldsymbol{u}\|_\infty \leq c\|\nabla^{2k+1}\boldsymbol{u}\|_\infty \text{ with } c \text{ independent of } \alpha \\ &\text{and } \boldsymbol{u}; \end{aligned}\right\} \tag{8.1}$$

here

$$\mathfrak{N}^{(s)}\,\boldsymbol{u} = (\lambda_s + \mu_s)\,\Delta^{s-1}\,\nabla\,\mathrm{div}\,\boldsymbol{u} + \mu_s\,\Delta^s\,\boldsymbol{u}$$

are the Navier operators of order $2s$ with the Lamé moduli of order s given by the equations

$$\lambda_s = \imath_s\big((2s-1)\,\omega_s - (n+2s)\,\psi_s\big), \quad \mu_s = \imath_s\big(\omega_s + (n+2s)\,\psi_s\big)$$

that involve a normalization constant $\imath_s$ and moments ψ_s and ω_s,

$$\iota_s = \kappa_{n-1} \Big/ 2^s\,!\prod_{i=0}^{s-1}(2i+n), \quad \eta_s := \int_0^\infty \eta(r) r^{n+2s+1}\, dr, \quad \textit{with} \quad \eta := \psi,\ \omega.$$

Remark 8. The first member of the sum in $_1$ is the classical Navier operator $\mathfrak{N}^{(1)} \equiv \mathfrak{N}$ from , with the Lamé moduli

$$\lambda,\ \mu = \frac{\kappa_{n-1} \int_0^\infty \big(\omega(r) \mp (n+2)\psi(r)\big) r^{n+3}\, dr}{2n(2+n)}.$$

9 References

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