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solution to the Riemann problem
for the full Euler system in 2D**

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Abstract: The question of well- and ill-posedness of entropy admissible solutions to the multi-dimensional systems of conservation laws has been studied recently in the case of isentropic Euler equations. In this context special initial data were considered, namely the 1D Riemann problem which is extended trivially to a second space dimension. It was shown that there exist infinitely many bounded entropy admissible weak solutions to such a 2D Riemann problem for isentropic Euler equations, if the initial data give rise to a 1D self-similar solution containing a shock. In this work we study such a 2D Riemann problem for the full Euler system in two space dimensions and prove the existence of infinitely many bounded entropy admissible weak solutions in the case that the Riemann initial data give rise to the 1D self-similar solution consisting of two shocks and possibly a contact discontinuity.

Contents

1	Introduction	2
2	The 1D Riemann solution	4
3	Main results	5
4	Existence of infinitely many admissible weak solutions	7
4.1	Sufficient condition for non-uniqueness	8
4.2	Algebraic equations and inequalities	9
5	Proof of Theorem 3.5	11
5.1	Reduction to special case	11
5.2	Proof of Proposition 5.1	12
5.2.1	Shock speeds	13
5.2.2	Constants C_i and γ_i	15
5.2.3	Subsolution and admissibility inequalities	17

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6	Proof of Theorem 3.6	18
6.1	The case $R < 0$	19
6.1.1	Solution to algebraic equations	19
6.1.2	Positivity of p_1, ε_1 and ε_2	20
6.1.3	Admissibility inequalities	24
6.2	The case $R > 0$	25
6.3	The case $R = 0$	26
6.3.1	Solution to algebraic equations	26
6.3.2	Positivity of p_1, ε_1 and ε_2	26
6.3.3	Admissibility inequalities	28

1 Introduction

In this paper we consider the full compressible Euler system in the whole two-dimensional space, i.e.

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}_{\mathbf{x}}(\varrho \mathbf{v}) &= 0, \\
\partial_t(\varrho \mathbf{v}) + \operatorname{div}_{\mathbf{x}}(\varrho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p &= 0, \\
\partial_t \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, p) \right) + \operatorname{div}_{\mathbf{x}} \left[\left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, p) + p \right) \mathbf{v} \right] &= 0,
\end{aligned} \tag{1.1}$$

where the density $\varrho = \varrho(t, \mathbf{x}) \in \mathbb{R}^+$, the pressure $p = p(t, \mathbf{x}) \in \mathbb{R}^+$ and the velocity $\mathbf{v} = \mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^2$ are unknown functions of the time $t \in [0, \infty)$ and the position $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

Furthermore we consider an ideal gas, i.e.

$$e(\varrho, p) = c_v \frac{p}{\varrho},$$

where $c_v > 0$ is a constant called the specific heat at constant volume. We recall that from a physical point of view $c_v = \frac{f}{2}$, where f is the number of degrees of freedom.

The system of equations (1.1) is complemented with the initial condition

$$(\varrho, \mathbf{v}, p)(0, \mathbf{x}) = (\varrho^0, \mathbf{v}^0, p^0)(\mathbf{x}) \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Further, we supplement the equations (1.1) with the entropy condition

$$\partial_t \left(\varrho s(\varrho, p) \right) + \operatorname{div}_{\mathbf{x}} \left(\varrho s(\varrho, p) \mathbf{v} \right) \geq 0, \tag{1.3}$$

where $s(\varrho, p)$ is the entropy. For the ideal gas we have

$$s(\varrho, p) = \log \left(\frac{p^{c_v}}{\varrho^{c_v+1}} \right).$$

By *admissible weak solution* - we will also use the term *weak entropy solution* - we mean a triple of L^∞ -functions (ϱ, \mathbf{v}, p) that fulfill (1.1)-(1.3) in the sense of distributions, see Section 3 for precise definitions.

In this paper we consider Riemann initial data

$$(\varrho^0, \mathbf{v}^0, p^0)(\mathbf{x}) := \begin{cases} (\varrho_-, \mathbf{v}_-, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, \mathbf{v}_+, p_+) & \text{if } x_2 > 0 \end{cases}, \tag{1.4}$$

where $\varrho_\pm \in \mathbb{R}^+$, $\mathbf{v}_\pm \in \mathbb{R}^2$ and $p_\pm \in \mathbb{R}^+$ are constants. We write $\mathbf{v} = (v_1, v_2)^T$ for the components of the velocity \mathbf{v} . Throughout this paper we additionally assume $v_{-,1} = v_{+,1} = 0$, where $v_{\pm,1}$ are the first components of the velocities \mathbf{v}_\pm .

Following the groundbreaking results of De Lellis and Székelyhidi [8, 9] concerning ill-posedness of the incompressible Euler equations, the question of well- and ill-posedness of admissible weak solutions to the isentropic compressible Euler equations was raised. In fact, De Lellis and Székelyhidi themselves showed in [9] the existence of initial data $(\varrho^0, \mathbf{v}^0) \in L^\infty(\mathbb{R}^2)$ for which there exist infinitely many admissible weak solutions (such initial data is called *wild initial data*). In the context of hyperbolic conservation laws this result in particular shows that the notion of admissible weak solution, i.e. solution satisfying the system of conservation laws and an inequality arising from the companion law, is probably not suitable for the problem in the case of more than one space dimension.

This result was further improved by Chiodaroli [3] and Feireisl [10] who showed that the ill-posedness is related to the irregularity of the velocity field, proving that there exist wild initial data $(\varrho^0, \mathbf{v}^0)$ with $\varrho^0 \in C^1(\mathbb{R}^2)$. Finally, De Lellis, Chiodaroli and Kreml [4] showed the existence of Lipschitz wild initial data. The method of the proof was motivated by the observation of Székelyhidi [17] that for the incompressible Euler system, vortex sheet initial data are wild. Hence, the authors in [4] studied the Riemann problem for the isentropic Euler system and found an example of Riemann initial data allowing for existence of infinitely many admissible weak solutions while being generated by a backwards rarefaction wave (i.e. a compression wave).

Since the Riemann problem is a basic building block in the theory of one-dimensional systems of conservation laws, it was only natural to study its properties in multi-dimensional case further. In fact, Chen and Chen [2] already proved that if the solution to the Riemann problem consists only of rarefaction waves, such solution is unique in the class of multi-dimensional admissible weak solutions. The same was later proved also in [12].

On the non-uniqueness side, Chiodaroli and Kreml [5] proved that whenever the Riemann initial data give rise to the self-similar solution consisting of two shocks, there exist also infinitely many other bounded admissible weak solutions. Moreover they also showed that the self-similar solution may not be entropy rate admissible, meaning that some of the other solutions constructed may produce more total entropy. The study of the Riemann problem for the isentropic Euler system continued with results of Chiodaroli and Kreml [6] and Klingenberg and Markfelder [14] who independently showed ill-posedness in the case of Riemann initial data giving rise to the self-similar solution consisting of one rarefaction wave and one shock. More precisely in [6] the authors need a certain smallness condition for the initial data, whereas the result in [14] covers all cases of such Riemann initial data and also solves the case of Riemann initial data giving rise to self-similar solution consisting of a single shock.

Finally, Březina, Chiodaroli and Kreml [1] showed that the above mentioned results of ill-posedness of admissible weak solutions hold also in the presence of a contact discontinuity in the self-similar solution arising from the multidimensionality of the problem. However, the question, whether a single contact discontinuity is a unique admissible weak solution to the isentropic Euler system or not, is still an open problem; it is worth mentioning that this is actually a direct analog of the vortex sheet initial data considered by Székelyhidi in [17].

In view of results for the isentropic system a natural question arises, whether these results can be transferred to the case of full Euler equations. It was shown by Chen and Chen in [2] that uniqueness of rarefaction waves actually holds also for the full system, see also [13]. On the non-uniqueness side, we note the result of Markfelder and Klingenberg [15], who showed that for the isentropic system there exist infinitely many solutions conserving the energy. This however can be viewed as a proof of existence of infinitely many solutions to the full system with a proper choice of c_v ; all these solutions also conserve the entropy. In addition to that we want to mention the result by Feireisl et al. [11], who proved existence of wild initial data for the full Euler equations. Their solutions conserve the entropy and have piecewise constant density and pressure. In this paper we prove non-uniqueness of admissible weak solutions for a larger class of Riemann initial data, namely for initial data that give rise to self-similar solution consisting of two shocks and possibly a contact discontinuity.

The paper is structured as follows. In Section 2 we recall some results for the one-dimensional Riemann problem for the full Euler system and present a condition on the initial states $(\varrho_\pm, \mathbf{v}_\pm, p_\pm)$ such that the self-similar solution consists of two shocks and possibly a contact discontinuity. In

Section 3 we present our two main results, Theorem 3.5 and Theorem 3.6. In Section 4 we provide the technical tools used in the proof, in particular we present a key lemma relating the existence of infinitely many admissible weak solutions to the existence of a single so-called admissible fan subsolution. Finally, in Sections 6 and 5 we present proofs of our two main results.

2 The 1D Riemann solution

The initial data (1.2) of the problem we study are one-dimensional, we have $(\varrho^0, \mathbf{v}^0, p^0)(x_2)$ only with no dependence on the x_1 variable. Therefore it is reasonable to search for solutions which share this property, i.e. for solutions satisfying $(\varrho, \mathbf{v}, p)(t, \mathbf{x}) = (\varrho, \mathbf{v}, p)(t, x_2)$. Moreover since we assume $v_{-,1} = v_{+,1} = 0$, we search for solutions with the property $v_1(t, \mathbf{x}) \equiv 0$.

Hence we obtain the 1D Riemann problem

$$\begin{aligned} \partial_t \varrho + \partial_2(\varrho v_2) &= 0, \\ \partial_t(\varrho v_2) + \partial_2(\varrho(v_2)^2 + p) &= 0, \\ \partial_t \left(\frac{1}{2} \varrho(v_2)^2 + \varrho e(\varrho, p) \right) + \partial_2 \left[\left(\frac{1}{2} \varrho(v_2)^2 + \varrho e(\varrho, p) + p \right) v_2 \right] &= 0, \end{aligned} \tag{2.1}$$

for the unknown functions $(\varrho, v_2, p)(t, x_2)$ with initial data

$$(\varrho, v_2, p)(0, x_2) = (\varrho^0, v_2^0, p^0)(x_2) := \begin{cases} (\varrho_-, v_{-,2}, p_-) & \text{if } x_2 < 0 \\ (\varrho_+, v_{+,2}, p_+) & \text{if } x_2 > 0 \end{cases}. \tag{2.2}$$

The problem (2.1)-(2.2) is a model example of a one-dimensional Riemann problem for a system of conservation laws and has been studied extensively by several authors, e.g. by Smoller [16, Chapter 18 §B]. It is well known that this problem admits a self-similar solution (i.e. solution consisting of functions of a single variable $\frac{x_2}{t}$) belonging to the class of BV functions. This solution to (2.1)-(2.2) is unique in the class of BV self-similar functions and consists of three waves connected by constant states, where the first wave is either an admissible shock or a rarefaction wave, the second wave is a contact discontinuity and the third wave is again either an admissible shock or a rarefaction wave. In the rest of this paper we will call this solution the *1D Riemann solution*. Table 1 shows all the possibilities of the structure of the 1D Riemann solution.

1-wave	2-wave	3-wave	1-wave	2-wave	3-wave
-	-	-	-	contact	-
-	-	shock	-	contact	shock
-	-	rarefaction	-	contact	rarefaction
shock	-	-	shock	contact	-
shock	-	shock	shock	contact	shock
shock	-	rarefaction	shock	contact	rarefaction
rarefaction	-	-	rarefaction	contact	-
rarefaction	-	shock	rarefaction	contact	shock
rarefaction	-	rarefaction	rarefaction	contact	rarefaction

Table 1: All the 18 possibilities of the structure of the 1D Riemann solution

It is easy to observe that the 1D Riemann solution is an admissible weak solution to the original 2D problem (1.1)-(1.2).

In this paper we will focus on the cases, where the 1D Riemann solution contains two admissible shocks and possibly a contact discontinuity. Therefore we recall the following proposition stating conditions for the initial data under which such 1D Riemann solution emerges.

Proposition 2.1. *Assume that either*

- $p_- \leq p_+$ and $v_{+,2} - v_{-,2} < -(p_+ - p_-) \sqrt{\frac{2c_v}{\varrho_-(p_- + (2c_v + 1)p_+)}}$ or
- $p_+ \leq p_-$ and $v_{+,2} - v_{-,2} < -(p_- - p_+) \sqrt{\frac{2c_v}{\varrho_+(p_+ + (2c_v + 1)p_-)}}$.

Then the 1D Riemann solution to problem (2.1)-(2.2) consists of a 1-shock, possibly a 2-contact discontinuity and a 3-shock. The intermediate states $(\varrho_{M-}, \mathbf{v}_M, p_M)$ and $(\varrho_{M+}, \mathbf{v}_M, p_M)$ are given by

- p_M is the unique solution of

$$-\sqrt{2c_v} \left(\frac{p_M - p_-}{\sqrt{\varrho_-(p_- + (2c_v + 1)p_M)}} + \frac{p_M - p_+}{\sqrt{\varrho_+(p_+ + (2c_v + 1)p_M)}} \right) = v_{+,2} - v_{-,2};$$

- $v_{M,2} = v_{-,2} - \sqrt{2c_v} \frac{p_M - p_-}{\sqrt{\varrho_-(p_- + (2c_v + 1)p_M)}}$;
- $\varrho_{M-} = \varrho_- \frac{p_- + (2c_v + 1)p_M}{p_M + (2c_v + 1)p_-}$;
- $\varrho_{M+} = \varrho_+ \frac{p_+ + (2c_v + 1)p_M}{p_M + (2c_v + 1)p_+}$;

and the following properties hold.

- If $\varrho_{M-} = \varrho_{M+}$ then the 2-contact discontinuity does not appear.
- $p_M > \max\{p_-, p_+\}$, $\varrho_{M-} > \varrho_-$ and $\varrho_{M+} > \varrho_+$.
- The shock speeds denoted by σ_-, σ_+ satisfy $\sigma_{\pm}(\varrho_{\pm} - \varrho_{M\pm}) = \varrho_{\pm} v_{\pm,2} - \varrho_{M\pm} v_{M,2}$.
- The speed of the contact discontinuity is given by $v_{M,2}$.

For the proof of Proposition 2.1 we refer to [16]. More details about self-similar solutions to 1D systems of conservation laws can be also found in [7].

3 Main results

In this section we state the definition of solutions to the problem (1.1)-(1.2) and our main results.

Definition 3.1. *Let $(\varrho^0, \mathbf{v}^0, p^0) \in L^\infty(\mathbb{R}^2)$. The triplet $(\rho, \mathbf{v}, p) \in L^\infty((0, \infty) \times \mathbb{R}^2; \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+)$ is called a weak solution of (1.1)-(1.2) if the following system of identities is satisfied*

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\varrho \partial_t \psi + \varrho \mathbf{v} \cdot \nabla_{\mathbf{x}} \psi \right) d\mathbf{x} dt + \int_{\mathbb{R}^2} \varrho^0(\mathbf{x}) \psi(0, \mathbf{x}) d\mathbf{x} = 0 \quad (3.1)$$

for all test functions $\psi \in C_c^\infty([0, \infty) \times \mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\varrho \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + (\varrho \mathbf{v} \otimes \mathbf{v}) : \nabla_{\mathbf{x}} \boldsymbol{\varphi} + p \operatorname{div}_{\mathbf{x}} \boldsymbol{\varphi} \right) d\mathbf{x} dt + \int_{\mathbb{R}^2} \varrho^0(\mathbf{x}) \mathbf{v}^0(\mathbf{x}) \cdot \boldsymbol{\varphi}(0, \mathbf{x}) d\mathbf{x} = 0 \quad (3.2)$$

for all test functions $\boldsymbol{\varphi} \in C_c^\infty([0, \infty) \times \mathbb{R}^2; \mathbb{R}^2)$,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, p) \right) \cdot \partial_t \phi + \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + \varrho e(\varrho, p) + p \right) \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi \right] d\mathbf{x} dt \\ + \int_{\mathbb{R}^2} \left(\frac{1}{2} \varrho^0(\mathbf{x}) |\mathbf{v}^0(\mathbf{x})|^2 + \varrho^0(\mathbf{x}) e(\varrho^0(\mathbf{x}), p^0(\mathbf{x})) \right) \phi(0, \mathbf{x}) d\mathbf{x} = 0 \end{aligned} \quad (3.3)$$

for all test functions $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^2)$.

Definition 3.2. Let $(\varrho^0, \mathbf{v}^0, p^0) \in L^\infty(\mathbb{R}^2)$. A weak solution of (1.1)-(1.2) is called *admissible* (or *weak entropy solution*), if the following inequality is satisfied for all non-negative test functions $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2; \mathbb{R}_0^+)$:

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\varrho s(\varrho, p) \partial_t \varphi + \varrho s(\varrho, p) \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \right) dx dt + \int_{\mathbb{R}^2} \varrho^0(\mathbf{x}) s(\varrho^0(\mathbf{x}), p^0(\mathbf{x})) \varphi(0, \mathbf{x}) dx \leq 0. \quad (3.4)$$

The solutions which we are going to construct will have a special structure, partially motivated by the structure of the 1D Riemann solution. Therefore we introduce first the notion of N -fan partition ($N = 1, 2$) and then N -fan solutions.

Definition 3.3. Let $N \in \{1, 2\}$ and $\mu_0 < \mu_1$ ($\mu_0 < \mu_1 < \mu_2$) real numbers. An N -fan partition of $(0, \infty) \times \mathbb{R}^2$ is a set of $N + 2$ open sets $\Omega_-, \Omega_1, \Omega_+$ ($\Omega_-, \Omega_1, \Omega_2, \Omega_+$) of the form

$$\begin{aligned} \Omega_- &= \{(t, \mathbf{x}) : t > 0 \text{ and } x_2 < \mu_0 t\}; \\ \Omega_i &= \{(t, \mathbf{x}) : t > 0 \text{ and } \mu_{i-1} t < x_2 < \mu_i t\}; \\ \Omega_+ &= \{(t, \mathbf{x}) : t > 0 \text{ and } x_2 > \mu_N t\}. \end{aligned}$$

Definition 3.4. An admissible weak solution (ϱ, \mathbf{v}, p) of the full Euler system (1.1)-(1.2) with Riemann initial data (1.4) is called an N -fan solution, if there exists $N \in \{1, 2\}$ and an N -fan partition of $(0, \infty) \times \mathbb{R}^2$ such that

- (ϱ, p) are constant on each set $\Omega_-, \Omega_i, \Omega_+$ of the N -fan partition;
- $|\mathbf{v}|^2$ is constant on each set $\Omega_-, \Omega_i, \Omega_+$ of the N -fan partition;
- $(\varrho, \mathbf{v}, p) = (\varrho_\pm, \mathbf{v}_\pm, p_\pm)$ on the sets Ω_\pm .

See also Fig. 1.

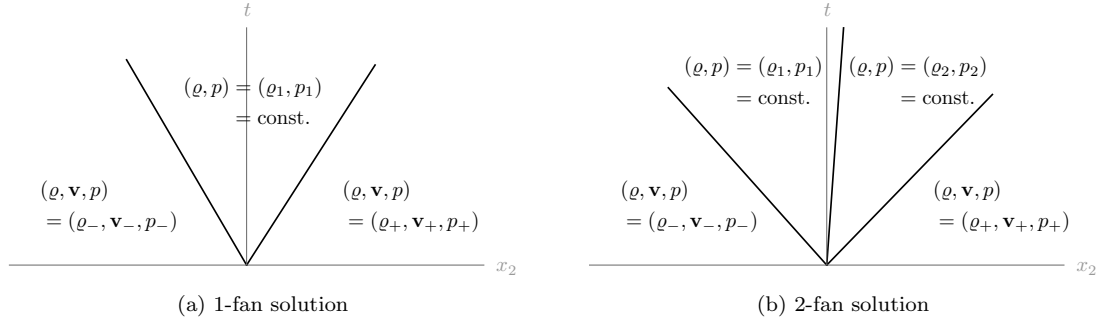


Figure 1: Structure of an N -fan solution. On the leftmost and rightmost sets (i. e. on Ω_\pm), the solution coincides with the initial states. On each set in the middle ($\Omega_i, i \in \{1, N\}$), the density ϱ and pressure p are constant, whereas the velocity \mathbf{v} does not need to be constant.

Now we are ready to state our main theorems.

Theorem 3.5. Let $c_v > \frac{1}{2}$ and let the Riemann initial data (1.4) be such that the 1D Riemann solution consists either

- of a 1-shock and a 3-shock or
- of a 1-shock, a 2-contact discontinuity and a 3-shock.

Then there exist infinitely many admissible weak solutions to (1.1)-(1.4). These solutions are all 2-fan solutions.

Theorem 3.6. *Let $c_v > 0$. Let $\varrho_{\pm}, p_{\pm} > 0$ and $V \in \mathbb{R}$ be given.*

- (i) *If $\varrho_- < \varrho_+$ or $\varrho_- = \varrho_+$ and $p_- > p_+$ let $v_{-,2} = V$. Then there exists $U = U(\varrho_{\pm}, p_{\pm}, V)$ such that for all $v_{+,2} < U$ there exist infinitely many admissible weak solutions to (1.1)-(1.4). These solutions are all 1-fan solutions.*
- (ii) *If $\varrho_- > \varrho_+$ or $\varrho_- = \varrho_+$ and $p_- < p_+$ let $v_{+,2} = V$. Then there exists $U = U(\varrho_{\pm}, p_{\pm}, V)$ such that for all $v_{-,2} > U$ there exist infinitely many admissible weak solutions to (1.1)-(1.4). These solutions are all 1-fan solutions.*

Remark 3.7. *Theorem 3.5 is the most general result presented here, covering all the cases of Riemann initial data giving rise to the self-similar solution containing two shocks. Solutions constructed in the proof of this Theorem are all 2-fan solutions. It might be of interest that at least if the difference between the velocities $v_{-,2}$ and $v_{+,2}$ is large enough, one can construct also another type of solutions, namely 1-fan solutions, this is stated in Theorem 3.6. We emphasize that the set of initial data in Theorem 3.6 is a strict subset of the set of initial data in Theorem 3.5. Finally, we refer the reader to Remark 6.7 for the reason why Theorem 3.6 does not cover the case $\varrho_- = \varrho_+$ and $p_- = p_+$.*

Remark 3.8. *Note that the general Theorem 3.5 requires the specific heat at constant volume coefficient c_v to be strictly larger than $\frac{1}{2}$, whereas the results of Theorem 3.6 hold for all positive c_v .*

4 Existence of infinitely many admissible weak solutions

The content of this section shows the method to prove non-uniqueness of admissible weak solutions. As in the isentropic case (see [4] and following works) the idea is to work with a so called *admissible fan subsolution* - here a quintuple of piecewise constant functions satisfying certain system of partial differential equations. Then, Theorem 4.2 tells us that the existence of such an admissible fan subsolution implies existence of infinitely many weak entropy solutions to (1.1)-(1.4).

In what follows we denote by $\mathcal{S}_0^{2 \times 2}$ the space of symmetric traceless 2×2 matrices and \mathbb{I} denotes the 2×2 identity matrix.

Definition 4.1. *An admissible N -fan subsolution to the Euler system (1.1)-(1.3) with Riemann initial data (1.4) is a quintuple $(\bar{\varrho}, \bar{\mathbf{v}}, \bar{U}, \bar{C}, \bar{p}) : (0, \infty) \times \mathbb{R}^2 \rightarrow (\mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{S}_0^{2 \times 2} \times \mathbb{R}^+ \times \mathbb{R}^+)$ of piecewise constant functions, which satisfies the following properties:*

1. *There exists an N -fan partition $\Omega_-, \Omega_1, \Omega_+$ (or $\Omega_-, \Omega_1, \Omega_2, \Omega_+$) of $(0, \infty) \times \mathbb{R}^2$ and for $i \in \{1, N\}$ there exist constants $\varrho_i \in \mathbb{R}^+$, $\mathbf{v}_i \in \mathbb{R}^2$, $U_i \in \mathcal{S}_0^{2 \times 2}$, $C_i \in \mathbb{R}^+$ and $p_i \in \mathbb{R}^+$, such that*

$$(\bar{\varrho}, \bar{\mathbf{v}}, \bar{U}, \bar{C}, \bar{p}) = \sum_{i \in \{-, +\}} \left(\varrho_i, \mathbf{v}_i, \mathbf{v}_i \otimes \mathbf{v}_i - \frac{|\mathbf{v}_i|^2}{2} \mathbb{I}, |\mathbf{v}_i|^2, p_i \right) \mathbf{1}_{\Omega_i} + \sum_{i=1}^N (\varrho_i, \mathbf{v}_i, U_i, C_i, p_i) \mathbf{1}_{\Omega_i},$$

where $\varrho_{\pm}, \mathbf{v}_{\pm}, p_{\pm}$ are the constants given by the initial condition (1.4).

2. *For $i \in \{1, N\}$ the following inequality holds in the sense of definiteness:*

$$\mathbf{v}_i \otimes \mathbf{v}_i - U_i < \frac{C_i}{2} \mathbb{I}.$$

3. For all test functions $(\psi, \boldsymbol{\varphi}, \phi) \in C_c^\infty([0, \infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$ the following identities hold:

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} [\bar{\varrho} \partial_t \psi + \bar{\varrho} \bar{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \psi] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \psi(0, \cdot) \, d\mathbf{x} = 0, \\ & \int_0^\infty \int_{\mathbb{R}^2} \left[\bar{\varrho} \bar{\mathbf{v}} \cdot \partial_t \boldsymbol{\varphi} + \bar{\varrho} \bar{U} : \nabla_{\mathbf{x}} \boldsymbol{\varphi} + \left(\bar{p} + \frac{1}{2} \bar{\varrho} \bar{C} \right) \operatorname{div}_{\mathbf{x}} \boldsymbol{\varphi} \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \mathbf{v}^0 \cdot \boldsymbol{\varphi}(0, \cdot) \, d\mathbf{x} = 0, \\ & \int_0^\infty \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} \bar{\varrho} \bar{C} + c_v \bar{p} \right) \partial_t \phi + \left(\frac{1}{2} \bar{\varrho} \bar{C} + (c_v + 1) \bar{p} \right) \bar{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \phi \right] \, d\mathbf{x} \, dt \\ & \quad + \int_{\mathbb{R}^2} \left(\varrho^0 \frac{|\mathbf{v}^0|^2}{2} + c_v p^0 \right) \phi(0, \cdot) \, d\mathbf{x} = 0. \end{aligned}$$

4. For every non-negative test function $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2, \mathbb{R}_0^+)$ the inequality

$$\int_0^\infty \int_{\mathbb{R}^2} \left[\bar{\varrho} s(\bar{\varrho}, \bar{p}) \partial_t \varphi + \bar{\varrho} s(\bar{\varrho}, \bar{p}) \bar{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 s(\varrho^0, p^0) \varphi(0, \cdot) \, d\mathbf{x} \leq 0$$

is fulfilled.

4.1 Sufficient condition for non-uniqueness

The key tool in the proof of Theorems 3.5 and 3.6 is the following theorem relating existence of infinitely many admissible N -fan solutions to the existence of a single admissible N -fan subsolution

Theorem 4.2. *Let $(\varrho_\pm, \mathbf{v}_\pm, p_\pm)$ be such that there exists an admissible N -fan subsolution $(\bar{\varrho}, \bar{\mathbf{v}}, \bar{U}, \bar{C}, \bar{p})$ to the Cauchy problem (1.1)-(1.4). Then there are infinitely many admissible weak N -fan solutions (ϱ, \mathbf{v}, p) to (1.1)-(1.4) with the following properties:*

- $(\varrho, p) = (\bar{\varrho}, \bar{p})$,
- $\mathbf{v} = \mathbf{v}_\pm$ on Ω_\pm ,
- $|\mathbf{v}|^2 = C_i$ a.e. on Ω_i , $i = 1, N$.

The proof of Theorem 4.2 is based on a convex integration technique for the pressureless incompressible Euler equations which was introduced in seminal works of De Lellis and Székelyhidi [8, 9] and is summarized in the following Proposition.

Proposition 4.3. *Let $(\tilde{\mathbf{v}}, \tilde{U}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2 \times 2}$ and $C > 0$ such that $\tilde{\mathbf{v}} \otimes \tilde{\mathbf{v}} - \tilde{U} < \frac{C}{2} \mathbb{I}$. Furthermore let $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ open. Then there exist infinitely many maps $(\underline{\mathbf{v}}, \underline{U}) \in L^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2 \times 2})$ with the following properties.*

1. $\underline{\mathbf{v}}$ and \underline{U} vanish outside Ω .
2. For all test functions $(\psi, \boldsymbol{\varphi}) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2)$ it holds that

$$\begin{aligned} & \iint_{\Omega} \underline{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \psi \, d\mathbf{x} \, dt = 0, \\ & \iint_{\Omega} (\underline{\mathbf{v}} \cdot \partial_t \boldsymbol{\varphi} + \underline{U} : \nabla_{\mathbf{x}} \boldsymbol{\varphi}) \, d\mathbf{x} \, dt = 0. \end{aligned}$$

3. $(\tilde{\mathbf{v}} + \underline{\mathbf{v}}) \otimes (\tilde{\mathbf{v}} + \underline{\mathbf{v}}) - (\tilde{U} + \underline{U}) = \frac{C}{2} \mathbb{I}$ is fulfilled almost everywhere in Ω .

For the proof of Proposition 4.3 we refer to [4, Lemma 3.7].

The proof of Theorem 4.2 using Proposition 4.3 is now quite straightforward.

Proof. Let $(\bar{\varrho}, \bar{\mathbf{v}}, \bar{U}, \bar{C}, \bar{p})$ an admissible N -fan subsolution. Furthermore for $i = 1, N$ let $(\mathbf{v}_i, U_i) \in L^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2 \times \mathcal{S}_0^{2 \times 2})$ pairs of functions as in Proposition 4.3 with $\tilde{\mathbf{v}} = \mathbf{v}_i, \tilde{U} = U_i, C = C_i$ and $\Omega = \Omega_i$. It suffices to prove that $(\varrho, \mathbf{v}, p) = \left(\bar{\varrho}, \bar{\mathbf{v}} + \sum_{i=1}^N \mathbf{v}_i, \bar{p}\right)$ is an admissible weak solution to (1.1)-(1.4). The fact that it is indeed a N -fan solution is clear.

Let $(\psi, \varphi, \phi) \in C_c^\infty([0, \infty) \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R})$ be test functions. Then we get

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} [\varrho \partial_t \psi + \varrho \mathbf{v} \cdot \nabla_{\mathbf{x}} \psi] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \psi(0, \cdot) \, d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{R}^2} \left[\bar{\varrho} \partial_t \psi + \bar{\varrho} \left(\bar{\mathbf{v}} + \sum_{i=1}^N \mathbf{v}_i \right) \cdot \nabla_{\mathbf{x}} \psi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \psi(0, \cdot) \, d\mathbf{x} \\ &= \sum_{i=1}^N \iint_{\Omega_i} \varrho_i \mathbf{v}_i \cdot \nabla_{\mathbf{x}} \psi \, d\mathbf{x} \, dt = 0, \end{aligned}$$

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \left[\varrho \mathbf{v} \cdot \partial_t \varphi + \varrho \mathbf{v} \otimes \mathbf{v} : \nabla_{\mathbf{x}} \varphi + p \operatorname{div}_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \mathbf{v}^0 \cdot \varphi(0, \cdot) \, d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{R}^2} \left[\bar{\varrho} \bar{\mathbf{v}} \cdot \partial_t \varphi + \bar{\varrho} \bar{U} : \nabla_{\mathbf{x}} \varphi + \left(\bar{p} + \frac{1}{2} \bar{\varrho} \bar{C} \right) \operatorname{div}_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 \mathbf{v}^0 \cdot \varphi(0, \cdot) \, d\mathbf{x} \\ &+ \sum_{i=1}^N \iint_{\Omega_i} \left[\varrho_i \mathbf{v}_i \cdot \partial_t \varphi + \varrho_i U_i : \nabla_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt = 0, \end{aligned}$$

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} \varrho |\mathbf{v}|^2 + c_v p \right) \partial_t \phi + \left(\frac{1}{2} \varrho |\mathbf{v}|^2 + (c_v + 1) p \right) \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi \right] \, d\mathbf{x} \, dt \\ &+ \int_{\mathbb{R}^2} \left(\varrho^0 \frac{|\mathbf{v}^0|^2}{2} + c_v p^0 \right) \phi(0, \cdot) \, d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} \bar{\varrho} \bar{C} + c_v \bar{p} \right) \partial_t \phi + \left(\frac{1}{2} \bar{\varrho} \bar{C} + (c_v + 1) \bar{p} \right) \bar{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \phi \right] \, d\mathbf{x} \, dt \\ &+ \int_{\mathbb{R}^2} \left(\varrho^0 \frac{|\mathbf{v}^0|^2}{2} + c_v p^0 \right) \phi(0, \cdot) \, d\mathbf{x} + \sum_{i=1}^N \iint_{\Omega_i} \left(\varrho_i \frac{C_i}{2} + (c_v + 1) p_i \right) \mathbf{v}_i \cdot \nabla_{\mathbf{x}} \phi \, d\mathbf{x} \, dt = 0. \end{aligned}$$

For every non-negative test function $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2, \mathbb{R}_0^+)$ it holds

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \left[\varrho s(\varrho, p) \partial_t \varphi + \varrho s(\varrho, p) \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 s(\varrho^0, p^0) \varphi(0, \cdot) \, d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{R}^2} \left[\bar{\varrho} s(\bar{\varrho}, \bar{p}) \partial_t \varphi + \bar{\varrho} s(\bar{\varrho}, \bar{p}) \bar{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \varphi \right] \, d\mathbf{x} \, dt + \int_{\mathbb{R}^2} \varrho^0 s(\varrho^0, p^0) \varphi(0, \cdot) \, d\mathbf{x} \\ &+ \sum_{i=1}^N \iint_{\Omega_i} \varrho_i s(\varrho_i, p_i) \mathbf{v}_i \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} \, dt \leq 0. \end{aligned}$$

□

4.2 Algebraic equations and inequalities

Since the admissible N -fan subsolution consists of a quintuple of piecewise constant functions which satisfy a certain set of partial differential equations and inequalities, it is easy to observe that these differential constraints are equivalent to a system of algebraic equations and inequalities which arise from the Rankine-Hugoniot conditions on the interfaces between the sets of the N -fan partition. More precisely we have the following.

Proposition 4.4. *Let $\varrho_{\pm}, p_{\pm} \in \mathbb{R}^+$, $\mathbf{v}_{\pm} \in \mathbb{R}^2$ be given. The constants $\mu_0, \mu_1 \in \mathbb{R}$ (resp. $\mu_0, \mu_1, \mu_2 \in \mathbb{R}$) and $\varrho_i, p_i \in \mathbb{R}^+$,*

$$\mathbf{v}_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \in \mathbb{R}^2, \quad U_i = \begin{pmatrix} \gamma_i & \delta_i \\ \delta_i & -\gamma_i \end{pmatrix} \in \mathcal{S}_0^{2 \times 2},$$

and $C_i \in \mathbb{R}^+$ (for $i = 1, N$) define an admissible N -fan subsolution to the Cauchy problem (1.1)-(1.4) if and only if they fulfill the following algebraic equations and inequalities:

- Order of the speeds:

$$\mu_0 < \mu_1 \quad \text{resp.} \quad \mu_0 < \mu_1 < \mu_2 \quad (4.1)$$

- Rankine Hugoniot conditions on the left interface:

$$\mu_0 (\varrho_- - \varrho_1) = \varrho_- v_{-,2} - \varrho_1 \beta_1 \quad (4.2)$$

$$\mu_0 (\varrho_- v_{-,1} - \varrho_1 \alpha_1) = \varrho_- v_{-,1} v_{-,2} - \varrho_1 \delta_1 \quad (4.3)$$

$$\mu_0 (\varrho_- v_{-,2} - \varrho_1 \beta_1) = \varrho_- v_{-,2}^2 - \varrho_1 \left(\frac{C_1}{2} - \gamma_1 \right) + p_- - p_1 \quad (4.4)$$

$$\begin{aligned} \mu_0 \left(\frac{1}{2} \varrho_- |v_-|^2 + c_v p_- - \varrho_1 \frac{C_1}{2} - c_v p_1 \right) = \\ \left(\frac{1}{2} \varrho_- |v_-|^2 + (c_v + 1) p_- \right) v_{-,2} - \left(\varrho_1 \frac{C_1}{2} + (c_v + 1) p_1 \right) \beta_1 \end{aligned} \quad (4.5)$$

- Rankine Hugoniot conditions on the right interface:

$$\mu_N (\varrho_N - \varrho_+) = \varrho_N \beta_N - \varrho_+ v_{+,2} \quad (4.6)$$

$$\mu_N (\varrho_N \alpha_N - \varrho_+ v_{+,1}) = \varrho_N \delta_N - \varrho_+ v_{+,1} v_{+,2} \quad (4.7)$$

$$\mu_N (\varrho_N \beta_N - \varrho_+ v_{+,2}) = \varrho_N \left(\frac{C_N}{2} - \gamma_N \right) - \varrho_+ v_{+,2}^2 + p_N - p_+ \quad (4.8)$$

$$\begin{aligned} \mu_N \left(\varrho_N \frac{C_N}{2} + c_v p_N - \frac{1}{2} \varrho_+ |v_+|^2 - c_v p_+ \right) = \\ \left(\varrho_N \frac{C_N}{2} + (c_v + 1) p_N \right) \beta_N - \left(\frac{1}{2} \varrho_+ |v_+|^2 + (c_v + 1) p_+ \right) v_{+,2} \end{aligned} \quad (4.9)$$

- If $N = 2$ we additionally have Rankine Hugoniot conditions on the middle interface:

$$\mu_1 (\varrho_1 - \varrho_2) = \varrho_1 \beta_1 - \varrho_2 \beta_2 \quad (4.10)$$

$$\mu_1 (\varrho_1 \alpha_1 - \varrho_2 \alpha_2) = \varrho_1 \delta_1 - \varrho_2 \delta_2 \quad (4.11)$$

$$\mu_1 (\varrho_1 \beta_1 - \varrho_2 \beta_2) = \varrho_1 \left(\frac{C_1}{2} - \gamma_1 \right) - \varrho_2 \left(\frac{C_2}{2} - \gamma_2 \right) + p_1 - p_2 \quad (4.12)$$

$$\begin{aligned} \mu_1 \left(\varrho_1 \frac{C_1}{2} + c_v p_1 - \varrho_2 \frac{C_2}{2} - c_v p_2 \right) = \\ \left(\varrho_1 \frac{C_1}{2} + (c_v + 1) p_1 \right) \beta_1 - \left(\varrho_2 \frac{C_2}{2} + (c_v + 1) p_2 \right) \beta_2 \end{aligned} \quad (4.13)$$

- Subsolution conditions for $i = 1, N$:

$$C_i - \alpha_i^2 - \beta_i^2 > 0 \quad (4.14)$$

$$\left(\frac{C_i}{2} - \alpha_i^2 + \gamma_i \right) \left(\frac{C_i}{2} - \beta_i^2 - \gamma_i \right) - (\delta_i - \alpha_i \beta_i)^2 > 0 \quad (4.15)$$

- *Admissibility condition on the left interface:*

$$\mu_0 \left(\varrho_1 s(\varrho_1, p_1) - \varrho_- s(\varrho_-, p_-) \right) \leq \varrho_1 s(\varrho_1, p_1) \beta_1 - \varrho_- s(\varrho_-, p_-) v_{-,2} \quad (4.16)$$

- *Admissibility condition on the right interface:*

$$\mu_N \left(\varrho_+ s(\varrho_+, p_+) - \varrho_N s(\varrho_N, p_N) \right) \leq \varrho_+ s(\varrho_+, p_+) v_{+,2} - \varrho_N s(\varrho_N, p_N) \beta_N \quad (4.17)$$

- *If $N = 2$ we additionally have an admissibility condition on the middle interface:*

$$\mu_1 \left(\varrho_2 s(\varrho_2, p_2) - \varrho_1 s(\varrho_1, p_1) \right) \leq \varrho_2 s(\varrho_2, p_2) \beta_2 - \varrho_1 s(\varrho_1, p_1) \beta_1 \quad (4.18)$$

Proof. There is really nothing to prove in this proposition, differential constraints for piecewise constant functions translate to Rankine-Hugoniot conditions in a standard way and the subsolution conditions (4.14)-(4.15) state that the matrix $\frac{C_i}{2} \mathbb{I} - \mathbf{v}_i \otimes \mathbf{v}_i + U_i$ is positive definite, which holds if and only if its trace and its determinant are both positive. \square

As we already mentioned, in order to prove our main theorems, due to Theorem 4.2 it is always enough to find a single N -fan subsolution. Moreover, we are going to make a special ansatz for the subsolutions. Recall that we assume throughout this paper that $v_{-,1} = v_{+,1} = 0$. We will look for N -fan subsolutions satisfying

$$\alpha_i = 0 \quad \& \quad \delta_i = \alpha_i \beta_i = 0, \quad (4.19)$$

$i = 1, N$. One can easily check that this choice implies that equations (4.3), (4.7) and (4.11) are trivially satisfied. We also note that in the case $N = 1$ this is in fact a consequence of the set of Rankine-Hugoniot conditions, for more details see [5, Lemma 4.2].

For simplicity of notation, we will use in the rest of the paper notation v_{\pm} in place of $v_{\pm,2}$ and similarly v_M instead of $v_{M,2}$.

5 Proof of Theorem 3.5

5.1 Reduction to special case

We recall that the proof of Theorem 3.5 is finished as soon as we find a single admissible 2-fan subsolution. Since the structure of the 2-fan subsolution is similar to the 1D Riemann solution to the problem, we will look for the 2-fan subsolution as a suitable small perturbation of the 1D Riemann solution.

Moreover, we claim that we can make the following choice without loss of generality.

Proposition 5.1. *Let the assumption of Theorem 3.5 hold and assume furthermore that $v_M = 0$, where v_M is the second component of the velocity of the 1D Riemann solution (see Proposition 2.1). Then there exist infinitely many admissible weak 2-fan solutions to the Riemann problem (1.1)-(1.4).*

Proposition 5.1 implies Theorem 3.5 using the Galilean invariance of the Euler equations. Indeed, if we consider new initial data as follows:

$$(\varrho^{0,\text{new}}, \mathbf{v}^{0,\text{new}}, p^{0,\text{new}}) := (\varrho^0, \mathbf{v}^0 - (0, v_M)^T, p^0). \quad (5.1)$$

the 1D Riemann solution to the problem (1.1)-(1.3) with (5.1) will fulfill $v_M^{\text{new}} = 0$. We apply Proposition 5.1 and find infinitely many admissible weak 2-fan solutions (ϱ, \mathbf{v}, p) to the Riemann problem with initial data (5.1). Then, again by Galilean invariance, $(\varrho, \mathbf{v} + (0, v_M)^T, p)$ are admissible weak 2-fan solutions to the Riemann problem with original initial data (1.4), and the proof of Theorem 3.5 is finished.

What remains now is to prove Proposition 5.1.

5.2 Proof of Proposition 5.1

Let the assumptions of Proposition 5.1 be true. As we mentioned above, we will look for an admissible 2-fan subsolution as a suitable perturbation of the 1D Riemann solution. This perturbation will be quantified by a small parameter $\varepsilon > 0$.

For convenience we define functions $A, B, D : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} A(\varepsilon) &:= \varrho_-(\varrho_{M-} + \varepsilon)(\varrho_{M+} - \varepsilon - \varrho_+) - \varrho_+(\varrho_{M+} - \varepsilon)(\varrho_{M-} + \varepsilon - \varrho_-); \\ B(\varepsilon) &:= \varrho_- \varrho_+ (\varrho_{M-} + \varepsilon)(\varrho_{M+} - \varepsilon) (v_- - v_+)^2 - (p_- - p_+) A(\varepsilon); \\ D(\varepsilon) &:= v_- \varrho_- (\varrho_{M-} + \varepsilon)(\varrho_{M+} - \varepsilon - \varrho_+) - v_+ \varrho_+ (\varrho_{M+} - \varepsilon)(\varrho_{M-} + \varepsilon - \varrho_-). \end{aligned}$$

First of all we want to show some properties of the functions A and B . It is easy to deduce from Proposition 2.1 and the assumption $v_M = 0$, that

$$v_- = \sqrt{2c_v} \frac{p_M - p_-}{\sqrt{\varrho_-(p_- + (2c_v + 1)p_M)}}, \quad (5.2)$$

$$v_+ = -\sqrt{2c_v} \frac{p_M - p_+}{\sqrt{\varrho_+(p_+ + (2c_v + 1)p_M)}}. \quad (5.3)$$

Since $p_M > \max\{p_-, p_+\}$, we have $v_+ < 0 < v_-$. Furthermore we obtain from Proposition 2.1 and (5.2), (5.3) that

$$\begin{aligned} \frac{\varrho_{M-} - \varrho_-}{\varrho_- \varrho_{M-}} &= \frac{2c_v(p_M - p_-)}{\varrho_-(p_- + (2c_v + 1)p_M)} = \frac{(v_-)^2}{p_M - p_-}, \\ \frac{\varrho_{M+} - \varrho_+}{\varrho_+ \varrho_{M+}} &= \frac{2c_v(p_M - p_+)}{\varrho_+(p_+ + (2c_v + 1)p_M)} = \frac{(v_+)^2}{p_M - p_+}. \end{aligned}$$

This leads to

$$\begin{aligned} (v_-)^2 + (p_- - p_+) \frac{\varrho_{M-} - \varrho_-}{\varrho_- \varrho_{M-}} &= (v_-)^2 \frac{p_M - p_+}{p_M - p_-}, \\ (v_+)^2 - (p_- - p_+) \frac{\varrho_{M+} - \varrho_+}{\varrho_+ \varrho_{M+}} &= (v_+)^2 \frac{p_M - p_-}{p_M - p_+}, \end{aligned}$$

and finally

$$\begin{aligned} B(0) &= \\ &= \varrho_- \varrho_+ \varrho_{M-} \varrho_{M+} (v_- - v_+)^2 - (p_- - p_+) \left(\varrho_- \varrho_{M-} (\varrho_{M+} - \varrho_+) - \varrho_+ \varrho_{M+} (\varrho_{M-} - \varrho_-) \right) \\ &= \varrho_- \varrho_+ \varrho_{M-} \varrho_{M+} \left[(v_-)^2 + (p_- - p_+) \frac{\varrho_{M-} - \varrho_-}{\varrho_- \varrho_{M-}} + (v_+)^2 - (p_- - p_+) \frac{\varrho_{M+} - \varrho_+}{\varrho_+ \varrho_{M+}} - 2v_- v_+ \right] \\ &= \varrho_- \varrho_+ \varrho_{M-} \varrho_{M+} \left[(v_-)^2 \frac{p_M - p_+}{p_M - p_-} + (v_+)^2 \frac{p_M - p_-}{p_M - p_+} - 2v_- v_+ \right] > 0. \end{aligned}$$

Now, by continuity of the function B , there exists an $\varepsilon_{\max,1} > 0$ such that $B(\varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_{\max,1}]$. Because $\varrho_{M+} > \varrho_+$, there exists $\varepsilon_{\max,2} > 0$ such that $\varrho_{M+} - \varepsilon - \varrho_+ > 0$ for all $\varepsilon \in (0, \varepsilon_{\max,2}]$.

Next we want to show that there is an $\varepsilon_{\max,3} > 0$ such that $A(\varepsilon) \neq 0$ for all $\varepsilon \in (0, \varepsilon_{\max,3}]$. To this end, let us first assume, that $A(0) \neq 0$. Then, by continuity of the function A , there exists such an $\varepsilon_{\max,3} > 0$. Now consider the case where $A(0) = 0$. Then we obtain

$$\begin{aligned} A(\varepsilon) &= \varrho_-(\varrho_{M-} + \varepsilon)(\varrho_{M+} - \varepsilon - \varrho_+) - \varrho_+(\varrho_{M+} - \varepsilon)(\varrho_{M-} + \varepsilon - \varrho_-) \\ &= \varepsilon^2(\varrho_+ - \varrho_-) - \varepsilon(2\varrho_- \varrho_+ + (\varrho_- - \varrho_+)(\varrho_{M-} - \varrho_{M+})) + \underbrace{A(0)}_{=0} \\ &= \varepsilon((\varrho_+ - \varrho_-)\varepsilon - 2\varrho_- \varrho_+ - (\varrho_- - \varrho_+)(\varrho_{M-} - \varrho_{M+})), \end{aligned}$$

which has at most two zeros: If $\varrho_- = \varrho_+$ then

$$A(\varepsilon) = 0 \quad \Longleftrightarrow \quad \varepsilon = 0;$$

and if $\varrho_- \neq \varrho_+$ then

$$A(\varepsilon) = 0 \quad \Longleftrightarrow \quad \varepsilon = 0 \quad \text{or} \quad \varepsilon = \frac{2\varrho_- \varrho_+ + (\varrho_- - \varrho_+)(\varrho_{M-} - \varrho_{M+})}{\varrho_+ - \varrho_-}.$$

Hence there exists $\varepsilon_{\max,3} > 0$ such that $A(\varepsilon) \neq 0$ for all $\varepsilon \in (0, \varepsilon_{\max,3}]$.

We set $\varepsilon_{\max} := \min\{\varepsilon_{\max,1}, \varepsilon_{\max,2}, \varepsilon_{\max,3}\}$ and then we have

$$\begin{aligned} A(\varepsilon) &\neq 0, & \varrho_{M+} - \varepsilon - \varrho_+ &> 0, \\ B(\varepsilon) &> 0, & \varrho_{M-} + \varepsilon - \varrho_- &> 0, \end{aligned} \tag{5.4}$$

for all $\varepsilon \in (0, \varepsilon_{\max}]$.

5.2.1 Shock speeds

Next we define the functions $\mu_0, \mu_1, \mu_2 : (0, \varepsilon_{\max}] \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mu_0(\varepsilon) &:= \frac{1}{A(\varepsilon)} \left[D(\varepsilon) + \varrho_- \varrho_+ (\varrho_{M+} - \varepsilon)(v_- - v_+) - \sqrt{(\varrho_{M-} + \varepsilon)^2 \frac{\varrho_{M+} - \varepsilon - \varrho_+}{\varrho_{M-} + \varepsilon - \varrho_-} B(\varepsilon)} \right]; \\ \mu_1(\varepsilon) &:= \frac{1}{A(\varepsilon)} \left[D(\varepsilon) - \sqrt{(\varrho_{M-} + \varepsilon - \varrho_-)(\varrho_{M+} - \varepsilon - \varrho_+) B(\varepsilon)} \right]; \\ \mu_2(\varepsilon) &:= \frac{1}{A(\varepsilon)} \left[D(\varepsilon) + \varrho_- \varrho_+ (\varrho_{M-} + \varepsilon)(v_- - v_+) - \sqrt{(\varrho_{M+} - \varepsilon)^2 \frac{\varrho_{M-} + \varepsilon - \varrho_-}{\varrho_{M+} - \varepsilon - \varrho_+} B(\varepsilon)} \right]. \end{aligned}$$

Note first that the functions μ_0, μ_1, μ_2 are well-defined because of (5.4). We claim that these functions define perturbations of the shock speeds σ_- , v_M and σ_+ of the 1D Riemann solution. More precisely we have

Proposition 5.2. *It holds that*

$$\lim_{\varepsilon \rightarrow 0} \mu_0(\varepsilon) = \sigma_-, \quad \lim_{\varepsilon \rightarrow 0} \mu_1(\varepsilon) = v_M, \quad \lim_{\varepsilon \rightarrow 0} \mu_2(\varepsilon) = \sigma_+.$$

Proof. We start with the Rankine-Hugoniot conditions for the 1D Riemann solution

$$\sigma_- (\varrho_- - \varrho_{M-}) = \varrho_- v_- - \varrho_{M-} v_M; \tag{5.5}$$

$$\sigma_- (\varrho_- v_- - \varrho_{M-} v_M) = \varrho_- v_-^2 - \varrho_{M-} v_M^2 + p_- - p_M; \tag{5.6}$$

$$\sigma_+ (\varrho_{M+} - \varrho_+) = \varrho_{M+} v_M - \varrho_+ v_+; \tag{5.7}$$

$$\sigma_+ (\varrho_{M+} v_M - \varrho_+ v_+) = \varrho_{M+} v_M^2 - \varrho_+ v_+^2 + p_M - p_+ \tag{5.8}$$

and we obtain by eliminating σ_- , σ_+ and p_M that

$$\begin{aligned} &(\varrho_{M+} - \varrho_+)(\varrho_- v_- - \varrho_{M-} v_M)^2 + (\varrho_- - \varrho_{M-})(\varrho_{M+} v_M - \varrho_+ v_+)^2 \\ &= \left(\varrho_- v_-^2 - \varrho_+ v_+^2 - (v_M)^2 (\varrho_{M-} - \varrho_{M+}) + p_- - p_+ \right) (\varrho_{M+} - \varrho_+) (\varrho_- - \varrho_{M-}). \end{aligned}$$

In order to solve this equation for v_M , we write it as follows

$$A(0) (v_M)^2 - 2D(0) v_M + E = 0, \tag{5.9}$$

where the constant

$$E := (p_- - p_+)(\varrho_{M-} - \varrho_-)(\varrho_{M+} - \varrho_+) + (v_-)^2 \varrho_- \varrho_{M-} (\varrho_{M+} - \varrho_+) - (v_+)^2 \varrho_+ \varrho_{M+} (\varrho_{M-} - \varrho_-)$$

only depends on the initial states. Now we have to consider two cases, namely $A(0) = 0$ and $A(0) \neq 0$.

Let us start with $A(0) = 0$. Then we easily deduce that

$$D(0) = \varrho_- \varrho_{M-} (\varrho_{M+} - \varrho_+) (v_- - v_+),$$

which does not vanish because $v_+ < 0 < v_-$ and $\varrho_{M+} > \varrho_+$. Hence we get from (5.9), that

$$v_M = \frac{E}{2D(0)}. \quad (5.10)$$

Next we want to compute $\lim_{\varepsilon \rightarrow 0} \mu_1(\varepsilon)$ and compare it with (5.10). Keeping in mind that we are considering the case $A(0) = 0$, we get

$$\begin{aligned} & D(0) - \sqrt{(\varrho_{M-} - \varrho_-)(\varrho_{M+} - \varrho_+) B(0)} \\ &= (v_- - v_+) \varrho_- \varrho_{M-} (\varrho_{M+} - \varrho_+) - \sqrt{(\varrho_{M-} - \varrho_-)(\varrho_{M+} - \varrho_+) \varrho_- \varrho_+ \varrho_{M-} \varrho_{M+} (v_- - v_+)^2} \\ &= (v_- - v_+) \varrho_- \varrho_{M-} (\varrho_{M+} - \varrho_+) - \sqrt{(\varrho_{M+} - \varrho_+)^2 (\varrho_+)^2 (\varrho_{M+})^2 (v_- - v_+)^2} = 0. \end{aligned}$$

Hence we can apply L'Hospital's rule. We obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} A'(\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \left(\varrho_- (\varrho_{M+} - \varepsilon - \varrho_+) - \varrho_- (\varrho_{M-} + \varepsilon) - \varrho_+ (\varrho_{M+} - \varepsilon) + \varrho_+ (\varrho_{M-} + \varepsilon - \varrho_-) \right) \\ &= -2\varrho_- \varrho_+ - (\varrho_- - \varrho_+) (\varrho_{M-} - \varrho_{M+}). \end{aligned}$$

A short calculation shows that this is non-zero: From $A(0) = 0$ we can deduce that

$$\frac{\varrho_- - \varrho_+}{\varrho_- \varrho_+} = \frac{\varrho_{M-} - \varrho_{M+}}{\varrho_{M-} \varrho_{M+}}.$$

This means that $\varrho_- - \varrho_+$ and $\varrho_{M-} - \varrho_{M+}$ have the same sign, which implies $(\varrho_- - \varrho_+) (\varrho_{M-} - \varrho_{M+}) \geq 0$. Since $\varrho_- \varrho_+ > 0$, we have $A'(0) < 0$, in particular $A'(0) \neq 0$

Furthermore a long but straightforward computation yields

$$\lim_{\varepsilon \rightarrow 0} \left[D(\varepsilon) - \sqrt{(\varrho_{M-} + \varepsilon - \varrho_-)(\varrho_{M+} - \varepsilon - \varrho_+) B(\varepsilon)} \right]' = \frac{A'(0) E}{2D(0)}.$$

Hence by L'Hospital's rule we obtain $\lim_{\varepsilon \rightarrow 0} \mu_1(\varepsilon) = \frac{E}{2D(0)}$ and recalling (5.10), we deduce $\lim_{\varepsilon \rightarrow 0} \mu_1(\varepsilon) = v_M$.

Let us now consider the case $A(0) \neq 0$. Then we obtain from (5.9) that

$$v_M = \frac{1}{A(0)} \left[D(0) \pm \sqrt{D(0)^2 - A(0) E} \right].$$

The correct sign in the equation above is “-” because¹ $v_M = 0$ and $D(0) > 0$, which easily follows from $v_+ < 0 < v_-$. Furthermore it is simple to check that

$$D(0)^2 - A(0) E = (\varrho_{M-} - \varrho_-)(\varrho_{M+} - \varrho_+) B(0).$$

Then it is easy to conclude $\mu_1(0) = v_M$.

To finish the proof of Proposition 5.2 we have to show that $\lim_{\varepsilon \rightarrow 0} \mu_1(\varepsilon) = v_M$ implies that $\lim_{\varepsilon \rightarrow 0} \mu_0(\varepsilon) = \sigma_-$ and $\lim_{\varepsilon \rightarrow 0} \mu_2(\varepsilon) = \sigma_+$. It is straightforward to deduce that

$$\begin{aligned} \mu_0(\varepsilon) &= v_- + \frac{\varrho_{M-} + \varepsilon}{\varrho_{M-} + \varepsilon - \varrho_-} (\mu_1(\varepsilon) - v_-); \\ \mu_2(\varepsilon) &= v_+ + \frac{\varrho_{M+} - \varepsilon}{\varrho_{M+} - \varepsilon - \varrho_+} (\mu_1(\varepsilon) - v_+). \end{aligned}$$

¹Alternatively, this can be verified by considering the admissibility criterion.

On the other hand we get from (5.5) and (5.7), that

$$\sigma_- = v_- + \frac{\varrho_{M-}}{\varrho_{M-} - \varrho_-} (v_M - v_-); \quad \sigma_+ = v_+ + \frac{\varrho_{M+}}{\varrho_{M+} - \varrho_+} (v_M - v_+).$$

Hence we easily deduce $\lim_{\varepsilon \rightarrow 0} \mu_0(\varepsilon) = \sigma_-$ and $\lim_{\varepsilon \rightarrow 0} \mu_2(\varepsilon) = \sigma_+$. \square

Because of $\sigma_- < v_M < \sigma_+$ and the continuity of the functions μ_0, μ_1, μ_2 , we may assume that $\mu_0(\varepsilon) < \mu_1(\varepsilon) < \mu_2(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_{\max}]$. If this is not the case we redefine ε_{\max} to be a bit smaller than the smallest positive value of ε for which $\mu_0(\varepsilon) < \mu_1(\varepsilon) < \mu_2(\varepsilon)$ is violated.

5.2.2 Constants C_i and γ_i

In order to proceed further we need to introduce a second positive parameter $\delta > 0$. We define the functions $C_1, C_2, \gamma_1, \gamma_2 : (0, \varepsilon_{\max}] \times (0, p_M)$ by

$$\begin{aligned} C_1(\varepsilon, \delta) &:= \frac{2}{(\varrho_{M-} + \varepsilon)(\mu_0(\varepsilon) - \mu_1(\varepsilon))} \left[-\mu_0(\varepsilon) \left(c_v(p_M - \delta - p_-) - \frac{1}{2} \varrho_- |v_-|^2 \right) \right. \\ &\quad \left. + \mu_1(\varepsilon)(c_v + 1)(p_M - \delta) - \left(\frac{1}{2} \varrho_- |v_-|^2 + (c_v + 1)p_- \right) v_- \right]; \\ C_2(\varepsilon, \delta) &:= \frac{2}{(\varrho_{M+} - \varepsilon)(\mu_2(\varepsilon) - \mu_1(\varepsilon))} \left[-\mu_2(\varepsilon) \left(c_v(p_M - \delta - p_+) - \frac{1}{2} \varrho_+ |v_+|^2 \right) \right. \\ &\quad \left. + \mu_1(\varepsilon)(c_v + 1)(p_M - \delta) - \left(\frac{1}{2} \varrho_+ |v_+|^2 + (c_v + 1)p_+ \right) v_+ \right]; \\ \gamma_1(\varepsilon, \delta) &:= \frac{1}{(\varrho_{M-} + \varepsilon)} \left[(\varrho_{M-} + \varepsilon) \frac{C_1(\varepsilon, \delta)}{2} - \varrho_- v_-^2 + p_M - \delta - p_- - \mu_0(\varepsilon) \left((\varrho_{M-} + \varepsilon) \mu_1(\varepsilon) - \varrho_- v_- \right) \right]; \\ \gamma_2(\varepsilon, \delta) &:= \frac{1}{(\varrho_{M+} - \varepsilon)} \left[(\varrho_{M+} - \varepsilon) \frac{C_2(\varepsilon, \delta)}{2} - \varrho_+ v_+^2 + p_M - \delta - p_+ - \mu_2(\varepsilon) \left((\varrho_{M+} - \varepsilon) \mu_1(\varepsilon) - \varrho_+ v_+ \right) \right]. \end{aligned}$$

Note that these functions are well-defined because of the arguments above. More precisely, it holds that $\mu_0(\varepsilon) - \mu_1(\varepsilon) \neq 0$ and $\mu_2(\varepsilon) - \mu_1(\varepsilon) \neq 0$ for all $\varepsilon \in (0, \varepsilon_{\max}]$.

Proposition 5.3. *It holds that*

$$\begin{aligned} \lim_{(\varepsilon, \delta) \rightarrow 0} C_1(\varepsilon, \delta) &= (v_M)^2; & \lim_{(\varepsilon, \delta) \rightarrow 0} \gamma_1(\varepsilon, \delta) &= -\frac{(v_M)^2}{2}; \\ \lim_{(\varepsilon, \delta) \rightarrow 0} C_2(\varepsilon, \delta) &= (v_M)^2; & \lim_{(\varepsilon, \delta) \rightarrow 0} \gamma_2(\varepsilon, \delta) &= -\frac{(v_M)^2}{2}. \end{aligned}$$

Proof. To prove this, we need the Rankine Hugoniot conditions of the 1D Riemann solution in the energy equation

$$\begin{aligned} \sigma_- \left(\frac{1}{2} \varrho_- |v_-|^2 + c_v p_- - \frac{1}{2} \varrho_{M-} |v_M|^2 - c_v p_M \right) \\ = \left(\frac{1}{2} \varrho_- |v_-|^2 + (c_v + 1) p_- \right) v_- - \left(\frac{1}{2} \varrho_{M-} |v_M|^2 + (c_v + 1) p_M \right) v_M; \end{aligned} \tag{5.11}$$

$$\begin{aligned} \sigma_+ \left(\frac{1}{2} \varrho_{M+} |v_M|^2 + c_v p_M - \frac{1}{2} \varrho_+ |v_+|^2 - c_v p_+ \right) \\ = \left(\frac{1}{2} \varrho_{M+} |v_M|^2 + (c_v + 1) p_M \right) v_M - \left(\frac{1}{2} \varrho_+ |v_+|^2 + (c_v + 1) p_+ \right) v_+. \end{aligned} \tag{5.12}$$

We obtain that $\lim_{(\varepsilon, \delta) \rightarrow 0} C_1(\varepsilon, \delta) = (v_M)^2$ and $\lim_{(\varepsilon, \delta) \rightarrow 0} C_2(\varepsilon, \delta) = (v_M)^2$ by using Proposition 5.2 and (5.11), (5.12).

The fact that $\lim_{(\varepsilon, \delta) \rightarrow 0} \gamma_1(\varepsilon, \delta) = -\frac{(v_M)^2}{2}$ and $\lim_{(\varepsilon, \delta) \rightarrow 0} \gamma_2(\varepsilon, \delta) = -\frac{(v_M)^2}{2}$ can be shown analogously by using the Rankine Hugoniot conditions (5.6), (5.8). \square

We continue the proof of the Proposition 5.1 by observing that the perturbations defined above indeed help to define an admissible 2-fan subsolution.

Proposition 5.4. *If there exists $(\varepsilon, \delta) \in (0, \varepsilon_{\max}] \times (0, p_M)$ such that the following inequalities are fulfilled, then there exists an admissible 2-fan subsolution to the Cauchy problem (1.1)-(1.4).*

- *Order of the speeds:*

$$\mu_0(\varepsilon) < \mu_1(\varepsilon) < \mu_2(\varepsilon) \quad (5.13)$$

- *Subsolution conditions:*

$$C_1(\varepsilon, \delta) - \mu_1(\varepsilon)^2 > 0 \quad (5.14)$$

$$C_2(\varepsilon, \delta) - \mu_1(\varepsilon)^2 > 0 \quad (5.15)$$

$$\left(\frac{C_1(\varepsilon, \delta)}{2} + \gamma_1(\varepsilon, \delta) \right) \left(\frac{C_1(\varepsilon, \delta)}{2} - \mu_1(\varepsilon)^2 - \gamma_1(\varepsilon, \delta) \right) > 0 \quad (5.16)$$

$$\left(\frac{C_2(\varepsilon, \delta)}{2} + \gamma_2(\varepsilon, \delta) \right) \left(\frac{C_2(\varepsilon, \delta)}{2} - \mu_1(\varepsilon)^2 - \gamma_2(\varepsilon, \delta) \right) > 0 \quad (5.17)$$

- *Admissibility condition on the left interface:*

$$\begin{aligned} & \mu_0(\varepsilon) \left((\varrho_{M-} + \varepsilon) s(\varrho_{M-} + \varepsilon, p_M - \delta) - \varrho_- s(\varrho_-, p_-) \right) \\ & \leq (\varrho_{M-} + \varepsilon) s(\varrho_{M-} + \varepsilon, p_M - \delta) \mu_1(\varepsilon) - \varrho_- s(\varrho_-, p_-) v_- \end{aligned} \quad (5.18)$$

- *Admissibility condition on the right interface:*

$$\begin{aligned} & \mu_2(\varepsilon) \left(\varrho_+ s(\varrho_+, p_+) - (\varrho_{M+} - \varepsilon) s(\varrho_{M+} - \varepsilon, p_M - \delta) \right) \\ & \leq \varrho_+ s(\varrho_+, p_+) v_+ - (\varrho_{M+} - \varepsilon) s(\varrho_{M+} - \varepsilon, p_M - \delta) \mu_1(\varepsilon) \end{aligned} \quad (5.19)$$

Proof. Let there be $(\varepsilon, \delta) \in (0, \varepsilon_{\max}] \times (0, p_M)$ such that (5.13)-(5.19) hold. In order to show that there exists an admissible 2-fan subsolution, we use Proposition 4.4, i.e. we define the constants appearing in Proposition 4.4 as follows:

$$\mu_0 := \mu_0(\varepsilon); \quad \mu_1 := \mu_1(\varepsilon); \quad \mu_2 := \mu_2(\varepsilon);$$

$$\begin{aligned} \varrho_1 &:= \varrho_{M-} + \varepsilon; & \varrho_2 &:= \varrho_{M+} - \varepsilon; \\ \mathbf{v}_1 &:= \mathbf{v}_2 := \begin{pmatrix} 0 \\ \mu_1(\varepsilon) \end{pmatrix}; & p_1 &:= p_2 := p_M - \delta; \\ U_1 &:= \begin{pmatrix} \gamma_1(\varepsilon, \delta) & 0 \\ 0 & -\gamma_1(\varepsilon, \delta) \end{pmatrix}; & U_2 &:= \begin{pmatrix} \gamma_2(\varepsilon, \delta) & 0 \\ 0 & -\gamma_2(\varepsilon, \delta) \end{pmatrix}; \\ C_1 &:= C_1(\varepsilon, \delta); & C_2 &:= C_2(\varepsilon, \delta). \end{aligned}$$

It is a matter of straightforward calculation to check that with this choice (4.1)-(4.17) hold. \square

5.2.3 Subsolution and admissibility inequalities

In order to finish the proof of Proposition 5.1, we have to find $(\varepsilon, \delta) \in (0, \varepsilon_{\max}] \times (0, p_M)$ such that the conditions (5.13)-(5.19) are satisfied.

We start with noting that we already have (5.13) fulfilled for all $\varepsilon \in (0, \varepsilon_{\max}]$.

Let us now investigate the subsolution conditions (5.14)-(5.17). We start with the terms in the first parenthesis in (5.16)-(5.17). We obtain by using that $\delta \in (0, p_M)$

$$\begin{aligned} \frac{C_1(\varepsilon, \delta)}{2} + \gamma_1(\varepsilon, \delta) &= \frac{(2c_v - 1)\delta}{\varrho_{M-} + \varepsilon} - \frac{2\mu_1(\varepsilon)\delta}{(\varrho_{M-} + \varepsilon)(\mu_0(\varepsilon) - \mu_1(\varepsilon))} + \frac{C_1(\varepsilon, 0)}{2} + \gamma_1(\varepsilon, 0) \\ &\geq \frac{(2c_v - 1)\delta}{\varrho_{M-} + \varepsilon} - \underbrace{\frac{2|\mu_1(\varepsilon)|p_M}{(\varrho_{M-} + \varepsilon)|\mu_0(\varepsilon) - \mu_1(\varepsilon)|}}_{=:R_1(\varepsilon)} + \frac{C_1(\varepsilon, 0)}{2} + \gamma_1(\varepsilon, 0) \end{aligned} \quad (5.20)$$

$$\begin{aligned} \frac{C_2(\varepsilon, \delta)}{2} + \gamma_2(\varepsilon, \delta) &= \frac{(2c_v - 1)\delta}{\varrho_{M+} - \varepsilon} - \frac{2\mu_1(\varepsilon)\delta}{(\varrho_{M+} - \varepsilon)(\mu_2(\varepsilon) - \mu_1(\varepsilon))} + \frac{C_2(\varepsilon, 0)}{2} + \gamma_2(\varepsilon, 0) \\ &\geq \frac{(2c_v - 1)\delta}{\varrho_{M+} - \varepsilon} - \underbrace{\frac{2|\mu_1(\varepsilon)|p_M}{(\varrho_{M+} - \varepsilon)|\mu_2(\varepsilon) - \mu_1(\varepsilon)|}}_{=:R_2(\varepsilon)} + \frac{C_2(\varepsilon, 0)}{2} + \gamma_2(\varepsilon, 0) \end{aligned} \quad (5.21)$$

where Propositions 5.2 and 5.3 together with the fact that $v_M = 0$ imply that

$$\lim_{\varepsilon \rightarrow 0} R_1(\varepsilon) = 0; \quad \lim_{\varepsilon \rightarrow 0} R_2(\varepsilon) = 0.$$

Therefore $|R_1(\varepsilon)|$ and $|R_2(\varepsilon)|$ become arbitrary small if we choose ε small. Because of $c_v > \frac{1}{2}$, there exists $\tilde{\varepsilon}_1(\delta) \in (0, \varepsilon_{\max}]$ for each $\delta \in (0, p_M)$, such that

$$\frac{C_1(\varepsilon, \delta)}{2} + \gamma_1(\varepsilon, \delta) > 0 \quad \& \quad \frac{C_2(\varepsilon, \delta)}{2} + \gamma_2(\varepsilon, \delta) > 0 \quad (5.22)$$

hold for all $\varepsilon \in (0, \tilde{\varepsilon}_1(\delta))$.

Similarly we handle terms in the second parenthesis in inequalities (5.16)-(5.17). We obtain

$$\frac{C_1(\varepsilon, \delta)}{2} - \mu_1(\varepsilon)^2 - \gamma_1(\varepsilon, \delta) = \frac{\delta}{\varrho_{M-} + \varepsilon} + \underbrace{\frac{C_1(\varepsilon, 0)}{2} - \mu_1(\varepsilon)^2 - \gamma_1(\varepsilon, 0)}_{=:R_3(\varepsilon)} \quad (5.23)$$

$$\frac{C_2(\varepsilon, \delta)}{2} - \mu_1(\varepsilon)^2 - \gamma_2(\varepsilon, \delta) = \frac{\delta}{\varrho_{M+} - \varepsilon} + \underbrace{\frac{C_2(\varepsilon, 0)}{2} - \mu_1(\varepsilon)^2 - \gamma_2(\varepsilon, 0)}_{=:R_4(\varepsilon)}. \quad (5.24)$$

With the same arguments as above, we obtain that for each $\delta \in (0, p_M)$ there exists $\tilde{\varepsilon}_2(\delta) \in (0, \varepsilon_{\max}]$ such that

$$\frac{C_i(\varepsilon, \delta)}{2} - \mu_1(\varepsilon)^2 - \gamma_i(\varepsilon, \delta) > 0, \quad i = 1, 2 \quad (5.25)$$

hold for all $\varepsilon \in (0, \tilde{\varepsilon}_2(\delta))$.

Combining (5.22) and (5.25) we obtain (5.16) and (5.17) while summing together (5.22) and (5.25) we obtain (5.14) and (5.15).

To finish the proof we have to show that we can achieve that in addition the admissibility conditions (5.18) and (5.19) hold. Note that in the limit $(\varepsilon, \delta) \rightarrow (0, 0)$ the admissibility conditions (5.18) and (5.19) turn into the admissibility conditions of the 1D Riemann solution (according to Proposition 5.2). Since the latter are fulfilled strictly, we can choose $\delta > 0$ sufficiently small and also $\varepsilon \in (0, \min\{\tilde{\varepsilon}_1(\delta), \tilde{\varepsilon}_2(\delta)\})$ sufficiently small such that (5.18) and (5.19) hold. This finishes the proof of Proposition 5.1.

6 Proof of Theorem 3.6

Unlike in the case of 2-fan solutions as in Theorem 3.5, we cannot use here the 1D Riemann solution and try to perturb it in a suitable way in order to find a 1-fan subsolution. On the other hand, since we don't have the middle interface, the number of equations and inequalities in Proposition 4.4 is lower than in the case $N = 2$.

Since i could be only equal to 1 in Proposition 4.4, for simplicity of notation we write $\alpha, \beta, \gamma, \delta$ and C instead of $\alpha_1, \beta_1, \gamma_1, \delta_1$ and C_1 . We also recall that we set $\alpha = 0$ and $\delta = \alpha\beta = 0$, see (4.19).

The subsolution conditions (4.14)-(4.15) then simplify to

$$C - \beta^2 > 0; \quad (6.1)$$

$$\left(\frac{C}{2} + \gamma\right)\left(\frac{C}{2} - \beta^2 - \gamma\right) > 0. \quad (6.2)$$

It is not difficult to observe (see also [5, Lemma 4.3]) that the necessary condition for (6.1)-(6.2) to be satisfied is $\frac{C}{2} - \gamma > \beta^2$ which motivates us to introduce instead of C and γ new unknowns

$$\varepsilon_1 := \frac{C}{2} - \gamma - \beta^2; \quad (6.3)$$

$$\varepsilon_2 := C - \beta^2 - \varepsilon_1. \quad (6.4)$$

The set of algebraic identities and inequalities from Proposition 4.4 then simplifies into

- Order of the speeds

$$\mu_0 < \mu_1 \quad (6.5)$$

- Rankine Hugoniot conditions on the left interface:

$$\mu_0(\varrho_- - \varrho_1) = \varrho_- v_- - \varrho_1 \beta \quad (6.6)$$

$$\mu_0(\varrho_- v_- - \varrho_1 \beta) = \varrho_- v_-^2 - \varrho_1(\beta^2 + \varepsilon_1) + p_- - p_1 \quad (6.7)$$

$$\begin{aligned} \mu_0(2c_v p_- + \varrho_- v_-^2 - 2c_v p_1 - \varrho_1(\beta^2 + \varepsilon_1 + \varepsilon_2)) \\ = ((2c_v + 2)p_- + \varrho_- v_-^2)v_- - ((2c_v + 2)p_1 + \varrho_1(\beta^2 + \varepsilon_1 + \varepsilon_2))\beta; \end{aligned} \quad (6.8)$$

- Rankine-Hugoniot conditions on the right interface:

$$\mu_1(\varrho_1 - \varrho_+) = \varrho_1 \beta - \varrho_+ v_+ \quad (6.9)$$

$$\mu_1(\varrho_1 \beta - \varrho_+ v_+) = \varrho_1(\beta^2 + \varepsilon_1) - \varrho_+ v_+^2 + p_1 - p_+ \quad (6.10)$$

$$\begin{aligned} \mu_1(2c_v p_1 + \varrho_1(\beta^2 + \varepsilon_1 + \varepsilon_2) - 2c_v p_+ - \varrho_+ v_+^2) \\ = ((2c_v + 2)p_1 + \varrho_1(\beta^2 + \varepsilon_1 + \varepsilon_2))\beta - ((2c_v + 2)p_+ + \varrho_+ v_+^2)v_+; \end{aligned} \quad (6.11)$$

- Subsolution conditions:

$$\varepsilon_1 > 0 \quad (6.12)$$

$$\varepsilon_2 > 0; \quad (6.13)$$

- Admissibility conditions

$$\mu_0(L_- - L_1) \geq L_- v_- - L_1 \beta \quad (6.14)$$

$$\mu_1(L_1 - L_+) \geq L_1 \beta - L_+ v_+. \quad (6.15)$$

where for simplicity of notation we introduced L_i as

$$L_i = \varrho_i s(\varrho_i, p_i) = \varrho_i \log \left(\frac{p_i^{c_v}}{\varrho_i^{c_v+1}} \right), \quad i = -, 1, +.$$

We observe that we have 6 equations and 5 inequalities for 7 unknowns $\mu_0, \mu_1, \varrho_1, p_1, \beta, \varepsilon_1, \varepsilon_2$. Moreover to these 5 inequalities we have to keep in mind other inequalities which have to be satisfied, namely $\varrho_1 > 0$ and $p_1 > 0$.

We define the following quantities, functions of the Riemann initial data

$$R = \varrho_- - \varrho_+ \tag{6.16}$$

$$A = \varrho_- v_- - \varrho_+ v_+ \tag{6.17}$$

$$H = \varrho_- v_-^2 - \varrho_+ v_+^2 + p_- - p_+ \tag{6.18}$$

$$u = v_- - v_+ \tag{6.19}$$

$$B = A^2 - RH = \varrho_- \varrho_+ u^2 - (\varrho_+ - \varrho_-)(p_+ - p_-). \tag{6.20}$$

Since we have six equations for seven unknowns it is reasonable to choose one unknown as a parameter and express other unknowns as functions of Riemann initial data and this parameter. We choose the density in the middle region ϱ_1 as a parameter. Note that we may assume $B > 0$, this can be achieved by taking u large enough as in the assumptions of Theorem 3.6.

6.1 The case $R < 0$

6.1.1 Solution to algebraic equations

Recalling that $R < 0$ means $\varrho_- < \varrho_+$ we express μ_0 and μ_1 from equations (6.6), (6.7), (6.9), (6.10) as

$$\mu_0 = \frac{A}{R} - \frac{1}{R} \sqrt{B \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-}} \tag{6.21}$$

$$\mu_1 = \frac{A}{R} - \frac{1}{R} \sqrt{B \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} \tag{6.22}$$

and we observe that $\mu_0 < \mu_1$ if $\varrho_1 > \varrho_+ > \varrho_-$. In what follows we therefore always assume that the set of possible values of ϱ_1 is $\varrho_1 \in (\varrho_+, \infty)$. There exists also another solution μ_0, μ_1 , which we do not use, see Remark 6.6 at the end of section 6.1.3.

Next we express β in two ways which will be useful later. Using (6.9) and (6.6) respectively we obtain

$$\beta = \frac{\varrho_-}{\varrho_1} v_- + \mu_0 \frac{\varrho_1 - \varrho_-}{\varrho_1}; \tag{6.23}$$

$$\beta = \frac{\varrho_+}{\varrho_1} v_+ + \mu_1 \frac{\varrho_1 - \varrho_+}{\varrho_1}. \tag{6.24}$$

It is technically more difficult to express p_1, ε_1 and ε_2 . We start with rewriting (6.10) as

$$p_1 = p_+ + \varrho_+ v_+^2 - \varrho_1 (\beta^2 + \varepsilon_1) + \mu_1^2 (\varrho_1 - \varrho_+). \tag{6.25}$$

Then, a lengthy yet straightforward computation yields the following equations as a consequence of (6.8), (6.11)

$$\begin{aligned} & (\beta + v_-) \varrho_1 \varepsilon_1 (\varrho_- - \varrho_1) - \varrho_- \varrho_1 (\beta - v_-) (\varepsilon_1 + \varepsilon_2) \\ & = (\beta - v_-) [(\varrho_- - \varrho_1)(p_1 + p_-) + 2c_v \varrho_- p_1 - 2c_v \varrho_1 p_-]; \end{aligned} \tag{6.26}$$

$$\begin{aligned} & -(v_+ + \beta) \varrho_1 \varepsilon_1 (\varrho_1 - \varrho_+) + \varrho_+ \varrho_1 (v_+ - \beta) (\varepsilon_1 + \varepsilon_2) \\ & = (v_+ - \beta) [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v \varrho_1 p_+ - 2c_v \varrho_+ p_1]. \end{aligned} \tag{6.27}$$

At this point we assume that $v_+ - \beta$ and $v_- - \beta$ are nonzero and refer to Lemma 6.1 for proof. We continue by expressing $\varepsilon_1 + \varepsilon_2$ from (6.27) and plugging this to (6.26). We have

$$\varepsilon_1 + \varepsilon_2 = \frac{1}{\varrho_1 \varrho_+} [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v \varrho_1 p_+ - 2c_v \varrho_+ p_1] + \frac{v_+ + \beta}{v_+ - \beta} \varepsilon_1 \frac{\varrho_1 - \varrho_+}{\varrho_+} \quad (6.28)$$

and consequently (6.26) becomes

$$\begin{aligned} & \left(\frac{\beta + v_-}{\beta - v_-} \varrho_1 \varrho_+ (\varrho_- - \varrho_1) - \frac{v_+ + \beta}{v_+ - \beta} \varrho_1 \varrho_- (\varrho_1 - \varrho_+) \right) \varepsilon_1 \\ & = \varrho_+ [(\varrho_- - \varrho_1)(p_1 + p_-) + 2c_v (\varrho_- p_1 - \varrho_1 p_-)] + \varrho_- [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v (\varrho_1 p_+ - \varrho_+ p_1)]. \end{aligned} \quad (6.29)$$

Expressing the right hand side of (6.29) we end up with

$$\begin{aligned} & \left(\frac{\beta + v_-}{\beta - v_-} \varrho_1 \varrho_+ (\varrho_- - \varrho_1) - \frac{v_+ + \beta}{v_+ - \beta} \varrho_1 \varrho_- (\varrho_1 - \varrho_+) \right) \varepsilon_1 \\ & = \varrho_1 p_1 R + \varrho_+ \varrho_- (p_- - p_+) + (2c_v + 1) \varrho_1 (\varrho_- p_+ - \varrho_+ p_-). \end{aligned} \quad (6.30)$$

Once again it is useful to introduce further notation to simplify resulting expressions. We define

$$X = \varrho_+ \varrho_- (p_- - p_+) + (2c_v + 1) \varrho_1 (\varrho_- p_+ - \varrho_+ p_-) \quad (6.31)$$

$$Y = \frac{\beta + v_-}{\beta - v_-} \varrho_+ (\varrho_- - \varrho_1) - \frac{v_+ + \beta}{v_+ - \beta} \varrho_- (\varrho_1 - \varrho_+) \quad (6.32)$$

$$Z = p_+ + \varrho_+ v_+^2 - \varrho_1 \beta^2 + \mu_1^2 (\varrho_1 - \varrho_+). \quad (6.33)$$

This way, assuming $Y \neq 0$ which will be justified in Lemma 6.2, we rewrite (6.30) as

$$\varepsilon_1 = \frac{\varrho_1 p_1 R + X}{\varrho_1 Y} \quad (6.34)$$

and plugging this into (6.25) and assuming $Y + \varrho_1 R \neq 0$ which will be justified in Lemma 6.2, too, we get

$$p_1 = \frac{YZ - X}{Y + \varrho_1 R}, \quad (6.35)$$

this in turn yields

$$\varepsilon_1 = \frac{\varrho_1 RZ + X}{\varrho_1 (Y + \varrho_1 R)} \quad (6.36)$$

and finally ε_2 is expressed using (6.28)

$$\varepsilon_2 = \frac{1}{\varrho_1 \varrho_+} [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v (\varrho_1 p_+ - \varrho_+ p_1)] + \left(\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 \right) \varepsilon_1. \quad (6.37)$$

6.1.2 Positivity of p_1 , ε_1 and ε_2

Before we continue further, we remind the following useful expressions which can be derived from (6.6) and (6.9)

$$\beta - \mu_0 = \frac{\varrho_-}{\varrho_1} (v_- - \mu_0) \quad (6.38)$$

$$v_- - \beta = \frac{\varrho_1 - \varrho_-}{\varrho_1} (v_- - \mu_0) \quad (6.39)$$

$$\mu_1 - \beta = \frac{\varrho_+}{\varrho_1} (\mu_1 - v_+) \quad (6.40)$$

$$\beta - v_+ = \frac{\varrho_1 - \varrho_+}{\varrho_1} (\mu_1 - v_+). \quad (6.41)$$

In particular we observe that the signs of $v_- - \beta$ and $\beta - v_+$ are the same as $v_- - \mu_0$ and $\mu_1 - v_+$ respectively.

Lemma 6.1. For $u = v_- - v_+$ sufficiently large it holds

$$v_- - \mu_0 > 0 \quad \& \quad \mu_1 - v_+ > 0. \quad (6.42)$$

Proof. We have

$$\begin{aligned} v_- - \mu_0 &= -\frac{\varrho_+ u}{R} + \frac{1}{R} \sqrt{(\varrho_- \varrho_+ u^2 - (\varrho_+ - \varrho_-)(p_+ - p_-)) \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-}} \\ &= \frac{\varrho_+ u}{|R|} - \frac{1}{|R|} \sqrt{(\varrho_- \varrho_+ u^2 - (\varrho_+ - \varrho_-)(p_+ - p_-)) \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-}}. \end{aligned} \quad (6.43)$$

Now we distinguish two cases. First, if $p_+ \geq p_-$, then we easily observe (recall $\varrho_+ > \varrho_-$) that

$$v_- - \mu_0 > \frac{1}{|R|} (\varrho_+ u - \sqrt{\varrho_- \varrho_+} u) > 0. \quad (6.44)$$

However if $p_+ < p_-$ we get

$$v_- - \mu_0 > \frac{1}{|R|} (\varrho_+ u - \sqrt{\varrho_- \varrho_+ u^2 + (\varrho_+ - \varrho_-)(p_- - p_+)}) \quad (6.45)$$

and the expression on the right hand side can be made positive assuming u is large enough. Similarly we proceed with quantity $\mu_+ - v_+$. \square

We recall that Lemma 6.1 justifies the assumptions $v_+ - \beta$ and $v_- - \beta$ nonzero we made in the process of deriving (6.35) and (6.36).

We continue by analyzing whether one can find ϱ_1 such that p_1 , ε_1 and ε_2 are all positive. This part starts with rewriting the expression (6.32) for Y . We have

$$\begin{aligned} Y &= \varrho_+ (\varrho_1 - \varrho_-) \frac{v_- + \beta}{v_- - \beta} + \varrho_- (\varrho_1 - \varrho_+) \frac{\beta + v_+}{\beta - v_+} \\ &= \varrho_1 \varrho_+ \frac{v_- + \beta}{v_- - \mu_0} + \varrho_1 \varrho_- \frac{\beta + v_+}{\mu_1 - v_+} \\ &= \varrho_1 \varrho_+ \frac{v_- + \mu_0}{v_- - \mu_0} + \varrho_1 \varrho_- \frac{\mu_1 + v_+}{\mu_1 - v_+}, \end{aligned} \quad (6.46)$$

where the last equality holds although obviously $\beta \notin \{\mu_0, \mu_1\}$. In what follows our strategy is to express certain quantities as functions of ϱ_1 and u while treating ϱ_{\pm} , p_{\pm} and v_{\pm} as data. This way we obtain

$$Y = \varrho_1 R \left[1 + \frac{2}{\sqrt{\frac{B}{\varrho_-^2 u^2} \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} - 1} + 2v_- \left(\frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_+)}{\varrho_+^2 (\varrho_1 - \varrho_-)} - u}} - \frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2 (\varrho_1 - \varrho_+)} - u}} \right) \right]. \quad (6.47)$$

Lemma 6.2. For $u = v_- - v_+$ large enough it holds $Y < 0$ and $Y + \varrho_1 R < 0$ for all $\varrho_1 > \varrho_+$.

Proof. To prove Lemma 6.2 we examine the limit of Y as $u \rightarrow \infty$. First we observe

$$\sqrt{\frac{B}{\varrho_-^2 u^2} \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} \rightarrow \sqrt{\frac{\varrho_+ (\varrho_1 - \varrho_-)}{\varrho_- (\varrho_1 - \varrho_+)}} > 1 \quad (6.48)$$

and therefore the second term in the square brackets of (6.47) is positive. We continue by proving

that the last term converges to zero. We write

$$\begin{aligned}
& 2v_- \left(\frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_+)}{\varrho_+^2(\varrho_1 - \varrho_-)} - u}} - \frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2(\varrho_1 - \varrho_+)} - u}} \right) \\
&= \frac{2v_-}{u} \left(\frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_+)}{u^2 \varrho_+^2(\varrho_1 - \varrho_-)} - 1}} - \frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2 u^2(\varrho_1 - \varrho_+)} - 1}} \right) \\
&= \frac{2v_-}{u} \left(\frac{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2 u^2(\varrho_1 - \varrho_+)} - 1} - \sqrt{\frac{B(\varrho_1 - \varrho_+)}{u^2 \varrho_+^2(\varrho_1 - \varrho_-)} - 1}}{\left(\sqrt{\frac{B(\varrho_1 - \varrho_+)}{u^2 \varrho_+^2(\varrho_1 - \varrho_-)} - 1} \right) \left(\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2 u^2(\varrho_1 - \varrho_+)} - 1} \right)} \right) \tag{6.49}
\end{aligned}$$

and observe that both terms in the numerator of the fraction converge to distinct quantities and the denominator has finite limit as $u \rightarrow \infty$. Therefore the whole fraction has a finite nonzero limit and the whole term therefore can be made arbitrarily small by choosing u sufficiently large.

Altogether we have proved that the expression in the square brackets in (6.47) is positive at least for u sufficiently large and since $R < 0$ we conclude that $Y < 0$ for large u . Also we observe that Y has a finite limit as $u \rightarrow \infty$. The same two properties obviously hold also for $Y + \varrho_1 R$. \square

Remark 6.3. *At this point we emphasize the need to take v_- as fixed and by taking u large enough recover v_+ . This procedure would not work the other way round, i.e. taking v_+ fixed. If we would consider v_+ fixed, the expression (6.47) would become*

$$Y = \varrho_1 R \left[1 + \frac{2}{\sqrt{\frac{B}{\varrho_+^2 u^2} \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-} - 1}} + 2v_+ \left(\frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_+)}{\varrho_+^2(\varrho_1 - \varrho_-)} - u}} - \frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2(\varrho_1 - \varrho_+)} - u}} \right) \right]. \tag{6.50}$$

It is not difficult to figure out that

$$\lim_{u \rightarrow \infty} \left(1 + \frac{2}{\sqrt{\frac{B}{\varrho_+^2 u^2} \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-} - 1}} \right) < 0, \tag{6.51}$$

which yields Y positive and this would cause troubles later when studying the sign of ε_1 .

Now let us turn our attention to expressions X and Z . We recall here the definition (6.31) of X

$$X = \varrho_+ \varrho_- (p_- - p_+) + (2c_v + 1) \varrho_1 (\varrho_- p_+ - \varrho_+ p_-). \tag{6.52}$$

We see that X does not depend on u and may be positive or negative depending on values of ϱ_{\pm}, p_{\pm} and ϱ_1 .

Concerning Z we have from (6.33)

$$\begin{aligned}
Z &= p_+ + \varrho_+ v_+^2 - \varrho_1 \beta^2 + \mu_1^2 (\varrho_1 - \varrho_+) \\
&= p_+ + \frac{(\varrho_1 - \varrho_+) \varrho_+}{\varrho_1} (v_+ - \mu_1)^2 \tag{6.53}
\end{aligned}$$

and we see that Z is always positive. Moreover we plug in the expression (6.22) to obtain

$$\begin{aligned}
Z &= p_+ + \frac{(\varrho_1 - \varrho_+) \varrho_+}{\varrho_1} (v_+ - \mu_1)^2 \\
&= p_+ + \frac{(\varrho_1 - \varrho_+) \varrho_+}{\varrho_1 R^2} \left(\sqrt{B \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} - \varrho_- u \right)^2 \\
&= p_+ + \frac{(\varrho_1 - \varrho_+) \varrho_+ u^2}{\varrho_1 R^2} \left(\sqrt{\left(\varrho_- \varrho_+ + \frac{R(p_+ - p_-)}{u^2} \right) \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} - \varrho_- \right)^2. \tag{6.54}
\end{aligned}$$

Observing that the quantity in parenthesis on the last line of (6.54) has finite nonzero limit as $u \rightarrow \infty$ we conclude that Z grows as u^2 .

We are now ready to study the signs of p_1 and ε_1 .

Lemma 6.4. *For $u = v_- - v_+$ large enough it holds $p_1 > 0$ and $\varepsilon_1 > 0$ for all $\varrho_1 > \varrho_+$.*

Proof. Since we have from (6.35)

$$p_1 = \frac{YZ - X}{Y + \varrho_1 R}, \quad (6.55)$$

we easily see that for large u the leading term is $\frac{YZ}{Y + \varrho_1 R}$ which grows like u^2 , whereas $\frac{X}{Y + \varrho_1 R}$ has a finite limit as $u \rightarrow \infty$. Since both Y and $Y + \varrho_1 R$ are negative at least for large u and $Z > 0$, we conclude that $p_1 > 0$ for u large enough.

Similarly we have from (6.36)

$$\varepsilon_1 = \frac{\varrho_1 RZ + X}{\varrho_1(Y + \varrho_1 R)} \quad (6.56)$$

and using similar arguments as above for p_1 we see that the term $\frac{RZ}{Y + \varrho_1 R}$ grows like u^2 and is positive whereas $\frac{X}{\varrho_1(Y + \varrho_1 R)}$ has finite limit as $u \rightarrow \infty$ and therefore $\varepsilon_1 > 0$ for u large enough. \square

Next we study the sign of ε_2 . We have

Lemma 6.5. *For*

$$\rho_1 > \frac{(2c_v + 1)\varrho_- + \varrho_+}{2} + \frac{\sqrt{\varrho_+^2 + (4c_v^2 - 1)\varrho_+\varrho_-}}{2} \quad (6.57)$$

and $u = v_- - v_+$ large enough it holds $\varepsilon_2 > 0$.

Proof. We start from (6.37)

$$\begin{aligned} \varepsilon_2 &= \frac{1}{\varrho_1 \varrho_+} [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v (\varrho_1 p_+ - \varrho_+ p_1)] + \varepsilon_1 \left(\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 \right) \\ &= \frac{p_+ ((2c_v + 1)\varrho_1 - \varrho_+)}{\varrho_1 \varrho_+} + \frac{p_1 (\varrho_1 - (2c_v + 1)\varrho_+)}{\varrho_1 \varrho_+} \\ &\quad + \varepsilon_1 \left(-\frac{\varrho_1}{\varrho_+} + \frac{2\varrho_1 R}{\varrho_+ \left(\varrho_- - \sqrt{\frac{B(\varrho_1 - \varrho_-)}{u^2(\varrho_1 - \varrho_+)}} \right)} + \frac{2\varrho_1 R v_-}{\varrho_+ \left(\sqrt{B \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} - \varrho_- u \right)} \right). \end{aligned} \quad (6.58)$$

Keeping in mind that both p_1 and ε_1 grow as u^2 we identify the leading terms of ε_2 as $u \rightarrow \infty$. We have

$$\begin{aligned} \varepsilon_2 &= \frac{YZ}{Y + \varrho_1 R} \frac{\varrho_1 - (2c_v + 1)\varrho_+}{\varrho_1 \varrho_+} + \frac{RZ}{Y + \varrho_1 R} \left(-\frac{\varrho_1}{\varrho_+} + \frac{2\varrho_1 R}{\varrho_+ \left(\varrho_- - \sqrt{\frac{B(\varrho_1 - \varrho_-)}{u^2(\varrho_1 - \varrho_+)}} \right)} \right) + l.o.t. \\ &= \frac{Z}{\varrho_+(Y + \varrho_1 R)} \left(\frac{Y(\varrho_1 - (2c_v + 1)\varrho_+)}{\varrho_1} + R \left(-\varrho_1 + \frac{2\varrho_1 R}{\varrho_+ \left(\varrho_- - \sqrt{\frac{B(\varrho_1 - \varrho_-)}{u^2(\varrho_1 - \varrho_+)}} \right)} \right) \right) + l.o.t. \\ &=: \frac{Z}{\varrho_+(Y + \varrho_1 R)} E + l.o.t., \end{aligned} \quad (6.59)$$

where *l.o.t.* stands for lower order terms with respect to u^2 . We know that $\frac{Z}{\varrho_+(Y + \varrho_1 R)}$ is negative so we continue by studying the remaining part E of the leading term of ε_2 and to make $\varepsilon_2 > 0$ we want to show that $E < 0$.

Recall that from (6.47)-(6.49) we already know that

$$\lim_{u \rightarrow \infty} Y = \varrho_1 R \left(1 + \frac{2}{\sqrt{\frac{\varrho_+ (\varrho_1 - \varrho_-)}{\varrho_- (\varrho_1 - \varrho_+)}} - 1} \right) \quad (6.60)$$

and therefore a straightforward calculation yields

$$\lim_{u \rightarrow \infty} E = R \left[-(2c_v + 1)\varrho_+ + \left(\frac{2\varrho_+ \varrho_1}{\varrho_-} - 2(2c_v + 1)\varrho_+ \right) \frac{1}{\sqrt{\frac{\varrho_+ (\varrho_1 - \varrho_-)}{\varrho_- (\varrho_1 - \varrho_+)}} - 1} \right] \quad (6.61)$$

Since we have $R < 0$, we want the term in square parenthesis of (6.61) to be positive. It is not difficult to show that this can be done by choosing ϱ_1 large enough with respect to ϱ_{\pm} , more specifically if

$$\varrho_1 > \frac{(2c_v + 1)\varrho_- + \varrho_+}{2} + \frac{\sqrt{\varrho_+^2 + (4c_v^2 - 1)\varrho_+ \varrho_-}}{2} \quad (6.62)$$

then $\lim_{u \rightarrow \infty} E < 0$. □

6.1.3 Admissibility inequalities

Finally we have to check whether the admissibility inequalities (6.14)-(6.15) are satisfied. We start by plugging the expression (6.23) into (6.14) and we obtain

$$(v_- - \mu_0)(\varrho_- L_1 - \varrho_1 L_-) \geq 0. \quad (6.63)$$

We already know that $v_- - \mu_0 > 0$ at least for u large enough, see Lemma 6.1. Therefore we need to ensure that

$$\varrho_- L_1 \geq \varrho_1 L_-. \quad (6.64)$$

Using the definition of L_i the inequality (6.64) is equivalent to

$$\frac{p_1^{c_v}}{\varrho_1^{c_v+1}} \geq \frac{p_-^{c_v}}{\varrho_-^{c_v+1}}. \quad (6.65)$$

We proceed similarly with (6.15) where we plug in the expression (6.24) for β to get

$$(\mu_1 - v_+)(\varrho_+ L_1 - \varrho_1 L_+) \geq 0. \quad (6.66)$$

Again by Lemma 6.1 we know that $\mu_1 - v_+ > 0$ at least for u large enough and thus we need

$$\varrho_+ L_1 \geq \varrho_1 L_+, \quad (6.67)$$

which is equivalent to

$$\frac{p_1^{c_v}}{\varrho_1^{c_v+1}} \geq \frac{p_+^{c_v}}{\varrho_+^{c_v+1}}. \quad (6.68)$$

Combining (6.65) and (6.68) we end up with

$$p_1^{c_v} \geq \varrho_1^{c_v+1} \max \left\{ \frac{p_-^{c_v}}{\varrho_-^{c_v+1}}, \frac{p_+^{c_v}}{\varrho_+^{c_v+1}} \right\}. \quad (6.69)$$

However we already know that p_1 grows as u^2 , so regardless of the choice of ϱ_1 we can always ensure that (6.69) is satisfied by choosing u large enough.

We have showed that for any ϱ_1 satisfying (6.57) there exists some $\bar{u}(\varrho_1)$ such that if $u > \bar{u}(\varrho_1)$ then we can construct a 1-fan subsolution. Then the point (i) of Theorem 3.6 in the case $R < 0$ is proved by defining

$$U = U(\varrho_{\pm}, p_{\pm}, v_-) := v_- - \inf_{\rho_1} \bar{u}(\rho_1), \quad (6.70)$$

where the inf is taken among ρ_1 such that (6.57) holds.

Remark 6.6. *There exists also another solution to equations (6.6), (6.7), (6.9), (6.10), namely*

$$\mu_0 = \frac{A}{R} + \frac{1}{R} \sqrt{B \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-}} \quad (6.71)$$

$$\mu_1 = \frac{A}{R} + \frac{1}{R} \sqrt{B \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} \quad (6.72)$$

and here it holds $\mu_0 < \mu_1$ for $\varrho_1 < \varrho_- < \varrho_+$. However this solution is not convenient to work with, since it violates Lemma 6.1. More precisely, one of the inequalities (6.42) holds with the opposite sign. This causes troubles in the analysis of the admissibility inequalities (6.14)–(6.15), because these then yield one upper bound and one lower bound for p_1 instead of two lower bounds. Then naturally the argument with taking u sufficiently large fails.

6.2 The case $R > 0$

The proof in the case $R > 0$ follows the same steps. The expressions (6.21)–(6.22) for μ_0, μ_1 stay the same and again we see that $\mu_0 < \mu_1$ if $\varrho_1 > \varrho_- > \varrho_+$. Since we treat v_+ as fixed in this case, instead of (6.47) we get

$$Y = \varrho_1 R \left[1 + \frac{2}{\sqrt{\frac{B}{\varrho_+^2 u^2} \frac{\varrho_1 - \varrho_+}{\varrho_1 - \varrho_-}} - 1} + 2v_+ \left(\frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_+)}{\varrho_+^2 (\varrho_1 - \varrho_-)} - u}} - \frac{1}{\sqrt{\frac{B(\varrho_1 - \varrho_-)}{\varrho_-^2 (\varrho_1 - \varrho_+)} - u}} \right) \right]. \quad (6.73)$$

Similarly as in the case $R < 0$ we show that the term in the square brackets is positive at least for u sufficiently large, so in this case we conclude that $Y > 0$ and obviously also $Y + \varrho_1 R > 0$. Since nothing changes in terms X and Z , this yields that p_1 and ε_1 are both positive at least for u sufficiently large.

The analysis of ε_2 changes again with respect to the fact that v_+ is now fixed instead of v_- . Instead of (6.58) we obtain

$$\varepsilon_2 = \frac{p_+((2c_v + 1)\varrho_1 - \varrho_+)}{\varrho_1 \varrho_+} + \frac{p_1(\varrho_1 - (2c_v + 1)\varrho_+)}{\varrho_1 \varrho_+} + \varepsilon_1 \left(-\frac{\varrho_1}{\varrho_+} + \frac{2\varrho_1 R v_+}{\varrho_+ \left(\sqrt{B \frac{\varrho_1 - \varrho_-}{\varrho_1 - \varrho_+}} - \varrho_- u \right)} \right) \quad (6.74)$$

and hence

$$\varepsilon_2 = \frac{Z}{\varrho_+(Y + \varrho_1 R)} \left(\frac{Y(\varrho_1 - (2c_v + 1)\varrho_+)}{\varrho_1} - \varrho_1 R \right) + l.o.t. =: \frac{Z}{\varrho_+(Y + \varrho_1 R)} \tilde{E} + l.o.t.. \quad (6.75)$$

Since the limit of Y as $u \rightarrow \infty$ changes to

$$\lim_{u \rightarrow \infty} Y = \varrho_1 R \left(1 + \frac{2}{\sqrt{\frac{\varrho_- (\varrho_1 - \varrho_+)}{\varrho_+ (\varrho_1 - \varrho_-)} - 1}} \right), \quad (6.76)$$

a direct calculation yields

$$\lim_{u \rightarrow \infty} \tilde{E} = R \left[-(2c_v + 1)\varrho_+ + \frac{2\varrho_1 - 2(2c_v + 1)\varrho_+}{\sqrt{\frac{\varrho_- (\varrho_1 - \varrho_+)}{\varrho_+ (\varrho_1 - \varrho_-)} - 1}} \right]. \quad (6.77)$$

It is not difficult to obtain that $\lim_{u \rightarrow \infty} \tilde{E} > 0$ if

$$\varrho_1 > \frac{\varrho_- + (2c_v + 1)\varrho_+}{2} + \frac{\sqrt{\varrho_-^2 + (4c_v^2 - 1)\varrho_+ \varrho_-}}{2} \quad (6.78)$$

and thus choosing such ϱ_1 we always find u large enough such that $\varepsilon_2 > 0$. Since nothing changes regarding the admissibility inequalities, we conclude the proof of point (ii) of Theorem 3.6 in the case $R > 0$ by the same arguments as in the case of point (i) in the case $R < 0$.

6.3 The case $R = 0$

6.3.1 Solution to algebraic equations

This case has to be treated separately since even the expression for the speeds of interfaces μ_0, μ_1 (6.21)-(6.22) are considerably different. We have

$$\mu_0 = \frac{1}{2} \left(-\frac{\varrho_+ u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u} + v_- + v_+ \right) \quad (6.79)$$

$$\mu_1 = \frac{1}{2} \left(\frac{\varrho_+ u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u} + v_- + v_+ \right) \quad (6.80)$$

and we immediately see that since we examine the case where $u > 0$ we are forced to assume $\varrho_1 > \varrho_+ = \varrho_-$ in order to ensure $\mu_0 < \mu_1$.

The relations (6.23)-(6.29) stay the same in the case $R = 0$, however the expressions for quantities X and Y simplify a little bit to

$$X = \varrho_+^2 (p_- - p_+) + (2c_v + 1) \varrho_1 \varrho_+ (p_+ - p_-) = \varrho_+ (p_+ - p_-) ((2c_v + 1) \varrho_1 - \varrho_+) \quad (6.81)$$

$$Y = \varrho_+ (\varrho_1 - \varrho_+) \left(\frac{v_- + \beta}{v_- - \beta} + \frac{\beta + v_+}{\beta - v_+} \right) \quad (6.82)$$

and consequently we have

$$\varepsilon_1 = \frac{X}{\varrho_1 Y} \quad (6.83)$$

and

$$p_1 = Z - \frac{X}{Y}, \quad (6.84)$$

with Z defined as in (6.33). Finally, ε_2 is given by (6.37).

6.3.2 Positivity of p_1, ε_1 and ε_2

We claim that Lemma 6.1 holds also in the case $R = 0$. Indeed, it is even easier here to see that

$$v_- - \mu_0 = \frac{1}{2} \left(\frac{\varrho_1 u}{\varrho_1 - \varrho_+} - \frac{p_- - p_+}{\varrho_+ u} \right) \quad (6.85)$$

$$\mu_1 - v_+ = \frac{1}{2} \left(\frac{\varrho_1 u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u} \right) \quad (6.86)$$

and both expressions on the right hand sides are positive at least for u large enough.

Next, we observe that the sign of X depends on the sign of $p_+ - p_-$, namely

$$\text{sign } X = \text{sign } (p_+ - p_-) \quad (6.87)$$

and in particular this shows that this presented construction simply does not work in the case $\varrho_- = \varrho_+$ and $p_- = p_+$, because in this case we end up with $\varepsilon_1 = 0$. Also, knowing the sign of X and having in mind the expression (6.83) for ε_1 we see that in order to ensure $\varepsilon_1 > 0$ we need to have

$$\text{sign } Y = \text{sign } X = \text{sign } (p_+ - p_-). \quad (6.88)$$

Therefore we study now the sign of Y in a similar manner as in the proof of Lemma 6.2. We start this part by claiming that the expression (6.46) still holds and can be written as

$$Y = \varrho_1 \varrho_+ \left(\frac{v_- + \mu_0}{v_- - \mu_0} + \frac{\mu_1 + v_+}{\mu_1 - v_+} \right). \quad (6.89)$$

Using (6.79), (6.80), keeping v_- fixed and expressing $v_+ = v_- - u$ we end up after a simple calculation with

$$Y = 4\varrho_1\varrho_+ \left(\frac{2v_-}{\frac{\varrho_1 u}{\varrho_1 - \varrho_+} - \frac{(\varrho_1 - \varrho_+)(p_- - p_+)^2}{\varrho_1 \varrho_+^2 u^3}} - \frac{1}{\frac{\varrho_1}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u^2}} \right). \quad (6.90)$$

We observe that in the limit $u \rightarrow \infty$ we get $Y \rightarrow -4\varrho_+(\varrho_1 - \varrho_+) < 0$. Therefore we conclude that $\varepsilon_1 > 0$ for u large enough if we keep v_- fixed and if $p_- > p_+$.

Similarly if we keep v_+ fixed, express $v_- = v_+ + u$ we end up instead of (6.90) with

$$Y = 4\varrho_1\varrho_+ \left(\frac{2v_+}{\frac{\varrho_1 u}{\varrho_1 - \varrho_+} - \frac{(\varrho_1 - \varrho_+)(p_- - p_+)^2}{\varrho_1 \varrho_+^2 u^3}} + \frac{1}{\frac{\varrho_1}{\varrho_1 - \varrho_+} - \frac{p_- - p_+}{\varrho_+ u^2}} \right) \quad (6.91)$$

and in the limit $u \rightarrow \infty$ we get $Y \rightarrow 4\varrho_+(\varrho_1 - \varrho_+) > 0$. Therefore we conclude that $\varepsilon_1 > 0$ for u large enough if we keep v_+ fixed and if $p_- < p_+$.

Concerning the sign of p_1 we have similarly as in (6.54)

$$Z = \frac{\varrho_1\varrho_+u^2}{4(\varrho_1 - \varrho_+)} + \frac{p_- + p_+}{2} + \frac{(\varrho_1 - \varrho_+)(p_- - p_+)^2}{4\varrho_1\varrho_+u^2}, \quad (6.92)$$

thus we see that $Z > 0$ and $Z \sim u^2$ as $u \rightarrow \infty$. Therefore using (6.84) we conclude $p_1 > 0$.

Remark 6.7. *From the considerations above it is also clear why we do not state anything about the case $\varrho_- = \varrho_+$ and $p_- = p_+$ in Theorem 3.6. In this case $X = 0$ and the only way how one could choose $\varepsilon_1 > 0$ and satisfy (6.30) would be to have also $Y = 0$. However, even if $p_- = p_+$ we observe that Y has nonzero finite limit as $u \rightarrow \infty$ and hence this construction fails in this case.*

We finish this section with the study of the sign of ε_2 .

Lemma 6.8. *For*

$$\rho_1 > (2c_v + 1)\varrho_+ \quad (6.93)$$

and $u = v_- - v_+$ large enough it holds $\varepsilon_2 > 0$.

Proof. Recalling (6.37) we have

$$\begin{aligned} \varepsilon_2 &= \frac{1}{\varrho_1\varrho_+} [(\varrho_1 - \varrho_+)(p_1 + p_+) + 2c_v(\varrho_1 p_+ - \varrho_+ p_1)] + \varepsilon_1 \left(\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 \right) \\ &= p_1 \frac{\varrho_1 - (1 + 2c_v)\varrho_+}{\varrho_1\varrho_+} + p_+ \frac{(2c_v + 1)\varrho_1 - \varrho_+}{\varrho_1\varrho_+} + \varepsilon_1 \left(\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 \right). \end{aligned} \quad (6.94)$$

We already know, that p_1 grows as u^2 as $u \rightarrow \infty$. The second term on the right hand side of (6.94) does not depend on u at all. Concerning the third term, we know that ε_1 has a finite limit as $u \rightarrow \infty$ and now we are going to show that the same property holds also for the term $\left(\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 \right)$. Indeed, by a simple calculation using (6.24) we have

$$\frac{v_+ + \beta}{v_+ - \beta} \frac{\varrho_1 - \varrho_+}{\varrho_+} - 1 = \frac{\varrho_1}{\varrho_+} \frac{v_+ + \mu_1}{v_+ - \mu_1}. \quad (6.95)$$

Plugging in (6.80) we obtain

$$\begin{aligned} \frac{\varrho_1}{\varrho_+} \frac{v_+ + \mu_1}{v_+ - \mu_1} &= \frac{\varrho_1}{\varrho_+} \left(-\frac{3v_+ + v_-}{\frac{\varrho_1 u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u}} - 1 \right) \\ &= \frac{\varrho_1}{\varrho_+} \left(-\frac{4v_+ + u}{\frac{\varrho_1 u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u}} - 1 \right) \end{aligned} \quad (6.96)$$

$$= \frac{\varrho_1}{\varrho_+} \left(-\frac{4v_- - 3u}{\frac{\varrho_1 u}{\varrho_1 - \varrho_+} + \frac{p_- - p_+}{\varrho_+ u}} - 1 \right) \quad (6.97)$$

If we assume v_- is fixed we see from (6.97) that the expression on the left hand side has a finite nonzero limit as $u \rightarrow \infty$, whereas if we assume v_+ is fixed, we conclude the same using (6.96).

Altogether we have that the leading term in the expression (6.94) as $u \rightarrow \infty$ is $p_1 \frac{\varrho_1 - (1+2c_v)\varrho_+}{\varrho_1 \varrho_+}$. In particular if we assume $\varrho_1 > (2c_v + 1)\varrho_+$, then $\varepsilon_2 > 0$ provided u is taken large enough. \square

6.3.3 Admissibility inequalities

Nothing changes in the analysis of the admissibility inequalities (6.14)–(6.15) with respect to the case $R \neq 0$ and we can follow word by word section 6.1.3, in particular also here when we assume $R = 0$ we have

$$p_1^{c_v} \geq \varrho_1^{c_v+1} \max \left\{ \frac{p_-^{c_v}}{\varrho_-^{c_v+1}}, \frac{p_+^{c_v}}{\varrho_+^{c_v+1}} \right\}. \quad (6.98)$$

and we know that p_1 grows as u^2 , so we can choose u large enough in order to (6.98) to be satisfied.

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