Lower bounds on eigenvalues of symmetric elliptic operators

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Outline

Model problem

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\
u_i &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Lower bounds on eigenvalues:

$$? \leq \lambda_i \leq \Lambda_{h,i}$$

- Motivation
- Classical Weinstein's lower bound
- Method 1: Weinstein's bound in weak setting
- Method 2: Kato's bound in weak setting
- Method 3: Lehmann–Goerisch method
- Method 4: Crouzeix–Raviart elements based lower bounds
- Numerical comparison

Goal

Solve problems

- reliably – with guaranteed accuracy
- efficiently – as fast as possible

Example – dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
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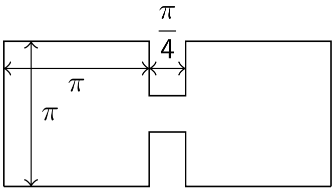

[Trefethen, Betcke 2006]

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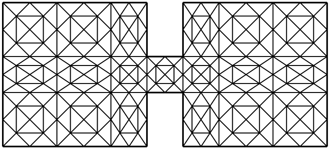
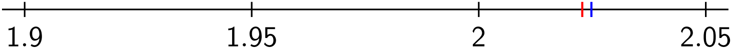
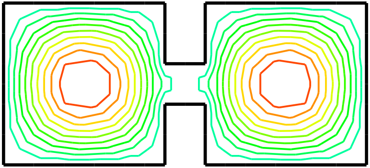
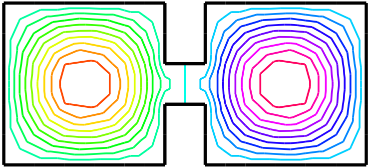




$\lambda_1 \approx 2.02280$ $\lambda_2 \approx 2.02481$





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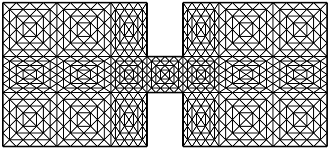

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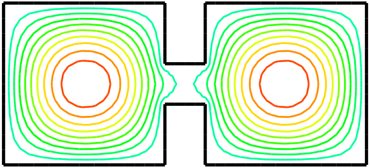

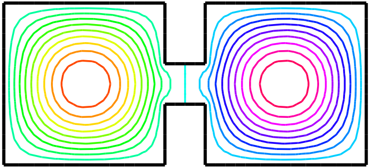

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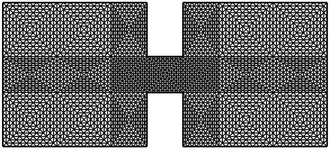

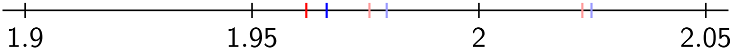

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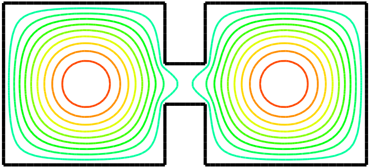
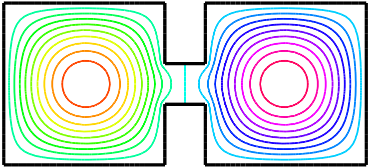
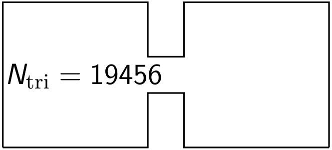
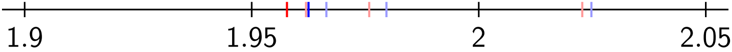




Example – dumbbell

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \partial\Omega \end{aligned}$$





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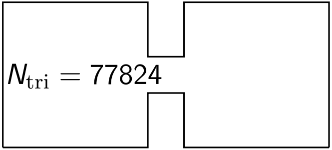

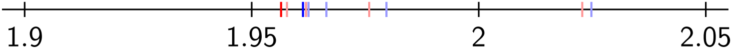

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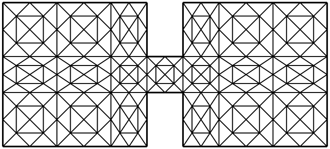
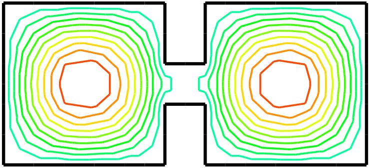
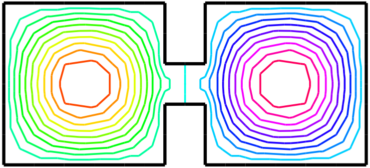


$$1.91067 \leq \lambda_1 \leq 2.02280 \qquad 1.91981 \leq \lambda_2 \leq 2.02481$$





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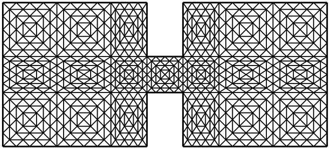

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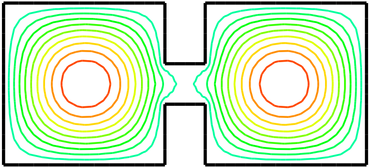

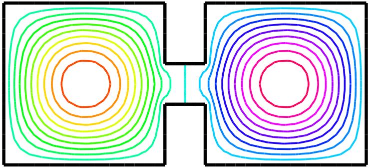

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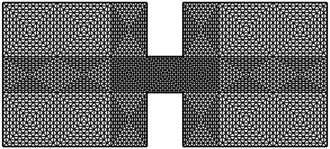

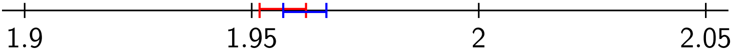

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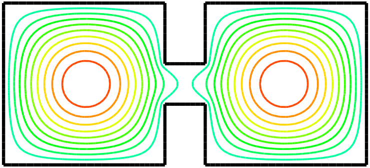

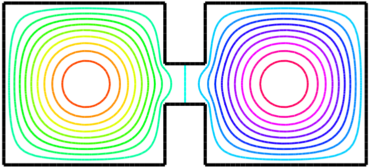

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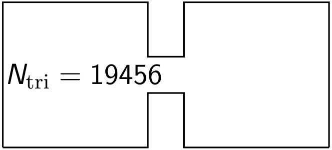

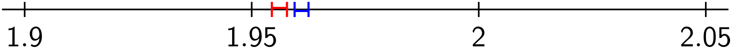

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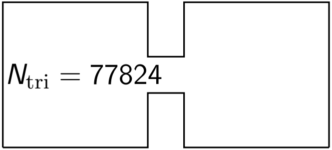

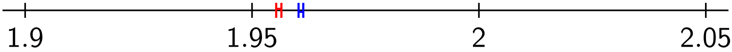

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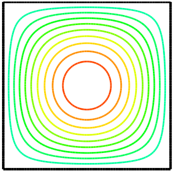
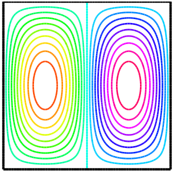
$1.95532 \le \lambda_1 \le 1.95646$	$1.96025 \le \lambda_2 \le 1.96129$

Example: Square

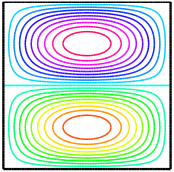
$\lambda_1 = 2, \; u_1(x,y) = \sin(x)\sin(y)$

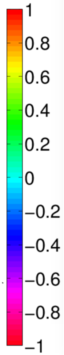


$\lambda_2 = 5, \; u_2(x,y) = \sin(2x)\sin(y)$



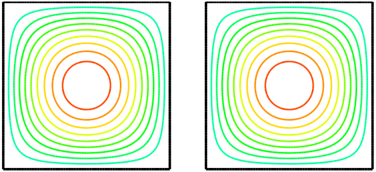
$\lambda_3 = 5, \; u_3(x,y) = \sin(x)\sin(2y)$



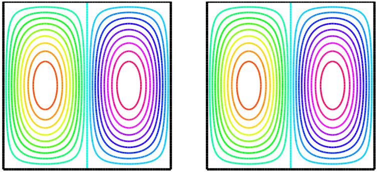


Example: Two squares

$\lambda_1 = 2$



$\lambda_2 = 5$



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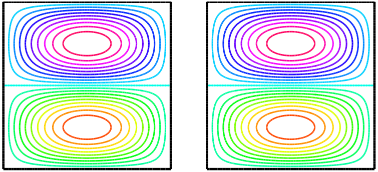

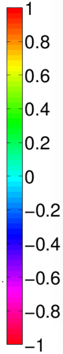

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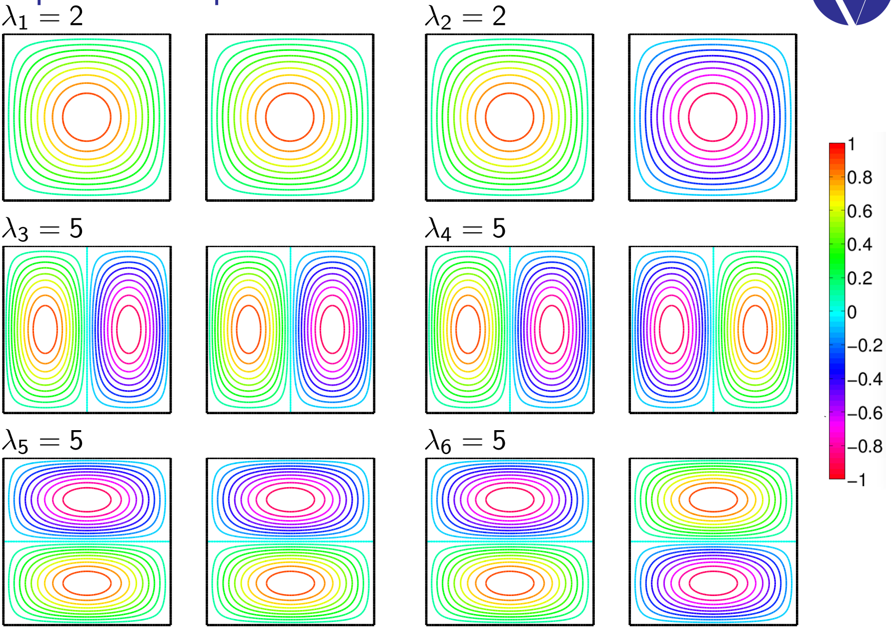
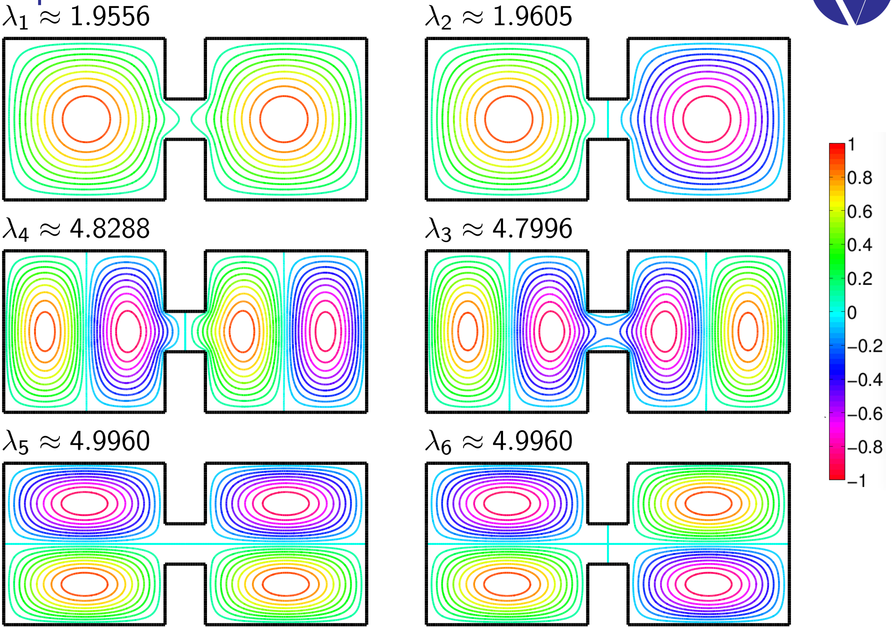


Example: Dumbbell



Lower bounds on eigenvalues

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, 1950, ...

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, C. Carstensen, R.G. Duran, D. Galistl, J. Gedicke, F. Goerisch, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, Xuefeng Liu, M. Plum, S.I. Repin, V.G. Sigillito, Hehu Xie, Yidu Yang, Zhimin Zhang, ... *many others*

Weinstein's bounds

Eigenvalue problem: Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$Au_i = \lambda_i u_i$$

Setting:

- V ... Hilbert space
- $A : D(A) \to V$ linear, symmetric operator
- $\{u_i\}$ form orthonormal basis in V
- $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$

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Theorem 1 (Weinstein 1937):

- Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $\varepsilon = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
- Let $\dfrac{\lambda_{n-1} + \lambda_n}{2} \leq \lambda_* \leq \dfrac{\lambda_n + \lambda_{n+1}}{2}$ for some n.

Then $\lambda_* - \varepsilon \leq \lambda_n$.

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Then $\lambda_* - \varepsilon \leq \lambda_n$.

Proof: $\|Au_* - \lambda_* u_*\|^2 = \sum_{j=1}^{\infty} \langle Au_* - \lambda_* u_*, u_j \rangle^2$

$$= \sum_{j=1}^{\infty} |\lambda_j - \lambda_*|^2 \langle u_*, u_j \rangle^2 \geq \min_j |\lambda_j - \lambda_*|^2 \|u_*\|^2$$

Thus, $|\lambda_n - \lambda_*| = \min_j |\lambda_j - \lambda_*| \leq \dfrac{\|Au_* - \lambda_* u_*\|}{\|u_*\|} = \varepsilon.$ □

Weak form

Eigenvalue problem: Find $u_i \in V \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Setting:

- V is a Hilbert space
- $a(\cdot,\cdot)$ is a symmetric, continuous, V-elliptic bilinear form
- $b(\cdot,\cdot)$ is a symmetric, continuous, positive semidefinite bilinear form
- $\{u_i\}$ form orthonormal basis in V, i.e. $b(u_i, u_j) = \delta_{ij}$
- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

Example:

- $a(u, v) = (\nabla u, \nabla v)$
- $b(u, v) = (u, v)$

Method 1: Weinstein's bound in the weak form

Theorem 2:

- Let $u_* \in V \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

- Let $\|w\|_a \leq \eta$.
- Let $\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+1}}$

Then

$$\ell_n^{\mathrm{W}} \leq \lambda_n, \quad \text{where } \ell_n^{\mathrm{W}} = \frac{1}{4|u_*|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_*|u_*|_b^2}\right)^2.$$

[Vejchodský, Šebestová 2017]

Method 2: Kato's bound in the weak form

Theorem 3:

- Let $1 \le n \le s$.
- Let $u_{*,i} \in V$ and $\lambda_{*,i} \in \mathbb{R}$, $i = n, \dots, s$, satisfy

$$a(u_{*,i}, v_*) = \lambda_{*,i} b(u_{*,i}, v_*) \quad \forall v_* \in V_*, \quad |u_{*,i}|_b = 1,$$

  where $V_* = \operatorname{span}\{u_{*,i},\ i = n, \dots, s\}$.
- Let $w_i \in V$, $i = n, \dots, s$, be given by

$$a(w_i, v) = a(u_{*,i}, v) - \lambda_{*,i} b(u_{*,i}, v) \quad \forall v \in V.$$

- Let $\|w_i\|_a \le \eta_i$ for all $i = n, \dots, s$.
- Let $\lambda_{s-1} \le \lambda_{*,s} < \nu \le \lambda_{s+1}$.

Then

$$\ell_n^{\mathrm{K}} \le \lambda_n, \quad \text{where } \ell_n^{\mathrm{K}} = \lambda_{*,n} \left(1 + \nu \lambda_{*,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right)^{-1}.$$

[Vejchodský, Šebestová 2017]

Complementary upper bound on the residual

Theorem 4:

- Let $V = H_0^1(\Omega)$, $a(u,v) = (\nabla u, \nabla v)$, and $b(u,v) = (u,v)$.
- Let $u_* \in V$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $w \in V$ satisfy

$$a(w,v) = a(u_*,v) - \lambda_* b(u_*,v) \quad \forall v \in V.$$

- Let $\mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega)$ be such that $-\operatorname{div}\mathbf{q} = \lambda_* u_*$.

Then

$$\|\nabla w\|_{L^2(\Omega)} \leq \eta = \|\nabla u_* - \mathbf{q}\|_{L^2(\Omega)}.$$

[Synge 1957], [Haslinger, Hlaváček 1976], [Křížek, Hlaváček 1984], [Neittaanmäki, Repin 2004], [Braess 2007], ...

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h$, $\|u_{h,n}\|_{L^2(\Omega)} = 1$, $n = r, \dots, s$
- Flux reconstruction: $\mathbf{q}_{h,n} = \sum_{\boldsymbol{z} \in \mathcal{N}_h} \mathbf{q}_{\boldsymbol{z},n}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\boldsymbol{z},n} \in \boldsymbol{W}_{\boldsymbol{z}}$, $d_{\boldsymbol{z},n} \in P_1^*(\mathcal{T}_{\boldsymbol{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\boldsymbol{z},n}, \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} - (d_{\boldsymbol{z},n}, \operatorname{div} \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} &= (\psi_{\boldsymbol{z}} \nabla u_{h,n}, \mathbf{w}_h)_{\omega_{\boldsymbol{z}}} \quad && \forall \mathbf{w}_h \in \boldsymbol{W}_{\boldsymbol{z}} \\
-(\operatorname{div} \mathbf{q}_{\boldsymbol{z},n}, \varphi_h)_{\omega_{\boldsymbol{z}}} &= (r_{\boldsymbol{z},n}, \varphi_h)_{\omega_{\boldsymbol{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\boldsymbol{z}})
\end{aligned}
$$

where

- $\omega_{\boldsymbol{z}}$ is the patch of elements around vertex $\boldsymbol{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\boldsymbol{z}}$ is the set of elements in $\omega_{\boldsymbol{z}}$
- $\boldsymbol{W}_{\boldsymbol{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\boldsymbol{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\boldsymbol{z}}$ and $\mathbf{w}_h \cdot \boldsymbol{n}_{\omega_{\boldsymbol{z}}} = 0$ on $\Gamma^{\text{ext}}_{\omega_{\boldsymbol{z}}}\}$
- $P_1^*(\mathcal{T}_{\boldsymbol{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\boldsymbol{z}}) : \int_{\omega_{\boldsymbol{z}}} v_h \,\mathrm{d}x = 0\} & \text{for } \boldsymbol{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\boldsymbol{z}}) & \text{for } \boldsymbol{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\boldsymbol{z},n} = \Lambda_{h,n} \psi_{\boldsymbol{z}} u_{h,n} - \nabla \psi_{\boldsymbol{z}} \cdot \nabla u_{h,n}$

Flux reconstruction

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- Error estimator: $\eta_n = \|\nabla u_{h,n} - \mathbf{q}_{h,n}\|_{L^2(\Omega)}$
- Weinstein's bound: $\ell_n^{\mathrm{W}} = \left(-\eta_n + \sqrt{\eta_n^2 + 4\Lambda_{h,n}}\right)^2 / 4$

  provided $\Lambda_{h,n} \leq \sqrt{\lambda_n \lambda_{n+1}}$.
- Kato's bound: $\ell_n^{\mathrm{K}} = \Lambda_{h,n} \left(1 + \nu \Lambda_{h,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\Lambda_{h,i}^2 (\nu - \Lambda_{h,i})}\right)^{-1}$

  provided $\Lambda_{h,s} < \nu \leq \lambda_{s+1}$.

How to get ν?

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\ u_i &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Theorem: If $\Omega \subset \mathcal{L}$ then $\lambda_n^{(\mathcal{L})} \leq \lambda_n^{(\Omega)}$.

Proof:

- $H_0^1(\Omega) \subset H_0^1(\mathcal{L})$
- $\lambda_1^{(\mathcal{L})} = \min_{v \in H_0^1(\mathcal{L})} \frac{(\nabla v, \nabla v)_{\mathcal{L}}}{(v, v)_{\mathcal{L}}} \leq \min_{v \in H_0^1(\Omega)} \frac{(\nabla v, \nabla v)_{\Omega}}{(v, v)_{\Omega}} = \lambda_1^{(\Omega)}$
- Use Courant minimax principle for λ_n.

$\square$

How to get ν?

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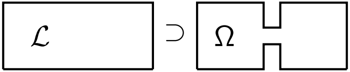
Theorem: If $\Omega \subset \mathcal{L}$ then $\lambda_n^{(\mathcal{L})} \leq \lambda_n^{(\Omega)}$.

Example: 10 eigenvalues in the dumbbell



$\Longrightarrow \quad 8.93827 \leq \lambda_{11}$

But $\lambda_8 \approx 7.986$, $\lambda_9 \approx 9.353$, $\lambda_{10} \approx 9.510$, $\lambda_{11} \approx 9.998$.

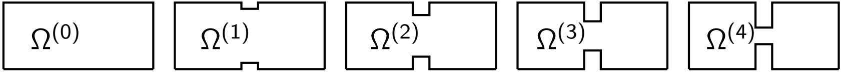
Homotopy method

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

- Let $\Omega = \Omega^{(m)} \subset \Omega^{(m-1)} \subset \cdots \subset \Omega^{(1)} \subset \Omega^{(0)}$.
- Let exact eigenvalues are known on $\Omega^{(0)}$.
- Theorem $\Rightarrow \lambda_n^{(k-1)} \leq \lambda_n^{(k)}$, $k = 1, 2, \ldots, m$.

[Plum 1990, 1991]

Example:



Analytically:	$\nu = 12.16$	$\nu = 11.39$	$\nu = 10.77$	$\nu = 9.988$
$12.16 \leq \lambda_{17}^{(0)}$	$\ell_{15}^{\mathrm{K}} \doteq 11.39$	$\ell_{13}^{\mathrm{K}} \doteq 10.77$	$\ell_{11}^{\mathrm{K}} \doteq 9.988$	

Method 3. Lehmann–Goerisch

Input: A priori lower bound: $\nu \le \lambda_{s+1}$

Algorithm:

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, s$
- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \boldsymbol{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \dots, s$
  $\boldsymbol{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\}$
  $Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in P_k(K) \quad \forall K \in \mathcal{T}_h\}$

$$\begin{aligned} (\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + (q_{h,i}, \operatorname{div} \mathbf{w}_h) &= 0 && \forall \mathbf{w}_h \in \boldsymbol{W}_h, \\ (\operatorname{div} \boldsymbol{\sigma}_{h,i}, \varphi_h) &= (-u_{h,i}, \varphi_h) && \forall \varphi_h \in Q_h, \end{aligned}$$

[Behnke, Mertins, Plum, Wieners 2000]

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Algorithm:

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- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \boldsymbol{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \dots, s$
- Set:

$$\gamma = \|u_{h,s} + \operatorname{div} \boldsymbol{\sigma}_{h,s}\|_{L^2(\Omega)}$$
$$\rho = \nu + \gamma$$
$$\boldsymbol{M}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - \rho)(u_{h,i}, u_{h,j})$$
$$\boldsymbol{N}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - 2\rho)(u_{h,i}, u_{h,j}) + \rho^2(\boldsymbol{\sigma}_{h,i}, \boldsymbol{\sigma}_{h,j}) + (\rho^2/\gamma)(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_{h,i}, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_{h,j})$$

- Solve:

$$\mu_1 \le \dots \le \mu_s : \quad \boldsymbol{M}\mathbf{y}_i = \mu_i \boldsymbol{N}\mathbf{y}_i, \quad i = 1, 2, \dots, s$$

- If $\boldsymbol{N}$ is s.p.d. and if $\mu_{s+1-n} < 0$ then
$$\ell_n^{\mathrm{LG}} = \rho - \gamma - \rho / \left(1 - \mu_{s+1-n}\right) \le \lambda_n, \quad n = 1, 2, \dots, s.$$

[Behnke, Mertins, Plum, Wieners 2000]

Method 4. Crouzeix–Raviart elements

Crouzeix–Raviart finite elements

$V_h^{\mathrm{CR}} = \{v_h \in P_1(\mathcal{T}_h) : v_h$ continuous in midpoints of all $\gamma \in \mathcal{E}_h\}$

Find $0 \neq u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}}$, $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Lower bound (no round-off errors)

$$\ell_i^{\mathrm{CR}} = \frac{\lambda_{h,i}^{\mathrm{CR}}}{1 + \kappa^2 \lambda_{h,i}^{\mathrm{CR}} h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \ldots$$

where

- $\kappa = 0.1893$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$

[Carstensen, Gedicke 2013], [Xuefeng LIU 2015]

Method 4. Crouzeix–Raviart elements

Crouzeix–Raviart finite elements

$V_h^{\mathrm{CR}} = \{v_h \in P_1(\mathcal{T}_h) : v_h \text{ continuous in midpoints of all } \gamma \in \mathcal{E}_h\}$
Find $0 \neq u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}}$, $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Lower bound (inexact solver: $\boldsymbol{A}\tilde{\boldsymbol{u}}_i^{\mathrm{CR}} \approx \tilde{\lambda}_{h,i}^{\mathrm{CR}} \boldsymbol{B}\tilde{\boldsymbol{u}}_i^{\mathrm{CR}}$)

$$\tilde{\ell}_i^{\mathrm{CR}} = \frac{\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\boldsymbol{r}\|_{\boldsymbol{B}^{-1}}}{1 + \kappa^2 \left(\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\boldsymbol{r}\|_{\boldsymbol{B}^{-1}}\right) h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \dots$$

where

- $\kappa = 0.1893$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$
- $\boldsymbol{r} = \boldsymbol{A}\tilde{\boldsymbol{u}}_i^{\mathrm{CR}} - \tilde{\lambda}_{h,i}^{\mathrm{CR}} \boldsymbol{B}\tilde{\boldsymbol{u}}_i^{\mathrm{CR}}$

Provided

- $\|\boldsymbol{r}\|_{\boldsymbol{B}^{-1}} < \tilde{\lambda}_{h,i}^{\mathrm{CR}}$
- $\tilde{\lambda}_{h,i}^{\mathrm{CR}}$ is closer to $\lambda_{h,i}^{\mathrm{CR}}$ than to any other discrete eigenvalue $\lambda_{h,j}^{\mathrm{CR}}$, $j \neq i$

Method 4. Crouzeix–Raviart elements

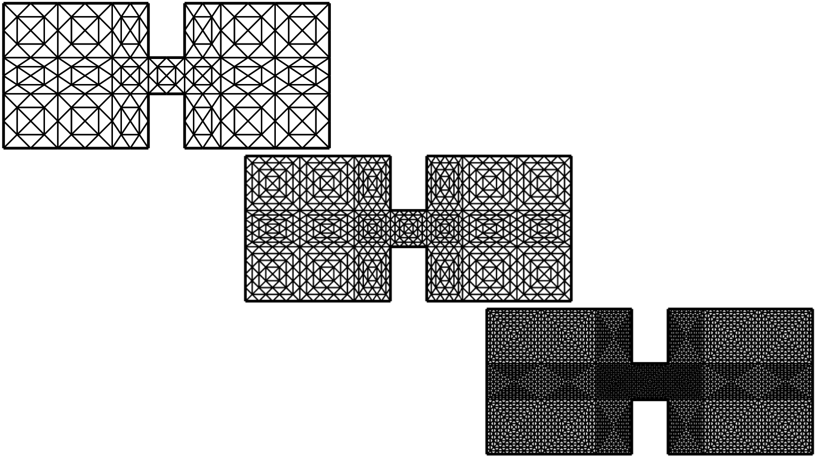
Upper bound

- $\mathcal{T}_h^*$ is the red refinement of $\mathcal{T}_h$
- $u_{h,i}^* = \mathcal{I}_{\mathrm{CM}} \tilde{u}_{h,i}^{\mathrm{CR}}$ for $i = 1, 2, \ldots, m$
- $\boldsymbol{S}, \boldsymbol{Q} \in \mathbb{R}^{m \times m}$ with entries $\boldsymbol{S}_{j,k} = (\nabla u_{h,j}^*, \nabla u_{h,k}^*)$ and $\boldsymbol{Q}_{j,k} = (u_{h,j}^*, u_{h,k}^*)$
- $\boldsymbol{S}\mathbf{y}_i = \Lambda_i^* \boldsymbol{Q}\mathbf{y}_i, \quad i = 1, 2, \ldots, m$
- $\Lambda_1^* \leq \Lambda_2^* \leq \cdots \leq \Lambda_m^*$
- $\lambda_i \leq \Lambda_i^* \quad$ for $i = 1, 2, \ldots, m$

Example: Dumbbell – convergence

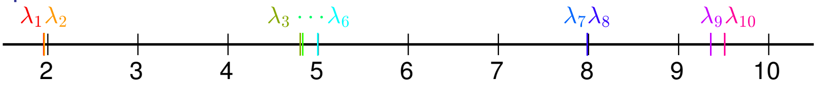
$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Uniformly refined meshes:

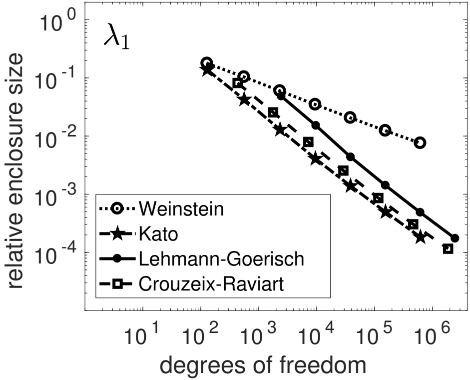


Example: Dumbbell – convergence

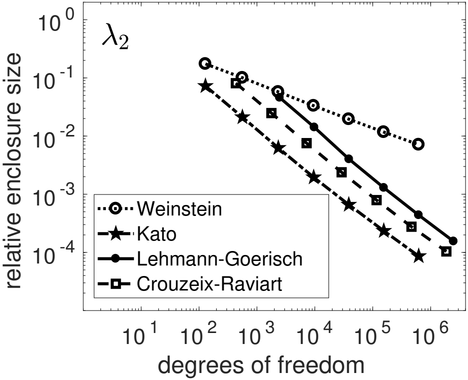
Spectrum:



$1.95569_{(\mathrm{LG})} \leq \lambda_1 \leq 1.95591_{(\mathrm{CR})}$

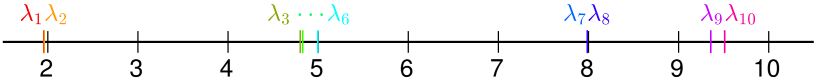


$1.96059_{(\mathrm{LG})} \leq \lambda_2 \leq 1.96079_{(\mathrm{CR})}$

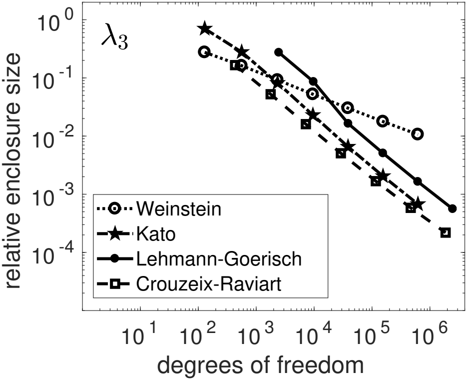


Example: Dumbbell – convergence

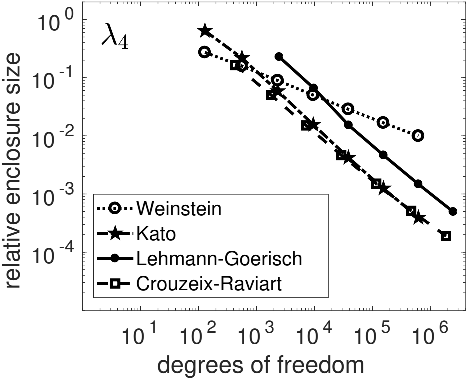
Spectrum:



$4.80024_{(\mathrm{CR})} \leq \lambda_3 \leq 4.80129_{(\mathrm{CR})}$

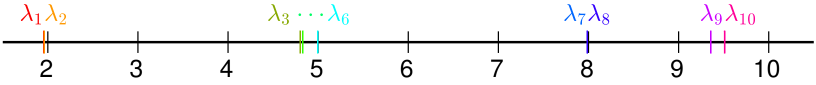


$4.82944_{(\mathrm{CR})} \leq \lambda_4 \leq 4.83036_{(\mathrm{CR})}$

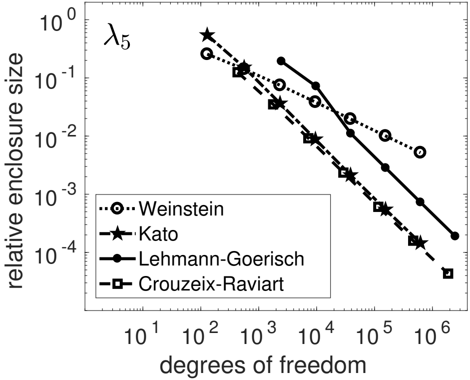


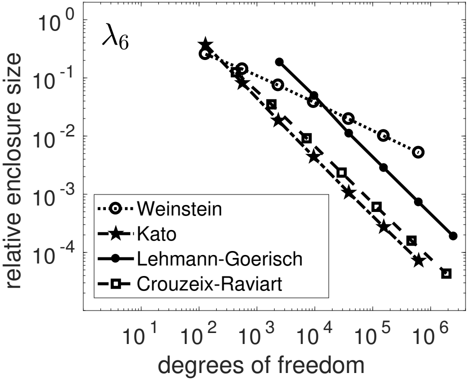
Example: Dumbbell – convergence

Spectrum:



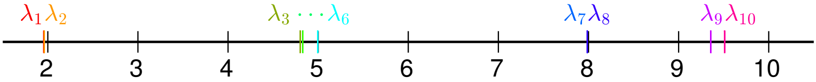
$4.99671_{(\mathrm{CR})} \leq \lambda_5 \leq 4.99693_{(\mathrm{CR})}$ $\quad$ $4.99672_{(\mathrm{CR})} \leq \lambda_6 \leq 4.99694_{(\mathrm{CR})}$



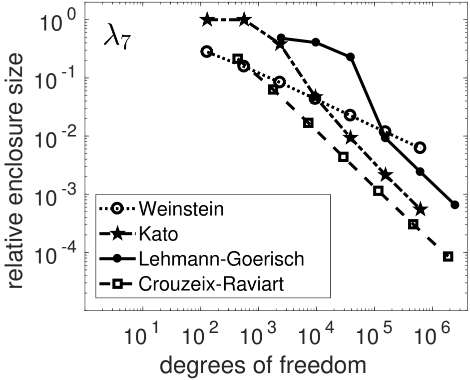


Example: Dumbbell – convergence

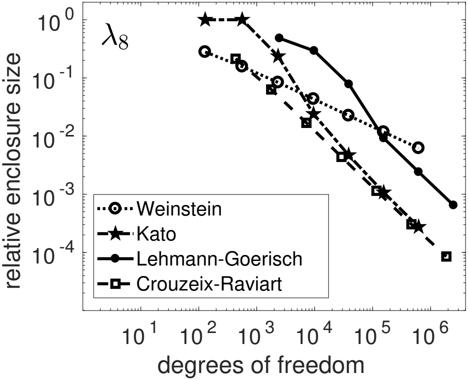
Spectrum:



$$7.98657_{(\mathrm{CR})} \le \lambda_7 \le 7.98725_{(\mathrm{CR})}$$

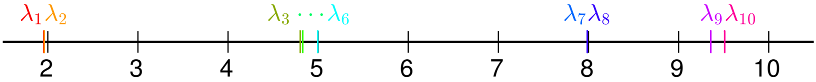


$$7.98664_{(\mathrm{CR})} \le \lambda_8 \le 7.98732_{(\mathrm{CR})}$$

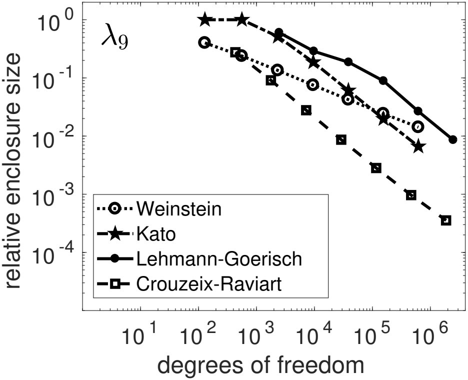


Example: Dumbbell – convergence

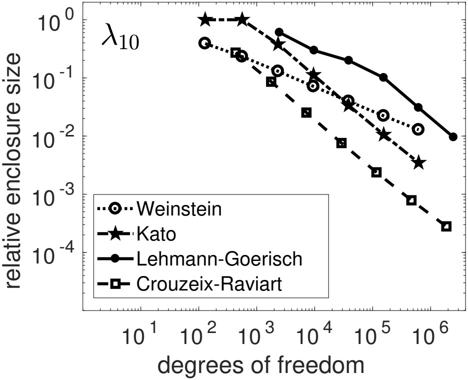
Spectrum:



$9.35556_{(\mathrm{CR})} \le \lambda_9 \le 9.35888_{(\mathrm{CR})}$



$9.50943_{(\mathrm{CR})} \le \lambda_{10} \le 9.51210_{(\mathrm{CR})}$



Conclusions

	Weinstein	Kato	Lehmann– –Goerisch	Crouzeix– –Raviart
speed of convergence	$-$	$+$	$+$	$+$
a priori information	$-$	$-$	$-$	$\pm$
algebraic error	$+$	$-$	$+$	$+$
generality	$+$	$+$	$+$	$-$
higher-order	$\pm$	$+$	$+$	$-$
robustness	$+$	$-$	$-$	$+$
local problems	$+$	$+$	$-$	$+$
error indicator	$+$	$+$	$+$	$-$

Thank you for your attention

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