Mathematical thermodynamics of viscous fluids

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This course is a short introduction to the mathematical theory of the motion of viscous fluids. We introduce the concept of weak solution to the Navier-Stokes-Fourier system and discuss its basic properties. In particular, we construct the weak solutions as a suitable limit of a mixed numerical scheme based on a combi- nation of the finite volume and finite elements method. The question of stability and robustness of various classes of solutions is addressed with the help of the relative (modulated) energy functional. Related results concerning weak-strong uniqueness and conditional regularity of weak solutions are presented. Finally, we discuss the asymptotic limit when viscosity of the fluid tends to zero. Several examples of ill- posedness for the limit Euler system are given and an admissibility criterion based on the viscous approximation is proposed.

1 Fluids in continuum mechanics

We start by introducing some basic concepts of the mathematical theory of fluids in motion in the framework of continuum mechanics.

1.1 Fluids in equilibrium

A fluid in equilibrium (at rest) is characterized by two fundamental state variables: the mass density ϱ and the (specific) internal energy e. In addition, we introduce another state variable - the specific entropy s - a function of and e enjoying the following properties, see Callen [10]:

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1. The entropy $s = s(\varrho; e)$ is an increasing function of the internal energy e,
$$\frac{\partial s}{\partial e} = \frac{1}{\vartheta},$$
where $\vartheta > 0$ is the absolute temperature.
2. For a thermally and mechanically insulated fluid occupying a physical domain Ω, maximization of the total entropy
$$S = \int_{\Omega} \varrho s \ \mathrm{d}x$$
yields the equilibrium state of the system.
3. The Third law of thermodynamics:
$$s \to 0 \text{ whenever } \vartheta \to 0.$$

In what follows, it will be more convenient to use ϱ and ϑ as the basic variables characterizing the state of a fluid, while the other thermodynamic quantities $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ are interrelated through *Gibbs' equation*:

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right) \tag{1}$$

where $p = p(\varrho, \vartheta)$ is a new thermodynamic function called pressure. In addition to (1), other hypotheses will be imposed throughout the text, notably the so-called thermodynamics stability condition discussed in Section 3.6. Besides, there will be also various purely technical conditions resulting as a suitable compromise between the underlying physical background and the needs of the mathematical theory.

1.2 Fluids in motion

Suppose a fluid occupies a part of the physical space R^3 represented by a domain Ω. For the sake of simplicity, we also suppose that Ω does not change with time. We adopt the *Eulerian description of motion* taking the coordinate system attached to physical space Ω rather than to the fluid itself. The motion is characterized by the macroscopic velocity field $\mathbf{u} = \mathbf{u}(t; x)$ - a function of the time $t \in (0; T)$ and the spatial position $x \in \Omega$. The so-called streamlines - the hypothetical paths $\mathbf{X} = \mathbf{X}(t)$ of the fluid "particles" - are determined by the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{u}(t, \mathbf{X}), \ \mathbf{X}(0) = x \in \Omega. \tag{2}$$

Accordingly, a streamline $\mathbf{X} = \mathbf{X}(t; x)$ may be viewed as a function of time and the initial position $x \in \Omega$.

Remark 1. I In accordance with the bulk of the reference material cited in the text, we use the symbol $x \in R^3$ to denote the position vector rather than the more consistent notation $\mathbf{x} \in R^3$.

Obviously, certain regularity of the velocity field $\mathbf{u}$ is needed for the streamlines to be well-defined through (2). In particular, $\mathbf{u} = \mathbf{u}(t; x)$ should be Lipschitz continuous in the $x-$variable for $\mathbf{X}$ to be uniquely determined by the initial position x. Unfortunately, such a degree of smoothness is in general not accessible by the available mathematical apparatus, and, as a result, most problems arising in mathematical fluid dynamics are not (known to be) well–posed unless certain smallness conditions on the data are imposed.

Remark 2. Solvability of system (2) is a crucial issue in the mathematical theory. Generalized solutions based on a reformulation of (2) in terms of transport theory were introduced in the seminal paper by DiPerna and Lions [26], a more elaborate treatment allowing u to be merely a BV function (but still of bounded divergence) was developed by Ambrosio [1]. As we shall see below, the regularity of solutions to problems in fluid dynamics indicated by the available a priori bounds is not sufficient for $\mathbf{u}$ to enter any of the admissibility classes specified in [1] or [26].

1.3 Field equations

A suitable mathematical description of fluids in continuum mechanics is given by a system of field equations expressing the basic physical principles: The balance of mass, momentum, and energy.

1.3.1 Mass conservation

Mass conservation in fluid dynamics is formulated through *equation of continuity*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{3}$$

or, if the velocity field is smooth,

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}, \tag{4}$$

where the left-hand side describes the mass transport along the stream-lines while the "source" term represents its changes due to compressibility.

1.3.2 Momentum balance

Momentum balance is enforced through *Newton's second law*. Introducing the Cauchy stress tensor T we get

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}_x\mathsf{T} + \varrho\mathbf{f}, \tag{5}$$

where $\mathbf{f}$ is the mass density of the external volume forces that may be acting on the fluid. *Fluids* as materials in continuum mechanics are characterized by *Stokes law*:

$$\mathsf{T} = \mathsf{S} - p\mathsf{I}, \tag{6}$$

where S is the viscous stress tensor, the basic properties of which will be discussed below.

1.3.3 Energy and entropy

The kinetic energy balance is obtained by taking the scalar product of the velocity with the momentum equation (5):

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + p\right)\mathbf{u}\right] + \mathrm{div}_x\left(\mathsf{S}\cdot\mathbf{u}\right) \tag{7}$$

$$= p\,\mathrm{div}_x\mathbf{u} - \mathsf{S} : \nabla_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}.$$

Even in the absence of external forces, equation (7) is not a conservation law - there is a source term

$$-\mathsf{T} : \nabla_x\mathbf{u} = -\mathsf{S} : \nabla_x\mathbf{u} + p\,\mathrm{div}_x\mathbf{u}.$$

Thus the kinetic energy $\frac{1}{2}\varrho|\mathbf{u}|^2$ does not represent the total energy of the fluid and must be augmented by the internal energy density ϱe.

In accordance with the *First law of thermodynamics*, the total energy is conserved or its changes are only due to the external sources. In view of (7), the internal energy equation takes the form

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathsf{S} : \nabla_x\mathbf{u} - p\,\mathrm{div}_x\mathbf{u}, \tag{8}$$

where $\mathbf{q}$ denotes the heat flux, whereas the total energy balance reads

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\mathbf{u}\right] + \mathrm{div}_x(\mathsf{S}\cdot\mathbf{u} + \mathbf{q}) = \varrho\mathbf{f}\cdot\mathbf{u}. \tag{9}$$

Here, we have deliberately omitted the effect of external heat (energy) sources, and, hereafter, we shall also ignore the external force $\mathbf{f}$ unless specified otherwise.

There are several alternative ways how to express the total energy balance, all of them equivalent to (9) within the class of *smooth* solutions. Introducing the thermal pressure P_ϑ and the specific heat at constant volume c_v,

$$p_\vartheta = \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta},\ c_v = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta},$$

we may use Gibss' relation (1) to rewrite (8) in the form of thermal energy balance frequently used in the literature:

$$\varrho c_v \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) + \mathrm{div}_x \mathbf{q} = \mathsf{S} : \nabla_x \mathbf{u} - \vartheta p_\vartheta \mathrm{div}_x \mathbf{u}. \tag{10}$$

Another consequence of (1), (8) is the entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma,\ \sigma = \frac{1}{\vartheta} \left(\mathsf{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \tag{11}$$

that may be seen as a mathematical formulation of the *Second law of thermodynamics.* Accordingly, the entropy production rate σ must be non-negative for any physically admissible process yielding the restrictions

$$\mathsf{S} : \nabla_x \mathbf{u} \geq 0,\ -\mathbf{q} \cdot \nabla_x \vartheta \geq 0. \tag{12}$$

In the above discussion we have systematically used the equation of continuity (3) writing

$$\varrho \left(\partial_t G + \mathbf{u} \cdot \nabla_x G \right) = \partial_t(\varrho G) + \mathrm{div}_x(\varrho \mathbf{u} G)$$

and vice versa. Note that right-hand side is the so-called divergence form more convenient for the weak formulation discussed below.

1.4 Boundary behavior

Any real physical domain is bounded although some problems may be conveniently posed on unbounded domains. In both cases, the boundary behavior of the fluid is relevant for determining the motion inside Ω. We focus on very simple boundary conditions yielding energetically insulated fluid systems. Specifically, we impose the impermeability of the boundary

$$\mathbf{u} \cdot \mathbf{N}|_{\partial \Omega} = 0,\ \mathbf{n} - \text{the outer normal vector to } \partial \Omega, \tag{13}$$

supplemented, in the case of viscous fluids, with either the no-slip

$$\mathbf{u} \times \mathbf{n}|_{\partial \Omega} = 0, \tag{14}$$

or the complete slip

$$[\mathsf{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial \Omega} = 0. \tag{15}$$

In addition, the no- flux boundary conditions will be imposed on the heat flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{16}$$

Accordingly, in the absence of external forces, the total mass as well as the total energy energy are conserved quantities:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \varrho \ \mathrm{d}x = 0, \ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x = 0. \tag{17}$$

Remark 3. The boundary behavior is definitely more complex in the real world applications that may include in and/or out flux source terms on the boundary, the boundary itself may be another unknown of the problem etc. The resulting problems are mathematically very complicated and their resolution leans on highly non-trivial tools, see e.g. Coutand and Shkoller [21], [22], Secchi et al. [11] [59], among others.

2 Mathematics of viscous compressible fluids

In this section, we introduce the concept of *weak solution* to the system of field equations of fluid thermodynamics. We start by fixing the initial state of the system:

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \varrho_0 > 0, \ \vartheta_0 > 0 \tag{18}$$

in the physical domain Ω occupied by the fluid. Note that only integrability but not particular smoothness of the data is needed in the weak formulation introduced below.

2.1 Equation of continuity

We say that ϱ, $\mathbf{u}$ satisfy equation (5) in a weak sense if

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t = \left[\int_{\Omega} \varrho \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} \tag{19}$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and any $\varphi \in C^\infty([0,T] \times \overline{\Omega}$.

We also introduce the *renormalized solutions* satisfying

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left[b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + \left(b(\varrho) - b'(\varrho) \varrho \right) \mathrm{div}_x \mathbf{u} \right] \ \mathrm{d}x \tag{20}$$

$$= \left[\int_{\Omega} b(\varrho)\varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2}$$

for any smooth function b satisfying suitable growth conditions. Renormalized solutions to transport equations were introduced in the seminal paper by DiPerna and Lions [26].

Note that (19) can be seen as a special case of (20) with $b(\varrho) = \varrho$. Both (19) and (20) require certain continuity of ϱ as a function of time. Taking $\varphi(t,x) = \psi(t)\phi(x)$ in (19) we easily observe that the function

$$t \in [0,T] \mapsto \int_{\Omega} \varrho(t,\cdot)\phi \ \mathrm{d}x$$

can be redefined on a set of times of zero measure so that the resulting function belongs to $C[0;T]$ for any $\phi \in C_c^\infty(\Omega)$. Thus the density may be viewed as a weakly continuous function of the time variable; in particular, the initial condition makes sense. The situation becomes more delicate for the renormalized equation (20) as a composition with nonlinear function b does not in general commute with the weak topology. Fortunately, however, the densities satisfying (20) with a suitable velocity field $\mathbf{u}$ are in fact strongly continuous as functions of the time with values in the Lebesgue space $L^1(\Omega)$, see DiPerna and Lions [26].

2.2 Momentum equation

A weak formulation of the momentum balance (5), (6) (with $\mathbf{f} = 0$) reads

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} [\varrho\mathbf{u} \cdot \partial_t \varphi + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p \mathrm{div}_x \varphi - \mathsf{S} : \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t \tag{21}$$

$$= \left[\int_{\Omega} \varrho\mathbf{u} \cdot \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2}$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and any test function $\varphi \in C^\infty([0,T] \times \overline{\Omega}; R^3)$. In addition, we require $\varphi \in C_c^\infty([0,T] \times \Omega)$ in the case of the no-slip boundary conditions (13), (14), and $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ for the complete slip (13), (15).

Similarly to the above, it is easy to observe that the mapping $t \mapsto \varrho\mathbf{u}(t,\cdot)$ is weakly continuous so the instantaneous values of the momentum are well defined.

2.3 Energy–entropy

In order to close the system of field equations, a weak formulation of the energy and/or entropy is needed. Accordingly, we have to choose one among the equations (9–11) as a suitable "representative" keeping in mind they may not be equivalent in the weak setting.

2.3.1 Entropy based weak formulation

Of course, the most natural candidate would be the total energy balance (9) expressing the First law. In view of the technical difficulties discussed later in this text, however, we opted for the Second law encoded in (11), where, in addition, we allow the entropy production rate σ to be a non-negative measure satisfying

$$\sigma \geq \frac{1}{\vartheta}\left(\mathsf{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{22}$$

Accordingly, a weak formulation of (11), (22) reads

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t \tag{23}$$

$$+ \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\vartheta} \left[\mathsf{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \varphi \ \mathrm{d}x \ \mathrm{d}t \leq \left[\int_{\Omega} \varrho s \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2}$$

for a.a. $0 \leq \tau_1 \leq \tau_2 \leq T$ including $\tau_1 = 0$, and any $\varphi \in C^\infty([0,T] \times \overline{\Omega})$, $\varphi \geq 0$. Here it is worth noting that the entropy density ϱS unlike ϱ and $\varrho \mathbf{u}$, *may not* be weakly continuous as a function of time.

Replacing *equation* by *inequality* may result in a lost of information that must be compensated by adding an extra constraint. Here, we supplement (23) with the total energy balance

$$\left[\int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = 0 \tag{24}$$

for a.a. $0 \leq \tau_1 \leq \tau_2 \leq T$, including $\tau_1 = 0$.

Although the entropy formulation may seem rather awkward and unnecessarily complicated, it turns out to be quite convenient to deal with in the framework of weak solutions. In particular, it gives rise to the relative energy inequality with the associated concept of dissipative solution discussed below.

To conclude, we remark that the weak formulation based on the integral identities (19–24) complies with an obligatory principle of compatibility namely any weak solution that is sufficiently smooth solves the system of equations (3), (5), (9) together with the relevant boundary conditions in the classical sense. The interested reader may consult the monograph [37, Chapter 2].

2.3.2 Thermal energy weak formulation

If c_v is constant, we may replace (23) by the thermal energy balance (10) written again as an inequality

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left[c_v \varrho\vartheta \partial_t \varphi + c_v \varrho\vartheta \mathbf{u} \cdot \nabla_x \varphi + \mathbf{q} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \,\mathrm{d}t \tag{25}$$

$$+ \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathsf{S} : \nabla_x \mathbf{u} - \varrho p_\vartheta \mathrm{div}_x \mathbf{u} \right] \varphi \ \mathrm{d}x \,\mathrm{d}t \leq \left[\int_{\Omega} c_v \varrho \vartheta \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2}$$

for a.a. $0 \leq \tau_1 \leq \tau_2 \leq T$ including $\tau_1 = 0$, and any $\varphi \in C^\infty([0,T] \times \overline{\Omega})$, $\varphi \geq 0$. Similarly to the preceding section, the inequality (25) is supplemented by the total energy balance

$$\left[\int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} \tag{26}$$

for a.a. $0 \leq \tau_1 \leq \tau_2 \leq T$ including $\tau_1 = 0$. We may go even further replacing (26) by an inequality

$$\int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \ \mathrm{d}x \leq \int_{\Omega} \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \ \mathrm{d}x \tag{27}$$

for a.a. $\tau \in [0, T]$.

The weak formulation based on the thermal energy balance is simpler than (23), (24) and can be used in the analysis of the associated *numerical schemes*. Similarly to the entropy formulation introduced in the preceding part, the resulting weak formulation is compatible with the strong one as soon as all quantities in question are smooth enough.

2.4 Constitutive relations, Navier-Stokes-Fourier system

In order to obtain, at least formally, a mathematically well-posed problem, the constitutive relations for the viscous stress tensor S as well as the heat flux $\mathbf{q}$ must be specified. They must obey the general principle stated in (12) in order to comply with the Second law of thermodynamics.

We consider the simplest possible situation when S is a linear function of the velocity gradient $\nabla_x \mathbf{u}$, while, analogously, $\mathbf{q}$ depends linearly on $\nabla_x \vartheta$.

More specifically, we impose Newton's rheological law

$$\mathsf{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathsf{I} \right] + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathsf{I}, \tag{28}$$

where $\mu(\vartheta)$, $\nu(\vartheta)$ are non-negative scalar functions representing the shear and bulk viscosity coefficient, respectively.

Similarly, the heat flux will obey Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \tag{29}$$

with the heat conductivity coefficient κ.

The system of field equations supplemented with the constitutive relations (28), (29) will be referred to as *Navier-Stokes-Fourier system*.

Remark 4. In general, the so-called transport coefficients μ, η, and κ should depend on the density as well. For technical reasons, however, we are able to handle the temperature dependent case only. Note that this is physically relevant at least for gases, see e.g. Becker [6].

3 Well-posedness, approximation scheme

The question of *well-posedness* of a system of equations, including the problem of existence, uniqueness, and stability with respect to the data, is crucial in the mathematical theory. As is well known, well posedness for the equations and systems arising in fluid mechanics features sofar unsurmountable mathematical difficulties due to the occurrence of possible singularities, in particular in the velocity field (cf. Fefferman [29]). From this point of view, the concept of weak solution offers a suitable framework to attack the problem. Moreover, the weak solutions seem indispensable in the theory of inviscid fluids where the singularities are known to occur in finite time no matter how smooth and even small the initial data might be, see e.g. the classical texts by Smoller [61], or the more recent treatment by Majda [50].

Although the question of mere *existence* of solutions to problems like the Navier-Stokes-Fourier system seems extremely difficult, a more ambitious task is to design a suitable *approximation scheme* usable in effective numerical implementations. In accordance with the philosophy proposed in the nowadays classical monograph by J.-L.Lions [47], solutions should be obtained as limits of a finite number of algebraic equations solvable by means of a suitable numerical method.

3.1 An approximation scheme for the Navier-Stokes-Fourier system

Following [35], we propose a discrete approximation scheme for solving the Navier-Stokes-Fourier system. To this end, it seems more convenient to employ the thermal energy formulation based on the relations (19), (21), (25), and (27).

3.1.1 Hypotheses

Several mostly technical restrictions must be imposed on the constitutive equations to make the problem tractable by means of the available analytical tools. Specifically, we suppose that:

- The internal energy $e(\varrho, \vartheta)$ can be decomposed in the form

$$e(\varrho, \vartheta) = c_v \vartheta + P(\varrho)$$

  where the specific heat at constant volume c_v is a positive constant. Accordingly, we suppose that the pressure takes the form

$$p(\varrho, \vartheta) = a\varrho^\gamma + b\varrho + \varrho\vartheta, \ a, b > 0, \ \gamma > 3, \tag{30}$$

  therefore the (specific) internal energy reads

$$e(\varrho, \vartheta) = c_v \vartheta + P(\varrho), \ c_v > 0, \ P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma + b\varrho \log(\varrho). \tag{31}$$

- The viscosity coefficients in (28) are constant, in particular, we may write

$$\mathrm{div}_x \mathsf{S}(\nabla_x \mathbf{u}) = \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}, \lambda = \frac{1}{3}\mu + \eta > 0. \tag{32}$$

- The heat flux $\mathbf{q}$ obeys Fourier's law (29) where the heat conductivity coefficient κ is a continuously differentiable function of the temperature satisfying

$$\kappa = \kappa(\vartheta), \ \underline{\kappa}(1 + \vartheta^2) \le \kappa(\vartheta) \le \overline{\kappa}(1 + \vartheta^2), \ \underline{\kappa} > 0. \tag{33}$$

Remark 5. The constitutive relations (30–33) are by no means optimal and must be seen as a suitable compromise between the physical background and the need of suitable *a priori bounds* required by the mathematical theory. Note that showing *converge* of a numerical method is mathematically more involved than giving a mere existence proof, for which (30–33) are sofar indispensable, see [30].

3.2 Time discretization

We propose to approach the Navier-Stokes-Fourier by Rothe's method or the method of *time discretization*. We fix the time step $\Delta t > 0$, and, supposing that the approximate solutions $[\varrho^j, \vartheta^j, \mathbf{u}^j]$ at the times $j\Delta t$, $j = 0, \dots, k-1$ is already known, we define $[\varrho^k, \vartheta^k, \mathbf{u}^k]$ as a solution of the system of "stationary" problems

$$D_t \varrho^k \equiv \frac{\varrho^k - \varrho^{k-1}}{\Delta t} = \mathcal{C}(\varrho^k, \vartheta^k, \mathbf{u}^k), \tag{34}$$

$$D_t(\varrho^k \mathbf{u}^k) \equiv \frac{\varrho^k \mathbf{u}^k - \varrho^{k-1}\mathbf{u}^{k-1}}{\Delta t} = \mathcal{M}(\varrho^k, \vartheta^k, \mathbf{u}^k), \tag{35}$$

$$c_v D_t(\varrho^k \vartheta^k) \equiv \frac{\varrho^k \vartheta^k - \varrho^{k-1}\vartheta^{k-1}}{\Delta t} = \mathcal{T}(\varrho^k, \vartheta^k, \mathbf{u}^k), \tag{36}$$

for certain operators $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{T}$. Such a scheme is called implicit as we have to solve a system of (non-linear) equations to determine $[\varrho^k, \vartheta^k, \mathbf{u}^k]$ at each time step.

3.3 Space discretization

Our next goal is to approximate each equation in (34–36) by a finite system of algebraic equations. This is usually done by applying a suitable projection onto a finite dimensional space. Accordingly, we replace $\varrho^k \approx \varrho^k_h$, $\vartheta^k \approx \vartheta^k_h$, $\mathbf{u}^k \approx \mathbf{u}^k_h$ by finite vectors, where the parameter $h > 0$ characterizes the degree of space discretization.

Similarly to the weak formulation discussed in Section 2, we multiply the corresponding equations by a test function belonging to a suitable *finite-dimensional space* and perform the by-parts integration. In such a way, we may, for instance, replace the differential operator

$$-\mathrm{div}_x \mathsf{S}(\nabla_x \mathbf{u}^k) \approx \mu \int_\Omega \nabla_x \mathbf{u}^k_h : \nabla_x \phi \ \mathrm{d}x + \lambda \int_\Omega \mathrm{div}_x \mathbf{u}^k_h \mathrm{div}_x \phi \ \mathrm{d}x.$$

In numerical analysis, such a step can be performed via a finite element method (FEM). The physical domain Ω is approximated by a numerical domain Ω_h, the latter being divided into small elementary pieces by *triangulation*. The test functions restricted to these elementary pieces (elements) are usually polynomials of finite degree enjoying certain continuity on the faces common to two neighboring elements. In order to specify our approximation scheme we start by a short excursion in numerical analysis.

3.3.1 Mesh, triangulation

The physical space Ω is approximated by a polyhedral bounded domain $\Omega_h \subset R^3$ that admits a *tetrahedral* mesh E_h; the individual elements in the mesh will be denoted by $E \in E_h$,

$$\Omega_h = \cup_{E \in E_h} E.$$

Faces in the mesh are denoted as Γ, whereas Γ_h is the set of all faces. Moreover, the set of faces $\Gamma \subset \partial\Omega_h$ is denoted $\Gamma_{h,\text{ext}}$, while $\Gamma_{h,\text{int}} = \Gamma \setminus \Gamma_{h,\text{ext}}$. The size (diameter h_E of its elements E in the mesh) is proportional to a positive parameter h.

In addition, the mesh enjoys certain additional properties (cf. Eymard et al. [28, Chapter 3]):

- The intersection $E \cap F$ of two elements $E, F \in E_h$, $E \neq F$ is either empty or their common face, edge, or vertex.
- For any $E \in E_h$, $\text{diam}[E] \approx h$, $r[E] \approx h$, where r denotes the radius of the largest sphere contained in E.
- There is a family of control points $x_E \in E$, $E \in E_h$ such that if E and F are two neighboring elements sharing a common face Γ, then the segment $[x_E, x_F]$ is perpendicular to Γ. We denote

$$d_\Gamma = |x_E - x_F| > 0.$$

Each face $\Gamma \subset \Gamma_h$ is associated with a normal vector $\mathbf{n}$. We shall write Γ_E whenever a face $\Gamma \subset \partial E$ is considered as a part of the boundary of the element E. In such a case, the normal vector to Γ_E is always the *outer* normal vector with respect to E. Moreover, for any function g continuous on each element E, we set

$$g^{\text{out}}|_\Gamma = \lim_{\delta \to 0+} g(\cdot + \delta \mathbf{n}),\ g^{\text{int}}|_\Gamma = \lim_{\delta \to 0+} g(\cdot - \delta \mathbf{n}),$$

$$[[g]] = g^{\text{out}} - g^{\text{in}},\ \{g\}_\Gamma = \frac{1}{2}\left(g^{\text{out}} + g^{\text{in}}\right).$$

3.3.2 FEM structure

The velocity field $\mathbf{u}^k$ will be approximated by the so-called Crouzeix-Raviart finite elements (see for instance Brezzi and Fortin [8]) belonging to the space

$$V_h(\Omega_h) = \Big\{ v \in L^2(\Omega_h) \ \Big| \ v|_E = \text{affinefunction},\ E \in E_h,$$

$$\int_\Gamma [[v]]\ \text{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\text{int}} \Big\}.$$

For the sake of definiteness, we focus on the no-slip boundary conditions (13), (14). Accordingly, we introduce the space

$$V_{h,0}(\Omega_h) = \left\{ v \in V_H \ \Big| \ \int_\Gamma v \ \mathrm{dS}_x = 0 \text{ for any } \Gamma \in \Gamma_{h,\mathrm{ext}} \right\}.$$

The finite element method is suitable for approximating the viscous stress, however, the Navier-Stokes-Fourier system features also hyperbolic aspects expressed through equations like (3). The convective terms, appearing in any of the field equations are better approximated by means of the finite volume method (FVM) introduced in the next section.

3.3.3 FVM structure

Roughly speaking, the finite volume method replaces integration over the elements by integration over faces. To this end, we introduce the space of piece-wise constant functions

$$Q_h(\Omega_h) = \left\{ v \in L^2(\Omega_h) \ \Big| \ v|_E = a_E \in R \text{ for any } E \in E_h \right\},$$

along with the associated projection operator

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h(\Omega_h), \ \Pi_h^Q[v] \equiv \hat{v}, \ \Pi_h^Q[v]|_E = \frac{1}{|E|} \int_E v \ \mathrm{d}x \text{ for anye} \in E_h.$$

The convective terms are discretized by means of the so-called upwind defined as follows:

$$\mathrm{Up}[r, \mathbf{u}] = r^{\mathrm{in}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^+ + r^{\mathrm{out}}[\tilde{\mathbf{u}} \cdot \mathbf{n}]^-,$$

where we have denoted

$$[c]^+ = \max\{c, 0\}, \ [c]^- = \min\{c, 0\}, \ \tilde{v} = \frac{1}{\Gamma} \int_\Gamma v \ \mathrm{dS}_x.$$

Note carefully that such a definition makes sense as soon as $r \in Q_h(\Omega_h)$, $\mathbf{u} \in V_h(\Omega_h; R^3)$ and $\Gamma \in \Gamma_{h,\mathrm{int}}$. We approximate

$$\int_{\Omega_h} \varrho Z \mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x \approx \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho Z, \mathbf{u}] \, [[\phi]] \ \mathrm{dS}_x.$$

3.4 Approximation scheme

We introduce the notation ∇_h, div_h to denote the restriction of the differential operators ∇_x, div_x to any element $E \in E_h$, specifically

$$\nabla_h v|E = \nabla(v|_E),\ \mathrm{div}_h \mathbf{v}|E = \mathrm{div}_x(\mathbf{v}|_E)$$

for functions v, $\mathbf{v}$ differentiable on any element $E \in E_h$.

The zero-th order terms $[\varrho^0, \vartheta^0, \mathbf{u}^0]$ being determined the initial data, we define $\varrho_h^k, \vartheta_h^k, \mathbf{u}_h^k]$ successively as a solution to the approximation scheme - numerical method:

$$\int_{\Omega_h} D_t \varrho_h^k \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k]\, [[\phi]] \ \mathrm{dS}_x \tag{37}$$

$$+h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] [[\phi]] \ \mathrm{dS}_x = 0$$

for any $\phi \in Q_h(\Omega_h)$, with a parameter $0 < \alpha < 1$;

$$\int_{\Omega_h} D_t(\varrho_h^k \hat{\mathbf{u}}_h^k) \cdot \phi \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \hat{\mathbf{u}}_h^k] \cdot \left[\left[\hat{\phi}\right]\right] \ \mathrm{dS}_x \tag{38}$$

$$+ \int_{\Omega_h} \left[\mu \nabla_h \mathbf{u}_h^k : \nabla_h \phi + \lambda \mathrm{dih}_h \mathbf{u}_h^k \mathrm{div}_h \phi\right] \ \mathrm{d}x - \int_{\Omega_h} p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \phi \ \mathrm{d}x$$

$$+h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left[\left[\varrho_h^k\right]\right] \left\{\hat{\mathbf{u}}_h^k\right\} \left[\left[\hat{\phi}\right]\right] \ \mathrm{dS}_x = 0$$

for any $\phi \in V_{h,0}(\Omega_h; R^3)$;

$$c_v \int_{\Omega_h} D_t(\varrho_h^k \vartheta_h^k)\phi \ \mathrm{d}x - c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k]\, [[\phi]] \ \mathrm{dS}_x \tag{39}$$

$$+ \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} \left[\left[K(\vartheta_h^k)\right]\right] [[\phi]] \ \mathrm{dS}_x$$

$$= \int_{\Omega_h} \left[\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right] \phi \ \mathrm{d}x - \int_{\Omega_h} \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x$$

for any $\phi \in Q_h(\Omega_h)$, where we have set

$$K(\vartheta) = \int_0^\vartheta \kappa(z) \mathrm{d}z.$$

Here, the terms proportional to h^α represent numerical counterparts of the artificial viscosity regularization used in [30, Chapter 7] and were introduced by Eymard et al. [29] to prove convergence of the momentum method.

3.5 Existence of weak solutions via the numerical scheme

The physical domain Ω will be approximated by the polyhedral domains Ω_h so that

$$\Omega \subset \overline{\Omega} \subset \Omega_h \subset \left\{ x \in R^3 \ \Big| \ \mathrm{dist}[x, \overline{\Omega}] < h \right\}. \tag{40}$$

The discrete solutions $[\varrho_h^k, \vartheta_h^k, \mathbf{u}_h^k]$ can be extended to be defined at any time t setting

$$\varrho_h(t,\cdot) = \varrho_h^0,\ \vartheta_h(t,\cdot) = \vartheta_h^0,\ \mathbf{u}^h(t,\cdot) = \mathbf{u}_h^0 \text{ for } t \leq 0,$$

$$\varrho_h(t,\cdot) = \varrho_h^k,\ \vartheta_h(t,\cdot) = \vartheta_h^k,\ \mathbf{u}_h(t,\cdot) = \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t),\ k = 1, 2, \ldots,$$

and, accordingly, the discrete time derivative of a quantity v_h is

$$D_t v_h(t,\cdot) = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t},\ t > 0.$$

We claim the following result:

Theorem 1. *Let $\Omega \subset R3$ be a bounded domain of class C^1 approximated by a family of polyhedral domains $\{\Omega_h\}_{h>0}$ in the sense (40), where each Ω_h admits a tetrahedral mesh satisfying the hypotheses specified in Section 3.3. Suppose that $\mu > 0$, $\lambda > 0$, and that the pressure $p = p(\varrho;\vartheta)$ and the heat conductivity coefficient $\kappa = \kappa(\vartheta)$ comply with (30–33). Suppose that*

$$\Delta t \approx h$$

and

$$\varrho_h^0 > 0,\ \vartheta_h^0 > 0 \text{ for all } h > 0.$$

Then

- *the numerical scheme (37), (38), (39) admits a solution*

$$\varrho_h^k > 0,\ \vartheta_h^k > 0,\ \mathbf{u}_h^k \text{ for any finite } k = 1, 2, \ldots;$$

-

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly in } L^2(0,T;L^6(\Omega)),$$

$$\mathbf{u}_h \to \mathbf{u} \ \textit{weakly in} \ L^2(0,T; L^6(\Omega; R^3)),$$
$$\nabla_h \mathbf{u}_h^k \to \Delta_x \mathbf{u} \ \textit{weakly in} \ L^2((0,T)\times\Omega; R^{3\times 3}),$$

at least for suitable subsequences, where $[\varrho, \vartheta, \mathbf{u}]$ *is a weak solution of the thermal energy formulation of the Navier-Stokes-Fourier system (18), (21), (25), (26) in* $(0,T)\times\Omega$.

Remark 6. As a matter of fact, the thermal energy balance (25) is satisfied in the following *very weak sense*:

$$\int_{\tau_1}^{\tau_2} \int_\Omega \left[c_v \varrho\vartheta \partial_t \varphi + c_v \varrho\vartheta \mathbf{u} \cdot \nabla_x \varphi - \overline{K(\vartheta)} \Delta\varphi \right] \ \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_{\tau_1}^{\tau_2} \int_\Omega [\mathsf{S} : \nabla_x \mathbf{u} - \varrho p_\vartheta \mathrm{div}_x \mathbf{u}] \varphi \ \mathrm{d}x \, \mathrm{d}t \le \left[\int_\Omega c_v \varrho \vartheta \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2}$$

for a.a. $0 \le \tau_1 \le \tau_2 \le T$, including $\tau_1 = 0$, and any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$, $\varphi \ge 0$ where

$$\varrho \overline{K(\vartheta)} = \varrho K(\vartheta),$$

meaning the "correct" constitutive relation between K and ϑ holds only out of the vacuum.

The existence part in Theorem 1 was established in [35, Section 8.1], convergence of the numerical solutions was shown in [35, Theorem 3.1].

3.6 Existence for the entropy formulation

In this part we shortly recall the available existence theory for the Navier-Stokes-Fourier system in the entropy formulation (19–24). Unfortunately, to the best of our knowledge, there is only an "analytical" proof available without any numerical counterpart of the associated approximation scheme.

3.6.1 Constitutive relations

We start with a list of hypotheses imposed on the constitutive relations:

- In addition to Gibbs' equation (1), the pressure $p = p(\varrho; \vartheta)$ and the specific internal energy $e = e(\varrho; \vartheta)$ satisfy the *hypothesis of thermodynamic stability*
$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ c_v(\varrho,\vartheta) = \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \tag{41}$$
for all $\varrho > 0$, $\vartheta > 0$, cf. [5].

- The internal energy and the pressure take the form

$$e(\varrho,\vartheta) = e_m(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4, \ p(\varrho,\vartheta) = p_m(\varrho,\vartheta) + \frac{a}{4}\vartheta^4, \ a > 0, \tag{42}$$

where e_m, p_m represent molecular components augmented by radiation, see [37, Chapter 1]. Moreover, p_m and e_m satisfy the monoatomic gas equation of state

$$p_m(\varrho,\vartheta) = \frac{2}{3}\varrho e_m(\varrho,\vartheta). \tag{43}$$

In this context, Gibbs' equation (1) yields

$$p_m(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right); \text{ whence } e_m(\varrho,\vartheta) = \frac{3}{2}\vartheta\frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right). \tag{44}$$

- The thermodynamic stability hypothesis (41) then implies that

$$P(0) = 0, \ P'(Z) > 0, \ 0 < \frac{\frac{5}{3}P'(Z)Z - P(Z)}{Z} < c \text{ for any } Z > 0, \tag{45}$$

where, in addition, we require the specific heat at constant volume to be uniformly bounded.
- It follows from (45) that the function $Z \mapsto P(Z)/Z^{5/3}$ is non-increasing, and we suppose

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{46}$$

- The transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ in (28), (29) depend on the absolute temperature,

$$\underline{\mu}(1+\vartheta^\alpha) \le \mu(\vartheta) \le \overline{\mu\mu}(1+\vartheta^\alpha), \ |\mu'(\vartheta)| \le c \text{ for all } \vartheta > 0, \ \frac{2}{5} < \alpha \le 1, \ \underline{\mu} > 0, \tag{47}$$

$$0 \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta^\alpha) \text{ for all } \vartheta > 0, \tag{48}$$

$$0 < \underline{\kappa}(1+\vartheta^3) \le \kappa(\vartheta) \le \overline{\vartheta}(1+\vartheta^3) \text{ for all } \vartheta > 0, \ \underline{\kappa} > 0. \tag{49}$$

3.6.2 Global existence

We report the following existence result proved in [37, Chapter 3, Theorem 3.1]:

Theorem 2. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the pressure p and the internal energy e are interrelated through (41–44), where $P \in C[0,\infty) \cap C^3(0,\infty)$ satisfies the structural hypotheses (45), (46). Let the transport coefficients μ, η, κ be continuously differentiable functions of the temperature ϑ satisfying (47–49). Let the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ be given such that*

$$\varrho_0,\ \vartheta_0 \in L^\infty(\Omega),\ \varrho_0,\ \vartheta_0 > 0 \ a.a.\ in\ \Omega,\ \mathbf{u}_0 \in L^2(\Omega; R^3).$$

Then Then the Navier-Stokes-Fourier system (19–24) admits a weak solution ϱ, ϑ, $\mathbf{u}$ in $(0;T) \times \Omega$ belonging to the class:

$$\varrho \geq 0 \ a.a.\ in\ (0,T) \times \Omega,$$

$$\varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{5/3}(\Omega)) \cap L^\beta((0,T) \times \Omega)$$

for some $\beta > \frac{5}{3}$;

$$\vartheta > 0 \ a.a.\ in\ (0,T) \times \Omega,\ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),$$

$$\vartheta^3,\ \log(\vartheta) \in L^2(0,T; W^{1,2}(\Omega));$$

$$\mathbf{u} \in L^2(0,T; W_0^{\Lambda}(\Omega; R^3)),\ \Lambda = \frac{8}{5-\alpha},\ \varrho\mathbf{u} \in C_{\rm weak}(0,T; L^{5/4}(\Omega; R^3)).$$

Remark 7. Note that integrability of the velocity gradient is related to the exponent α appearing in the structural hypotheses (47), (48). The restriction $\frac{2}{5} < \alpha \leq 1$ can be relaxed, see e.g. Poul [57].

4 A priori estimates for the entropy formulation

A priori estimates represent the heart of any mathematical theory related to non–linear partial differential equations. These are the natural bounds imposed on the solution set by the underlying system of differential equations (differential constraints) endowed with a family of data (initial and boundary conditions, driving forces as the case may be). *A priori* estimates are of purely formal character, being derived under the hypothesis that the solutions in question are smooth. However, as we shall see below, all a priori bounds that can be derived for the Navier-Stokes-Fourier system actually hold even within the much larger class of the weak solutions introduced in Section 2.3. This is mainly because all nowadays available a priori bounds follow directly from the underlying physical principles: The energy conservation or the entropy balance already included in the weak formulation. In this section, we review a complete list of known a priori bounds for the Navier-Stokes-Fourier system. The proofs of several estimates are mostly sketched whereas a more detailed analysis may be found in [32].

4.1 Total mass conservation

The total mass conservation follows directly by taking a spatially homogeneous test function in the equation of continuity (19):

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho(0,\cdot)\ \mathrm{d}x = M \text{ for any } t \in [0,T]. \tag{50}$$

Since ϱ is non-negative, we deduce that

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|_{L^1(\Omega)} \leq c(\text{data}). \tag{51}$$

Such a bound may be of particular interest on unbounded domains, where it provides a valuable piece of information concerning the asymptotic decay of the density ϱ for $|x| \to \infty$.

4.2 Energy estimates

The energy conservation principle (24) gives rise to

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x \leq c(\text{data}). \tag{52}$$

Under the hypotheses listed in (42–46) we therefore deduce that

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\Omega;R^3)} \leq c(\text{data}), \tag{53}$$

and

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|_{L^{5/3}(\Omega)} \leq c(\text{data}). \tag{54}$$

Finally, as a consequence of the presence of the radiation components:

$$\operatorname{ess}\sup_{t\in(0,T)} \|\vartheta(t,\cdot)\|_{L^4(\Omega)} \leq c(\text{data}). \tag{55}$$

Remark 8. It is important to notice that (55) yields a bound on ϑ and not on $\varrho\vartheta$, where the latter would be the best bound available in the absence of radiation.

4.3 A priori estimates based on energy dissipation

The entropy balance (23) evaluated for the test function $\varphi = 1$ yields

$$\int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\ \mathrm{d}x \tag{56}$$

$$\geq \int_0^\tau \int_\Omega \frac{1}{\vartheta}\left[\mathsf{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right]\ \mathrm{d}x\,\mathrm{d}t \text{ for a.a. } \tau \in (0,T).$$

Moreover, if Ω is a *bounded* domain, it can be shown that

$$\mathrm{ess}\sup_{t\in(0,T)} \int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x \leq c(\text{data}) \tag{57}$$

in terms of the energy estimates (54), (55).

In accordance with hypotheses (47–49), the transport coefficients μ and κ are bounded below away from zero; whence

$$\int_0^T \int_\Omega |\nabla_x \log(\vartheta)|^2\ \mathrm{d}x\,\mathrm{d}t \leq c(\text{data}), \tag{58}$$

and

$$\int_0^T \int_\Omega \frac{\mu(\vartheta)}{\vartheta}\left|\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right|^2\ \mathrm{d}x\,\mathrm{d}t \leq c(\text{data}), \tag{59}$$

together with

$$\int_0^T \int_\Omega |\nabla_x\vartheta|^2\ \mathrm{d}x\,\mathrm{d}t \leq c(\text{data}). \tag{60}$$

The estimates on the velocity gradient to be derived from (59) are more delicate. The easy way would be to assume

$$\mu(\vartheta) \approx \vartheta, \text{ for } \vartheta \gtrsim 1.$$

Under these circumstances, a generalized version of Korn's inequality could be used together with (53), (59) to obtain

$$\int_0^T \int_\Omega |\nabla_x\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t \leq c(\text{data}).$$

Unfortunately, in accordance with the physical background, a realistic behavior of μ is rather

$$\underline{\mu}(1+\sqrt{\vartheta}) \leq \mu(\vartheta) \leq \overline{\mu}(1+\sqrt{\vartheta}),$$

see e.g. Becker [6], yielding only

$$\int_0^T \int_\Omega \frac{1}{\sqrt{\vartheta}} \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 \ \mathrm{d}x \, \mathrm{d}t \leq c,$$

or the more general asymptotic behavior specified through (47). Accordingly, the resulting estimate must be "interpolated" with (55), (60) to obtain

$$\left\| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right\|_{L^2(0,T;L^\Lambda(\Omega;R^{3\times 3}))} \leq c(\mathrm{data}),$$

where, in general, $\Lambda < 2$. The specific value of the Lebesgue exponent Λ depends on α in hypothesis (47), more precisely,

$$\Lambda = \frac{8}{5-\alpha}$$

yielding the regularity of the velocity field claimed in Theorem 2:

$$\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{W^{1,\Lambda}(\Omega;R^3)} \ \mathrm{d}t \leq c, \ \Lambda = \frac{8}{5-\alpha}. \tag{61}$$

4.4 Pressure estimates

The pressure estimates available for the Navier-Stokes-Fourier system read

$$\int_0^T \int_\Omega p(\varrho,\vartheta)\varrho^\beta \ \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}) \tag{62}$$

for a certain $\beta > 0$. They can be derived by using the quantity

$$\psi(t)\mathcal{B}\left[\varrho^\beta - \frac{1}{|\Omega|}\int_\Omega \varrho^\beta \ \mathrm{d}x \right], \ \psi \in C_c^\infty(0,T)$$

as a test function in the momentum equation (21). Here, $\mathcal{B}$ is an operator enjoying the following properties:

- $\mathcal{B}$ is a bounded operator from $\tilde{L}^p(\Omega)$ to $W^{1,p}(\Omega;R^3)$ for any $1 < p < \infty$, where $\tilde{L}^p$ is the subspace of L^p of functions of zero mean;
- $$\mathrm{div}_x \mathcal{B}[v] = v, \ \mathcal{B}[v]|_{\partial\Omega} = 0 \text{ for any } v \in \tilde{L}^p(\Omega);$$
- if, in addition, $v = \mathrm{div}_x \mathbf{h}$, where $\mathbf{h}\cdot\mathbf{n}|_{\partial\Omega} = 0$, we have

$$\|\mathcal{B}[\mathrm{div}_x \mathbf{h}]\|_{L^q(\Omega;R^3)} \leq c(p,q)\|\mathbf{h}\|_{L^q(\Omega;R^3)} \text{ for any } 1 < q < \infty.$$

An example of operator $\mathcal{B}$ was constructed by Bogovskii [7], a detailed analysis of its basic properties may be found in Galdi [42], or Novotný and Straškraba [55]. The proof of (62) can be found in [37, Chapter 3].

5 Weak sequential stability of the solution set of the Navier-Stokes-Fourier system

The problem of weak sequential stability represents a central issue in the mathematical analysis of the Navier-Stokes-Fourier system. Having established all *a priori* bounds in the previous section we consider a family $\{\varrho_n, \mathbf{u}_n, \vartheta_n\}_{n=1}^{\infty}$ of solutions of the full Navier-Stokes-Fourier, assuming, in accordance with (53–55), (58–61), that

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^{\infty}(0,T;L^{5/3}(\Omega)), \tag{63}$$

$$\vartheta_n \to \vartheta \text{ weakly-(*) in } L^{\infty}(0,T;L^{4}(\Omega)) \text{ and weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{64}$$

and

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,\Lambda}(\Omega;R^3)), \tag{65}$$

with $\Lambda = 8/(5-\alpha)$, $2/5 < \alpha \leq 1$. Our goal is to show that the limit triple of functions $[\varrho, \vartheta, \mathbf{u}]$ represents another weak solution of the same problem. Such a property is called *weak sequential stability* of the solution set.

5.1 Div-Curl lemma

Div-Curl lemma developed in the framework of the theory of compensated compactness became one of the most efficient tools of the modern theory of partial differential equations (see Murat [54], Tartar [62], Yi [64]).

Lemma 1. *Let* $\{\mathbf{U}_n\}_{n=1}^{\infty}$, $\{\mathbf{V}_n\}_{n=1}^{\infty}$ *be two sequences of vector fields such that*

$$\mathbf{U}_n \to \mathbf{U} \textit{ weakly in } L^p(R^N;R^N),$$

$$\mathbf{V}_n \to \mathbf{V} \textit{ weakly in } L^q(R^N;R^N),$$

where

$$1 < p,q < \infty,\ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Assume, in addition, that

$$\{\mathrm{div}\mathbf{U}_n\}_{n=1}^{\infty} \textit{ is precomapct in } W^{-1,s}(R^N),$$

$$\{\mathsf{Curl}\mathbf{V}_n\}_{n=1}^{\infty} \text{ is precomapct in } W^{-1,s}(R^{N\times N}),$$

for a certain $s > 1$.

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(R^N).$$

Now, we observe that any sequence of (weak) solutions of a conservation law

$$\partial_t r_n + \mathrm{div}_x \mathbf{F}_n = s_n$$

can be written in the form

$$\mathrm{div}_{t,x}[r_n, \mathbf{F}_n] = s_n$$

in the 4-dimensional space-time cylinder $(0;T) \times \Omega$. Note also that the arguments of Lemma 1 can be easily localized. Next, consider a family of functions $\{V_n\}_{n=1}^{\infty}$ such that

$$\|\nabla_x V_n\|_{L^q(0,T;L^q(\Omega;R^3))} \le c \text{ for some } q > 1,$$

in particular

$$\|\mathsf{Curl}_{t,x}[H(V_n), 0, 0, 0]\| \le c$$

for any $H \in W^{1,\infty}(R)$. A direct application of Div-Curl lemma yields

$$r_n H(V_n) \to r\overline{H(V)} \text{ weakly in } L^r((0,T) \times \Omega) \text{ for any } H \in W^{1,\infty}(R) \tag{66}$$

as soon as

$$r_n \to r,\ \mathbf{F}_n \to \mathbf{F} \text{ weakly in } L^r((0,T) \times \Omega) \text{ for a certain } r > 1,$$

$$H(V_n) \to \overline{H}(V) \text{ weakly-(*) in } L^\infty((0,T) \times \Omega),$$

and

$$\{s_n\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{M}([0,T] \times \overline{\Omega}).$$

Relation (66) can be seen as a kind of "biting limit" (see Brooks and Chacon [9]) yielding

$$r_n V_n \to rV \text{ weakly in } L^1((0,T) \times \Omega) \tag{67}$$

as soon as both $\{r_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ are equi-integrable (weakly precompact) in $L^1((0,T) \times \Omega)$. Here V denotes the limit of $\{V_n\}_{n=1}^{\infty}$.

Div-Curl lemma can be used to identify the weak limit of all convective terms in the weak formulation of the Navier-Stokes-Fourier system. Accordingly, letting $n \to \infty$ we obtain, in the weak sense,

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{68}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho, \vartheta)} = \mathrm{div}_x \overline{\mathsf{S}} \tag{69}$$

$$\partial_t \left(\overline{\varrho s(\varrho, \vartheta)} \right) + \mathrm{div}_x \left(\overline{\varrho s(\varrho, \vartheta)} \mathbf{u} \right) + \mathrm{div}_x \left(\frac{\overline{\mathbf{q}}}{\vartheta} \right) \geq \overline{\frac{1}{\vartheta} \left(\mathsf{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)}, \tag{70}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \overline{\varrho e(\varrho, \vartheta)} \right) \ \mathrm{d}x = 0, \tag{71}$$

where *bar* denotes a weak L^1-limit of composed functions.

In addition, passing to the limit in the renormalized equation of continuity (20) gives rise to

$$\partial_t \overline{b(\varrho)} + \mathrm{div}_x \left(\overline{b(\varrho)} \mathbf{u} \right) + \overline{(b'(\varrho)\varrho - b(\varrho)) \, \mathrm{div}_x \mathbf{u}} = 0. \tag{72}$$

5.2 Strong convergence of the temperature

Our goal is to show that the sequence $\{\vartheta_n\}_{n=1}^\infty$ converges a.a. in $(0,T) \times \Omega$. The presence of the radiation component of the entropy will play a crucial role in the proof. We start with a preliminary result that can be considered as a fundamental theorem of the theory of Young measures (see Ball [3], Pedregal [56]).

Theorem 3. *Let $\{\mathbf{U}_n\}_{n=1}^\infty$ be equi-integrable (weakly precompact) sequence of functions in $L^1(Q; R^M)$, $Q \subset R^N$.*

Then $\{\mathbf{U}_n\}_{n=1}^\infty$ possesses a subsequence (not relabeled) such that there exists a parameterized family of probability measures $\{\nu_y\}_{y \in Q}$ on R^M enjoying the following property:

$$\overline{F(\cdot, \mathbf{U})}(y) = \langle \nu_y, F(y, \cdot) \rangle \ \textit{for a.a. } y \in Q,$$

whenever $F = F(y, \mathbf{U})$ is a Caratheodory function on $Q \times R^M$, and

$$F(\cdot, \mathbf{U}_n) \to \overline{\mathbf{F}(\cdot, \mathbf{U})} \ \textit{weakly in } L^1(Q).$$

In view of the hypothesis of thermodynamics stability (41), the entropy is a strictly increasing function of ϑ, more specifically,

$$\int_0^T \int_\Omega \left(\varrho_n s(\varrho_n, \vartheta_n) - \varrho_n s(\varrho_n, \vartheta) \right) (\vartheta_n - \vartheta) \ \mathrm{d}x \, \mathrm{d}t \geq \frac{4a}{3} \int_0^T \int_\Omega |\vartheta_n - \vartheta|^4 \ \mathrm{d}x \, \mathrm{d}t. \tag{73}$$

Note that we have exploited the presence of the radiation component ϱs_R.

Consequently, to show strong convergence of $\{\vartheta_n\}_{n=1}^\infty$, it is enough to show that the left-hand side of (73) tends to zero. To this end, we start by repeating the arguments applied in the preceding section to the convective terms to show that

$$\overline{\varrho s(\varrho, \vartheta) \vartheta} = \overline{\varrho s(\varrho, \vartheta)} \vartheta. \tag{74}$$

Moreover, by the same token, we can use the piece of information provided by the renormalized equation of continuity to deduce

$$\overline{b(\varrho)h(\vartheta)} = \overline{b(\varrho)}\ \overline{h(\vartheta)} \text{ for all bounded continuous functions } b, h.$$

In terms of the Young measures, this means

$$\mathbf{u}_{t,x}^{(\varrho,\vartheta)} = \mathbf{u}_{t,x}^{\varrho} \otimes \nu_{t,x}^{\vartheta} \text{ for a.a. } (t,x) \in (0,T) \times \Omega, \tag{75}$$

where $\nu^{(\varrho,\vartheta)}$, ν^{ϱ}, and ν^{ϑ} denote the Young measure associated to the family $\{(\varrho_n, \vartheta_n)\}_{n=1}^{\infty}$, $\{\varrho_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$, respectively.

Finally, Theorem 3 yields

$$\varrho_n s(\varrho_n, \vartheta_n)(\vartheta_n - \vartheta) \to 0 \text{ weakly in } L^1((0,T) \times \Omega),$$

which, together with (74), gives rise to the desired conclusion (73). Thus we have shown that

$$\vartheta_n \to \vartheta \text{ (strongly) in } L^4((0,T) \times \Omega). \tag{76}$$

5.3 Strong convergence of the density

To show strong (pointwise a.a.) convergence of the family of densities we introduce the cut-off functions

$$T_k(z) = kT\left(\frac{z}{k}\right),\ z \geq 0,\ k \geq 1, \tag{77}$$

where $T \in C^{\infty}[0,\infty)$ satisfies

$$T(z) = \begin{cases} z \text{ for } 0 \leq z \leq 1, \\ \\ \text{concave for } 1 \leq z \leq 3, \\ \\ 2 \text{ for } z \geq 3. \end{cases}$$

By virtue of (72), we have

$$\partial_t \overline{\varrho L_k(\varrho)} + \operatorname{div}_x \left(\overline{\varrho L_k(\varrho)} \mathbf{u} \right) + \overline{T_k(\varrho) \operatorname{div}_x \mathbf{u}} = 0, \tag{78}$$

where we have set

$$L_k(\varrho) = \int_1^{\varrho} \frac{T_k(z)}{z^2}\ \mathrm{d}z.$$

Next, following the approach of Lions [49], we need also the renormalized equation to hold for the limit ϑ, $\mathbf{u}$, namely

$$\partial_t \varrho L_k(\varrho) + \mathrm{div}_x\,(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\mathrm{div}_x\mathbf{u} = 0. \tag{79}$$

Note that this step is not completely obvious since the regularizing technique introduced by DiPerna and Lions [26] does not apply here because of the low degree of integrability of ϱ (and also of $\mathbf{u}$). Fortunately, this problem can be solved the method developed in [30] introducing the oscillations defect measure associated to the sequence $\{\varrho_n\}_{n=1}^{\infty}$, specifically,

$$\mathbf{osc}_p[\varrho_n \to \varrho](Q) = \sup_{k\geq 1}\left(\limsup_{n\to\infty}\int_Q |T_k(\varrho_n) - T_k(\varrho)|^p \ \ \mathrm{d}x\ \ \mathrm{d}t\right). \tag{80}$$

We report the following result proved in [32], [39].

Proposition 1. *Let $Q \subset (0,T)\times\Omega$ be a domain. Suppose that*

$$\varrho_n \to \varrho \textit{ weakly in } L^1(Q),$$

$$\mathbf{u}_n \to \mathbf{u} \textit{ weakly in } L^r(Q;R^3),$$

$$\nabla_x\mathbf{u}_n \to \nabla_x\mathbf{u} \textit{ weakly in } L^r(Q;R^{3\times 3})$$

and

$$\mathbf{osc}_p[\varrho_n \to \varrho](Q) < \infty \textit{ for a certain } p \textit{ such that } \frac{1}{p} + \frac{1}{r} < 1.$$

Finally, let $\{\varrho_n, \mathbf{u}_n\}_{n=1}^{\infty}$ satisfy the renormalized equation (20) in $\mathcal{D}'(Q)$.

Then ϱ, $\mathbf{u}$ also satisfy (20) in $\mathcal{D}'(Q)$.

In order to apply Proposition 1 to justify (79) we have to find bounds on the oscillation defect measure associated to the family of densities. To this end, we take

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x\Delta^{-1}[1_\Omega T_k(\varrho_n)],\ \psi \in C_c^\infty(0,T),\ \phi \in C_c^\infty(\Omega)$$

as a test function in the momentum balance (21). Here, the symbol Δ^{-1} denotes the inverse of the Laplace operator defined on the whole space R^3 by means of convolution with the Poisson kernel. After a bit lengthy but entirely straightforward manipulation, we obtain

$$\int_0^T \int_\Omega \psi\phi\Big(p(\varrho_n,\vartheta_n)T_k(\varrho_n) - \mathsf{S}_n : \mathcal{R}[1_\Omega T_k(\varrho_n)]\Big)\ \mathrm{d}x\,\mathrm{d}t \tag{81}$$

$$= \int_0^T \int_\Omega \psi\phi\Big(\varrho_n\mathbf{u}_n\cdot\mathcal{R}[1_\Omega T_k(\varrho_n)\mathbf{u}_n] - \varrho_n\mathbf{u}_n\otimes\mathbf{u}_n : \mathcal{R}[1_\Omega T_k(\varrho_n)]\Big)\ \mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{j=1}^{5} I_{j,n},$$

where the symbol $\mathcal{R} = \mathcal{R}_{i,j}$ denotes the pseudodifferential operator of zero order $\mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j}$, or, in terms of the Fourier symbols

$$\mathcal{R}_{i,j}[v] = \mathcal{F}^{-1}_{\xi \to x}\left[\frac{\xi_i \xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\right],$$

with $\mathcal{F}$ denoting the standard Fourier transform. The integrals on the right-hand side of (81) read:

$$I_{1,n} = -\int_0^T \int_\Omega \psi\phi\varrho_n \mathbf{u}_n \cdot \nabla_x \Delta^{-1}\left[1_\Omega \Big(T_k(\varrho_n) - T_k'(\varrho_n)\varrho_n\Big)\mathrm{div}_x \mathbf{u}_n\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_{2,n} = -\int_0^T \int_\Omega \psi p(\varrho_n, \vartheta_n)\nabla_x\phi \cdot \nabla_x \Delta^{-1}\left[1_\Omega T_k(\varrho_n)\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_{3,n} = \int_0^T \int_\Omega \psi \mathsf{S}_n : \nabla_x\phi \otimes \nabla_x \Delta^{-1}\left[1_\Omega T_k(\varrho_n)\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_{4,n} = \int_0^T \int_\Omega \psi \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x\phi \otimes \nabla_x \Delta^{-1}\left[1_\Omega T_k(\varrho_n)\right] \ \mathrm{d}x\,\mathrm{d}t,$$

and

$$I_{5,n} = -\int_0^T \int_\Omega \partial_t\psi\phi\varrho_n \mathbf{u}_n \cdot \nabla_x \Delta^{-1}[1_\Omega T_k(\varrho_n)] \ \mathrm{d}x\,\mathrm{d}t.$$

Similarly, using

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x \Delta^{-1}[1_\Omega \overline{T_k(\varrho)}], \ \psi \in C_c^\infty(0,T), \ \phi \in C_c^\infty(\Omega)$$

as a test function in the weak formulation of the limit equation (69) we obtain

$$\int_0^T \int_\Omega \psi\phi\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \mathsf{S} : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}]\Big) \ \mathrm{d}x\,\mathrm{d}t \tag{82}$$

$$= \int_0^T \int_\Omega \psi\phi\Big(\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega \overline{T_k(\varrho)}\mathbf{u}] - \varrho\mathbf{u}\otimes\mathbf{u} : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}]\Big) \ \mathrm{d}x\,\mathrm{d}t$$

$$+ \sum_{j=1}^5 I_j,$$

with

$$I_1 = -\int_0^T \int_\Omega \psi\phi\varrho\mathbf{u} \cdot \nabla_x \Delta^{-1}\left[1_\Omega \overline{\Big(T_k(\varrho) - T_k'(\varrho)\varrho\Big)\mathrm{div}_x \mathbf{u}}\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_2 = -\int_0^T \int_\Omega \psi\overline{p(\varrho,\vartheta)}\nabla_x\phi \cdot \nabla_x \Delta^{-1}\left[1_\Omega \overline{T_k(\varrho)}\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_3 = \int_0^T \int_\Omega \psi \mathsf{S} : \nabla_x\phi \otimes \nabla_x \Delta^{-1}\left[1_\Omega \overline{T_k(\varrho)}\right] \ \mathrm{d}x\,\mathrm{d}t,$$

$$I_4 = \int_0^T \int_\Omega \psi \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \phi \otimes \nabla_x \Delta^{-1} \left[1_\Omega \overline{T_k(\varrho)} \right] \ \mathrm{d}x \, \mathrm{d}t,$$

and

$$I_5 = - \int_0^T \int_\Omega \partial_t \psi \phi \varrho \mathbf{u} \cdot \nabla_x \Delta^{-1} [1_\Omega \overline{T_k(\varrho)}] \ \mathrm{d}x \, \mathrm{d}t.$$

We claim that all integrals on the right-hand side of (81) converge to their counterparts in (82), in particular, we may infer that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \psi \phi \Big(p(\varrho_n, \vartheta_n) T_k(\varrho_n) - \mathsf{S}_n \mathcal{R}[1_\Omega T_k(\varrho_n)] \Big) \ \mathrm{d}x \, \mathrm{d}t \tag{83}$$

$$= \int_0^T \int_\Omega \psi \phi \Big(\overline{p(\varrho, \vartheta) T_k(\varrho)} - \mathsf{S} : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \Big) \ \mathrm{d}x \, \mathrm{d}t.$$

To see this, observe first that, by virtue of the regularizing effect of the operator $\nabla_x \Delta^{-1}$, $I_{j,n} \to I_j$ as $n \to \infty$ for any $j = 1, \dots, 5$, see [32], [39] for details. To handle the remaining term, we report the following result that can be seen as a direct consequence of Div-Curl lemma:

Lemma 2. *Let*

$$\mathbf{U}_n \to \mathbf{U} \ \textit{weakly in} \ L^p(R^3; R^3),$$

$$\mathbf{V}_n \to \mathbf{V} \ \textit{weakly in} \ L^q(R^3; R^3),$$

where $p, q \geq 1$,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Then

$$\mathbf{U}_n \cdot \mathcal{R}[\mathbf{V}_n] - \mathbf{V}_n \cdot \mathcal{R}[\mathbf{U}_n] \to \mathbf{U} \cdot \mathcal{R}[\mathbf{V}] - \mathbf{V} \cdot \mathcal{R}[\mathbf{U}] \ \textit{weakly in} \ L^r(R^3).$$

To see the conclusion of Lemma 2, it is enough to rewrite

$$\mathbf{U}_n \cdot \mathcal{R}[\mathbf{V}_n] - \mathbf{V}_n \cdot \mathcal{R}[\mathbf{U}_n] = (\mathbf{U}_n - \mathcal{R}[\mathbf{U}_n]) \, \mathcal{R}[\mathbf{V}_n] - (\mathbf{V}_n - \mathcal{R}[\mathbf{V}_n]) \, \mathcal{R}[\mathbf{U}_n]$$

and to apply Lemma 1. Indeed

$$\mathrm{div}_x \left(\mathbf{U}_n - \mathcal{R}[\mathbf{U}_n] \right) = \mathrm{div}_x \left(\mathbf{V}_n - \mathcal{R}[\mathbf{V}_n] \right), \ \mathsf{Curl} \mathcal{R}[\mathbf{U}_n] = \mathsf{Curl} \mathcal{R}[\mathbf{V}_n] = 0.$$

Now, since

$$\int_0^T \int_\Omega \psi \phi \Big(\varrho_n \mathbf{u}_n \cdot \mathcal{R}[1_\Omega T_k(\varrho_n) \mathbf{u}_n] - \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \mathcal{R}[1_\Omega T_k(\varrho_n)] \Big) \ \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \psi \mathbf{u}_n \cdot \Big(\mathcal{R}[\phi \varrho_n \mathbf{u}_n] 1_\Omega T_k(\varrho_n) - \mathcal{R}[1_\Omega T_k(\varrho_n)] \phi \varrho_n \mathbf{u}_n \Big) \ \mathrm{d}x \, \mathrm{d}t,$$

we deduce by means of Lemma 2 that

$$\mathcal{R}[\phi\varrho_n\mathbf{u}_n]1_\Omega T_k(\varrho_n) - \mathcal{R}[1_\Omega T_k(\varrho_n)]\phi\varrho_n\mathbf{u}_n$$

$$\to \mathcal{R}[\phi\varrho\mathbf{u}]1_\Omega\overline{T_k(\varrho)} - \mathcal{R}[1_\Omega\overline{T_k(\varrho)}]\phi\varrho\mathbf{u} \text{ in } L^2(0,T;W^{-1,q'}(R^3;R^3),$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,q}(R^3;R^3)$$

for certain conjugate exponents q, q' provided $\mathbf{u}_n$ was extended as a function in $W^{1,q}(R^3;R3)$ outside Ω. Accordingly, the desired relation (83) follows.

At this stage, the crucial observation is that relation (83), rewritten in the form

$$\lim_{n\to\infty}\int_0^T\int_\Omega \psi\Big(\phi p(\varrho_n,\vartheta_n)T_k(\varrho_n) - T_k(\varrho_n)\mathcal{R}:[\phi\mathsf{S}_n]\Big)\ \mathrm{d}x\,\mathrm{d}t \tag{84}$$

$$= \int_0^T\int_\Omega \psi\Big(\phi\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{T_k(\varrho)}\mathcal{R}:[\phi\mathsf{S}]\Big)\ \mathrm{d}x\,\mathrm{d}t$$

gives rise to

$$\lim_{n\to\infty}\int_0^T\int_\Omega \psi\phi\Big(p(\varrho_n,\vartheta_n)T_k(\varrho_n) - T_k(\varrho_n)\left(\frac{4}{3}\mu(\vartheta_n)+\eta(\vartheta_n)\right)\mathrm{div}_x\mathbf{u}_n\Big)\ \mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T\int_\Omega \psi\phi\Big(\overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \overline{T_k(\varrho)}\left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}\Big)\ \mathrm{d}x\,\mathrm{d}t, \tag{85}$$

where the quantity $p-(4/3\mu+\eta)\mathrm{div}_x\mathbf{u}$ is the celebrated effective viscous flux introduced by Lions [49], see also Hoff [43], [44], Serre [60].

Next, observe that quantities appearing (84) and (85) differ by a commutator of $\mathcal{R}$ with the operator of multiplication on a scalar function μ. Consequently, in order to see how (84) yields (85), we need the following result that may be viewed as a particular application of the general theory developed by Coifman and Meyer [20] (see also Coifman et al. [19] and [31]).

Lemma 3. *Let $\mu \in W^{1,2}(R^3)$ be a scalar function and $\mathbf{V} \in L^r \cap L^1(R^3;R^3)$ a vector field, $r > \frac{6}{5}$.*

Then

$$\left\|(\nabla_x\Delta^{-1}\mathrm{div}_x)[\mu\mathbf{V}] - \mu(\nabla_x\Delta^{-1}\mathrm{div}_x)[\mathbf{V}]\right\|_{W^{\omega,p}(R^3)}$$

$$\leq c\,\|\mu\|_{W^{1,2}(R^3)}\,\|\mathbf{V}\|_{L^r\cap L^1(R^3;R^3)}$$

for certain $\omega > 0$, $p > 1$.

As

$$T_k(\varrho_n) \to \overline{T_k(\varrho)} \text{ in } C_{\mathrm{weak}}([0,T];L^q(\Omega)) \text{ for any } 1 < q < \infty,$$

Lemma 3, together with (84), imply (85).

Next, relation (85) yields

$$\overline{p_m(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} \tag{86}$$

$$= \overline{p_m(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}.$$

Now, as we have already established the strong convergence of the temperature, it can be shown that

$$\overline{p_m(\varrho,\vartheta)T_k(\varrho)} - \overline{p_m(\varrho,\vartheta)}\ \overline{T_k(\varrho)} \geq c\,\mathbf{osc}_{8/3}[\varrho_n \to \varrho]((0,T)\times\Omega); \tag{87}$$

whence, after some manipulation (see [32]), we deduce that

$$\mathbf{osc}_p[\varrho_n \to \varrho]((0,T)\times\Omega) < \infty \text{ for a certain } p > \frac{8}{3+\alpha}. \tag{88}$$

Indeed the "molecular" component of the pressure p_m can be written in the form $p_m = p_{\mathrm{mon}} + p_{\mathrm{conv}}$, where p_{mon} is non-decreasing in ϱ, while p_{conv} is a convex function, $p_{\mathrm{conv}}(\varrho) \approx \varrho^{5/3}$. Moreover, it can be checked by direct inspection that

$$\int_0^T \int_\Omega \left(\overline{\varrho^{5/3}T_k(\varrho)} - \overline{\varrho^{5/3}}\ \overline{T_k(\varrho)}\right)\ \mathrm{d}x\,\mathrm{d}t$$

$$\geq \lim_{n\to\infty}\int_0^T\int_\Omega |T_k(\varrho_n)-T_k(\varrho)|^{8/3}\ \mathrm{d}x\,\mathrm{d}t,$$

see [32] for details.

By virtue of Proposition 1, the limit functions ϱ, $\mathbf{u}$ therefore satisfy equation (79), in particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\right)\ \mathrm{d}x + \int_\Omega \left(\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\right)\ \mathrm{d}x$$

$$= \int_\Omega \left(T_k(\varrho)\mathrm{div}_x\mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\right)\ \mathrm{d}x.$$

Thus, letting $k \to \infty$ we may infer that

$$\int_\Omega \left(\overline{\varrho\log(\varrho)} - \varrho\log(\varrho)\right)(\tau,\cdot)\ \mathrm{d}x \leq \int_\Omega \left(\overline{\varrho_0\log(\varrho_0)} - \varrho_0\log(\varrho_0)\right)\ \mathrm{d}x, \tag{89}$$

which implies the desired conclusion

$$\varrho_n \to \varrho \text{ (up to a subsequence) a.a. in } (0,T)\times\Omega \tag{90}$$

as soon as the left-hand side of (89) vanishes, meaning as soon as the initial densities are precompact in $L^1(\Omega)$.

The strong (pointwise) convergence established in (76), (90) entails the property of weak sequential compactness for the Navier-Stokes-Fourier system in the entropy formulation and under the structural restrictions specified in Section 3.6. A similar property can be shown for the internal energy weak formulation, see [30, Chapter 5].

6 Relative energy, dissipative solutions, stability

In this section, we address the problem of stability of solutions to the Navier-Stokes-Fourier system. In particular, we find a convenient way how to measure the distance of a weak solution $[\varrho, \vartheta, \mathbf{u}]$ to an arbitrary trio of sufficiently smooth functions $[r, \Theta, \mathbf{U}]$. The hypothesis of thermodynamics stability (41) will play a crucial role.

6.1 Relative (modulated) energy

For an energetically isolated system, the total energy and mass

$$E = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x, \ M = \int_\Omega \varrho \ \mathrm{d}x$$

are constants of motion, while the total entropy

$$S = \int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x$$

is non-decreasing in time. In particular, the so-called *ballistic free energy*

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta), \ \Theta > 0 \text{ constant},$$

augmented by the kinetic energy gives rise to a *Lyapunov functional*

$$E_B = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta) \right] \ \mathrm{d}x.$$

As a direct consequence of the thermodynamic stability hypothesis (41), the ballistic free energy function enjoys two remarkable properties:

$$\begin{gathered} \varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is a convex function} \\ \vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ attains its global minimum at } \vartheta = \Theta \text{ for any fixed } \varrho. \end{gathered} \tag{91}$$

This motivates the following definition of the *relative (modulated) energy functional*

$$\mathcal{E}\left(\varrho, \vartheta \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right) \tag{92}$$
$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right] \ \mathrm{d}x.$$

It follows from (91) that

$$\mathcal{E}\left(\varrho, \vartheta \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right) \geq 0,\ \mathcal{E}\left(\varrho, \vartheta \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right) = 0 \text{ only if } \varrho = r,\ \vartheta = \Theta,\ \mathbf{u} = \mathbf{U}.$$

6.2 Dissipative solutions

The strength of the concept of relative energy lies in the fact that the time evolution of $\mathcal{E}\left(\varrho, \vartheta \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$ can be computed for any *weak* (entropy formulation) solution to the Navier-Stokes-Fourier system provided the trio of functions $[r, \Theta, \mathbf{U}]$ is smooth enough to be taken as admissible test functions in (19), (21), (23). Indeed the following *relative energy inequality*

$$\left[\mathcal{E}\left(\varrho, \vartheta \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \tag{93}$$
$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathsf{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \ \mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_0^\tau \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u}) \ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla_x\Theta \ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \varrho\Big(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u}) \ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[\mathsf{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{U} - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\Big] \ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[\varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\mathbf{U}\cdot\nabla_x\Theta\Big] \ \mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta \ \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[\Big(1 - \frac{\varrho}{r}\Big)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\Big] \ \mathrm{d}x\,\mathrm{d}t$$

holds for any weak solution of the Navier-Stokes-Fourier system (19–24) and any trio of (smooth) test functions satisfying the compatibility conditions

$$r > 0, \ \Theta > 0, \ \mathbf{U}|_{\partial\Omega} = 0 \text{ or } \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ as the case may be,} \tag{94}$$

see [38, Section 3].

Following Lions [48], who proposed a similar definition for the incompressible Euler system, we may say that $[\varrho, \vartheta, \mathbf{u}]$ is a *dissipative solution* to the Navier-Stokes-Fourier system (3), (7), (11) if **(i)** $[\varrho, \vartheta, \mathbf{u}]$ belong to the regularity class specified in Theorem 2, **(ii)** $[\varrho, \vartheta, \mathbf{u}]$ satisfy the relative energy inequality (93) for any trio $[r, \Theta, \mathbf{U}]$ of sufficiently smooth (for all integrals in (93) to be well defined) test functions satisfying the compatibility conditions (94). As observed in [38, Section 3], any weak solution of the Navier-Stokes-Fourier system (19–24) is a dissipative solution. An existence theory in the framework of dissipative solutions was developed and applied to a vast class of physical spaces, including unbounded domains in R^3, see Jesslé, Jin, Novotný [45].

6.3 Weak–strong uniqueness

An important property of the dissipative solutions is that they coincide with the strong solution of the same problem as long as the latter exists. Since weak solutions are dissipative, this remains true also for the weak solutions. The proof of the following statement can be found in [33, Theorem 6.2], and [38, Theorem 2.1]:

Theorem 4. *In addition to the hypotheses of Theorem 2, suppose that*

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \ \textit{with } S(Z) \to 0 \textit{ as } Z \to \infty. \tag{95}$$

Let $[\varrho, \vartheta, \mathbf{u}]$ be a dissipative (weak) solution to the Navier-Stokes-Fourier system (19–24) in the set $(0,T) \times \Omega$. Suppose that the Navier-Stokes-Fourier system admits a strong solution $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ in the time interval $(0,T)$, emanating from the same initial data and belonging to the class

$$\partial_t \tilde{\varrho}, \ \partial_t \tilde{\vartheta}, \ \partial_t \tilde{\mathbf{u}}, \partial_x^m \tilde{\varrho}, \ \partial_x^m \tilde{\vartheta}, \ \partial_x^m \tilde{\mathbf{u}} \in L^\infty((0,T) \times \Omega), \ m = 0, 1, 2.$$

Then

$$\varrho = \tilde{\varrho}, \ \vartheta = \tilde{\vartheta}, \ \mathbf{u} = \tilde{\mathbf{u}}.$$

The extra hypothesis (95) reflects the Third law of thermodynamics and can be possibly relaxed. Whether or not the Navier-Stokes-Fourier system admits global-in-time strong solutions is an interesting open question, for small data results see Matsumura and Nishida [52], [53].

6.4 Weak solutions based on the thermal energy formulation

All results of this section sofar applied to the entropy weak formulation (19–24) of the Navier-Stokes-Fourier system. A natural question to ask is to which extent the same idea may be used to the weak solution based on the thermal energy balance (25). As (23), (25) are apparently not equivalent in the weak framework, this is a non-trivial issue we want to address in this section. To this end, we consider smooth initial data, specifically,

$$\varrho_0,\ \vartheta_0 \in W^{3,2}(\Omega),\ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3),\ \varrho_0 > 0,\ \vartheta_0 > 0. \tag{96}$$

Our first result provides necessary conditions for a weak solution of (19), (21), (25), and (26) to satisfy the entropy balance (23), see [41, Lemmas 2.3, 2.4]

Proposition 2. *Under the hypotheses of Theorem 1, let* $[\varrho, \vartheta, \mathbf{u}]$ *be a weak solution of the Navier-Stokes-Fourier system (19), (21), (25), (26) originating from the initial data satisfying (96) and enjoying the extra regularity*

$$\varrho, \vartheta, \mathrm{div}_x \mathbf{u} \in L^\infty((0,T)\times\Omega),\ \mathbf{u} \in L^\infty((0,T)\times\Omega; R^3).$$

Then

$$\varrho > 0,\ \vartheta > 0 \ \textit{a.a. in}\ (0,T)\times\Omega$$

and the entropy balance (23) holds.

With the entropy inequality (23) at hand we may use the technique based on the relative energy functional $\mathcal{E}(\varrho, \vartheta, \mathbf{u} | r, \Theta, \mathbf{U})$ developed in the previous section. In particular, we have (see [41, Lemma 3.2]):

Proposition 3. *Under the hypotheses of Proposition 2, let* $[\tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}]$ *be a strong solution of the Navier-Stokes-Fourier system defined in* $(0T)$*, emanating from the initial data*

$$\tilde\varrho(0,\cdot) = \varrho(0,\cdot),\ \tilde\vartheta(0,\cdot) = \vartheta(0,\cdot),\ \tilde{\mathbf{u}}(0,\cdot) = \mathbf{u}(0,\cdot),$$

and belonging to the regularity class

$$\left\{ \begin{array}{c} \varrho,\ \vartheta \in C([0,T]; W^{3,2}(\Omega)),\ \mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)) \\ \\ \vartheta \in L^2(0,T; W^{4,2}(\Omega)),\ \mathbf{u} \in L^2(0,T; W^{4,2}(\Omega; R^3)), \\ \\ \partial_t \vartheta \in L^2(0,T; W^{2,2}(\Omega)),\ \partial_t \mathbf{u} \in L^2(0,T; W^{2,2}(\Omega; R^3)). \end{array} \right\}$$

Then $[\tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}]$ *coincides with the weak solution* $[\varrho, \vartheta, \mathbf{u}]$ *in* $(0,T)\times\Omega$*.*

Finally, we claim a conditional regularity result concerning the weak solutions of the Navier-Stokes-Fourier system, see [41, Theorem 2.2].

Theorem 5. *Under the hypotheses of Proposition 2, let* $[\varrho, \vartheta, \mathbf{u}]$ *be a weak solution of the Navier-Stokes-Fourier system (19), (21), (25), (26), emanating from regular initial data satisfying (96), and enjoying the extra regularity*

$$\varrho,\ \vartheta,\ \mathrm{div}_x \mathbf{u} \in L^\infty((0,T)\times\Omega),\ \mathbf{u} \in L^\infty((0,T)\times\Omega; R^3).$$

The $[\varrho, \vartheta, \mathbf{u}]$ *is a strong (classical) solution of the problem in* $(0,T)\times\Omega$*.*

6.5 Synergy analysis-numerics

From the practical point of view, the convergence of the numerical scheme established in Theorem 1 is not very satisfactory as it holds up to a suitable subsequence. As the weak solutions to the Navier-Stokes-Fourier system are not (known to be) unique, it is therefore not a priori excluded that there is another subsequence converging to a different solution of the same problem. However, combining Theorem 1 with Theorem 5 we may deduce the following unconditional convergence result that can be seen as an example of "synergy" between analysis and numerics:

Theorem 6. *Under the hypotheses of Theorem 1, let* $[\varrho_h, \vartheta_h, \mathbf{u}_h]_{h>0}$ *be a family of approximate solutions resulting from the numerical scheme (37–39), emanating from the initial data (96), such that*

$$\varrho_h > 0,\ \vartheta_h > 0,$$

and, in addition,

$$\varrho_h,\ \vartheta_h,\ |\mathbf{u}_h|,\ |\mathrm{div}_h \mathbf{u}_h| \leq M$$

a.a. in $(0,T)\times\Omega$ *for a certain* M *independent of* h*.*

Then

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty((0,T)\times\Omega)) \text{ and strongly in } L^1((0,T)\times\Omega),$$

$$\vartheta_h \to \vartheta \text{ weakly-(*) in } L^\infty((0,T)\times\Omega),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega; R^3),$$

$$\nabla_h \mathbf{u}_h \to \nabla_x \mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega; R^{3\times 3}),$$

where $[\varrho, \vartheta, \mathbf{u}]$ *is the (strong) solution of the Navier-Stokes-Fourier system in* $(0,T)\times\Omega$*.*

7 Viscosity solutions, inviscid limits

The Navier-Stokes-Fourier system describes the motion of a viscous and heat conducting fluid; the shear viscosity coefficient μ as well as the heat conductivity coefficient κ are (strictly) positive. Accordingly, the entropy production rate is strictly positive till the system reaches a thermodynamic equilibrium. The *inviscid* fluids, described by means of the Euler system, may be seen as the limit case of their viscous counterparts, where the viscosity and/or the heat conductivity vanishes. Solutions of the purely hyperbolic systems of conservation laws describing the motion of inviscid fluids exhibit very irregular behavior including the appearance of singularities - shock waves - in a finite time lap.

The concept of *weak* or even more general *measure-valued solution* is therefore indispensable in the mathematical theory of inviscid fluids. In the absence of a sufficiently strong dissipative mechanism, solutions of non-linear systems of conservation laws may develop fast oscillations and/or concentrations that inevitably give rise to singularities of various types. As shown in the nowadays classical work of Tartar [62], oscillations are involved in many problems, in particular in those arising in the context of inviscid fluids.

The well know deficiency of weak solutions is that they may not be uniquely determined in terms of the data and suitable admissibility criteria must be imposed in order to pick up the physically relevant ones, cf. Dafermos [23]. Although most of the admissibility constraints are derived from fundamental physical principles as the Second law of thermodynamics, their efficiency in eliminating the nonphysical solutions is still dubious, cf. Dafermos [24]. Recently, DeLellis and Székelyhidi [25] developed the method previously known as *convex integration* in the context of fluid mechanics, in particular for the Euler system. Among other interesting results, they show the existence of infinitely many solutions to the incompressible Euler system violating many of the standard admissibility criteria. Later, the method was adapted to the compressible case by Chiodaroli [16].

7.1 Euler–Fourier system

The class of weak solutions is apparently much larger than required by the classical theory. In other words, it might be easier to establish existence but definitely more delicate to show uniqueness among all possible weak solutions emanating form the same initial data. Adapting the technique of DeLellis and Székelyhidi [25] we show a rather illustrative but at the same time disturbing example of non-uniqueness in the context of fluid thermodynamics. To this end, consider the so-called Euler-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{97}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x(\varrho\vartheta) = 0, \tag{98}$$

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u} \tag{99}$$

System (97–99) The system can be viewed as a "special" case of the Navier-Stokes- Fourier system with $p = \varrho\vartheta$, $c_v = 3/2$, $\mu = \eta - 0$, and $\kappa = 1$. Although a correct physical justification of an inviscid, and, at the same time heat conducting fluid my be dubious, the system has been used as a suitable approximation in certain models, see Wilcox [63].

7.1.1 Infinitely many weak solutions

For the sake of simplicity, we consider the spatially periodic boundary conditions, meaning the underlying spatial domain

$$\Omega = \mathcal{T}^3 = \left([-1,1]|_{\{-1,1\}}\right)^3$$

is the "flat" torus.

We report the following result, see [18, Theorem 3.1]:

Theorem 7. *Let $T > 0$ be given. Let the initial data satisfy*

$$\varrho_0,\ \vartheta_0 \in C^3(\mathcal{T}^3),\ \mathbf{u}_0 \in C^3(\mathcal{T}^3;R^3),\ \varrho_0 > 0,\ \vartheta_0 > 0 \text{ in } \mathcal{T}^3.$$

Then the initial-value problem for the Euler-Fourier system (97–99) admits infinitely many weak solutions in $(0;T)\times\mathcal{T}^3$ belonging to the class

$$\varrho \in C^2([0,T]\times\mathcal{T}^3),\ \partial_t\vartheta \in L^p(0,T;L^p(\mathcal{T}^3)),\ \nabla_x^2\vartheta L^p(0,T;L^p(\mathcal{T}^3;R^3))$$

for any $1 \le p < \infty$,

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\mathcal{T}^3;R^3))\cap L^\infty((0,T)\times\mathcal{T}^3;R^3),\ \mathrm{div}_x\mathbf{u} \in C^2([0,T]\times\mathcal{T}^3).$$

7.1.2 Infinitely many admissible weak solutions

The infinitely many solutions claimed in Theorem 7 are obtained in a non-constructive way by applying a variant of the method of convex integration in the spirit of DeLellis and Székelyhidi [25]. Apparently, many of them are non-physical since they violate the First law of thermodynamics, notably

$$\mathrm{ess}\liminf_{t\to 0+}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\vartheta\right](t,\cdot)\ \mathrm{d}x > \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + c_v\varrho_0\vartheta_0\right]\ \mathrm{d}x.$$

This fact motivates the following admissibility criterion:

We say that a weak solution $[\varrho, \vartheta, \mathbf{u}]$ *of the Euler-Fourier system (97–99), supplemented with the initial data* $[\varrho_0, \vartheta_0, \mathbf{u}_0]$*, is admissible, if the energy inequality*

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta \right] (t, \cdot) \ \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + c_v \varrho_0 \vartheta_0 \right] \ \mathrm{d}x$$

holds for a.a. $t \in (0, T)$*.*

Similarly to the Navier-Stokes-Fourier system, it can be shown that admissible solutions satisfy the relative energy inequality (93) (with $\mathsf{S} = 0$) and enjoy the weak-strong uniqueness property, cf. [18]. Still the following result holds true, see [18, Theorem 4.2]:

Theorem 8. *Let* $T > 0$ *and the initial data*

$$\varrho_0, \ \vartheta_0 \in C^3(\mathcal{T}^3), \ \varrho_0 > 0, \ \vartheta_0 > 0 \ \textit{in} \ \mathcal{T}^3.$$

be given.

The there exists

$$\mathbf{u}_0 \in L^\infty(\mathcal{T}^3; R^3)$$

such that the initial-value problem for the Euler-Fourier system (97–99) admits infinitely many admissible weak solutions in $(0; T) \times \mathcal{T}^3$ *belonging to the class specified in Theorem 7.*

7.2 Riemann problem

At first glance, the velocity field $\mathbf{u} \in L^\infty$, the existence of which is claimed in Theorem 8, may seem rather irregular and possibly never reachable by trajectories emanating from "nice" initial data. Unfortunately, the situation is more delicate as illustrated by the following example due to Chiodaroli, DeLellis, and Kreml [17]. They consider the barotropic Euler system:

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{100}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho^2 = 0 \tag{101}$$

in $(0, T) \times R^2$, endowed with the 1D Riemann initial data data

$$\varrho_0 = \begin{cases} \varrho_- \ \text{for} \ x_1 < 0, \\ \\ \varrho_+ \ \text{for} \ x_1 > 0 \end{cases} \tag{102}$$

$$u_0^1 = \begin{cases} v_- \ \text{for} \ x_1 < 0, \\ \\ v_+ \ \text{for} \ x_1 > 0 \end{cases}, \ u_0^2 = 0. \tag{103}$$

It can be shown (see Chiodaroli et al. [17, Theorem 1.1]) that there are initial data (102), (103) such that the Riemann problem (100–103) admits infinitely many weak solutions satisfying the standard entropy admissibility criterion

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho^2 \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho^2 \right) \mathbf{u} \right] \leq 0. \tag{104}$$

What is more, such solutions may be extended backward in time as Lipschitz functions yielding regular initial data for which system (100), (101), supplemented with (104), admits infinitely many solutions, see Chiodaroli et al. [17, Corollary 1.2]:

Theorem 9. *There exist Lipschitz initial data* $[\varrho_0, \mathbf{u}_0]$ *for which the barotropic Euler system (100), (101) admits infinitely many weak solutions in* $(0,T) \times R^2$ *satisfying (104). In addition, the initial data are independent of the* x_2 *variable and* $u_0^2 = 0$.

7.2.1 Riemann problem for the full Euler system

Motivated by the previous results we consider the Riemann problem for the full Euler system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{105}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (\varrho \vartheta) = 0, \tag{106}$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + c_v \right) \mathbf{u} \right] = 0 \tag{107}$$

with the associated entropy inequality

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) \geq 0, \ \ s = s(\varrho, \vartheta) = \log \left(\frac{\vartheta^{c_v}}{\varrho} \right). \tag{108}$$

Similarly to the preceding section, we consider the Cauchy problem for the system (105–108) in the 2D-case in the spatial domain

$$\Omega = R^1 \times \mathcal{T}^1, \ \text{where } \mathcal{T}^1 = [0,1]|_{\{0,1\}} \text{ is the "flat" sphere,}$$

meaning meaning all functions of (t, x_1, x_2) are 1-periodic with respect to the second spatial coordinate x_2.

We introduce 1D Riemannian data

$$\varrho(0, x_1, x_2) = R_0(x_1) = \begin{cases} R_L \text{ for } x_1 < 0, \\ \\ R_R \text{ for } x_1 > 0, \end{cases} \tag{109}$$

$$\vartheta(0,x_1,x_2) = \Theta_0(x_1) = \begin{cases} \Theta_L \text{ for } x_1 < 0, \\ \\ \Theta_R \text{ for } x_1 > 0, \end{cases} \tag{110}$$

$$u_0^1(0,x_1,x_2) = U_0(x_1) = \begin{cases} U_L \text{ for } x_1 < 0, \\ \\ U_R \text{ for } x_1 > 0, \end{cases} \quad u_0^2(0,x_1,x_2) = 0. \tag{111}$$

As is well known, see for instance Chang and Hsiao [12], the Riemann problem (105–111) admits a solution

$$\varrho(t,x) = R(t,x_1) = R(\xi),$$

$$\vartheta(t,x) = \Theta(t,x_1) = \Theta(\xi),$$

$$\mathbf{u}(t,x) = [U(t,x_1),0] = [U(\xi),0]$$

depending solely on the self-similar variable $\xi = x_1/t$. Such a solution is unique in the class of BV solutions of the 1D problem, see Chen and Frid [14], [15].

We focus on the class of Riemann data producing shock-free solutions (rarefaction waves), more specifically, solutions that are locally Lipschitz in the open set $(0,T) \times \Omega$. We claim the following, see Chen and Chen [13], and [36, Theorem 2.1]:

Theorem 10. *Let $\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Euler system (105–108) in $(0;T) \times \Omega$ originating from the Riemann data (109–111) and satisfying the associated far field conditions. Suppose in addition that the Riemann data (109–111) give rise to a shock-free solution $[R,\Theta,U]$ of the 1D Riemann problem.*

Then

$$\varrho = R, \ \vartheta = \Theta, \ \mathbf{u} = [U,0] \ \textit{a.a. in } (0,T) \times \Omega.$$

In the light of this result, we may conjecture that the possibility of infinitely many solutions provided by the method of convex integration occurs only if the weak solution dissipates mechanical energy. A definitive answer to this question, however, remains completely open.

7.3 Viscosity solutions

The aforementioned examples reopened the old problem of suitable admissibility criteria to be imposed on the weak solutions to inviscid fluid systems. A natural one, advocated by Bardos et al. [4], admits only those solutions obtained as an inviscid limit of the associated viscous ow represented in our framework by the Navier-Stokes-Fourier system. In certain cases, indeed, such a selection process may eliminate the “wild” solutions constructed by the method of convex integration. To provide some support to this conjecture, we consider the barotropic Navier-Stokes system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{112}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathsf{S}(\nabla_x \mathbf{u}), \tag{113}$$

supplemented with the constitutive relations for the pressure

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1, \tag{114}$$

and the viscous stress

$$\mathsf{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathsf{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathsf{I},\ \mu > 0,\ \eta \geq 0, \tag{115}$$

along with its one-dimensional version:

$$\partial_t R + \partial_y (RV) = 0, \tag{116}$$

$$\partial_t (RV) + \partial_y (RV^2) + \partial_y p(R) = \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \partial^2_{y,y} V. \tag{117}$$

With the obvious identification $x_1 = y$, $\varrho(x) = R(x_1)$, $\mathbf{u}(x) = [V(x_1), 0, \dots, 0]$, any solution of problem (116), (117) satisfies also the extended system (112–115).

The 1D-dimensional fluid motion is nowadays well-understood, see Antontsev, Kazhikhov and Monakhov [2]. In particular, problem (116), (117) considered in the interval $(0,1)$, and supplemented with the boundary conditions

$$V(t,0) = V(t,1) = 0,\ t \in (0,T), \tag{118}$$

and the initial conditions

$$R(0,\cdot) = R_0 > 0,\ V(0,\cdot) = V_0 \tag{119}$$

admits a (unique) weak solution for a fairly vast class of initial data, see Amosov and Zlotnik [65]. Moreover, the solutions are regular provided the initial data are smooth enough, see Kazhikhov [46].

Our goal is to show that, unlike their "Eulerian" counterparts discussed in the previous section, solutions of the 1D-problem (116), (117) are stable in the class of weak solutions system (112–115). To this end, we consider a domain $\Omega \subset R^N$, $N = 2, 3$,

$$\Omega = (0,1) \times \mathcal{T}^{N-1},\ \text{with } \mathcal{T}^{N-1} = \left([0,1]|_{\{0,1\}} \right)^{N-1},$$

specifically all functions defined in Ω are 1-periodic with respect to the variables x_j, $j > 1$. Accordingly, any solution r, V of problem (116), (117) can be extended to be constant in x_j, $j > 1$.

We say that a pair of functions $[\varrho, \mathbf{u}]$ represent a *finite energy weak solution* to the Navier-Stokes system (112–115)) in the space-time cylinder $(0,T)\times\Omega$, supplemented with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{120}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \tag{121}$$

if:

- the density ϱ is a non-negative function, $\varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^N))$, $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma-1)}(\Omega; R^N))$;
- $$\left[\int_\Omega \varrho\varphi \ {\rm d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right] \ {\rm d}x\,{\rm d}t \tag{122}$$
  for any $0 \le \tau_1 \le \tau_2 \le T$, and any $\varphi \in C^\infty([0,T]\times\overline{\Omega})$;
- $$\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ {\rm d}x\right]_{t=\tau_1}^{t=\tau_2} \tag{123}$$
  $$= \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho){\rm div}_x\varphi - \mathsf{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\right] \ {\rm d}x\,{\rm d}t$$
  for any $0 \le \tau_1 \le \tau_2 \le T$, and any $\varphi \in C_c^\infty([0,T]\times\Omega; R^N)$;
- the *energy inequality*
  $$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot) \ {\rm d}x \le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right] \ {\rm d}x, \tag{124}$$
  with
  $$P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma,$$
  holds for a.a. $\tau \in (0,T)$.

Finite energy weak solutions to the barotropic Navier-Stokes system are known to exist for any finite energy initial data whenever $\gamma > N/2$, see Lions [49] and [39]. We claim the following stability result, see [40, Theorem 2.1]:

Theorem 11. *Let*

$$\gamma > \frac{N}{2},\ q > \max\{2,\gamma'\},\ \frac{1}{\gamma}+\frac{1}{\gamma'} = 1 \ \textit{if}\ N = 2,$$

$$q > \max\left\{3, \frac{6\gamma}{5\gamma-6}\right\} \ \textit{if}\ N = 3.$$

Let $[R, V]$ be a (strong) solution of the one-dimensional problem (116–119), with the initial data in the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0 \text{ in } [0,1],\ V_0 \in W_0^{1,q}(0,1).$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier–Stokes system (122–124) in $(0,T) \times \Omega$ satisfying

$$\varrho_0 \in L^\infty(\Omega),\ \varrho_0 > 0 \text{ a.a. in } \Omega, \mathbf{u}_0 \in L^2(\Omega; R^N).$$

Then

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right] (\tau, \cdot)\ \mathrm{d}x \tag{125}$$

$$\leq c(T) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + P(\varrho_0) - P'(R)(\varrho_0 - R_0) - P(R_0) \right]\ \mathrm{d}x$$

for a.a. $\tau \in (0,T)$.

We easily recognize a variant of the relative energy functional appearing in (125).

7.4 Vanishing dissipation limit for the Navier-Stokes-Fourier system

We conclude this part by a short discussion of the vanishing dissipation limit for the complete Navier-Stokes-Fourier system discussed in Section 3.6. To this end we suppose that the thermodynamics functions p, e, and s are given through (41–46), where the "radiation" coefficient $a > 0$ will be sent to zero in the asymptotic limit. More specifically, the target Euler system takes the form

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{126}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x P_m(\varrho, \vartheta) = 0, \tag{127}$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_m(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_m(\varrho, \vartheta) \right) \mathbf{u} + p_m(\varrho, \vartheta) \mathbf{u} \right] = 0, \tag{128}$$

considered on a bounded and smooth domain $\Omega \subset R^3$, and supplemented with the impermeability boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{129}$$

We remark We remark that the total energy balance (128) can be equivalently reformulated as the entropy balance equation

$$\partial_t(\varrho s_m(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) = 0, \tag{130}$$

or the thermal energy balance

$$c_v(\varrho,\vartheta)\Big(\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big) + \vartheta\frac{\partial p_m(\varrho,\vartheta)}{\partial\vartheta}\mathrm{div}_x\mathbf{u} = 0, \tag{131}$$

where

$$c_v(\varrho,\vartheta) = \frac{\partial e_m(\varrho,\vartheta)}{\partial\vartheta},$$

as long as the solution of the Euler system remains smooth.

A suitable existence result for the Euler system with the slip boundary condition (129) was obtained by Schochet [58, Theorem 1]. It asserts the local-in-time existence of a *classical* solution $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ of the Euler system (126), (127), (129), (130) if:

- $\Omega \subset R^3$ is a bounded domain with sufficiently smooth boundary, say $\partial\Omega$ of class C^∞;
- the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$ satisfy

$$\varrho_{0,E}, \vartheta_{0,E} \in W^{3,2}(\Omega),\ \mathbf{u}_{0,E} \in W^{3,2}(\Omega; R^3),\ \varrho_{0,E} > 0,\ \vartheta_{0,E} > 0 \text{ in } \overline{\Omega}; \tag{132}$$

- the compatibility conditions

$$\partial_t^k \mathbf{u}_{0,\varepsilon} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{133}$$

  hold for $k = 0, 1, 2$, where the “time derivative” of the initial data is computed from the equations.

7.4.1 Navier–Stokes–Fourier system

We consider a slight modification of the Navier–Stokes–Fourier system, namelu

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{134}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathsf{S}(\vartheta,\nabla_x\mathbf{u}) - \lambda\mathbf{u} \tag{135}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{136}$$

$$\sigma = \frac{1}{\vartheta}\left(\mathsf{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

where the viscous stress tensor $\mathsf{S}(\varrho,\nabla_x\mathbf{u})$ is given by Newton’s law

$$\mathsf{S}(\varrho,\nabla_x\mathbf{u}) = \nu\left[\mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathsf{I}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}tensI\right],\ \nu > 0, \tag{137}$$

and $\mathbf{q} = \mathbf{q}(\vartheta,\nabla_x\vartheta)$ is the heat flux determined by Fourier’s law

$$\mathbf{q} = -\omega\kappa(\vartheta)\nabla_x\vartheta,\ \omega > 0. \tag{138}$$

The scaling parameters a, ν, ω, and λ are positive quantities supposed to vanish in the asymptotic limit. The momentum equation (135) contains an extra "damping" term $-\lambda\mathbf{u}$.

System (134–136) is supplemented by the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathsf{S}(\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0, \tag{139}$$

accompanied with the no-flux condition

$$\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{140}$$

7.4.2 Relative energy inequality

Because of the presence of the extra term in the momentum equation (135), the relative energy inequality (93) takes the form

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \Big|\ r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau} \tag{141}$$

$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\mathsf{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\,\mathrm{d}t + \lambda\int_0^\tau\int_\Omega |\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^\tau\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \mathsf{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U}\ \mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\ \mathrm{d}x\,\mathrm{d}t + \lambda\int_0^\tau\int_\Omega \mathbf{u}\cdot\mathbf{U}\ \mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)(\mathbf{U}-\mathbf{u})\cdot\nabla_x\Theta\ \mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho\left(\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U}\right)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\ \mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \Big[\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{U}\cdot\nabla_x\Theta\Big]\ \mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \left[\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right]\ \mathrm{d}x\,\mathrm{d}t$$

for any trio of (smooth) test functions $[r,\Theta,\mathbf{U}]$ such that

$$r,\Theta > 0 \text{ in } \overline{\Omega},\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{142}$$

Similarly to the above, the relative energy inequality (141) holds for any weak solution $[\varrho, \vartheta, \mathbf{u}]$ of the Navier-Stokes-Fourier system specified through (19–23), where the total energy balance (24) is replaced by

$$\left[\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \lambda \int_0^\tau \int_\Omega |\mathbf{u}|^2 \ \mathrm{d}x \, \mathrm{d}t \leq 0 \tag{143}$$

for a.a. $\tau \in [0, T]$.

7.4.3 Vanishing dissipation limit

The obvious idea how to compare a weak solution $[\varrho, \vartheta, \mathbf{u}]$ of the Navier-Stokes-Fourier system to the strong solution $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ of the Euler system is to take the trio

$$r = \varrho_E, \ \Theta = \vartheta_E, \ \mathbf{U} = \mathbf{u}_E$$

as test functions in the relative energy inequality (141). Here we point out that such a step is essentially conditioned by our choice of the complete slip boundary condition (139) for the velocity field in the Navier-Stokes-Fourier system. Another choice of boundary behavior of u, in particular the no-slip conditions (13), (14), would lead to the well known and sofar unsurmountable difficulties connected with the presence of a boundary layers, see the surveys of E [27] or Masmoudi [51].

We report the following result, see [34, Theorem 3.1]:

Theorem 12. *Let the following hypotheses be satisfied:*

- *$\Omega \subset R^3$ is a bounded domain with smooth boundary.*
- *The thermodynamic functions p, e, and s are given by (42), (95), where p_m, e_m comply with (44–46), and, in addition,*

$$P \in C^1[0, \infty) \cap C^5(0, \infty), \ P'(Z) > 0 \textit{ for all } Z > 0.$$

- *The transport coefficients μ, η, and λ are given by (47–49), with $\alpha = 1$.*

Let $[\varrho_E, \vartheta_E, \mathbf{u}_e]$ be the classical (smooth) solution of the Euler system (126–128), (129) in a time interval $(0, T)$, originating from the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$ satisfying (132), (133).

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (134–138), (139), (140), where the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ satisfy

$$\varrho_0, \ \vartheta_0 > 0 \textit{ a.a. in } \Omega,$$

$$\int_\Omega \varrho_0 \ \mathrm{d}x \geq M, \ \|\varrho_0\|_{L^\infty(\Omega)} + \|\vartheta_0\|_{L^\infty(\Omega)} + \|\mathbf{u}_0\|_{L^\infty(\Omega; R^3)} \leq D.$$

Finally, let the scaling parameters a, ν, ω, and λ be positive numbers.

Then

$$\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\Big|\; \varrho_E, \vartheta_E, \mathbf{u}_E\right)\right]_{t=0}^{t=\tau}$$
$$\leq c(T, M, D) \max\left\{a, \nu, \omega, \lambda, \frac{\nu}{\sqrt{a}},\ \frac{\omega}{a}, \left(\frac{a}{\sqrt{\nu^3 \lambda}}\right)^{1/3}\right\}$$

for a.a. $\tau \in (0, T)$.

Corollary 1. *Under the hypotheses of Theorem 12, suppose that*

$$a, \nu, \omega, \lambda \to 0, \ \text{and} \ \frac{\omega}{a} \to 0,\ \frac{\nu}{\sqrt{a}} \to 0,\ \frac{a}{\sqrt{\nu^3 \lambda}} \to 0. \tag{144}$$

Then

$$\text{ess} \sup_{\tau \in (0,T)} \int_\Omega \left[\varrho |\mathbf{u} - \mathbf{u}_E|^2 + |\varrho - \varrho_E|^{5/3} + \varrho |\vartheta - \vartheta_E|^2\right] \ \mathrm{d}x \leq c(T, D, M) \times$$

$$\Lambda\Big(a, \nu, \omega, \lambda, \|\varrho_0 - \varrho_{0,E}\|_{L^\infty(\Omega)}, \|\vartheta_0 - \vartheta_{0,E}\|_{L^\infty(\Omega)}, \|\mathbf{u}_0 - \mathbf{u}_{0,E}\|_{L^\infty(\Omega; R^3)}\Big)$$

where Λ *is an explicitly computable function of its arguments,*

$$\Lambda\Big(a, \nu, \omega, \lambda, \|\varrho_0 - \varrho_{0,E}\|_{L^\infty(\Omega)}, \|\vartheta_0 - \vartheta_{0,E}\|_{L^\infty(\Omega)}, \|\mathbf{u}_0 - \mathbf{u}_{0,E}\|_{L^\infty(\Omega; R^3)}\Big) \to 0$$

provided a, ν, ω, λ *satisfy (144), and*

$$\|\varrho_0 - \varrho_{0,E}\|_{L^\infty(\Omega)},\ \|\vartheta_0 - \vartheta_{0,E}\|_{L^\infty(\Omega)},\ \|\mathbf{u}_0 - \mathbf{u}_{0,E}\|_{L^\infty(\Omega; R^3)} \to 0.$$

The convergence result stated in Corollary 1 is *path dependent*, the parameters a, ν, ω, λ are interrelated through (144). It is easy to check that (144) holds provided, for instance,

$$a \to 0,\ \mathbf{u} = a^\alpha, \omega = a^\beta,\ \lambda = a^\gamma,$$

where

$$\beta > 1,\ \frac{1}{2} < \alpha < \frac{2}{3},\ 0 < \gamma < 1 - \frac{3}{2}\alpha.$$

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