Convergence of a finite difference MAC scheme for the compressible Navier-Stokes

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Compressible barotropic Navier-Stokes

$$\begin{aligned} \rho_t + \mathrm{div}(\rho \mathbf{u}) &= 0. \\ (\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla p + \nabla \cdot \mathbb{S} \end{aligned} \qquad (1)$$

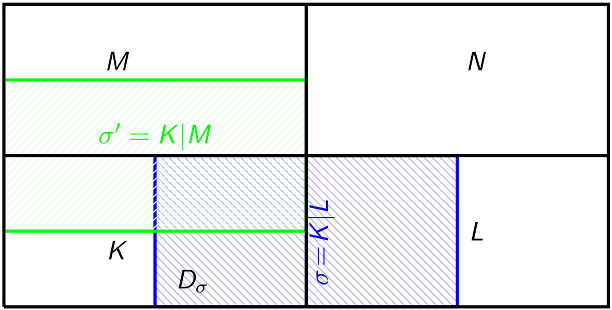
ρ : density
$\mathbf{u}$: velocity
p : pressure, $p = a\rho^\gamma$
$\mathbb{S}$: viscous stress, $\mathbb{S} = \mu \nabla \mathbf{u} + \lambda \mathrm{div} \mathbf{u}, \mu > 0, \lambda \geq 0$

Finite Volume-Finite Element by T. Karper, 2013, $\gamma > 3$

- E. Feireisl, R. Hošek, D. Maltese, A. Novotný, 2017
  bounded numerical solution
- E. Feireisl & M. Lukáčová-Medvid'ová, 2017
  dissipative measure-valued solution, $\gamma \in (1, 2)$
- R. Hošek & BS, 2018; H. Mizerová & BS, in preparation

Our interests: Simple FD, stability and convergence

Notations



- Elements: $\mathcal{M}$
- Faces: $\mathcal{E}$
- Dual Faces: $\widetilde{\mathcal{E}}$

$$\rho,\ p \in L_{\mathcal{M}}$$

$$\mathbf{u} = (u_1, \ldots, u_d) \in H_{\mathcal{E}} = (H_{\mathcal{E}}^{(1)}, \ldots, H_{\mathcal{E}}^{(d)})$$

Numerical scheme

$$D_t \rho_h^n + \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\rho_h^n \mathbf{u}_h^n) = 0$$

$$D_t(\widehat{\rho_h^n}^{(i)} u_{i,h}^n) + \mathrm{div}_{\mathcal{E}^{(i)}}^{\mathrm{up}}(\rho_h^n \mathbf{u}_h^n u_{i,h}^n) - \mu \Delta_{\mathcal{E}}^{(i)} u_{i,h}^n - \lambda \eth_{\mathcal{E}}^{(i)} \mathrm{div}_{\mathcal{M}} \mathbf{u}_h^n + \eth_{\mathcal{E}}^{(i)} p(\rho_h^n) = 0 \tag{2}$$

$$\sigma = \overrightarrow{K|L},\ K = [\overrightarrow{\sigma'' \sigma}],\ \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}^{(i)}$$

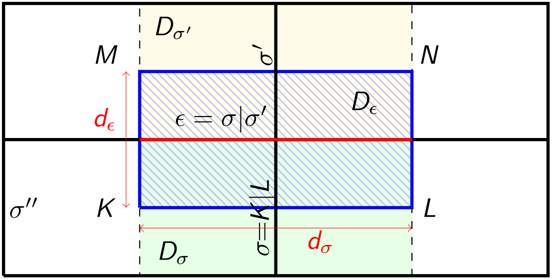
$$D_t f = \frac{f(t) - f(t - \Delta t)}{\Delta t}$$

$$\left(\Delta_{\mathcal{E}}^{(i)} v_i\right)_{D_\sigma} = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} \frac{|\epsilon|}{d_\epsilon}(v_{\sigma'} - v_\sigma)$$

$$(\eth_{\mathcal{M}}^{(i)} v_i)_K = \frac{v_{\sigma''} - v_\sigma}{h_i}$$

$$\widehat{\rho}^{(i)}(\mathbf{x}) = \sum_{\sigma \in \mathcal{E}^{(i)}} \rho_{D_\sigma} \mathcal{X}_{D_\sigma},\ \rho_{D_\sigma} = \frac{\rho_K |K| + \rho_L |L|}{|D_\sigma|}$$

$$\left(\eth_{\mathcal{E}}^{(i)} r\right)_{D_\sigma} = \frac{r_L - r_K}{d_\sigma}$$



Upwind flux

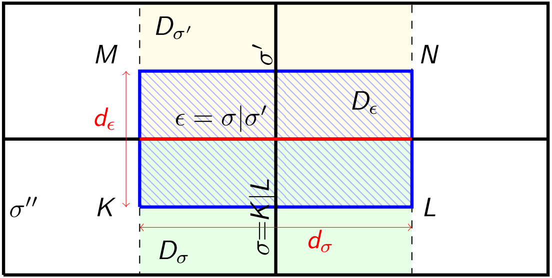
$$\mathrm{div}^{\mathrm{up}}_{\mathcal{M}}(\rho \mathbf{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{\sigma,K}(\rho, \mathbf{u}), \quad \mathrm{div}^{\mathrm{up}}_{\mathcal{E}^{(i)}}(\rho \mathbf{u} u_i)_{D_\sigma} = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\epsilon,\sigma}(\rho, \mathbf{u})\, u_\epsilon$$

$$F_{\sigma,K}(\rho, \mathbf{u}) = |\sigma|\ \rho^{\mathrm{up}}_\sigma\ u_{\sigma,K}, \qquad \forall\, \sigma = K|L \in \mathcal{E},$$

$$\rho^{\mathrm{up}}_\sigma = \begin{cases} \rho_K & \text{if } u_{\sigma,K} \geq 0, \\ \rho_L & \text{otherwise.} \end{cases} \qquad u_{\sigma,K} = u_\sigma\ \mathbf{e}^{(i)} \cdot \mathbf{n}_{\sigma,K},\ \sigma \in \mathcal{E}^{(i)}(K)$$

$$F_{\epsilon,\sigma} = \frac{1}{2} \begin{cases} F_{\sigma,K}(\rho,\mathbf{u})\ \mathbf{n}_{\epsilon,D_\sigma} \cdot \mathbf{n}_{\sigma,K} + F_{\sigma',K}(\rho,\mathbf{u})\ \mathbf{n}_{\epsilon,D_\sigma} \cdot \mathbf{n}_{\sigma',K} & \epsilon /\!/ \sigma \\ F_{\tau,K}(\rho,\mathbf{u}) + F_{\tau',L}(\rho,\mathbf{u}) & \epsilon \perp \sigma \end{cases}$$

$$u_\epsilon \equiv u_{i,\epsilon} = \frac{u_\sigma + u_{\sigma'}}{2}$$



Existence of num sol

Lemma 1

Let $\rho_h^{n-1} \in L_{\mathcal{M}}$, $\mathbf{u}_h^{n-1} \in H_{\mathcal{E}}$ *be given;* $\rho_K^{n-1} > 0$ *for all* $K \in \mathcal{M}$. *Then the numerical scheme (2) admits a solution*

$$\rho_h^n \in L_{\mathcal{M}}, \rho_K^n > 0 \text{ for all } K \in \mathcal{M}, \mathbf{u}_h^n \in H_{\mathcal{E}}.$$

Moreover, it satisfies the discrete conservation of mass

$$\sum_{K \in \mathcal{M}} \int_K \rho_K^n = \sum_{K \in \mathcal{M}} \int_K \rho_K^{n-1}.$$

Theorem 2 (Energy estimate)

Let $(\rho_h, \mathbf{u}_h)$ be a solution of (2). Then for any $n = 1, ..., N$, there exist $\rho^{n-1,n} \in [\min(\rho_h^{n-1}, \rho_h^n), \max(\rho_h^{n-1}, \rho_h^n)]$ and $\overline{\rho}_\sigma^n \in [\min(\rho_K^n, \rho_L^n), \max(\rho_K^n, \rho_L^n)]$ for any $\sigma = K|L \in \mathcal{E}$ such that

$$\frac{1}{\Delta t}\int_\Omega \left(\mathcal{H}(\rho_h^n) + \frac{1}{2}\rho_h^n|\mathbf{u}_h^n|^2\right)\mathrm{d}x - \frac{1}{\Delta t}\int_\Omega \left(\mathcal{H}(\rho_h^{n-1}) + \frac{1}{2}\rho_h^{n-1}|\mathbf{u}_h^{n-1}|^2\right)\mathrm{d}x$$
$$+ \mu||\nabla_h \mathbf{u}_h^n||_{L^2(\Omega)}^2 + \lambda||div_{\mathcal{M}}^{\mathrm{up}}\mathbf{u}_h^n||_{L^2(\Omega)}^2 + \frac{1}{2\Delta t}\int_\Omega \rho_h^{n-1}|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}|^2\mathrm{d}x$$
$$+ \int_\Omega \frac{1}{2\Delta t}\mathcal{H}''(\rho^{n-1,n})(\rho_h^n - \rho_h^{n-1})^2\mathrm{d}x + \frac{1}{2}\sum_{\sigma\in\mathcal{E}^{(i)}} |\sigma||\mathbf{u}_{\sigma,K}^n|\mathcal{H}''(\overline{\rho}_\sigma^n)[\![\rho_h]\!]_\sigma^2 = 0$$

where $\mathcal{H}(\rho) = \frac{1}{\gamma-1}p(\rho)$.

Consistency

Lemma 3

Let $\rho_h^n, \mathbf{u}_h^n$ be the solution to the numerical scheme (2) and suppose $\rho_h^n < \bar{\rho}$. Then

$$\int_\Omega \partial_h^t \rho_h^n \phi dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \cdot \nabla_x \phi dx = \mathcal{O}(h)$$

$$\int_\Omega \partial_h^t (\rho_h \mathbf{u}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \mathbf{u}_h^n \otimes \mathbf{u}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx + \mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx = \mathcal{O}(h)$$

Dissipative measure-valued solution

Definition 4 (DMVS[1])

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^{\infty}_{weak}\Big((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\Big)$$

is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau\in(0,T)$, for any $\psi\in C^1\big((0,T)\times\Omega; R^d\big)$

$$\left[\int_\Omega \langle \nu_{\tau,x};\rho\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\langle\nu_{t,x};\rho\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\rangle\cdot\nabla_x\psi]dxdt$$

$$\left[\int_\Omega \langle \nu_{\tau,x};\rho\mathbf{u}\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\langle\nu_{t,x};\rho\mathbf{u}\rangle\partial_t\psi + \langle\nu_{t,x};\rho\mathbf{u}\otimes\mathbf{u}+p(\rho)\mathbf{I}\rangle : \nabla_x\psi]dxdt$$

$$-\int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\psi dxdt, + \int_0^\tau\int_\Omega \mathcal{R}:\nabla_x\psi dxdt$$

$$\left[\int_\Omega \langle \nu_{\tau,x};E\rangle\psi(\tau,\cdot)dx\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\psi dxdt + \mathcal{D}(\tau) \le 0,$$

where

$$\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)}dt \le \int_0^\tau \mathcal{D}(\tau)dt$$

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Theorem 5

Let $[\rho_h, \mathbf{u}_h]$ be a family of numerical solutions obtained by the scheme (2) with suitable initial data and suppose $\rho_h^n < \bar{\rho}$.
Then any Young measure $\nu_{t,x}$ generated by the $[\rho_h, \mathbf{u}_h]$ for $h \to 0$ represents a dissipative measure-valued solution of NS (1).

Remark: Applying the weak-strong uniqueness (Feireisl et.al. 2016) we conclude the convergence to the strong solution.

Thank you for your attention!