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Bounded solutions of the Dirichlet problem for the Stokes resolvent system and for the Darcy-Forchheimer-Brinkman system

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BOUNDED SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE STOKES RESOLVENT SYSTEM AND FOR THE DARCY-FORCHHEIMER-BRINKMAN SYSTEM

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ABSTRACT. The paper studies the Dirichlet problem for the Stokes resolvent system for bounded boundary data on bounded and unbounded domains with compact Ljapunov boundary. (The boundary might be disconnected.) For a bounded domain we prove the existence of a unique solution $(\mathbf{u}, p)$ of the problem such that the velocity part $\mathbf{u}$ is bounded. For an unbounded domain we prove the existence of a such solution. But this solution is not unique. We characterize all solutions of the problem. Then we study bounded solutions of the nonlinear Dirichlet problem $-\Delta\mathbf{u}(x)+\lambda\mathbf{u}(x)+\nabla p(x) = F(x, \mathbf{u}(x))$, $\nabla\cdot\mathbf{u} = 0$ in Ω, $\mathbf{u}(x) = G(x, \mathbf{u}(x))$ on $\partial\Omega$, where F is bounded. As a consequence we study bounded solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system $-\Delta\mathbf{u} + \lambda\mathbf{u} + \beta|\mathbf{u}|\mathbf{u} + \nabla p = \mathbf{f}$, $\nabla \cdot \mathbf{u} = 0$. At last we prove a generalized maximum modulus principle for a solution $(\mathbf{u}, p)$ of the Stokes resolvent system such that the velocity part $\mathbf{u}$ is bounded.

1. INTRODUCTION

This paper studies the Dirichlet problem for the Stokes resolvent system

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{1.1}$$

with

$$\lambda \in \mathbb{C} \setminus \{z \leq 0\}. \tag{1.2}$$

This system is important in two situations. If λ is a positive constant then the system (1.1) (so called Brinkman system) is a model of a porous medium (see [26]). The system (1.1) for $\lambda = i\tau$ with $\tau > 0$ is utilized for a study of boundary value problems for the nonsteady Stokes system (see [3], [47]).

R. Farwig and H. Sohr studied the Dirichlet problem for the Stokes resolvent system (1.1) in homogeneous Sobolev spaces $D^{1,q}(\Omega)$ and in weighted Sobolev spaces $W^{1,q}(\Omega, \rho)$ on domains with compact boundary of class $\mathcal{C}^{1,1}$ (see [9], [10]). This problem was studied on bounded domains also by K. Schumacher in [42]. M. Geissert, M. Hess, C. Schwarz and K. Stavrakidis and also P. Deuring studied in [13], [5] the Dirichlet problem for the Stokes resolvent system (1.1) in homogeneous Sobolev spaces $D^{2,q}(\Omega)$ on domains with smooth compact boundary. M. Kohr, M. Lanza de Cristoforis and W. L. Wendland studied the Dirichlet problem for the Brinkman system and also for a nonlinear Brinkman system in $H^s(\Omega, R^m) \times H^{s-1}(\Omega)$ for $s \in (1/2, 3/2)$ in bounded domains $\Omega \subset R^m$ with connected Lipschitz boundary

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(see [24]). A classical solution of the Dirichlet problem for the Brinkman system was studied on domains with compact connected boundary of class $\mathcal{C}^2$ by W. Varnhorn ([47], [48]). He proved the unique solvability of the classical Dirichlet problem on bounded domains for a continuous boundary condition $\mathbf{g}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, d\sigma = 0. \tag{1.3}$$

For an unbounded domain $\Omega \subset R^m$ he proved the unique solvability of a classical solution $(\mathbf{u}, p)$ under the assumption $|\mathbf{u}(x)| \, |\nabla \mathbf{u}(x)| + |\mathbf{u}(x)| \, |p(x)| = o(|x|^{1-m})$ as $|x| \to \infty$. M. Kohr proved in [21] the existence of a classical solution for open sets with compact boundary of class $\mathcal{C}^{1,\alpha}$ (without assumption that $\partial\Omega$ is connected). The behaviour of solutions at infinity is the same like in the papers of W. Varnhorn.

We study the problem whether for a given boundary condition $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ there exists a solution $(\mathbf{u}, p)$ of (1.1) such that $\mathbf{u}$ is bounded in Ω and $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ in some sense. This problem is motivated by the same problem for the Laplace equation. For the Laplace equation a generalised solution of the Dirichlet problem is constructed as the infimum of the class of supersolutions of this problem - so called PWB-solution (see for example [2]). The value of a solution at a fixed point is given by the integral of the boundary condition with respect to the harmonic measure. Since the harmonic measure is a probabilistic measure, we obtain that the solution is bounded. The Dirichlet problem for the Laplace equation with boundary condition $f \in L^q(\partial\Omega)$ was studied by the integral equation method on domains with compact Lipschitz boundary ([16], [17], [19], [50]). It was studied so called L^q-solution of the Dirichlet problem, i.e. $\Delta u = 0$ in Ω, the nontangential maximal function of u is in $L^q(\partial\Omega)$ and the boundary condition is fulfilled in the sense of a nontangential limit at almost all points of the boundary. For Ω bounded it was shown that L^q-solution is also a PWB-solution (see [4]). So, the L^q-solution of the Dirichlet problem corresponding to a boundary condition $f \in L^\infty(\Omega)$ is bounded. R. Hunt and R. L. Wheeden proved that each bounded solution of the Laplace equation in a bounded domain with Lipschitz boundary has nontangential limit at almost all points of the boundary, i.e. it is an L^∞-solution of a Dirichlet problem for the Laplace equation (see [15]).

In hydrodynamics $(\mathbf{u}, p)$ is an L^q-solution of the Dirichlet problem if $(\mathbf{u}, p)$ is a solution of the corresponding system (Stokes system or Stokes resolvent system), the nontangential maximal function of $\mathbf{u}$ is in L^q on the boundary, and the boundary condition is fulfilled at almost all points of the boundary in the sense of a nontangential limit. L^2-solutions of the Dirichlet problem for the Stokes system on domains with compact Lipschitz boundary were studied by E. B. Fabes, C. E. Kenig and G. C. Verchota (see [8]). L^q-solutions of the Dirichlet problem for the Stokes system for q close to 2 on domains with compact Lipschitz boundary were studied by J. Kilty, M. Mitrea, Z. Shen and M. Wright ([20], [40], [43]). L^q-solutions of the Dirichlet problem for the Stokes system for $2 \le q < \infty$ on domains with compact Lipschitz boundary in $\mathbb{R}^2$ or $\mathbb{R}^3$ were studied by M. Mitrea and M. Wright ([40]). L^2-solutions of the Dirichlet problem for the Stokes resolvent system on bounded domains $\Omega \subset \mathbb{R}^3$ with connected Lipschitz boundary has been studied by H. J. Choe and H. Kozono in [3]. The same problem for a general Euclidean space was studied by Z. Shen in [44].

We study an L^∞-solution of the Dirichlet problem for the Stokes resolvent system on bounded domains with boundary of class $\mathcal{C}^{1,\alpha}$ (and general geometry). For a boundary condition $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ satisfying the compatibility condition (1.3) we show the existence of an L^∞-solution $(\mathbf{u}, p)$. A velocity $\mathbf{u}$ is unique, a pressure p is unique up to an additive constant. We show moreover that $\mathbf{u}$ is bounded. For Ω unbounded we show that there exist infinitely many L^∞-solutions of the Dirichlet problem for the Stokes resolvent system for all $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$. We describe all these solutions. We prove that if $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system then $\mathbf{u}$ is bounded. To be able to describe all solutions of the Dirichlet problem on unbounded domains, we study the behaviour at infinity of solutions $(\mathbf{u}, p)$ of the Stokes resolvent system with $\mathbf{u}$ bounded. Moreover we show that if $(\mathbf{u}, p)$ is a solution of the Stokes resolvent system such that $\mathbf{u}$ is bounded, then there exists a nontangential limit of $\mathbf{u}$ at almost all points of the boundary, i.e. $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system. Then we study an L^∞-solutions of the nonlinear Dirichlet problem for the Stokes resolvent system

$$-\Delta\mathbf{u}(x) + \lambda\mathbf{u}(x) + \nabla p(x) = F(x, \mathbf{u}(x)), \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega,$$

$$\mathbf{u}(x) = G(x, \mathbf{u}(x)) \quad \text{on } \partial\Omega.$$

Here F is bounded for Ω bounded and $F = 0$ for Ω unbounded. For Ω bounded we study also L^∞-solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \beta|\mathbf{u}|\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega.$$

M. Kohr, M. Lanza de Cristoforis and W. L. Wendland studied the Robin problem for the Darcy-Forchheimer-Brinkman system in $H^s(\Omega, \mathbb{R}^m) \times H^{s-1}(\Omega)$, the mixed Dirichlet-Robin problem for the Darcy-Forchheimer-Brinkman system in $H^{3/2}(\Omega, \mathbb{R}^m) \times H^{1/2}(\Omega)$ and the Navier problem in $H^1(\Omega, \mathbb{R}^m) \times L^2(\Omega)$ for the Darcy-Forchheimer-Brinkman system on a bounded domain with connected Lipschitz boundary ([23]). We prove a similar result for L^∞-solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system.

As a consequence of these results we prove the generalized maximum modulus principle. If u is a solution of the Laplace equation on a bounded domain Ω and $u \in \mathcal{C}(\overline{\Omega})$ then the classical maximum modulus principle holds:

$$\max_{x\in\overline{\Omega}} |u(x)| = \max_{x\in\partial\Omega} |u(x)|.$$

(See [2].) A similar result holds for elliptic partial differential equations of the second order (see [14]). It is well known that this result does not hold for hydrodynamical partial differential systems (see [49]). But a generalized maximum modulus principle has been studied. Maremonti and Russo proved in [33] a generalized maximum modulus principle for the Stokes system in the plane: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $\mathcal{C}^2$. Then there exists a constant C such that if $(\mathbf{u}, p)$ is a solution of the Stokes system

$$\Delta\mathbf{u} = \nabla p, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega$$

with $\mathbf{u} \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^2)$, then

$$\max_{x\in\overline{\Omega}} |\mathbf{u}(x)| \le C \max_{x\in\partial\Omega} |\mathbf{u}(x)|. \tag{1.4}$$

Then Kratz investigated the best constant in the generalized maximum modulus estimate (1.4) for the Stokes system in balls (see [28], [29], [30]).) Maremonti ([32]) proved the generalized maximum modulus principle (1.4) for the Stokes system for bounded domains with boundary of class $\mathcal{C}^2$ in $\mathbb{R}^m$. We show a similar (and more general) results for the Stokes resolvent system.

2. Definition of an L^q-solution of the Dirichlet problem

We now define an L^q-solution of the Dirichlet problem for the Stokes resolvent system.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. If $x \in \partial\Omega$, $a > 0$, denote the non-tangential approach region of opening a at the point x by

$$\Gamma_a^\Omega(x) := \{y \in \Omega; |x - y| < (1 + a) \operatorname{dist}(y, \partial\Omega)\}.$$

If now $\mathbf{v}$ is a vector function defined in Ω we denote the non-tangential maximal function of $\mathbf{v}$ on $\partial\Omega$ by

$$\mathbf{v}_\Omega^*(x) := \sup\{|\mathbf{v}(y)|; y \in \Gamma_a^\Omega(x)\}.$$

If $x \in \partial\Omega$, $\Gamma(x) = \Gamma_a^\Omega(x)$ then

$$\mathbf{v}(x) = \lim_{\Gamma(x) \ni y \to x} \mathbf{v}(y)$$

is the non-tangential limit of $\mathbf{v}$ with respect to Ω at x.

Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z < 0\}$, $1 < q \leq \infty$, $\mathbf{g} \in L^q(\partial\Omega, \mathbb{C}^m)$. *We say that* $(\mathbf{u}, p)$ *is an* L^q*-solution of the Dirichlet problem for the Stokes resolvent system*

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0,\ \nabla \cdot \mathbf{u} = 0 \ \text{in } \Omega, \qquad \mathbf{u} = \mathbf{g} \ \text{on } \partial\Omega, \tag{2.1}$$

if $(\mathbf{u}, p)$ *is a classical solution of the Stokes resolvent system* (1.1) *in* Ω, $\mathbf{u}_\Omega^* \in L^q(\partial\Omega)$, *there exists the nontangential limit of* $\mathbf{u}$ *at almost all points of* $\partial\Omega$, *and the Dirichlet boundary condition* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega$ *is fulfilled in the sense of the non-tangential limit at almost all points of* $\partial\Omega$.

3. Auxiliary lemmas

We shall look for an L^∞-solution of the Dirichlet problem for the Stokes resolvent system using the integral equation method. We shall study properties of corresponding boundary layer potentials. For these reasons we need the following technical auxiliary lemmas.

Lemma 3.1. *Let* A, D *be Borel subsets of* $\mathbb{R}^m$, μ *be a nonnegative Radon measure on* D, K *be a Borel-measurable function on* $A \times D$, *and* $\alpha, \beta > 0$. *Fix* $x \in A$. *Let* $|K(x,y)| \leq C_1 |x - y|^{-\alpha}$ *for* $y \in D$. *Denote* $B(x; r) = \{y \in \mathbb{R}^m; |x - y| < r\}$. *If* $\beta > \alpha$ *and* $\mu(B(x; r)) \leq C_2 r^\beta$ *for* $0 < r < \rho$, *then*

$$\left| \int_{B(x;\rho)} K(x,y) \, \mathrm{d}\mu(y) \right| \leq \frac{C_1 C_2 \beta}{\beta - \alpha} \rho^{\beta - \alpha}.$$

If $\beta < \alpha$ *and* $\mu(B(x; r)) \leq C_2 r^\beta$ *for* $\rho < r$, *then*

$$\left| \int_{D \setminus B(x;\rho)} K(x,y) \, \mathrm{d}\mu(y) \right| \leq \frac{C_1 C_2 \alpha}{|\beta - \alpha|} \rho^{\beta - \alpha}.$$

Proof.

$$\alpha \int_s^t r^{-\alpha-1}\mu(B(x;r)\setminus B(x;s))\mathrm{d}r = \alpha \int_s^t r^{-\alpha-1} \int_{B(x;r)\setminus B(x;s)} \mathrm{d}\mu(y)\mathrm{d}r$$

$$= \alpha \int_{B(x;t)\setminus B(x;s)} \int_{|x-y|}^t r^{-\alpha-1}\mathrm{d}r \ \mathrm{d}\mu(y)$$

$$= \ -t^{-\alpha}\mu(B(x;t)\setminus B(x;s)) + \int_{B(x;t)\setminus B(x;s)} |x-y|^{-\alpha} \ \mathrm{d}\mu(y).$$

Thus

$$\left| \int_{B(x;t)\setminus B(x;s)} K(x,y)\ \mathrm{d}\mu(y) \right| \le \int_{B(x;t)\setminus B(x;s)} C_1|x-y|^{-\alpha}\ \mathrm{d}\mu(y)$$

$$= C_1\alpha \int_s^t r^{-\alpha-1}\mu(B(x;r)\setminus B(x;s))\mathrm{d}r + C_1 t^{-\alpha}\mu(B(x;t)\setminus B(x;s)).$$

If $\mu(B(x;r)) \le C_2 r^\beta$ for $0<r<\rho$, $\beta-\alpha>0$, then for $s=0$ and $t=\rho$

$$\left| \int_{B(x;\rho)} K(x,y)\ \mathrm{d}\mu(y) \right| \le C_1C_2\alpha \int_0^\rho r^{\beta-\alpha-1}\ \mathrm{d}r + C_1C_2\rho^{\beta-\alpha} = \frac{C_1C_2\beta}{\beta-\alpha}\rho^{\beta-\alpha}.$$

If $\mu(B(x;r)) \le C_2 r^\beta$ for $\rho<r$, $\beta-\alpha<0$, then for $s=\rho$ and $t\to\infty$

$$\left| \int_{D\setminus B(x;\rho)} K(x,y)\ \mathrm{d}\mu(y) \right| \le C_1C_2\alpha \int_\rho^\infty r^{\beta-\alpha-1}\ \mathrm{d}r = \frac{C_1C_2\alpha}{|\beta-\alpha|}\rho^{\beta-\alpha}.$$

□

Lemma 3.2. *Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let $K(x,y)$ be a function defined on $\{[x,y]\in\mathbb{R}^m\times\partial\Omega; x\neq y\}$ such that $K(x,\cdot)$ is Borel-measurable for each $x\in R^m$. For $f\in L^\infty(\partial\Omega)$ define*

$$Kf(x) = \int_{\partial\Omega} K(x,y)f(y)\ \mathrm{d}\sigma(y). \tag{3.1}$$

Suppose that there exist positive constants ρ, C_1 and $\alpha\in(0,1)$ such that $|K(x,y)|\le C_1$ for $y\in\partial\Omega$, $\mathrm{dist}(x,\partial\Omega)\ge\rho$, and $|K(x,y)|\le C_1|x-y|^{\alpha+1-m}$ for $y\in\partial\Omega$, $\mathrm{dist}(x,\partial\Omega)<\rho$. Then there exists a constant C such that

$$|Kf|\le C\|f\|_{L^\infty(\partial\Omega)} \quad \text{on } \mathbb{R}^m, \qquad \forall f\in L^\infty(\partial\Omega). \tag{3.2}$$

Proof. Clearly, we can suppose that $\rho\ge\mathrm{diam}(\partial\Omega)$. Let $|f|\le c$. If $\mathrm{dist}(x,\partial\Omega)\ge\rho$ then

$$|Kf(x)|\le C_1 c\int_{\partial\Omega} 1\ \mathrm{d}\sigma.$$

Since $\partial\Omega$ is Lipschitz, there exists a constant C_2 such that $\sigma(B(x;r)\cap\partial\Omega) \le C_2 r^{m-1}$ for any $x \in \mathbb{R}^m$ and $r > 0$. If $\operatorname{dist}(x,\partial\Omega) < \rho$ then

$$|Kf(x)| \le \frac{cC_1C_2}{\alpha}(2\rho)^\alpha$$

by Lemma 3.1. □

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let $K(x,y)$ be a function defined on $\{[x,y] \in \mathbb{R}^m \times \partial\Omega; x \ne y\}$ such that $K(x,\cdot)$ is Borel-measurable for each $x \in R^m$. For $f \in L^\infty(\partial\Omega)$ define $Kf(x)$ by (3.1). Suppose that there exist positive constants ρ and C_1 such that $|K(x,y)| \le C_1$ for $y \in \partial\Omega$ and $|x| \ge \rho$. Suppose that there exists a constant C_2 such that*

$$|K(x,y) - K(z,y)| \le C_2 \frac{|x-z|}{|z-y|^m}$$

for all $x,z \in B(0;\rho)$ and $y \in \partial\Omega$ with $|z-y| > 2|x-z|$. Suppose that there exist constants $C_3 \in (0,\infty)$ and $\alpha \in (0,1)$ such that $|K(z,y)| \le C_3|z-y|^{\alpha+1-m}$ for each $z,y \in \partial\Omega$. Suppose that there exists a constant C_4 such that $|K(x,y)| \le C_4|x-y|^{1-m}$ for all $x \in B(0;\rho)$ and $y \in \partial\Omega$. Then there exists a constant C such that (3.2) holds.

Proof. We can suppose that $\partial\Omega \subset B(0;\rho)$. Let $|f| \le c$. According to Lemma 3.2 there exists a constant C_5 such that $|Kf(x)| \le C_5 c$ for $x \in \partial\Omega \cup (R^m \setminus B(0;\rho))$.

Let now $x \in B(0;\rho) \setminus \partial\Omega$. Since $\partial\Omega$ is Lipschitz, there exists a constant C_6 such that $\sigma(B(x;r)\cap\partial\Omega) \le C_6 r^{m-1}$ for any $x \in \mathbb{R}^m$ and $r > 0$. Put $r = \operatorname{dist}(x,\partial\Omega)$. Choose $z \in \partial\Omega$ such that $|z-x| = r$. Then

$$|Kf(x)| \le c\left[\int_{\partial\Omega\cap B(z;2r)} |K(x,y)|\,\mathrm{d}\sigma(y) + \int_{\partial\Omega\setminus B(z;2r)} |K(x,y)-K(z,y)|\,\mathrm{d}\sigma(y)\right.$$
$$\left. + \int_{\partial\Omega\setminus B(z;2r)} |K(z,y)|\,\mathrm{d}\sigma(y)\right] \le c\left[\int_{\partial\Omega\cap B(z;2r)} C_4 r^{1-m}\,\mathrm{d}\sigma(y)\right.$$
$$\left. + \int_{\partial\Omega\setminus B(z;2r)} C_2\frac{|x-z|}{|z-y|^m}\,\mathrm{d}\sigma(y) + \int_{\partial\Omega} C_3|z-y|^{\alpha+1-m}\,\mathrm{d}\sigma(y)\right].$$

By virtue of Lemma 3.1

$$|Kf(x)| \le c\left(C_6C_4 2^{m-1} + C_6C_2 rm(2r)^{-1} + C_6C_3\frac{m-1}{\alpha}(2\rho)^\alpha\right)$$
$$\le cC_6 2^m m\left(C_4 + C_2 + C_3\frac{\rho^\alpha}{\alpha}\right).$$

□

4. Boundary layer potentials

We shall look for an L^∞-solution of the Dirichlet problem in the form of a Stokes resolvent double layer potential with density from $L^\infty(\partial\Omega,\mathbb{C}^m)$. For this reason we need to know properties of boundary layer potentials.

Let $\lambda \in \mathbb{C}\setminus\{z<0\}$. Then there exists a unique fundamental solution $E^\lambda = (E^\lambda_{ij})$, $Q^\lambda = (Q^\lambda_j)$ of the system (1.1) such that $E^\lambda(x) = o(|x|)$, $Q^\lambda(x) = o(|x|)$ as $|x| \to \infty$. Remember that for $i,j \in \{1,\dots,m\}$ we have

$$-\Delta E^\lambda_{ij} + \lambda E^\lambda_{ij} + \partial_i Q^\lambda_j = \delta_{ij}\delta_0, \quad \partial_1 E^\lambda_{1j} + \dots \partial_m E^\lambda_{mj} = 0, \tag{4.1}$$

$$-\Delta E^\lambda_{i,m+1} + \lambda E^\lambda_{i,m+1} + \partial_i Q^\lambda_{m+1} = 0, \quad \partial_1 E^\lambda_{1,m+1} + \dots \partial_m E^\lambda_{m,m+1} = \delta_0. \tag{4.2}$$

If $j \in \{1,\dots,m\}$ then

$$Q^\lambda_j(x) = E^\lambda_{j,m+1}(x) = \frac{1}{\omega_n}\frac{x_j}{|x|^m},$$

$$Q^\lambda_{m+1} = \begin{cases} \delta_0(x) + (\lambda/\omega_m)\ln|x|^{-1}, & m=2,\\ \delta_0(x) + (\lambda/\omega_m)(m-2)^{-1}|x|^{2-m}, & m>2,\end{cases}$$

where ω_m is the area of the unit sphere in $\mathbb{R}^m$. (See [47, p. 60]. The expressions of E^λ can be found in the book [47, Chapter 2]. We omit them for the sake of brevity.

For $\lambda = 0$ we obtain the fundamental solution of the Stokes system. If $i,j \in \{1,\dots,m\}$, the components of E^0 are given by

$$E^0_{ij}(x) = \frac{1}{2\omega_m}\left\{\frac{\delta_{ij}}{(m-2)|x|^{m-2}} + \frac{x_i x_j}{|x|^m}\right\}, \quad m \ge 3 \tag{4.3}$$

$$E^0_{ij}(x) = \frac{1}{4\pi}\left\{\delta_{ij}\ln\frac{1}{|x|} + \frac{x_j x_k}{|x|^2}\right\}, \quad m=2, \tag{4.4}$$

(see, e.g., [47, p. 16]).

If $i,j \le m$ then

$$|E^\lambda_{ij}(x) - E^0_{ij}(x)| = O(1) \quad \text{as } |x| \to 0. \tag{4.5}$$

(See [47], p. 66.)

If $\lambda \neq 0$ and $i,j \le m$ then

$$E^\lambda_{ij}(x) = O(|x|^{-m}),\ \nabla E^\lambda_{ij}(x) = O(|x|^{-m}) \qquad \text{as } |x| \to \infty \tag{4.6}$$

by [47, Chapter 2].

We denote $Q(x) = (Q^0_1(x),\dots,Q^0_m(x)) = (Q^\lambda_1(x),\dots,Q^\lambda_m(x))$. By $\tilde E^\lambda$ we denote the matrix of the type $m \times m$, where $\tilde E^\lambda_{ij}(x) = E^\lambda_{ij}(x)$ for $i,j \le m$.

Let now $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. If $1<q<\infty$ and $\mathbf{g} \in L^q(\partial\Omega,\mathbb{C}^m)$ then the single-layer potential for the Stokes resolvent system $E^\lambda_\Omega \mathbf{g}$ and its associated pressure potential $Q_\Omega \mathbf{g}$ are given by

$$E^\lambda_\Omega \mathbf{g}(x) := \int_{\partial\Omega} \tilde E^\lambda(x-y)\mathbf{g}(y)\ \mathrm{d}\sigma(y),$$

$$Q_\Omega \mathbf{g}(x) := \int_{\partial\Omega} Q(x-y)\mathbf{g}(y)\ \mathrm{d}\sigma(y).$$

Remark that $(E^\lambda_\Omega \mathbf{g}, Q_\Omega \mathbf{g})$ is a solution of the Stokes resolvent system (1.1) in the set $\mathbb{R}^m \setminus \partial\Omega$.

We summarize boundary behaviour of single layer potentials. If $\mathbf{g} \in L^2(\partial\Omega,\mathbb{C}^m)$ then $E^\lambda_\Omega \mathbf{g}(x)$ is the nontangential limit of $E^\lambda_\Omega \mathbf{g}$ at x with respect to $\Omega_+ = \Omega$ and with respect to $\Omega_- = \mathbb{R}^\setminus\overline{\Omega}$ at almost all $x \in \partial\Omega$ (see [18, Lemma 2.1.4], [22, Lemma 3.1]). If $\lambda \neq 0$ then $(E^\lambda_\Omega \mathbf{g})^*, (\nabla E^\lambda_\Omega \mathbf{g})^* \in L^2(\partial\Omega)$ (see [18, Lemma 2.1.4]), and

$$|E^\lambda_\Omega \mathbf{g}(x)| + |\nabla E^\lambda_\Omega \mathbf{g}(x)| = O(|x|^{-m}) \qquad \text{as } |x| \to \infty.$$

Denote

$$K^\lambda_\Omega(y,x) = -T_x(\tilde{E}^\lambda(x-y), Q(x-y))\mathbf{n}^\Omega(x),$$

where

$$T(\mathbf{u},p) = 2\hat{\nabla}\mathbf{u} - pI, \quad \hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the stress tensor corresponding to a velocity $\mathbf{u}$ and a pressure p. For $\mathbf{\Psi} \in L^2(\partial\Omega,\mathbb{C}^m)$ define

$$K'_{\Omega,\lambda}\mathbf{\Psi}(x) = \lim_{\epsilon\searrow 0} \int\limits_{\partial\Omega\setminus B(x;\epsilon)} K^\lambda_\Omega(y,x)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y),$$

where $B(x;\epsilon) = \{y; |x-y| < \epsilon\}$. Then $K'_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega,\mathbb{R}^m)$. If $\mathbf{\Psi} \in L^2(\partial\Omega,\mathbb{C}^m)$, there are the non-tangential limits $[\nabla E^\lambda_\Omega\mathbf{\Psi}(x)]_\pm$, $[Q^\lambda_\Omega\mathbf{\Psi})(x)]_\pm$ of $\nabla E^\lambda_\Omega\mathbf{\Psi}$, $Q^\lambda_\Omega\mathbf{\Psi}$ with respect to $\Omega_\pm$ at almost all $x \in \partial\Omega$, and

$$[T(E^\lambda_\Omega\mathbf{\Psi}, Q_\Omega\mathbf{\Psi})]_\pm\mathbf{n}^\Omega = \pm\frac{1}{2}\mathbf{\Psi} - K'_{\Omega,\lambda}\mathbf{\Psi}. \tag{4.7}$$

(For $\lambda = 0$ see [40, Corollary 4.3.2], for $\lambda \neq 0$ see for example [23, Lemma 3.1]. See also [39, Theorem 3.1].)

Now we define a double layer potential. For $\mathbf{\Psi} \in L^2(\partial\Omega,\mathbb{C}^m)$ define in $\mathbb{R}^m \setminus \partial\Omega$

$$(D^\lambda_\Omega\mathbf{\Psi})(x) = \int\limits_{\partial\Omega} K^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y), \tag{4.8}$$

and the corresponding pressure by

$$(\Pi^\lambda_\Omega\mathbf{\Psi})(x) = \int\limits_{\partial\Omega} \Pi^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y). \tag{4.9}$$

If $m > 2$ then

$$\Pi^\lambda_\Omega(x,y) = \frac{1}{\omega_m}\left\{-(y-x)\frac{2m(y-x)\cdot\mathbf{n}^\Omega(y)}{|y-x|^{m+2}} + \frac{2\mathbf{n}^\Omega(y)}{|y-x|^m} - \lambda\frac{|x-y|^{2-m}}{m-2}\mathbf{n}^\Omega(y)\right\}.$$

If $m = 2$ then

$$\Pi^\lambda_\Omega(x,y) = \frac{1}{2\pi}\left\{-(y-x)\frac{4(y-x)\cdot\mathbf{n}^\Omega(y)}{|y-x|^4} + \frac{2\mathbf{n}^\Omega(y)}{|y-x|^m} - \lambda\left(\ln\frac{1}{|x-y|}\right)\mathbf{n}^\Omega(y)\right\}.$$

Remark that $D^\lambda_\Omega\mathbf{\Psi} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \partial\Omega,\mathbb{R}^m)$, $\Pi^\lambda_\Omega\mathbf{\Psi} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \partial\Omega,\mathbb{R}^1)$ and $\nabla\Pi^\lambda_\Omega\mathbf{\Psi} - \Delta D^\lambda_\Omega\mathbf{\Psi} + \lambda D^\lambda_\Omega\mathbf{\Psi} = 0$, $\nabla\cdot D^\lambda_\Omega\mathbf{\Psi} = 0$ in $\mathbb{R}^m \setminus \partial\Omega$.

Define

$$K_{\Omega,\lambda}\mathbf{\Psi}(x) = \lim_{\epsilon\searrow 0} \int\limits_{\partial\Omega\setminus B(x;\epsilon)} K^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\ \mathrm{d}\sigma(y), \quad x \in \partial\Omega.$$

Then $K_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega;\mathbb{C}^m)$ (adjoint to $K'_{\Omega,\lambda}$). There exists the nontangential limit $[D^\lambda_\Omega\mathbf{\Psi}]_+(x)$ of $D^\lambda_\Omega\mathbf{\Psi}$ with respect to Ω_+ and the nontangential limit $[D^\lambda_\Omega\mathbf{\Psi}]_-(x)$ of $D^\lambda_\Omega\mathbf{\Psi}$ with respect to Ω_- for almost all $x \in \partial\Omega$ and

$$[D^\lambda_\Omega\mathbf{\Psi}]_\pm(x) = \pm\frac{1}{2}\mathbf{\Psi}(z) + K_{\Omega,\lambda}\mathbf{\Psi}(z). \tag{4.10}$$

At infinity we have the estimate

$$D^\lambda_\Omega\mathbf{\Psi}(x) = O(|x|^{1-m}) \qquad \text{as } |x| \to \infty.$$

Lemma 4.1. *Let $\mu \in \mathbb{C} \setminus \{z \leq 0\}$. If $i, j \leq m$ then $|\nabla E^{\mu}_{ij}(x) - \nabla E^{0}_{ij}(x)| = O(|x|^{2-m})$ as $|x| \to 0$.*

Proof. For $m > 2$ see [44, Theorem 2.5]. Let now $m = 2$. Since $Q^{\mu}_i = Q^0_i$, we obtain subtracting (4.1) for $\lambda = \mu$ and $\lambda = 0$

$$-\Delta(E^0_{ij} - E^{\mu}_{ij}) + \mu(E^0_{ij} - E^{\mu}_{ij}) = \mu E^0_{ij}.$$

Fix $q \in (2, \infty)$. Since $\mu E^0_{ij} \in L^q_{loc}(\mathbb{R}^2)$, regularity results for elliptic equations yields that $(E^0_{ij} - E^{\mu}_{ij}) \in W^{2,q}_{loc}(\mathbb{R}^2)$.(See for example [31, Chapter 2, Théorème 3.2] and [38, Proposition 2.7].) Sobolev's embedding theorem gives that $(E^0_{ij} - E^{\mu}_{ij}) \in \mathcal{C}^1(R^2)$ (see for example [1, Theorem 4.12]). □

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. If $\lambda \in \mathbb{C} \setminus \{z \leq 0\}$ then $K_{\Omega,\lambda} - K_{\Omega,0}$ is a compact linear operator on $L^2(\partial\Omega, \mathbb{R}^m)$.*

(See [25, Theorem 3.3].)

Proposition 4.3. *Let $\Omega \subset R^m$ be an open set with compact boundary of class $\mathcal{C}^{1,\alpha}$, $m \geq 2$, $0 < \alpha < 1$, $\lambda \in \mathbb{C} \setminus \{z < 0\}$. Then $K_{\Omega,\lambda}$ is a compact linear operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$ and on $L^2(\partial\Omega, \mathbb{C}^m)$.*

Proof. According to [47] and [6, Chapter III, Lemma 2.1] there is a constant C such that $|K^{\lambda}_{\Omega}(x, y)| \leq C|x - y|^{\alpha+1-m}$ for each $x, y \in \partial\Omega$. So, $K_{\Omega,\lambda}$ is a compact linear operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$ by [11, § 4.5.2, Satz 2] and on $L^2(\partial\Omega, \mathbb{C}^m)$ by [46, Satz 12.1]. □

Now we show that the double layer potential D^{λ}_{Ω} is a bounded operator from $L^\infty(\partial\Omega, \mathbb{C}^m)$ to $L^\infty(\mathbb{R}^m \setminus \partial\Omega, \mathbb{C}^m)$. To use Lemma 3.3 we need the following lemma:

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let $r \in (0, \infty)$ be such that $\partial\Omega \subset B(0; r)$. Then there exists a constant C such that*

$$|K^0_{\Omega}(x, y) - K^0_{\Omega}(z, y)| \leq C \frac{|x - z|}{|z - y|^m} \tag{4.11}$$

for all $x, z \in B(0; r)$, $y \in \partial\Omega$ with $|z - y| > 2|x - z|$.

Proof. There exists a constant C_1 such that

$$|\nabla^2 E^0_{ij}(x - y)| + |\nabla Q^0_j(x - y)| \leq C_1|x - y|^{-m}, \quad i, j \leq m.$$

Let i, j, k be given. If $x, z \in B(0; r)$, $y \in \partial\Omega$, then there exists $\tilde{z}$ in the interval $\overline{xz}$ such that

$$|\partial_k E^0_{ij}(x - y) - \partial_k E^0_{ij}(z - y)| = |(x - z) \cdot \nabla\partial_k E^0_{ij}(\tilde{z} - y)| \leq C_1|x - z||y - \tilde{z}|^{-m}.$$

If $|z - y| > 2|x - z|$ then $|\tilde{z} - y| > |z - y|/2$. Thus

$$|\partial_k E^0_{ij}(x - y) - \partial_k E^0_{ij}(z - y)| \leq 2^m C_1|x - z||z - y|^{-m}.$$

By the same way we prove

$$|Q_j(x - y) - Q_j(z - y)| \leq 2^m C_1|x - z||z - y|^{-m}.$$

By the definition of K^0_{Ω} we obtain (4.11). □

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact boundary of class $C^{1,\alpha}$, $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z < 0\}$. Then there exists a constant C such that*

$$|D^{\lambda}_{\Omega}\mathbf{f}| \leq C\|\mathbf{f}\|_{L^\infty(\partial\Omega)} \quad \text{on } \mathbb{R}^m \setminus \partial\Omega, \qquad \forall \mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m). \tag{4.12}$$

Proof. Suppose first that $\lambda = 0$. Fix $\rho > 0$ such that $\partial\Omega \subset B(0;\rho)$. According to Lemma 4.4 there exists a constant C_1 such that

$$|K^0_\Omega(x,y) - K^0_\Omega(z,y)| \le C_1 \frac{|x-z|}{|z-y|^m}$$

for all $x, z \in B(0;\rho)$, $y \in \partial\Omega$ with $|z-y| > 2|x-z|$. According to [47] or [6, Chapter III, Lemma 2.1] there is a constant C_2 such that $|K^0_\Omega(z,y)| \le C_2|z-y|^{\alpha+1-m}$ for each $z, y \in \partial\Omega$. Clearly, there exists a constant C_3 such that $|K^0_\Omega(x,y)| \le C_3|x-y|^{1-m}$ for all $x \in B(0;\rho)$ and $y \in \partial\Omega$. Lemma 3.3 gives that there exists a constant C such that (4.12) holds.

Suppose now that $\lambda \neq 0$.

According to Lemma 4.1 and the definition of K^λ_Ω there exists a constant C_4 such that

$$|K^\lambda_\Omega(x,y) - K^0_\Omega(x,y)| \le C_4|x-y|^{3/2-m}, \qquad \forall x \in B(0;\rho), y \in \partial\Omega,$$

and $|K^\lambda_\Omega(x,y) - K^0_\Omega(x,y)| \le C_4$ for $|x| \ge \rho$ and $y \in \partial\Omega$. Lemma 3.2 gives that there exists a constant C_5 such that

$$|D^\lambda_\Omega \mathbf{f} - D^0_\Omega \mathbf{f}| \le C_5 \|\mathbf{f}\|_{L^\infty(\partial\Omega)} \quad \text{on } \mathbb{R}^m \setminus \partial\Omega, \qquad \forall \mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m).$$

□

5. Behaviour of bounded solutions of the Stokes resolvent system at infinity

In this section we study a behaviour of a solution $(\mathbf{u}, p)$ of the Stokes resolvent system at infinity, under assumption that $\mathbf{u}$ is bounded. It was shown in [36] that if $(\mathbf{u}, p)$ is a solution of the Brinkman system and both $\mathbf{u}$ and p are bounded, then $\mathbf{u}(x) \to 0$ as $|x| \to \infty$. It does not hold if we suppose only that $\mathbf{u}$ is bounded. (If $\mathbf{c}$ is a constant vector then $\mathbf{u} \equiv \mathbf{c}$, $p = -\lambda(\mathbf{c} \cdot x)$ is a solution of the Stokes resolvent system (1.1).) To describe a behaviour of $\mathbf{u}$ and p at infinity we find an integral representation formula for $(\mathbf{u}, p)$.

First we prove Liouville's theorem.

Proposition 5.1. *Let $\lambda \in \mathbb{C} \setminus \{z < 0\}$, p be a distribution, $u_1, \dots, u_m$ be tempered distributions, $\mathbf{u} = (u_1, \dots, u_m)$. If $-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$ in R^m in the sense of distribution, then $u_1, \dots, u_m$, p are polynomials.*

Proof. Suppose first that p is a tempered distribution. The proof is literally the same like the proof of [36, Proposition 4.1] for $\lambda \ge 0$.

Let now p be general. Since u_j is a tempered distribution, also $\partial_k u_j$, Δu_j are tempered distributions. Since $\partial_j p = \Delta u_j - \lambda u_j$, we deduce that $\partial_j p$ is a tempered distribution. Since $-\Delta(\partial_j \mathbf{u}) + \lambda(\partial_j \mathbf{u}) + \nabla(\partial_j p) = 0$ in R^m, we deduce that $\partial_j u_1, \dots, \partial_j u_m$, $\partial_j p$ are polynomials. This forces that $u_1, \dots, u_m$, p are polynomials, too. □

Proposition 5.2. *Let $\lambda \in \mathbb{C} \setminus \{z < 0\}$, $\Omega \subset R^m$ be an unbounded open set with compact boundary. Let $-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0$ in Ω. If $\mathbf{u}$ is bounded then there exists $\mathbf{u}_\infty \in \mathbb{C}^m$ such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. If $\lambda \neq 0$ then $|\mathbf{u}(x) - \mathbf{u}_\infty| = O(|x|^{1-m})$ as $|x| \to \infty$.*

Proof. Choose $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $\omega = B(0;2r) \setminus \overline{B(0;r)}$. Then

$$\mathbf{u} = E^\lambda_\omega[T(\mathbf{u},p)n] + D^\lambda_\omega \mathbf{u}, \quad p = Q_\omega[T(\mathbf{u},p)n] + \Pi^\lambda_\omega \mathbf{u} \quad \text{in } \omega \tag{5.1}$$

by [47, p. 60]. Define

$$\mathbf{v} = \begin{cases} \mathbf{u}(x) + E^\lambda_{B(0;r)}[T(\mathbf{u},p)n](x) + D^\lambda_{B(0;r)}\mathbf{u}(x), & x \in \mathbb{R}^m \setminus \overline{B(0;r)}, \\ E^\lambda_{B(0;2r)}[T(\mathbf{u},p)n](x) + D^\lambda_{B(0;2r)}\mathbf{u}(x), & x \in B(0;2r), \end{cases}$$

$$q = \begin{cases} p(x) + Q_{B(0;r)}[T(\mathbf{u},p)n](x) + \Pi^\lambda_{B(0;r)}\mathbf{u}(x), & x \in \mathbb{R}^m \setminus \overline{B(0;r)}, \\ Q_{B(0;2r)}[T(\mathbf{u},p)n](x) + \Pi^\lambda_{B(0;2r)}\mathbf{u}(x), & x \in B(0;2r). \end{cases}$$

Then $\mathbf{v}$, q are well defined by (5.1). Clearly, $-\Delta \mathbf{v} + \lambda \mathbf{v} + \nabla q = 0$, $\nabla \cdot \mathbf{v} = 0$ in $\mathbb{R}^m$. Proposition 5.1 gives that $v_1, \dots, v_m$ are polynomials. Since $\mathbf{v}$ is bounded, there exists $\mathbf{u}_\infty \in \mathbb{C}^m$ such that $\mathbf{v} \equiv \mathbf{u}_\infty$. Since $\mathbf{u} = \mathbf{u}_\infty + E^\lambda_\omega[T(\mathbf{u},p)n] + D^\lambda_\omega \mathbf{u}$ in Ω, we infer that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. □

Now we prove an integral representation formula for a solution $(\mathbf{u}, p)$ of the Stokes resolvent system with $\mathbf{u}$ bounded. We need the following lemma:

Lemma 5.3. *If $\Omega \subset \mathbb{R}^m$ is a bounded open set with Lipschitz boundary then there is a sequence of bounded open sets Ω_j with boundaries of class C^∞ such that*

- *$\overline{\Omega}_j \subset \Omega$.*
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(y) \in \Gamma^\Omega_a(y)$ for each j and each $y \in \partial\Omega$ and $\sup\{|y - \Lambda_j(y)|; y \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\sigma = \sigma(\Lambda_j(E))$, and so that $\omega_j \to 1$ pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$.*
- *The normal vectors to Ω_j, $n(\Lambda_j(y))$, converge pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$, to $n(y)$.*

(See [50, Theorem 1.12].)

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded open set with Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z < 0\}$, $1 < q < \infty$. Let $-\Delta \mathbf{u} + \lambda \mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$ in Ω. Suppose that there exists nontangential limits of $\mathbf{u}$, $\nabla \mathbf{u}$ and p at almost all points of $\partial\Omega$. Fix $r > 0$ such that $\partial\Omega \subset B(0;r)$. Denote $\omega = \Omega \cap B(0;r)$. Suppose that the nontangential maximal functions with respect to ω: $\mathbf{u}^*_\omega, (\nabla \mathbf{u})^*_\omega, p^*_\omega \in L^q(\partial\omega)$. If $\mathbf{u}(x) \to 0$ as $|x| \to \infty$, then there exists a constant p_∞ such that*

$$\mathbf{u} = E^\lambda_\Omega[T(\mathbf{u},p)n] + D^\lambda_\Omega \mathbf{u}, \quad p = Q_\Omega[T(\mathbf{u},p)n] + \Pi^\lambda_\Omega \mathbf{u} + p_\infty \quad \textit{in } \Omega. \tag{5.2}$$

Proof. Approximate ω from inside by open sets $\omega(k)$ as in Lemma 5.3. Then

$$\mathbf{u} = E^\lambda_{\omega(k)}[T(\mathbf{u},p)n] + D^\lambda_{\omega(k)}\mathbf{u}, \quad p = Q_{\omega(k)}[T(\mathbf{u},p)n] + \Pi^\lambda_{\omega(k)}\mathbf{u} \quad \text{in } \omega(k).$$

by [47, p. 60]. Letting $k \to \infty$ we obtain by Lebesgue's lemma

$$\mathbf{u} = E^\lambda_\omega[T(\mathbf{u},p)n] + D^\lambda_\omega \mathbf{u}, \quad p = Q_\omega[T(\mathbf{u},p)n] + \Pi^\lambda_\omega \mathbf{u} \quad \text{in } \omega. \tag{5.3}$$

Define

$$\mathbf{v} = \begin{cases} -E^\lambda_\Omega[T(\mathbf{u},p)n](x) - D^\lambda_\Omega \mathbf{u}(x) + \mathbf{u}(x), & x \in \Omega, \\ E^\lambda_{B(0;r)}[T(\mathbf{u},p)n](x) + D^\lambda_{B(0;r)}\mathbf{u}(x), & x \in B(0;r), \end{cases}$$

$$s = \begin{cases} -Q_\Omega[T(\mathbf{u},p)n](x) - \Pi^\lambda_\Omega \mathbf{u}(x) + p(x), & x \in \Omega, \\ Q_{B(0;r)}[T(\mathbf{u},p)n](x) + \Pi^\lambda_{B(0;r)}\mathbf{u}(x), & x \in B(0;r). \end{cases}$$

The functions $\mathbf{v}$, s are well defined by (5.3). Clearly, $-\Delta\mathbf{v}+\lambda\mathbf{v}+\nabla s = 0$, $\nabla\cdot\mathbf{v} = 0$ in $\mathbb{R}^m$. Proposition 5.1 gives that $v_1,\dots,v_m$ are polynomials. Since $\mathbf{v}(x)\to 0$ as $|x|\to\infty$, we deduce that $\mathbf{v} = 0$. Thus $\nabla s = \Delta\mathbf{v} - \lambda\mathbf{v} \equiv 0$. So, there exists a constant p_∞ such that $s \equiv p_\infty$. The definition of $\mathbf{v}$, s in Ω gives (5.2) . □

6. Uniqueness of an L^∞-solution of the Dirichlet problem

In this section we study the uniqueness of a solution of the Dirichlet problem for the Stokes resolvent system (2.1). For this reason we need the following regularity result:

Lemma 6.1. *Let $\Omega\subset\mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $\lambda\in\mathbb{C}\setminus\{z<0\}$. Let $(\mathbf{u},p)$ be an L^∞-solution of the Dirichlet problem (2.1). If $\mathbf{g}\in W^{1,2}(\partial\Omega,\mathbb{C}^m)$, then $(\nabla\mathbf{u})^*_\Omega, p^*_\Omega \in L^2(\partial\Omega)$ (and therefore $\mathbf{u}\in W^{1,2}(\Omega,\mathbb{C}^m)$, $p\in L^2(\Omega)$), and there exist nontangential limits of $\nabla\mathbf{u}$ and p at almost all points of $\partial\Omega$.*

Proof. For $\lambda = 0$ see [40, Theorem 9.2.2, Theorem 9.2.5].

Let now $\lambda\neq 0$. Fix $q\in(m,\infty)$. Since $\mathbf{u}^*_\Omega\in L^\infty(\partial\Omega)\subset L^q(\partial\Omega)$, we have $\mathbf{u}\in L^q(\Omega,\mathbb{C}^m)$ (see [37, Lemma 4.1]). Define $\mathbf{u}=0$ on $\mathbb{R}^m\setminus\Omega$,

$$\mathbf{v} = \tilde{E}^0 * \lambda\mathbf{u}, \qquad \pi = Q*\lambda\mathbf{u}.$$

Then $-\Delta\mathbf{v}+\nabla\pi = \lambda\mathbf{u}$, $\nabla\mathbf{v} = 0$ in $\mathbb{R}^m$. Since $\lambda\mathbf{u}\in L^q(\mathbb{R}^m,\mathbb{C}^m)$, we have $\mathbf{v}\in W^{2,q}(\Omega,\mathbb{C}^m)$, $\pi\in W^{1,q}(\Omega)$ (see [12, Chapter IV, Theorem 2.1]). The Sobolev embedding theorem [1, Theorem 4.12] gives that $\mathbf{v}\in\mathcal{C}^1(\overline{\Omega},\mathbb{C}^m)$, $\pi\in\mathcal{C}(\overline{\Omega})$. So, $(\mathbf{u}+\mathbf{v},p+\pi)$ is an L^∞-solution of the Dirichlet problem for the Stokes system

$$\Delta(\mathbf{u}+\mathbf{v}) = \nabla(p+\pi), \qquad \nabla\cdot(\mathbf{u}+\mathbf{v}) = 0 \quad \text{on } \Omega,$$

$$\mathbf{v}+\mathbf{u} = \tilde{\mathbf{g}} \qquad \text{on } \partial\Omega,$$

where $\tilde{\mathbf{g}} = \mathbf{g}+\mathbf{v}\in W^{1,2}(\partial\Omega,\mathbb{C}^m)$. So, $(\nabla\mathbf{u}+\nabla\mathbf{v})^*_\Omega, (p+\pi)^*_\Omega\in L^2(\partial\Omega)$, and there exist nontangential limits of $\nabla\mathbf{u}+\nabla\mathbf{v}$ and $p+\pi$ at almost all points of $\partial\Omega$. Since $\mathbf{v}\in\mathcal{C}^1(\overline{\Omega},\mathbb{C}^m)$, $\pi\in\mathcal{C}(\overline{\Omega})$, we deduce $(\nabla\mathbf{u})^*_\Omega, p^*_\Omega\in L^2(\partial\Omega)$, and there exist nontangential limits of $\nabla\mathbf{u}$ and p at almost all points of $\partial\Omega$. According to [37, Lemma 4.1] we have $\mathbf{u}\in W^{1,2}(\Omega,\mathbb{C}^m)$, $p\in L^2(\Omega)$. □

We prove the uniqueness of a solution of the Dirichlet problem using Green's formula:

Lemma 6.2. *Let $\Omega\subset\mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $\lambda\in\mathbb{C}\setminus\{z<0\}$. If $(\mathbf{u},p)$ is a solution of the Stokes resolvent system (1.1) in Ω such that $\mathbf{u}^*_\Omega, (\nabla\mathbf{u})^*_\Omega, p^*_\Omega\in L^2(\partial\Omega)$, and there exist nontangential limits of $\mathbf{u}$, $\nabla\mathbf{u}$ and p at almost all points of $\partial\Omega$, then*

$$\int_{\partial\Omega}\mathbf{u}\cdot T(\mathbf{u},p)\mathbf{n}^\Omega\,\mathrm{d}\sigma = \int_\Omega(2|\hat{\nabla}\mathbf{u}|^2+\lambda|\mathbf{u}|^2)\,\mathrm{d}x. \tag{6.1}$$

Proof. Let Ω_j be open sets from Lemma 5.3. Then Green's formula gives (6.1) for Ω_j (see [47, p. 14] or [26], Theorem 1.5.1). Letting $j\to\infty$ we obtain (6.1) for Ω by virtue of Lebesgue's Lemma. □

Proposition 6.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z \le 0\}$. Let $(\mathbf{u}, p)$ be an L^2-solution of the Dirichlet problem for the Stokes resolvent system (2.1) such that $(\nabla \mathbf{u})^*_\Omega, p^*_\Omega \in L^2(\partial\Omega)$ and there exist nontangential limits of $\nabla \mathbf{u}$, p at almost all points of $\partial\Omega$. If $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$ and p is constant.*

Proof. According to Lemma 6.2

$$0 = \int_{\partial\Omega} \mathbf{u} \cdot T(\mathbf{u}, p)\mathbf{n}^\Omega \, \mathrm{d}\sigma = \int_\Omega (2|\hat{\nabla} \mathbf{u}|^2 + \lambda |\mathbf{u}|^2) \, \mathrm{d}x.$$

Thus $\mathbf{u} \equiv 0$. Hence $\nabla p = \Delta \mathbf{u} - \lambda \mathbf{u} = 0$. This forces that p is constant. ☐

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z \le 0\}$. Let $(\mathbf{u}, p)$ be an L^2-solution of the Dirichlet problem for the Stokes resolvent system (2.1) such that $\mathbf{u}(x) \to 0$ as $|x| \to \infty$. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$ and set $\omega(r) = \Omega \cap B(0; r)$. Suppose that $(\nabla \mathbf{u})^*_{\omega(r)}, p^*_{\omega(r)} \in L^2(\partial\omega)$ and there exist nontangential limits of $\nabla \mathbf{u}$, p at almost all points of $\partial\Omega$. If $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$ and p is constant.*

Proof. Put $\mathbf{f} = T(\mathbf{u}, p)\mathbf{n}^\Omega$. According to Proposition 5.4

$$\mathbf{u} = E^\lambda_\Omega \mathbf{f} + D^\lambda_\Omega \mathbf{g} = E^\lambda_\Omega \mathbf{f}.$$

Lemma 6.2 gives

$$\int_{\partial\omega(r)} (E^\lambda_\Omega \mathbf{f}) \cdot T(E^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})\mathbf{n} \, \mathrm{d}\sigma = \int_{\omega(r)} (2|\hat{\nabla} E^\lambda_\Omega \mathbf{f}|^2 + \lambda |E^\lambda_\Omega \mathbf{f}|^2) \, \mathrm{d}x.$$

Since $E^\lambda_\Omega \mathbf{f}(x) = O(|x|^{-m})$, $T(E^\lambda_\Omega \mathbf{f}(x), Q_\Omega \mathbf{f}(x))\mathbf{n}(x) = O(|x|^{1-m})$ as $|x| \to \infty$, we have

$$0 = \int_{\partial\Omega} (E^\lambda_\Omega \mathbf{f}) \cdot T(E^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})\mathbf{n} \, \mathrm{d}\sigma$$

$$= \lim_{r \to \infty} \int_{\partial\omega(r)} (E^\lambda_\Omega \mathbf{f}) \cdot T(E^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})\mathbf{n} \, \mathrm{d}\sigma = \int_\Omega (2|\hat{\nabla} E^\lambda_\Omega \mathbf{f}|^2 + \lambda |E^\lambda_\Omega \mathbf{f}|^2) \, \mathrm{d}x.$$

Thus $\mathbf{u} = E^\lambda_\Omega \mathbf{f} = 0$ in Ω. Hence $\nabla p = \Delta \mathbf{u} - \lambda \mathbf{u} = 0$. This forces that p is constant. ☐

Corollary 6.5. *Let $\Omega \subset \mathbb{R}^m$ be a domain with compact Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z \le 0\}$, $(\mathbf{u}, p)$ be an L^∞-solution of the Dirichlet problem for the Stokes resolvent system (2.1). If Ω is unbounded suppose moreover that $\mathbf{u}(x) \to 0$ as $|x| \to \infty$. If $\mathbf{g} \equiv 0$, then $\mathbf{u} \equiv 0$ and p is constant.*

Proof. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$ and put $\omega = \Omega \cap B(0; r)$. Lemma 6.1 gives that $(\nabla \mathbf{u})^*_\omega, p^*_\omega \in L^2(\partial\omega)$, and there exist nontangential limits of $\nabla \mathbf{u}$ and p at almost all points of $\partial\omega$. So, $\mathbf{u} \equiv 0$ and p is constant by Proposition 6.3 and Proposition 6.4. ☐

7. L^∞-solutions of the Dirichlet problem

In this section we prove the existence of a bounded solution of the Dirichlet problem (2.1) for the Stokes resolvent system and a bounded boundary condition. We look for a particular solution of the problem in the form of a double layer potential

$$\mathbf{u} = D^\lambda_\Omega \mathbf{f}, \quad p = \Pi^\lambda_\Omega \mathbf{f}$$

with $\mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m)$. Then $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem (2.1) if and only if

$$\frac{1}{2}\mathbf{f} + K_{\Omega,\lambda}\mathbf{f} = \mathbf{g}.$$

Proposition 7.1. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z \le 0\}$. Then $\frac{1}{2}I + K_{\Omega,\lambda}$ is an isomorphism on $L^2(\partial\Omega, \mathbb{C}^m)$. If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$ then $\frac{1}{2}I + K_{\Omega,\lambda}$ is an isomorphism on $L^\infty(\partial\Omega, \mathbb{C}^m)$.*

Proof. The operator $\frac{1}{2}I + K_{\Omega,0}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$ by [40, Proposition 5.3.5]. Since $K_{\Omega,0} - K_{\Omega,\lambda}$ is a compact operator on $L^2(\partial\Omega, \mathbb{C}^m)$ by Lemma 4.2, the operator $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$. Its adjoint operator $\frac{1}{2}I + K'_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$, too. Let now $\mathbf{f} \in L^2(\partial\Omega, \mathbb{C}^m)$ be such that $\frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f} = 0$. Define $\mathbf{u} = E^\lambda_\Omega \mathbf{f}$, $p = Q_\Omega \mathbf{f}$ on $\omega = \mathbb{R}^m \setminus \overline{\Omega}$. Then $T(\mathbf{u}, p)\mathbf{n}^\omega = \frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f} = 0$ by (4.7). Let now G be a component of ω. Then G is bounded. Properties of single layer potentials and Lemma 6.2 give

$$0 = \int_{\partial G} \mathbf{u} \cdot T(\mathbf{u}, p)\mathbf{n}^G \, \mathrm{d}\sigma = \int_G (2|\hat{\nabla}\mathbf{u}|^2 + \lambda|\mathbf{u}|^2) \, \mathrm{d}x.$$

Thus $\mathbf{u} = 0$ in G. Therefore $\nabla p = \Delta\mathbf{u} - \lambda\mathbf{u} = 0$ in G and p is constant in G. Since $0 = T(\mathbf{u}, p)\mathbf{n}^G = -p\mathbf{n}^G$, we infer that $p = 0$ in G. Hence $E^\lambda_\Omega\mathbf{f} = 0$, $Q_\Omega\mathbf{f} = 0$ in $\mathbb{R}^m \setminus \overline{\Omega}$. Using a nontangential limit we get $E^\lambda_\Omega\mathbf{f} = 0$ on $\partial\Omega$. So, $(E^\lambda_\Omega\mathbf{f}, Q_\Omega\mathbf{f})$ is an L^2-solution of the Dirichlet problem $-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla\pi = 0$, $\nabla \cdot \mathbf{v} = 0$ in Ω, $\mathbf{v} = 0$ on $\partial\Omega$. Using properties of single layer potentials and Proposition 6.4 we obtain that $\mathbf{v} := E^\lambda_\Omega\mathbf{f} = 0$, $\pi := Q_\Omega\mathbf{f} = 0$ in Ω. So, $0 = T(\mathbf{v}, \pi)\mathbf{n}^\Omega = \frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}$ by (4.7). Therefore, $0 = [\frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}] + [\frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f}] = \mathbf{f}$. Since $\frac{1}{2}I + K'_{\Omega,\lambda}$ is an injective Fredholm operator with index 0, it is an isomorphism on $L^2(\partial\Omega, \mathbb{C}^m)$. So, $\frac{1}{2}I + K_{\Omega,\lambda}$ is an isomorphism on $L^2(\partial\Omega, \mathbb{C}^m)$.

We have proved that $\frac{1}{2}I + K_{\Omega,\lambda}$ is an injective operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$. If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$, then $K_{\Omega,\lambda}$ is a compact operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$ by Proposition 4.3. The Riesz theorem gives that $\frac{1}{2}I + K_{\Omega,\lambda}$ is an isomorphism on $L^\infty(\partial\Omega, \mathbb{C}^m)$. □

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z \le 0\}$. For $1 \le q \le \infty$ denote $L^q_{\mathbf{n}}(\partial\Omega; \mathbb{C}^m) = \{\mathbf{f} \in L^q(\partial\Omega; \mathbb{C}^m); \int_{\partial\Omega} \mathbf{n}^\Omega \cdot \mathbf{f} \, \mathrm{d}\sigma = 0\}$. Then $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$ and $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m)) = L^2_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. If $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, then $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^\infty(\partial\Omega, \mathbb{C}^m)$ and $[\frac{1}{2}I + K_{\Omega,\lambda}](L^\infty(\partial\Omega, \mathbb{C}^m)) = L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$.*

Proof. The operator $\frac{1}{2}I + K_{\Omega,0}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$ by [40, Proposition 5.3.5]. Since $K_{\Omega,0} - K_{\Omega,\lambda}$ is a compact operator on $L^2(\partial\Omega, \mathbb{C}^m)$ by Lemma 4.2, the operator $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^2(\partial\Omega, \mathbb{C}^m)$.

If $\mathbf{f} \in L^2(\partial\Omega, \mathbb{C}^m)$ then $\mathbf{u} = D^\lambda_\Omega\mathbf{f}$, $p = \Pi^\lambda_\Omega\mathbf{f}$ is a solution of the Dirichlet problem for the Stokes resolvent system (2.1) with $\mathbf{g} = (\frac{1}{2}I + K_{\Omega,\lambda})\mathbf{f}$. Since $\nabla \cdot \mathbf{u} = 0$, the Divergence theorem gives $\mathbf{g} \in L^2_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. Thus $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m)) \subset L^2_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$.

We now prove that the dimension of the kernel of the operator $\frac{1}{2}I + K'_{\Omega,\lambda}$ in $L^2(\partial\Omega, \mathbb{C}^m)$ is at most 1. Let $\mathbf{f} \in L^2(\partial\Omega, \mathbb{C}^m)$ be such that $\frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f} = 0$. Define $\mathbf{u} = E^\lambda_\Omega \mathbf{f}$, $p = Q_\Omega \mathbf{f}$ on $\omega = \mathbb{R}^m \setminus \overline{\Omega}$. Then $T(\mathbf{u}, p)\mathbf{n}^\omega = \frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f} = 0$ by (4.7). Let now G be a component of ω. Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$ and denote $G(r) = G \cap B(0; r)$. Properties of single layer potentials and Lemma 6.2 give

$$\int_{\partial G(r)} \mathbf{u} \cdot T(\mathbf{u}, p)\mathbf{n}^G \, d\sigma = \int_{G(r)} (2|\hat{\nabla}\mathbf{u}|^2 + \lambda|\mathbf{u}|^2) \, dx.$$

Since $E^\lambda_\Omega \mathbf{f}(x) = O(|x|^{-m})$, $T(E^\lambda_\Omega \mathbf{f}(x), Q_\Omega \mathbf{f}(x))\mathbf{n}(x) = O(|x|^{1-m})$ as $|x| \to \infty$, we get for $r \to \infty$

$$0 = \int_{\partial G} \mathbf{u} \cdot T(\mathbf{u}, p)\mathbf{n}^G \, d\sigma = \int_G (2|\hat{\nabla}\mathbf{u}|^2 + \lambda|\mathbf{u}|^2) \, dx.$$

Thus $\mathbf{u} = 0$ in G. Hence $\nabla p = \Delta\mathbf{u} - \lambda\mathbf{u} = 0$ in G and p is constant in G. Since $0 = T(\mathbf{u}, p)\mathbf{n}^G = -p\mathbf{n}^G$, we infer that $p = 0$ in G. Thus $E^\lambda_\Omega \mathbf{f} = 0$, $Q_\Omega \mathbf{f} = 0$ in $\mathbb{R}^m \setminus \overline{\Omega}$. Using a nontangential limit we get $E^\lambda_\Omega \mathbf{f} = 0$ on $\partial\Omega$. So, $(E^\lambda_\Omega \mathbf{f}, Q_\Omega \mathbf{f})$ is an L^2-solution of the Dirichlet problem $-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla\pi = 0$, $\nabla \cdot \mathbf{v} = 0$ in Ω, $\mathbf{v} = 0$ on $\partial\Omega$. Using properties of single layer potentials and Proposition 6.3 we obtain that there exists a constant c such that $\mathbf{v} := E^\lambda_\Omega \mathbf{f} = 0$, $\pi := Q_\Omega \mathbf{f} = c$ in Ω. Therefore $\frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f} = T(\mathbf{v}, \pi)\mathbf{n}^\Omega = -c\mathbf{n}^\Omega$ by (4.7). Hence $\mathbf{f} = [\frac{1}{2}\mathbf{f} - K'_{\Omega,\lambda}\mathbf{f}] + [\frac{1}{2}\mathbf{f} + K'_{\Omega,\lambda}\mathbf{f}] = -c\mathbf{n}^\Omega$. So, the dimension of the kernel of the operator $\frac{1}{2}I + K'_{\Omega,\lambda}$ in $L^2(\partial\Omega, \mathbb{C}^m)$ is at most 1.

The codimension of the range $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m))$ is equal to the dimension of the kernel of the operator $\frac{1}{2}I + K'_{\Omega,\lambda}$ by [7, Satz 8.26] or [41, §5.4]. Thus the codimension of $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m))$ is at most 1. Since $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m)) \subset L^2_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$, we infer that $[\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m)) = L^2_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$.

Suppose now that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$. Then $K_{\Omega,\lambda}$ is a compact operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$ by Proposition 4.3. Therefore, $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^\infty(\partial\Omega, \mathbb{C}^m)$. Since $K_{\Omega,\lambda}$ is a compact operator on $L^2(\partial\Omega, \mathbb{C}^m)$ by Proposition 4.3, we have $[\frac{1}{2}I + K_{\Omega,\lambda}](L^\infty(\partial\Omega, \mathbb{C}^m)) = [\frac{1}{2}I + K_{\Omega,\lambda}](L^2(\partial\Omega, \mathbb{C}^m)) \cap L^\infty(\partial\Omega, \mathbb{C}^m) = L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ by [35, Lemma 5]. □

Theorem 7.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \le 0\}$, $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$.*

- *Then there exists an L^∞-solution of the Dirichlet problem for the Stokes resolvent system (2.1) if and only if $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$, i.e. $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, d\sigma = 0$.*
- *If $(\mathbf{u}, p)$, $(\mathbf{v}, \pi)$ are two L^∞-solutions of the Dirichlet problem (2.1), then $\mathbf{v} = \mathbf{u}$, $p - \pi$ is constant.*
- *If $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem (2.1), then $\mathbf{u}$ is bounded on Ω and*

$$\sup_{x \in \Omega} |\mathbf{u}(x)| \le C\|\mathbf{g}\|_{L^\infty(\partial\Omega, \mathbb{C}^m)}, \tag{7.1}$$

  where a constant C depends only on Ω.

Proof. Suppose that $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system (2.1). Since $\nabla \cdot \mathbf{u} = 0$, the Divergence theorem gives that $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{R}^m)$. If $(\mathbf{v}, \pi)$ is another L^∞-solution of the Dirichlet problem (2.1), then $\mathbf{v} \equiv \mathbf{u}$ and $p - \pi$ is constant by Corollary 6.5.

Proposition 7.2 says that $\frac{1}{2}I + K_{\Omega,\lambda}$ is a Fredholm operator with index 0 on $L^\infty(\partial\Omega, \mathbb{C}^m)$ and $[\frac{1}{2}I + K_{\Omega,\lambda}](L^\infty(\partial\Omega, \mathbb{C}^m)) = L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. The kernel $\mathrm{Ker}[\frac{1}{2}I + K_{\Omega,\lambda}]$ is a one-dimensional subspace of $L^\infty(\partial\Omega, \mathbb{C}^m)$. So, there exists a closed subspace X of $L^\infty(\partial\Omega, \mathbb{C}^m)$ such that $L^\infty(\partial\Omega, \mathbb{C}^m) = X \oplus \mathrm{Ker}[\frac{1}{2}I + K_{\Omega,\lambda}]$ (see [41, Lemma 5.1]). Define $U\mathbf{f} = [\frac{1}{2}I + K_{\Omega,\lambda}]$ for $\mathbf{f} \in X$. Then $U : X \to L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ is an isomorphism. According to Proposition 4.5 there exists a constant C_1 such that

$$|D^\lambda_\Omega \mathbf{f}| \le C_1 \|\mathbf{f}\|_{L^\infty(\partial\Omega)} \quad \text{on } \mathbb{R}^m \setminus \partial\Omega, \qquad \forall \mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m).$$

Let now $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. Put $\mathbf{u} = D^\lambda_\Omega(U^{-1}\mathbf{g})$, $p = \Pi^\lambda_\Omega(U^{-1}\mathbf{g})$. Then $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem (2.1). Clearly,

$$\sup_{x\in\Omega} |\mathbf{u}(x)| \le C_1 \|U^{-1}\| \; \|\mathbf{g}\|_{L^\infty(\partial\Omega,\mathbb{C}^m)}.$$

□

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \le 0\}$, $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$.*

- *If $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system (2.1), then there exists $\mathbf{u}_\infty \in \mathbb{C}^m$ such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$.*
- *Let $\mathbf{u}_\infty \in \mathbb{C}^m$ be given. Then*

$$\mathbf{u}(x) = D^\lambda_\Omega \left[(1/2)I + K_{\Omega,\lambda}\right]^{-1} (\mathbf{g} - \mathbf{u}_\infty) + \mathbf{u}_\infty, \tag{7.2}$$

$$p(x) = \Pi^\lambda_\Omega \left[(1/2)I + K_{\Omega,\lambda}\right]^{-1} (\mathbf{g} - \mathbf{u}_\infty) - \lambda \mathbf{u}_\infty \cdot x \tag{7.3}$$

is an L^∞-solution $(\mathbf{u}, p)$ of the Dirichlet problem (2.1) such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. If $(\mathbf{v}, \pi)$ is another L^∞-solutions of the Dirichlet problem (2.1) such that $\mathbf{v}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$, then $\mathbf{v} = \mathbf{u}$, $p - \pi$ is constant. Moreover, $\mathbf{u}$ is bounded on Ω and

$$\sup_{x\in\Omega} |\mathbf{u}(x)| \le C \left[\|\mathbf{g}\|_{L^\infty(\partial\Omega,\mathbb{C}^m)} + |\mathbf{u}_\infty|\right], \tag{7.4}$$

with a constant C depending only on Ω.

Proof. Suppose that $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system (2.1). Proposition 5.2 gives that there exists $\mathbf{u}_\infty \in \mathbb{C}^m$ such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. If $(\mathbf{v}, \pi)$ is another L^∞-solution of the Dirichlet problem (2.1) such that $\mathbf{v}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$, then $\mathbf{v} = \mathbf{u}$ and $p - \pi$ is constant by Corollary 6.5.

Proposition 7.1 says that $\frac{1}{2}I + K_{\Omega,\lambda}$ is an isomorphism on $L^\infty(\partial\Omega, \mathbb{C}^m)$. Clearly, $(\mathbf{u}, p)$ given by (7.2), (7.3) is an L^∞-solution $(\mathbf{u}, p)$ of the Dirichlet problem (2.1) such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. According to Proposition 4.5 there exists a constant C_1 such that

$$|D^\lambda_\Omega \mathbf{f}| \le C_1 \|\mathbf{f}\|_{L^\infty(\partial\Omega)} \quad \text{on } \mathbb{R}^m \setminus \partial\Omega, \qquad \forall \mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m).$$

Thus

$$\sup_{x\in\Omega} |\mathbf{u}(x)| \le C_1 \|[(1/2)I + K_{\Omega,\lambda}]^{-1}\| \; \left[\|\mathbf{g}\|_{L^\infty(\partial\Omega,\mathbb{C}^m)} + |\mathbf{u}_\infty|\right] + |\mathbf{u}_\infty|.$$

□

8. Nonlinear Dirichlet problem

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded domain with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \le 0\}$. Let $G : \partial\Omega \times \mathbb{C}^m \to \mathbb{C}^m$ be a locally bounded measurable mapping. Suppose that there exists a constant $q \in (0,1)$ such that $|G(x,u) - G(x,v)| \le q|u - v|$ for all $x \in \partial\Omega$ and $u, v \in \mathbb{C}^m$. If $\mathbf{u}_\infty \in \mathbb{C}^m$ then there exists an L^∞-solution $(\mathbf{u}, p)$ of the nonlinear Dirichlet problem for the Stokes resolvent system*

$$-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0,\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \qquad \mathbf{u}(x) = G(x, \mathbf{u}(x)) \text{ on } \partial\Omega. \tag{8.1}$$

such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. If $(\mathbf{v}, \pi)$ is another solution of this problem then $\mathbf{v} \equiv \mathbf{u}$ and $\pi - p$ is constant. Moreover,

$$\sup_{x\in\Omega} |\mathbf{u}(x)| \le C \left[\frac{1}{1-q} \sup_{y\in\partial\Omega} |G(y,0)| + |\mathbf{u}_\infty| \right]$$

where a constant C depends only on Ω.

Proof. If $(\mathbf{u}, p)$ is an L^∞-solution of the linear Dirichlet problem (2.1), then $(\mathbf{u}, p)$ is an L^∞-solution of the nonlinear Dirichlet problem (8.1) if and only if $g(x) = G(x, g(x))$. Define $\Phi(g(x)) = G(x, g(x))$. Then Φ is a contractive operator on $L^\infty(\partial\Omega, \mathbb{C}^m)$. The fixed point theorem ([7, Satz 1.24]) gives that there exists a unique $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{g}(x) = G(x, g(x))$. Moreover,

$$\|\mathbf{g}\|_{L^\infty(\partial\Omega)} \le \frac{1}{1-q} \|0 - \Phi(0)\|_{L^\infty(\partial\Omega)} \le \frac{1}{1-q} \sup_{y\in\partial\Omega} |G(y,0)|.$$

According to Theorem 7.4 there exists an L^∞-solution $(\mathbf{u}, p)$ of the linear Dirichlet problem (2.1) such that $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. So, $(\mathbf{u}, p)$ is an L^∞-solution of the nonlinear Dirichlet problem (8.1). Moreover, there exists a constant C dependent only on Ω such that

$$\sup_{x\in\Omega} |\mathbf{u}(x)| \le C\| \left[\mathbf{g}\|_{L^\infty(\partial\Omega)} + |\mathbf{u}_\infty| \right].$$

Let now $(\mathbf{v}, \pi)$ be another L^∞-solution of the nonlinear Dirichlet problem (8.1) such that $\mathbf{v}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. Then $(\mathbf{v}, \pi)$ is an L^∞-solution of the linear Dirichlet problem (2.1). Theorem 7.4 forces that $\mathbf{v} \equiv \mathbf{u}$ and $\pi - p$ is constant. □

In the case of a bounded domain we must add some condition on G but we can study a nonhomogeneous system

$$-\Delta\mathbf{u}(x) + \lambda\mathbf{u}(x) + \nabla p(x) = F(x, \mathbf{u}(x)),\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \tag{8.2}$$

$$\mathbf{u}(x) = G(x, \mathbf{u}(x)) \text{ on } \partial\Omega. \tag{8.3}$$

We say that $(\mathbf{u}, p) \in \mathcal{C}(\Omega, \mathbb{C}^m) \times \mathcal{C}(\Omega)$ is an L^∞-solution of the problem (8.2), (8.3) if the equations (8.2) are fulfilled in the sense of distributions, $\mathbf{u}^*_\Omega \in L^\infty(\Omega)$, there exists the nontangential limit of $\mathbf{u}$ at almost all points of $\partial\Omega$ and the boundary condition (8.3) is fulfilled in the sense of a nontangential limit at almost all points of the boundary.

The following auxiliary lemma is probably well known but we cannot find a reference.

Lemma 8.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, $\lambda \in \mathbb{C} \setminus (-\infty, 0\rangle$. For $\mathbf{f} \in L^\infty(\Omega, \mathbb{C}^m)$ define $\tilde{\mathbf{f}} = \mathbf{f}$ in Ω, $\tilde{\mathbf{f}} = 0$ on $\mathbb{R}^m \setminus \Omega$, $V\mathbf{f} = \tilde{E}^\lambda * \tilde{\mathbf{f}}$. Then $V : L^\infty(\Omega, \mathbb{C}^m) \to \mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$ is a compact linear operator.*

Proof. According to Lemma 4.1, (4.3), (4.4), (4.5) and (4.6) there exists a constant C_1 such that $|\tilde{E}^\lambda(x)| + |\nabla \tilde{E}^\lambda(x)| \leq C_1 |x|^{1-m}$. So, $\mathbf{f} \mapsto V\mathbf{f}$, $\mathbf{f} \mapsto \nabla V\mathbf{f} = (\nabla \tilde{E}) * \tilde{\mathbf{f}}$ are bounded compact linear operators on $L^\infty(\Omega, \mathbb{C}^m)$. (See [27, Chapter II, Theorem 8.1 and Theorem 8.6].) Thus $V : L^\infty(\Omega, \mathbb{C}^m) \to W^{1,\infty}(\Omega, \mathbb{C}^m)$ is compact. The imbedding of $W^{1,\infty}(\Omega, \mathbb{C}^m)$ into $\mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$ is compact by the Sobolev imbedding theorem [34, Chapter I, §1.10]. Hence $V : L^\infty(\Omega, \mathbb{C}^m) \to \mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$ is a compact linear operator. □

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \leq 0\}$. Let $G : \partial\Omega \times \mathbb{C}^m \to \mathbb{C}^m$ be a locally bounded measurable mapping such that $G(x, u(x)) \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ for all $\mathbf{u} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. Suppose that there exists a constant $q \in (0, 1)$ such that $|G(x, u) - G(x, v)| \leq q|u - v|$ for all $x \in \partial\Omega$ and $u, v \in \mathbb{C}^m$. Let $F : \Omega \times \mathbb{C}^m \to \mathbb{C}^m$ be a bounded measurable mapping. Then there exists an L^∞-solution $(\mathbf{u}, p)$ of the nonlinear Dirichlet problem for the Stokes resolvent system (8.2), (8.3). If $(\mathbf{u}, p)$ is an L^∞-solution of the problem (8.2), (8.3) then $\mathbf{u}$ is bounded and*

$$\sup_{x \in \Omega} |\mathbf{u}(x)| \leq C \left[\frac{1}{1-q} \sup_{y \in \partial\Omega} |G(y, 0)| + \sup_{\Omega \times \mathbb{C}^m} |F| \right]$$

where a constant C depends only on Ω. If $F(x, y)$ does not depend on y and $(\mathbf{v}, \pi)$ is another L^∞-solution of the problem (8.1), (8.3), then $\mathbf{v} \equiv \mathbf{u}$ and $\pi - p$ is constant.

Proof. If $(\mathbf{u}, p)$ is an L^∞-solution of (8.2),

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \tag{8.4}$$

then $(\mathbf{u}, p)$ is an L^∞-solution of (8.2), (8.3) if and only if $g(x) = G(x, g(x))$. The Divergence theorem gives that $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. Define $\Phi(g(x)) = G(x, g(x))$. Then Φ is a contractive operator on $L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$. The fixed point theorem ([7, Satz 1.24]) gives that there exists a unique $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{g}(x) = G(x, g(x))$. Moreover,

$$\|\mathbf{g}\|_{L^\infty(\partial\Omega)} \leq \frac{1}{1-q} \|0 - \Phi(0)\|_{L^\infty(\partial\Omega)} \leq \frac{1}{1-q} \sup_{y \in \partial\Omega} |G(y, 0)|.$$

For $\mathbf{v} \in L^\infty(\Omega, \mathbb{C}^m)$ define

$$\mathbf{f}^{\mathbf{v}}(x) = F(x, \mathbf{v}(x)).$$

Extend $\mathbf{f}^{\mathbf{v}}$ by 0 outside Ω. Let

$$\mathbf{w}^{\mathbf{v}} = \tilde{E}^\lambda * \mathbf{f}^{\mathbf{v}}, \quad \tau^{\mathbf{v}} = Q * \mathbf{f}^{\mathbf{v}}$$

be the volume potential corresponding to $\mathbf{f}^{\mathbf{v}}$. Then

$$-\Delta \mathbf{w}^{\mathbf{v}} + \lambda \mathbf{w}^{\mathbf{v}} + \nabla \tau^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}, \quad \nabla \cdot \mathbf{w}^{\mathbf{v}} = 0 \quad \text{in } \Omega$$

in the sense of distributions. We have $\mathbf{w}^{\mathbf{v}} \in \mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$ by Lemma 8.2 and

$$\|\mathbf{w}^{\mathbf{v}}\|_{\mathcal{C}(\overline{\Omega})} \leq C_1 \|\mathbf{f}^{\mathbf{v}}\|_{L^\infty(\Omega)},$$

where C_1 depends only on Ω. If $\mathbf{u}^{\mathbf{v}} = \mathbf{w}^{\mathbf{v}} + \tilde{\mathbf{w}}^{\mathbf{v}}$, $p = \tau^{\mathbf{v}} + \tilde{\tau}^{\mathbf{v}}$, then $(\mathbf{u}^{\mathbf{v}}, p^{\mathbf{v}})$ is an L^∞-solution of the problem

$$-\Delta \mathbf{u}^{\mathbf{v}} + \lambda \mathbf{u}^{\mathbf{v}} + \nabla p^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}},\ \nabla\cdot\mathbf{u}^{\mathbf{v}} = 0 \quad \text{in } \Omega, \quad \mathbf{u}^{\mathbf{v}} = \mathbf{g} \quad \text{on } \Omega, \tag{8.5}$$

and only if $(\tilde{\mathbf{w}}^{\mathbf{v}}, \tilde{\tau}^{\mathbf{v}})$ is an L^∞-solution of the problem

$$-\Delta \tilde{\mathbf{w}}^{\mathbf{v}} + \lambda \tilde{\mathbf{w}}^{\mathbf{v}} + \nabla \tilde{\tau}^{\mathbf{v}} = 0,\ \nabla\cdot\tilde{\mathbf{w}}^{\mathbf{v}} = 0 \quad \text{in } \Omega, \quad \tilde{\mathbf{w}}^{\mathbf{v}} = \mathbf{g} - \mathbf{w}^{\mathbf{v}} \quad \text{on } \Omega. \tag{8.6}$$

According to Theorem 7.3 there exists an L^∞-solution $(\tilde{\mathbf{w}}^{\mathbf{v}}, \tilde{\tau}^{\mathbf{v}})$ of (8.6) and

$$\sup_{x\in\Omega} |\tilde{\mathbf{w}}^{\mathbf{v}}(x)| \le C_2 \sup_{y\in\partial\Omega} |\mathbf{g}(y) - \mathbf{w}^{\mathbf{v}}(y)|$$

where a constant C_2 depends only on Ω. So, $\mathbf{u}^{\mathbf{v}} = \mathbf{w}^{\mathbf{v}} + \tilde{\mathbf{w}}^{\mathbf{v}}$, $p = \tau^{\mathbf{v}} + \tilde{\tau}^{\mathbf{v}}$ solve the problem (8.5) and

$$\sup_{x\in\Omega} |\mathbf{u}^{\mathbf{v}}(x)| \le C_1(1 + C_2) \sup_{\Omega\times\mathbb{C}^m} |F| + C_2 \frac{1}{1-q} \sup_{y\in\partial\Omega} |G(y,0)|. \tag{8.7}$$

Let now $(\tilde{\mathbf{u}}, \tilde{p})$ be another L^∞-solution of the problem (8.5). Then

$$-\Delta(\mathbf{u}^{\mathbf{v}} - \tilde{\mathbf{u}}) + \lambda(\mathbf{u}^{\mathbf{v}} - \tilde{\mathbf{u}}) + \nabla(p^{\mathbf{v}} - \tilde{p}) = 0, \nabla\cdot(\mathbf{u}^{\mathbf{v}} - \tilde{\mathbf{u}}) = 0 \text{ in } \Omega, \mathbf{u}^{\mathbf{v}} - \tilde{\mathbf{u}} = 0 \text{ on } \Omega.$$

Theorem 7.3 forces that $\mathbf{u}^{\mathbf{v}} - \tilde{\mathbf{u}} \equiv 0$ and $p^{\mathbf{v}} - \tilde{p}$ is constant. Define $U\mathbf{v} = \mathbf{u}^{\mathbf{v}}$. Then U is an operator on $L^\infty(\Omega, \mathbb{C}^m)$. According to (8.7) there exists a constant M such that $|U\mathbf{v}| \le M$ for all $\mathbf{v} \in L^\infty(\Omega, \mathbb{C}^m)$.

If $\mathbf{u} \in L^\infty(\Omega, \mathbb{C}^m)$ then there exists p such that $(\mathbf{u}, p)$ is an L^∞-solution of the problem (8.2), (8.3) if and only $U\mathbf{u} = \mathbf{u}$. So, if $(\mathbf{u}, p)$ is an L^∞-solution of the problem (8.2), (8.3) then (8.7) holds for $\mathbf{u}^{\mathbf{u}} = \mathbf{u}$.

$\{\mathbf{f}^{\mathbf{v}}; \mathbf{v} \in L^\infty(\Omega, \mathbb{C}^m)\}$ is a bounded subset of $L^\infty(\Omega, \mathbb{C}^m)$. Since $\mathbf{f}^{\mathbf{v}} \mapsto \mathbf{w}^{\mathbf{v}}$ is a compact mapping by Lemma 8.2, $\{\mathbf{w}^{\mathbf{v}}; \mathbf{v} \in L^\infty(\Omega, \mathbb{C}^m)\}$ is a precompact subset of $\mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$. Since the mapping $\mathbf{g} - \mathbf{w}^{\mathbf{v}} \mapsto \tilde{\mathbf{w}}$ is continuous, $\{\mathbf{u}^{\mathbf{v}}; \mathbf{v} \in L^\infty(\Omega, \mathbb{C}^m)\}$ is a precompact subset of $L^\infty(\Omega, \mathbb{C}^m)$. So, U is a compact mapping on $L^\infty(\Omega, \mathbb{C}^m)$. If $\mathbf{u} \in L^\infty(\Omega, \mathbb{C}^m)$, $0 \le \alpha \le 1$ and $\mathbf{u} = \alpha U\mathbf{u}$, then $|\mathbf{u}| \le |\alpha||U\mathbf{u}| \le M$. According to [14, Theorem 11.3] there exists $\mathbf{u} \in L^\infty(\Omega, \mathbb{C}^m)$, such that $U\mathbf{u} = \mathbf{u}$. So, $(\mathbf{u}^{\mathbf{u}}, p^{\mathbf{u}})$ is an L^∞-solution of the problem (8.2), (8.3). □

Next we study L^∞-solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \le 0\}$, $\beta \in \mathbb{C}$. Then there exist $\delta, \epsilon \in (0, \infty)$ such that the following holds: If $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ and $\mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ such that*

$$\|\mathbf{g}\|_{L^\infty(\partial\Omega)} + \|\mathbf{f}\|_{L^\infty(\Omega)} < \delta \tag{8.8}$$

then there exists an L^∞-solution $(\mathbf{u}, p)$ of the problem

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + \beta|\mathbf{u}|\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega \tag{8.9}$$

such that $|\mathbf{u}| \le \epsilon$. If $\tilde{\mathbf{g}} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$ and $\tilde{\mathbf{f}} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ satisfy (8.8) and $(\mathbf{v}, \pi)$ is an L^∞-solution of the problem

$$-\Delta \mathbf{v} + \lambda \mathbf{v} + \beta|\mathbf{v}|\mathbf{v} + \nabla \pi = \tilde{\mathbf{f}}, \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v} = \tilde{\mathbf{g}} \text{ on } \partial\Omega \tag{8.10}$$

such that $|\mathbf{v}| \le \epsilon$ then

$$|\mathbf{u} - \mathbf{v}| \le C\left[\|\mathbf{f} - \tilde{\mathbf{f}}\|_{L^\infty(\Omega)} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{L^\infty(\partial\Omega)}\right], |\mathbf{u}| \le C\left[\|\mathbf{f}\|_{L^\infty(\Omega)} + \|\mathbf{g}\|_{L^\infty(\partial\Omega)}\right]$$

where a constant C depends only on Ω. If $\tilde{\mathbf{f}} = \mathbf{f}$, $\tilde{\mathbf{g}} = \mathbf{g}$, then $\mathbf{v} \equiv \mathbf{u}$ and $\pi - p$ is constant.

Proof. If $\mathbf{F} \in L^\infty(\Omega, \mathbb{C}^m)$, $\mathbf{G} \in L^\infty_{\mathbf{n}}(\partial\Omega)$, then there exists an L^∞-solution $(\mathbf{u}, p)$ of the problem (8.2), (8.3). Moreover, $\mathbf{u}$ is determined uniquely and

$$|\mathbf{u}| \le C_1 \left[\|\mathbf{F}\|_{L^\infty(\Omega)} + \|\mathbf{G}\|_{L^\infty(\partial\Omega)}\right] \tag{8.11}$$

where a constant $C_1 \in (1, \infty)$ depends only on Ω. (See Theorem 8.3.)

Fix ϵ and δ such that

$$0 < \epsilon < [4C_1(|\beta| + 1)]^{-1}, \quad 0 < \delta < \epsilon/(4C_1). \tag{8.12}$$

Let now $(\mathbf{u}, p)$, $(\mathbf{v}, \pi)$ be an L^∞-solution of the problem (8.9), (8.10), respectively. Then $(\mathbf{u} - \mathbf{v}, p - \pi)$ is an L^∞-solution of the problem (8.2), (8.3) with

$$\mathbf{G} = \mathbf{g} - \tilde{\mathbf{g}}, \quad \mathbf{F} = \mathbf{f} - \tilde{\mathbf{f}} - \beta|\mathbf{u}|\mathbf{u} + \beta|\mathbf{v}|\mathbf{v}.$$

By virtue of (8.11)

$$|\mathbf{u} - \mathbf{v}| \le C_1 \left[\|\mathbf{f} - \tilde{\mathbf{f}}\|_{L^\infty(\Omega)} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{L^\infty(\partial\Omega)} + |\beta| \||\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}\|_{L^\infty(\Omega)}\right]. \tag{8.13}$$

We now estimate $\||\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}\|_{L^\infty(\Omega)}$:

$$\begin{aligned}\||\mathbf{v}|\mathbf{v}| - |\mathbf{u}|\mathbf{u}\|_{L^\infty(\Omega)} &\le \||\mathbf{v}|(\mathbf{v} - \mathbf{u})\|_{L^\infty(\Omega)} + \|[|\mathbf{v}| - |\mathbf{u}|]\mathbf{u}\|_{L^\infty(\Omega)} \\ &\le \|\mathbf{v} - \mathbf{u}\|_{L^\infty(\Omega)} \left[\|\mathbf{u}\|_{L^\infty(\Omega)} + \|\mathbf{v}\|_{L^\infty(\Omega)}\right].\end{aligned}$$

If $|\mathbf{u}| \le \epsilon$, $|\mathbf{v}| \le \epsilon$, then

$$\||\mathbf{v}|\mathbf{v}| - |\mathbf{u}|\mathbf{u}\|_{L^\infty(\Omega)} \le 2\epsilon\|\mathbf{v} - \mathbf{u}\|_{L^\infty(\Omega)} \le [2C_1(|\beta| + 1)]^{-1}\|\mathbf{v} - \mathbf{u}\|_{L^\infty(\Omega)}. \tag{8.14}$$

Substituting into (8.13)

$$\sup_\Omega |\mathbf{u} - \mathbf{v}| \le C_1 \left[\|\mathbf{f} - \tilde{\mathbf{f}}\|_{L^\infty(\Omega)} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{L^\infty(\partial\Omega)}\right] + \frac{1}{2} \sup_\Omega |\mathbf{u} - \mathbf{v}|.$$

So,

$$\sup_\Omega |\mathbf{u} - \mathbf{v}| \le 2C_1 \left[\|\mathbf{f} - \tilde{\mathbf{f}}\|_{L^\infty(\Omega)} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{L^\infty(\partial\Omega)}\right]. \tag{8.15}$$

For $\mathbf{v} = 0$, $\tilde{\mathbf{f}} = 0$, $\tilde{\mathbf{g}} = 0$ and $\tilde{\pi} = 0$ we have

$$\sup_\Omega |\mathbf{u}| \le 2C_1 \left[\|\mathbf{f}\|_{L^\infty(\Omega)} + \|\mathbf{g}\|_{L^\infty(\partial\Omega)}\right].$$

If $\tilde{\mathbf{f}} = \mathbf{f}$, $\tilde{\mathbf{g}} = \mathbf{g}$ then (8.15) gives that $\mathbf{v} = \mathbf{u}$. Subtracting (8.9) and (8.10) we obtain $\nabla(p - \pi) = 0$. Therefore $p - \pi$ is constant.

Let now $\mathbf{g} \in L^\infty_{\mathbf{n}}(\partial\Omega, \mathbb{C}^m)$, $\mathbf{f} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ satisfying (8.8) be given. We show the existence of an L^∞-solution of the problem (8.9). Denote

$$X_\epsilon = \{\mathbf{u} \in L^\infty(\Omega; \mathbb{C}^m); \|\mathbf{u}\|_{L^\infty(\Omega)} \le \epsilon\}.$$

For $\mathbf{v} \in X_\epsilon$ there exists an L^∞-solution $(U\mathbf{v}, V\mathbf{v})$ of the problem (8.2), (8.3) with

$$\mathbf{G} = \mathbf{g}, \qquad \mathbf{F} = \mathbf{f} - \beta|\mathbf{v}|\mathbf{v}.$$

Moreover, $U\mathbf{v}$ is determined uniquely. (See Theorem 8.3.) Remark that $(\mathbf{v}, V\mathbf{v})$ is an L^∞-solution of the problem (8.9) if and only if $U\mathbf{v} = \mathbf{v}$. According to (8.11), (8.8) and (8.12)

$$|U\mathbf{u}| \le C_1 \left[\|\mathbf{f}\|_{L^\infty(\Omega)} + |\beta|\|\mathbf{v}\|^2_{L^\infty(\Omega)} + \|\mathbf{g}\|_{L^\infty(\partial\Omega)}\right] \le C_1(\delta + |\beta|\epsilon^2) \le \epsilon.$$

So, $U(X_\epsilon) \subset X_\epsilon$. If $\mathbf{v}, \mathbf{u} \in X_\epsilon$ then $(U\mathbf{v} - U\mathbf{u}, V\mathbf{v} - V\mathbf{u})$ is an L^∞-solution of the problem (8.2), (8.3) with

$$\mathbf{G} = 0, \qquad \mathbf{F} = \beta|\mathbf{u}|\mathbf{u} - \beta|\mathbf{v}|\mathbf{v}.$$

By virtue of (8.11) and (8.14)

$$|U\mathbf{u} - U\mathbf{u}| \le C_1|\beta| \| \mathbf{v}|\mathbf{v}| - |\mathbf{u}|\mathbf{u}\|_{L^\infty(\Omega)} \le \frac{1}{2}|\mathbf{u} - \mathbf{u}|.$$

Since U is a contraction on X_ϵ, the Fixed point theorem ([14, Corollary 11.2]) gives that there exists a unique $\mathbf{u} \in X_\epsilon$ such that $U\mathbf{u} = \mathbf{u}$. So, $(\mathbf{u}, V\mathbf{u})$ is an L^∞-solution of the problem (8.9). □

9. Generalized maximum principle

In this section we prove the generalized maximum principle. We have proved that if $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem for the Stokes resolvent system, then $\mathbf{u}$ is bounded. (Remark that p might be unbounded as the formula (7.3) shows.) Now we prove that if $(\mathbf{u}, p)$ is a solution of the Stokes resolvent system such that $\mathbf{u}$ is bounded, then $(\mathbf{u}, p)$ is an L^∞-solution of some Dirichlet problem for the Stokes resolvent system.

Proposition 9.1. *Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{C} \setminus \{z < 0\}$, $(\mathbf{u}, p)$ be a solution of the Stokes resolvent system (1.1) in Ω. If $\mathbf{u}$ is bounded then there exists a nontangential limit of $\mathbf{u}$ at almost all points of $\partial\Omega$.*

Proof. Suppose first that Ω is a bounded starshaped domain and $\lambda = 0$. We can suppose that Ω is starshaped with respect to 0. For $r \in (1/2, 1)$ define $\mathbf{u}_r(x) = \mathbf{u}(\sqrt{r}x)$, $p_r(x) = \sqrt{r}p(\sqrt{r}x)$. Easy calculation yields

$$\nabla \cdot \mathbf{u}_r(x) = \sqrt{r}\nabla \cdot \mathbf{u}(\sqrt{r}x) = 0,$$

$$\Delta \mathbf{u}_r(x) + \nabla p_r(x) = r\Delta\mathbf{u}(\sqrt{r}x) + r\nabla p(\sqrt{r}x) = 0.$$

Thus $\mathbf{u}_r \in L^2_{\mathbf{n}}(\partial\Omega; \mathbb{C}^m)$ by the Divergence theorem. Denote by T the restriction of $\frac{1}{2}I + K_\Omega$ onto $L^2_{\mathbf{n}}(\partial\Omega; \mathbb{C}^m)$. Then T is an isomorphism by [40, Theorem 5.3.6]. Put $\mathbf{f}_r = T^{-1}\mathbf{u}_r$. Since $\mathbf{u}_r \in W^{1,2}(\partial\Omega, \mathbb{C}^m)$ we have $\mathbf{f}_r \in W^{1,2}(\partial\Omega, \mathbb{C}^m)$ by [40, Theorem 5.3.6]. So, $\mathbf{v}_r = D_\Omega \mathbf{f}_r$, $q_r = \Pi_\Omega \mathbf{f}_r$ is an L^2-solution of the Dirichlet problem

$$\Delta \mathbf{v}_r + \nabla q_r = 0, \ \nabla \cdot \mathbf{v}_r = 0 \quad \text{in } \Omega, \qquad \mathbf{v}_r = \mathbf{u}_r \quad \text{on } \partial\Omega.$$

Thus $\mathbf{u}_r = D_\Omega \mathbf{f}_r$ by [40, Theorem 8.2.1]. Since $\{\mathbf{u}_r\}$ is a bounded subset of $L^2(\partial\Omega, \mathbb{C}^m)$, the set $\{\mathbf{f}_r\}$ is also bounded in $L^2(\partial\Omega, \mathbb{C}^m)$. According to [45, Chapter 4, Theorem 4.61.A] there exists a sequence $r(j) \uparrow 1$ and $\mathbf{f} \in L^2(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{f}_{r(j)} \to \mathbf{f}$ weakly in $L^2(\partial\Omega, \mathbb{C}^m)$. If $x \in \Omega$ then

$$\mathbf{u}(x) = \lim_{j\to\infty} \mathbf{u}_{r(j)}(x) = \lim_{j\to\infty} D_\Omega \mathbf{f}_{r(j)}(x)] = D_\Omega \mathbf{f}(x).$$

Behaviour of a Stokes double layer potential gives that there exists a nontangential limit of u at almost all points of $\partial\Omega$.

Let now Ω be a bounded starshaped domain and $\lambda \ne 0$. Define $\mathbf{u} = 0$ on $\mathbb{R}^m \setminus \Omega$, and

$$\begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} = (E, Q) * \begin{pmatrix} \lambda\mathbf{u} \\ 0 \end{pmatrix},$$

where $*$ denotes the convolution. Then $\nabla \cdot \mathbf{v} = 0$, $\Delta\mathbf{v} - \nabla q = \lambda\mathbf{u}$. If $1 < t < \infty$ then $\mathbf{u} \in L^t(\mathbb{R}^m, \mathbb{C}^m)$ and thus $\mathbf{v} \in W^{2,t}_{loc}(\mathbb{R}^m, \mathbb{C}^m)$ (see [12, Chapter IV, Theorem 4.1]).

Sobolev embedding theorem gives that $\mathbf{v} \in \mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$. Thus $(\mathbf{u} - \mathbf{v}, p - q)$ is a solution of the Stokes system in Ω and $\mathbf{u} - \mathbf{v}$ is bounded. We have proved that there exists a nontangential limit of $\mathbf{u} - \mathbf{v}$ at almost all points of $\partial\Omega$. Since $\mathbf{v} \in \mathcal{C}(\overline{\Omega}, \mathbb{C}^m)$, there exists a nontangential limit of $\mathbf{u}$ at almost all points of $\partial\Omega$.

Let now Ω be general, $\lambda \in \mathbb{C} \setminus \{z < 0\}$. Choose $r > 0$ such that $\partial\Omega \subset B(0; r)$. Put $\omega = \Omega \cap B(0; r)$. According to [34, Chapter I, § 1.3.2, § 1.3.3] there exist starshaped domains $\Omega_1, \dots, \Omega_k$ with Lipschitz boundary such that $\omega = \Omega_1 \cup \dots \cup \Omega_k$. Since there exist nontangential limits of $\mathbf{u}$ with respect to Ω_j at almost all points of $\partial\Omega_j$, there exists a nontangential limit of $\mathbf{u}$ with respect to Ω at almost all points of $\partial\Omega$. □

Theorem 9.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \leq 0\}$. Let $(\mathbf{u}, p)$ be a solution of the Stokes resolvent system (1.1) in Ω such that $\mathbf{u}$ is bounded. Then there exists $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{g}(z)$ is the nontangential limit of $\mathbf{u}$ for almost all $z \in \partial\Omega$. Moreover,*

$$\sup_{x \in \Omega} |\mathbf{u}(x)| \leq C \|\mathbf{g}\|_{L^\infty(\partial\Omega)},$$

where a constant C depends only on Ω.

Proof. According to Proposition 9.1 there exists $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{g}(z)$ is the nontangential limit of $\mathbf{u}$ for almost all $z \in \partial\Omega$. So, $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem (2.1). The rest is a consequence of Theorem 7.3. □

Theorem 9.3. *Let $\Omega \subset \mathbb{R}^m$ be an unbounded open set with compact boundary of class $\mathcal{C}^{1,\alpha}$ with $0 < \alpha < 1$, and $\lambda \in \mathbb{C} \setminus \{z \leq 0\}$. Let $(\mathbf{u}, p)$ be a solution of the Stokes resolvent system (1.1) in Ω such that $\mathbf{u}$ is bounded. Then there exists $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ and $\mathbf{u}_\infty \in \mathbb{C}^m$ such that $\mathbf{g}(z)$ is the nontangential limit of $\mathbf{u}$ for almost all $z \in \partial\Omega$ and $\mathbf{u}(x) \to \mathbf{u}_\infty$ as $|x| \to \infty$. Moreover,*

$$\sup_{x \in \Omega} |\mathbf{u}(x)| \leq C \left[\|\mathbf{g}\|_{L^\infty(\partial\Omega)} + |\mathbf{u}_\infty| \right],$$

where a constant C depends only on Ω.

Proof. According to Proposition 9.1 there exists $\mathbf{g} \in L^\infty(\partial\Omega, \mathbb{C}^m)$ such that $\mathbf{g}(z)$ is the nontangential limit of $\mathbf{u}$ for almost all $z \in \partial\Omega$. So, $(\mathbf{u}, p)$ is an L^∞-solution of the Dirichlet problem (2.1). The rest is a consequence of Theorem 7.4. □

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