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Finite difference MAC scheme for compressible Navier-Stokes equations

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Compressible barotropic Navier-Stokes

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1a}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \mathbb{S} \tag{1b}$$

ρ : density
$\mathbf{u}$: velocity
p : pressure, $p = a\rho^\gamma$
$\mathbb{S}$: viscous stress, $\mathbb{S} = \mu \nabla \mathbf{u}, \mu > 0$

Boundary condition for $\mathbf{u}$

$$\mathbf{u}|_{\partial\Omega} = 0 \quad \text{or periodic} \tag{1c}$$

Initial values

$$\rho(\mathbf{x}, 0) = \rho_0 > 0 \tag{1d}$$

Motivation: stability and convergence

Literature

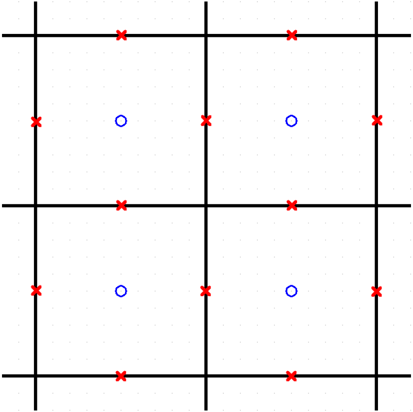
Finite Volume-Finite Element by T. Karper, 2013, $\gamma > 3$

- E. Feireisl, R. Hošek, D. Maltese, A. Novotný, 2017
  bounded numerical solution
- E. Feireisl, M. Lukáčová-Medvid'ová, 2017
  dissipative measure-valued solution, $\gamma \in (1, 2)$

Our interests: Finite Difference

Notations I

- Elements: $\Omega_h = \cup K$
- Faces: $\mathcal{E}$
- Exterior faces: $\mathcal{E}_{ext} = \partial\Omega \cup \mathcal{E}$.
- Interior faces:$\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$
- Faces of element K: $\mathcal{E}(K)$
- Primary grid $\circ$:
  density, pressure
- Dual grid $\times$: velocity



$$\mathcal{E}(K) := \left\{ \sigma = K \pm \frac{1}{2}\mathbf{e}_s, K \in \Omega_h, s = 1, \cdots, d \right\},$$

$$\sigma{,}s\pm = K \pm \frac{1}{2}\mathbf{e}_s.$$

Notations II

Between grids

$$\{f\}_\sigma = \frac{1}{2}(f_K + f_L), \quad \forall \sigma = K|L \in \mathcal{E}_{int}.$$

$$\bar{\mathbf{g}}_K = \frac{1}{2}\begin{pmatrix} g^1_{\sigma,1+} + g^1_{\sigma,1-} \\ g^2_{\sigma,2+} + g^2_{\sigma,2-} \\ g^3_{\sigma,3+} + g^3_{\sigma,3-} \end{pmatrix}.$$

The s-th component of vector $\mathbf{g}$ is defined on the face $\sigma \in \mathcal{E}$.

Functional spaces

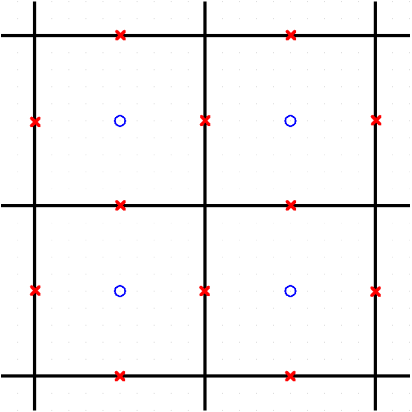
We denote the space for P0 functions with respect to the grid Ω_h by

$$X(\Omega_h) = \{f \in L^\infty(\Omega); f|_K \equiv f_K \in \mathcal{R}\}.$$

The s-th component of $\mathbf{g}$ is constant in the neighbourhood of the edge, not in the element K.

$$X(\mathcal{E}_{int})^d = \{\mathbf{g} \in X(\mathcal{E})^d; \mathbf{g}|_{\mathcal{E}_{ext}} = \mathbf{0}\}$$

Discrete differential operators I



Time

$$\partial_h^t \phi^n = \frac{\phi^n - \phi^{n-1}}{\Delta t}$$

Space

Let $\sigma = K|L, L = K + \mathbf{e}_s, s = 1, \cdots, d$.

$$(\partial_h^s f)_\sigma = \frac{f_L - f_K}{h}, \ f \in X(\Omega_h),$$

$$(\Delta_h f)_K = \frac{1}{h^2} \sum_{L \in \mathcal{N}(K)} (f_L - f_K),$$

$$(\Delta_h g)_\sigma = \frac{1}{h^2} \sum_{s=1}^{d} (g_{\sigma - \mathbf{e}_s} - 2g_\sigma + g_{\sigma + \mathbf{e}_s}).$$

Upwind flux

Let $f^+ = \max\{0, f\}, \quad f^- = \min\{0, f\}$. The upwind flux is given by

$$\mathrm{Up}[f, \mathbf{u}]_\sigma = f_K (u^s_\sigma)^+ + f_L (u^s_\sigma)^-,$$

Upwind discrete derivative and upwind divergence

$$\partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K = \frac{\mathrm{Up}[f, \mathbf{u}]_{\sigma,s+} - \mathrm{Up}[f, \mathbf{u}]_{\sigma,s-}}{h},$$

$$\mathrm{div}_{\mathrm{Up}}[g, \mathbf{u}]_K = \sum_{s=1}^{d} \partial_s^{\mathrm{Up}}[f, \mathbf{u}]_K.$$

Let $f \in X(\Omega_h), \mathbf{v} = [v^1, \cdots, v^d] \in X(\mathcal{E}_{int})^d$, then $\sum_{K \in \Omega_h} \int_K \mathrm{div}_{\mathrm{Up}}[f, \mathbf{v}]_K = 0.$

Numerical Scheme

$$\partial_h^t \rho_K^n + \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K - h^\alpha (\Delta_h \rho^n)_K = 0, \tag{2a}$$

$$\begin{aligned}\{\partial_h^t (\rho\bar{\mathbf{u}})^n\}_\sigma + \{\mathrm{div}_{\mathrm{Up}}[\rho^n \bar{\mathbf{u}}^n, \mathbf{u}^n]\}_\sigma + \big(\partial_h^s p(\rho^n)\big)_\sigma \mathbf{e}_s \\ - \mu\big(\Delta_h \mathbf{u}^n\big)_\sigma - h^\alpha \sum_{r=1}^{d} \big\{\partial_h^r\big(\{\hat{\mathbf{u}}^n\}\partial_h^r \rho^n\big)\big\}_\sigma = 0,\end{aligned} \tag{2b}$$

for all $K \in \Omega_h, \sigma \in \mathcal{E}_{int}$ and $n = \{1, \ldots, N\}$, with boundary conditions.

Renormalized continuity equation

Lemma 1

Let $(\rho_h, \mathbf{u}_h)$ *satisfy the discrete density equation* (2a). *Then for any* $B \in C^2(\mathcal{R})$ *it holds*

$$\sum_{K \in \Omega_h} \left(\partial_h^t B(\rho_K^n) + \Big(B'(\rho_K^n)\rho_K^n - B(\rho_K^n) \Big) (div_h \mathbf{u}^n)_K + \mathcal{P}_K \right) = 0, \quad (3)$$

where

$$\mathcal{P}_K = \frac{\Delta t}{2} B"(\overline{\rho_K^{n-1,n}}) |\partial_h^t \rho_K^n|^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \Big(h|\mathbf{u}_\sigma| + h^\alpha \Big) B"(\rho^{\star_\sigma}) |(\partial_h \rho)_\sigma|^2$$

Note that $\mathcal{P}_K \geq 0$ provided B is convex.

Lemma 2 (Existence of num sol)

Let $\rho_h^{n-1} \in X(\Omega_h)$, $\mathbf{u}_h^{n-1} \in X(\mathcal{E}_{int})^d$ *be given;* $\rho_K^{n-1} > 0$ *for all* $K \in \Omega_h$*.* *Then the numerical scheme (2a-2b) admits a solution*

$$\rho_h^n \in X(\Omega_h), \rho_K^n > 0 \text{ for all } K \in \Omega_h, \mathbf{u}_h^n \in X(\mathcal{E}_{int})^d.$$

Moreover, is satisfies the discrete conservation of mass

$$\sum_{K\in\Omega_h} \int_K \rho_K^n = \sum_{K\in\Omega_h} \int_K \rho_K^{n-1}.$$

Mass conservation

Positivity : non-negativity + strictly positivity

Positivity

Recall the renormalized continuity equation (3) with test function

$$B(z) = \begin{cases} (-z)^\eta & \text{for } z < 0, \\ 0 & \text{for } z \geq 0. \end{cases} \quad \eta \to 1^+.$$

$$B'(z)z - B(z) = 0. \sum_{K \in \Omega_h} \int_K B(\rho_K^n) = \sum_{K \in \Omega_h} \int_K \left(B(\rho_K^{n-1}) - P_K \right) \leq 0.$$

$$\sum_{K \in \Omega_h} \int_K \max\{-\rho_K^n, 0\} \leq 0. \to \rho_K^n \geq 0.$$

Choose $K \in \Omega_h$ such that $\rho_K^n \leq \rho_L^n$ for all $L \in \Omega_h$. Then we have

$$\rho_K^n - \rho_K^{n-1} = -\Delta t \mathrm{div}_{\mathrm{Up}}[\rho^n, \mathbf{u}^n]_K + \Delta t h^\alpha (\Delta_h \rho^n)$$

$$\geq -\frac{\Delta t}{h} \sum_{s=1}^{d} \left(\rho_K^n u_{\sigma_{s,+}}^s - \rho_K^n u_{\sigma_{s,-}}^s + (\rho_{K+\mathbf{e}_s}^n - \rho_K^n) u_{\sigma_{s,+}}^{s\,-} + (\rho_K^n - \rho_{K-\mathbf{e}_s}^n) u_{\sigma_{s,-}}^{s\,+} \right)$$

$$\geq -\Delta t \rho_K^n (\mathrm{div}_h \mathbf{u}^n)_K \geq -\Delta t \rho_K^n |(\mathrm{div}_h \mathbf{u}^n)_K|.$$

$$\rho_L^n \geq \rho_K^n \geq \frac{1}{1 + \Delta t |(\mathrm{div}_h \mathbf{u}^n)_K|} \rho_K^{n-1} > 0, \qquad \text{for any } L \in \Omega_h,$$

Lemma 3 (Energy stability)

Let $(\rho_h, \mathbf{u}_h)$ be the numerical solution obtained through the scheme (2a–2b). For any time step $m = 1, \cdots, N$ the following estimate holds,

$$\sum_{K\in\Omega_h}\int_K \left(\rho_K^m \frac{|\bar{\mathbf{u}}_K^m|^2}{2} + \frac{1}{\gamma-1}p(\rho_K^m)\right) + \Delta t \mu \sum_{n=1}^{m}\sum_{K\in\Omega_h}\int_K \sum_{r=1}^{3}\sum_{s=1}^{3} |(\partial_h^r(u^s)^n)_K|^2$$
$$+ \sum_{j=1}^{4}\mathcal{N}_j \le \sum_{K\in\Omega_h}\int_K \left(\rho_K^0 \frac{|\bar{\mathbf{u}}_K^0|^2}{2} + \frac{1}{\gamma-1}p(\rho_K^0)\right),$$

$$\mathcal{N}_1 = \Delta t \sum_{n=1}^{m}\sum_{K\in\Omega_h}\int_K \sum_{s=1}^{d}\frac{1}{2}\left((h^\alpha + h^2(u_{\sigma,s\mp}^{s,n})^{\pm})p''(\rho_{\sigma,s\mp}^{n,\star})|(\partial_h^s\rho^n)_{\sigma,s\mp}|^2\right),$$

$$\mathcal{N}_2 = \Delta t^2 \sum_{n=1}^{m}\sum_{K\in\Omega_h}\int_K \frac{p''(\rho_K^\eta)}{2}|\partial_t^h\rho_K^n|^2, \mathcal{N}_3 = \Delta t^2 \sum_{n=1}^{m}\sum_{K\in\Omega_h}\int_K \frac{\rho_K^{n-1}}{2}|\partial_t^h\bar{\mathbf{u}}_K^n|^2,$$

$$\mathcal{N}_4 = \Delta t \frac{h}{4}\sum_{n=1}^{m}\sum_{\Gamma\in\mathcal{E}_{int}}\int_\Gamma |Up[\rho^n, \mathbf{u}^n]_\sigma||(\partial_h^s\bar{\mathbf{u}}^n)_\sigma|^2.$$

Lemma 4 (Uniform bounds)

Let $(\rho_h, \mathbf{u}_h)$ be a numerical solution obtained through the scheme (2a)–(2b) and let the total initial energy E_0 be defined by

$$E_0 = \int_\Omega \frac{1}{2}\rho_0 \mathbf{u}_0^2 + \frac{1}{\gamma - 1} p(\rho_0) dx.$$

Then there exists a constant c only depend on E_0 (independent of $h, \Delta t$), such that

$$\|\rho_h\|_{L^\infty(L^\gamma(\Omega))} \lesssim c.$$

$$\|p(\rho_h)\|_{L^\infty(L^1(\Omega))} \lesssim c.$$

$$\|\nabla_h \mathbf{u}_h\|_{L^2(L^2(\Omega))} \lesssim c.$$

$$\|\mathbf{u}_h\|_{L^2(L^6(\Omega))} \lesssim c.$$

$$\|\sqrt{\rho_h}\bar{\mathbf{u}}_h\|_{L^\infty(L^2(\Omega))} \lesssim c.$$

$$\|\sqrt{\rho_h}\|_{L^2(L^\infty(\Omega))} \lesssim ch^{\theta-1},\ \theta = \theta(\alpha, \gamma) > 0.$$

Lemma 5 (Consistency for continuity)

Let $\rho_h^n, \hat{\mathbf{u}}_h^n$ be piecewise constant and piecewise affine representations, respectively, of the solution to the numerical scheme (2a–2b). Then for any $\phi \in C^1(\Omega)$ it holds that

$$\int_\Omega \partial_t^h \rho_h^n \phi dx - \int_\Omega \rho_h^n \hat{\mathbf{u}}_h^n \cdot \nabla_x \phi dx = h^\beta \int_\Omega \mathbf{R}_h \cdot \nabla_x \phi dx, \tag{4}$$

where $\beta > 0$ and $\|\mathbf{R}_h\|_{L^1(0,T;L^{3/2}(\Omega))} \lesssim 1$.

Multiply (2a) with $(\Pi^P \phi)_K$.

Lemma 6 (Consistency for momentum)

Let $(\rho_h^n, \mathbf{u}_h^n)$ *be piecewise constant representations of the solution to numerical scheme (2a–2b). Then for any* $\mathbf{v} \in C_0^1(\Omega) \cap W^{2,q}(\Omega), q > 1$ *it holds that*

$$\int_\Omega \partial_h^t(\rho_h \bar{\mathbf{u}}_h)^n \cdot \mathbf{v} dx - \int_\Omega \rho_h^n \bar{\mathbf{u}}_h^n \otimes \bar{\mathbf{u}}_h^n : \nabla_x \mathbf{v} dx - \int_\Omega p(\rho_h^n) div_x \mathbf{v} dx$$
$$+ \mu \int_\Omega (\nabla_h \mathbf{u}_h^n) : \nabla_x \mathbf{v} dx \quad = h^{\theta_1} \langle \mathbf{r_h^1}, \nabla_x \mathbf{v} \rangle + h^{\theta_2} \langle \mathbf{r}_h^2, \nabla_x^2 \mathbf{v} \rangle, \tag{5}$$

where $\mathbf{r_h^1} \in L^1(0, T; L^{\frac{6\gamma}{5\gamma+6}}(\Omega))$*,* $\mathbf{r_h^2} \in L^1(0, T; L^q(\Omega))$*,* $\theta_1, \theta_2 > 0$*.*

Multiply momentum scheme (2b) with $\Pi^D \mathbf{v}$.

Definition 7 (Dissipative measure-valued solution)

We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^{\infty}_{weak}\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times R^N\right)\right); \langle \nu_{t,x}; s\rangle \equiv \rho,\ \langle \nu_{t,x}; \mathbf{v}\rangle \equiv \mathbf{u},$$

is a dissipative measure-valued solution of the Navier-Stokes system in $(0,T)\times\Omega$, if the following holds for a.a. $\tau \in (0,T)$, for any $\psi \in C^1((0,T)\times\Omega; R^d)$

$$\left[\int_{\Omega} \langle \nu_{\tau,x}; s\rangle \psi(\tau,\cdot) dx\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega} [\langle \nu_{t,x}; s\rangle \partial_h^t \psi + \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \nabla_x \psi] dxdt + \int_0^{\tau} \langle r^C; \nabla_x \psi\rangle dt,$$

$$\left[\int_{\Omega} \langle \nu_{\tau,x}; s\mathbf{v}\rangle \psi(\tau,\cdot) dx\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega} [\langle \nu_{t,x}; s\mathbf{v}\rangle \partial_h^t \psi + \langle \nu_{t,x}; s\mathbf{v}\otimes\mathbf{v}\rangle : \nabla_x \psi + \langle \nu_{t,x}; p(s)\rangle] dxdt$$

$$- \int_0^{\tau}\int_{\Omega} \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dxdt, + \int_0^{\tau} \langle r^M; \nabla_x \psi\rangle dt$$

$$\left[\int_{\Omega} \langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + \frac{p(s)}{\gamma-1}\rangle \psi(\tau,\cdot)\Delta x\right]_{t=0}^{t=\tau} + \int_0^{\tau}\int_{\Omega} \mathcal{S}(\nabla \mathbf{u}) : \nabla_x \psi dxdt + \mathcal{D}(\tau) \leq 0,$$

where $r^C \in L^1([0,T]; \mathcal{M}(\bar{\Omega}))$, $r^M \in L^1([0,T]; \mathcal{M}(\bar{\Omega}))$ satisfying

$$|\langle r^C(\tau); \psi\rangle| \leq \chi(\tau)\mathcal{D}(\tau)\|\psi\|_{C(\bar{\Omega})}, |\langle r^M(\tau); \psi\rangle| \leq \xi(\tau)\mathcal{D}(\tau)\|\psi\|_{C(\bar{\Omega})}, \text{ for } \chi, \xi \in L^1(0,T).$$

In addition, for a.a. $\tau \in (0,T)$ it holds

$$\int_0^{\tau}\int_1 \langle \nu_{\tau,x}; |\mathbf{v}-\mathbf{u}|^2\rangle dxdt \leq c_P \mathcal{D}(\tau).$$

Theorem 8

Let $[\rho_h^k, \mathbf{u}_h^k]_{k=1}^{N_T}$ be a family of numerical solutions obtained by scheme (2) with $1 < \gamma < 2, \delta t \approx h,\ 1 < \alpha < 2\gamma - 1$ and suppose that the initial data satisfy $\rho_0 \in L^\infty(R^d),\ \rho_0 \geq \underline{\rho} > 0$ *a.a. in* $R^d,\ \mathbf{u}_0 \in L^2(R^d)$.
Then any Young measure $\{\nu_{t,x_{t,x\in(0,T)\times\Omega}}\}$ generated by $\rho_h^k, \mathbf{u}_h^k$ represents a dissipative measure-valued solution of the Navier-Stokes system (1) in the sense of Definition 7.

Applying the weak-strong uniqueness [1] we conclude

Theorem 9 (Convergence)

In addition the hypotheses of Theorem 8, suppose that the Navier-Stokes system (2) endowed with the initial data $[\rho_0, \mathbf{u}_0]$ and periodic boundary condition admits a regular solution $[\rho, \mathbf{u}]$ belonging to the class
$\rho, \nabla_x \rho, \mathbf{u}, \nabla_x \mathbf{u} \in C([0,T] \times \bar{\Omega},\ \partial_h^t \mathbf{u} \in L^2\left(0, T; C(\bar{\Omega}); R^d)\right),\ \rho > 0,\ \mathbf{u}|_{\partial\Omega} = 0.$ *Then*

$$\rho_h \to \rho \text{ (strongly) in } L^\gamma\left((0,T) \times K\right),$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2\left((0,T) \times K; R^d\right)$$

for any compact $K \subset \Omega$.

[1] Feireisl et.al. Dissipative measure-valued solutions to the compressible Navier–Stokes system. Calc. Vari. Partial Differ. Equ. 2016

Numerical test

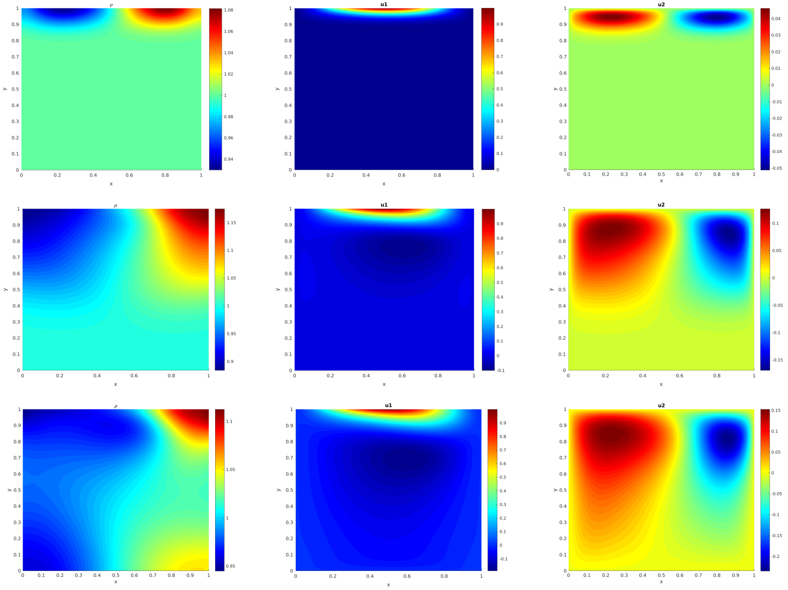
$$\Omega = [0,1]^2, \mu = 0.01, \ a = 1.0, \gamma = 1.4, \alpha = 0.83.$$

$$\Delta t = \text{CFL} \frac{h}{|u|_{\max}}$$

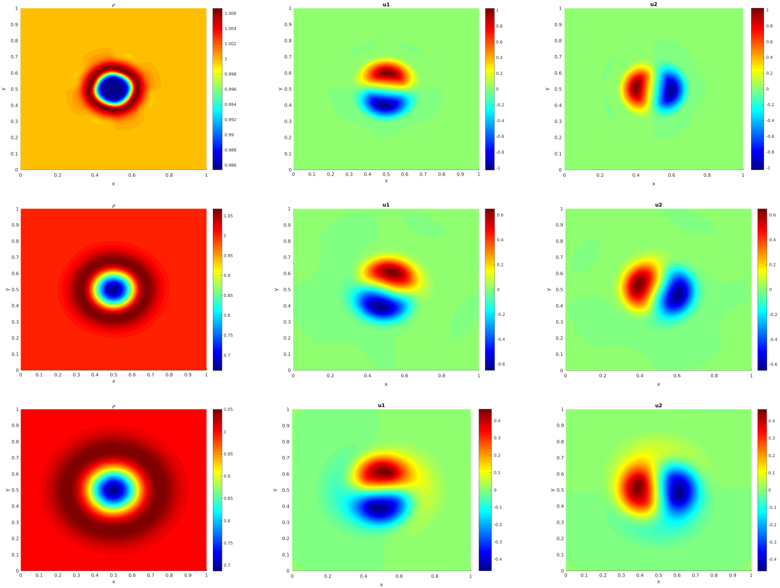
Cavity flow, upper boundary $\mathbf{u} = (16x^2(1-x)^2, 0)^T$.

Table : Convergence results of cavity flow

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	6.17e-01	–	4.65e-02	–	7.74e-03	–	4.94e-02	–
1/32	3.08e-01	1.00	2.32e-02	1.00	4.23e-03	0.87	3.19e-02	0.63
1/64	1.51e-01	1.03	1.12e-02	1.05	2.15e-03	0.97	1.96e-02	0.70
1/128	6.60e-02	1.19	4.75e-03	1.23	8.45e-04	1.35	9.97e-03	0.97



(a) density ρ **(b)** velocity U **(c)** velocity V



(a) density ρ **(b)** velocity U **(c)** velocity V

$$U(0,x,y) = u_r(r) * (y-0.5)/r,$$
$$V(0,x,y) = u_r(r) * (0.5-x)/r.$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$ and

$$u_r(r) = \sqrt{\gamma} \begin{cases} 2r/R & \text{if } 0 \le r < R/2, \\ 2(1-r/R) & \text{if } R/2 \le r < R, \\ 0 & \text{if } r \ge R, \end{cases}$$

Table : Convergence results of Gresho vortex test

h	$\|\nabla \mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\mathbf{u}\|_{L^2(L^2)}$	EOC	$\|\rho\|_{L^1(L^1)}$	EOC	$\|\rho\|_{L^\infty(L^\gamma)}$	EOC
1/16	2.23e-01	–	7.84e-03	–	3.19e-06	–	6.66e-03	–
1/32	1.19e-01	0.91	4.09e-03	0.94	1.63e-06	0.97	4.27e-03	0.64
1/64	6.04e-02	0.97	2.01e-03	1.03	5.92e-07	1.46	2.27e-03	0.91
1/128	2.66e-02	1.18	8.98e-03	1.16	2.24e-07	1.40	1.17e-03	0.96

Future work – A penalty method

$$(\rho \mathbf{u})_t + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \mathbf{1}_{\Omega_s} \mathbf{u} = -\nabla p + \nabla \cdot \mathbb{S}$$

$$\mathbf{1}_{\Omega_s} = \begin{cases} 1 & \mathbf{x} \in \overline{\Omega}_s \\ 0 & \mathbf{x} \in \Omega_f \end{cases}$$

Thank you for your attention!