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Abstract

We propose a simple model for a two phase flow with a diffuse interface. The model couples the compressible Navier–Stokes system governing the evolution of the fluid density and the velocity field with the Allen–Cahn equation for the order parameter. We show that the model is thermodynamically consistent, in particular a variant of the relative energy inequality holds. As a corollary, we show the weak-strong uniqueness principle, meaning any weak solution coincides with the strong solution emanating from the same initial data on the life span of the latter. Such a result plays a crucial role in the analysis of the associated numerical schemes. Finally we perform the low Mach number limit obtaining the standard incompressible model.

Keywords: Two phase flow, compressible Navier–Stokes system, Allen–Cahn equation

Contents

1 Introduction

We propose a diffuse interface model describing the motion of a binary mixture of compressible, viscous and macroscopically immiscible fluids occupying a bounded domain $\Omega \subset R^N$, $N = 1, 2, 3$. The state of the system at a given time $t \in [0, T]$ and a spatial position $x \in \Omega$ is described by means of the three field variables: the mass density $\varrho = \varrho(t, x)$, the macroscopic fluid velocity $\mathbf{u} = \mathbf{u}(t, x)$, and the order parameter $c = c(t, x)$, see the survey by Anderson, McFadden, and Wheeler [2]. The fundamental quantity is the free energy density of the mixture

$$E_{\rm free}(\varrho, c, \nabla_x c) = \frac{1}{2}\delta(\varrho)|\nabla_x c|^2 + \varrho f(\varrho, c). \tag{1.1}$$

Lowengrub and Truskinowski [15] propose a model, where $\delta(\varrho) = \varrho$ and the time evolution of the order parameter c is described by the Cahn–Hilliard equation. A similar approach has been adopted by Blesgen [3], where the Allen–Cahn equation is used.

Anderson et al. [2] proposed a slightly different approach corresponding to $\delta(\varrho) = 1$ in (1.1). Such a hypothesis gives rise to a model that is mathematically tractable, see [1], [8]. In this present paper, we proposed a variant of Blesgen's approach [3] adapted to the ansatz $\delta = 1$. Specifically, we consider the following system of equations:

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0, \tag{1.2}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho, c) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) - \mathrm{div}_x\left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right), \tag{1.3}$$

$$\partial_t c + \mathbf{u}\cdot\nabla_x c = \Delta_x c - \varrho\frac{\partial f(\varrho, c)}{\partial c}. \tag{1.4}$$

The viscous stress tensor $\mathbb{S}$ is given by the standard Newton's rheological law,

$$\mathbb{S}(\nabla_x\mathbf{u}) = \nu\left(\nabla_x\mathbf{u}^t + \nabla_x^t\mathbf{u} - \frac{2}{N}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \lambda\mathrm{div}_x\mathbf{u}\mathbb{I},\ \nu > 0,$$

while the pressure $p = p(\varrho, c)$ is derived from the free energy,

$$p(\varrho, c) = \varrho^2\frac{\partial f(\varrho, c)}{\partial\varrho}.$$

The problem is supplemented by the no–slip boundary condition for the velocity,

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.5}$$

and the Dirichlet boundary conditions for the order parameter,

$$c|_{\partial\Omega} = c_b. \tag{1.6}$$

Multiplying the momentum equation (1.3) by $\mathbf{u}$, the Allen–Cahn equation (1.4) by the expression $-\Delta_x c + \varrho\frac{\partial f(\varrho,c)}{\partial c}$, adding the results and integrating over Ω, we obtain the total energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + \varrho f(\varrho, c)\right]\ \mathrm{d}x + \int_\Omega\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x + \int_\Omega\left[\Delta_x c - \varrho\frac{\partial f(\varrho, c)}{\partial c}\right]^2\ \mathrm{d}x = 0. \tag{1.7}$$

Note that the total energy in (1.7) corresponds to Anderson et al. [2] ansatz, while the dissipation potential agrees with that of Blesgen [3] modulo a multiplicative factor $1/\varrho$ in the term depending on the order parameter. Besides its thermodynamics compatibility, the model (1.2–1.4) features significant mathematical properties, among which is the weak–strong uniqueness principle discussed in detail in the present paper. We show that a weak solution coincides with the strong one emanating from the same initial data as long as the strong solution exists. Such a property can facilitate enormously the numerical analysis as illustrated in [7]. In spite of the abundant amount of literature for the weak-strong uniqueness principle for various problems in fluid mechanics (see for example Wiedemann [16]), much less seems to be known for the problem in question. To the best of our knowledge, there is only one result concerning the weak-strong uniqueness for the Navier-Stokes-Allen-Cahn model under the incompressibility assumption, see Hošek and Mácha [11]. Our approach is based on a variant of the relative energy inequality that proves to be a versatile tool for identifying the asymptotic behavior of solutions. As a corollary, we perform the incompressible limit justifying the model studied in [11].

The paper is organized as follows. In Section 2, we collect the preliminary material and state the weak–strong uniqueness principle. As pointed out above, our approach is based on a variant of the relative energy approach introduced in Section 3. In Section 4, we complete the proof of the weak–strong uniqueness principle. Finally, in Section 5, we identify the incompressible limit.

2 Main results

To avoid technicalities, we consider the free energy in a simplified form

$$E_{\text{free}} = \frac{1}{2}|\nabla_x c|^2 + F_c(c) + F_e(\varrho), \tag{2.1}$$

yielding the pressure

$$p(\varrho, c) = p_e(\varrho) - F_c(c),\ p_e(\varrho) = \varrho F_e'(\varrho) - F_e(\varrho)\,. \tag{2.2}$$

Accordingly, the Allen–Cahn equation (1.4) reduces to

$$\partial_t c + \mathbf{u} \cdot \nabla_x c = \Delta_x c - F_c'(c), \tag{2.3}$$

and the total energy balance reads

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + F_e(\varrho) + F_c(c) \right] \ \mathrm{d}x \\ &\quad + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x + \int_\Omega \left[\Delta_x c - F_c'(c) \right]^2 \ \mathrm{d}x = 0. \end{aligned} \tag{2.4}$$

In addition, we suppose that

$$\begin{aligned} &p_e \in C[0,\infty) \cap C^\infty(0,\infty), \\ &p_e'(\varrho) > 0 \text{ for } \varrho > 0,\ \liminf_{\varrho \to \infty} p_e'(\varrho) > 0,\ p_e(\varrho) \leq c\,(1 + F_e(\varrho)) \ \text{ for all } \varrho \geq 0, \end{aligned} \tag{2.5}$$

and

$$F_c \in C^\infty(R),\ F_c'(c) > 0 \text{ for all } c \in (-\infty, -\overline{c}] \cup [\overline{c}, \infty),\ \overline{c} > 0. \tag{2.6}$$

Note that hypothesis (2.5) is satisfied for the iconic example of isentropic pressure $p(\varrho) = a\varrho^\gamma$, $a > 0$, $\gamma \geq 1$, while F_c can be the double well potential $F_c = (c^2 - 1)^2$.

In view of (2.6) we may apply the standard maximum principle to equation (2.3) to deduce

$$-K \leq c(t,x) \leq K \text{ for any } x \in \Omega,\ t \geq 0, \tag{2.7}$$

where K can be determined in terms of b_c, $\overline{c}$, and the norm of the initial data.

2.1 Weak solutions

Let the initial data,

$$\varrho(0,\cdot) = \varrho_0,\ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0,\ c(0,\cdot) = c_0, \tag{2.8}$$

be given in the class

$$\varrho_0 \geq 0 \text{ a.a. in } \Omega,\ \int_\Omega \left[\frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + F_e(\varrho_0) \right] \ \mathrm{d}x < \infty,\ c_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega),\ c_0|_{\partial\Omega} = c_b. \tag{2.9}$$

Definition 2.1. We shall say that the trio $[\varrho, \mathbf{u}, c]$ is a *dissipative weak solution* to the problem (1.2–1.6), (2.8) in the space–time cylinder $(0,T)\times\Omega$ if:

- the functions $[\varrho, \mathbf{u}, c]$ belong to the class

$$\begin{aligned}
&\varrho, F_e(\varrho) \in L^\infty(0,T;L^1(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega,\\
&\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^N)),\\
&c \in L^\infty(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty((0,T)\times\Omega),\ c|_{\partial\Omega} = c_b;
\end{aligned}$$

- the integral identity

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \tag{2.10}$$

  holds for any $\tau \geq 0$ and any test function $\varphi \in C^1([0,T]\times\overline{\Omega})$;

- the integral identity

$$\begin{aligned}
\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[\varrho\mathbf{u}\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p_e(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right] \ \mathrm{d}x\,\mathrm{d}t\\
&- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \int_\Omega \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right) : \nabla_x\boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \int_\Omega F_c(c)\mathrm{div}_x\boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{2.11}$$

  holds for any $\tau \geq 0$ and any test function $\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega};R^N)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$;

- the Allen–Cahn equation

$$\partial_t c + \mathbf{u}\cdot\nabla_x c = \Delta_x c - F_c'(c)$$

  is satisfied a.a. in $(0,T)\times\Omega$,

$$c(0,\cdot) = c_0,\ c|_{\partial\Omega} = c_b;$$

- the energy inequality

$$\begin{aligned}
&\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + F_e(\varrho) + F_c(c)\right] \ \mathrm{d}x\right]_{t=0}^{t=\tau}\\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \left[\Delta_x c - F_c'(c)\right]^2 \ \mathrm{d}x\,\mathrm{d}t \leq 0
\end{aligned} \tag{2.12}$$

  holds for $\tau \geq 0$.

To the best of our knowledge, the problem of existence of weak solution for the particular system (1.2–1.6), supplemented with suitable initial conditions, has not yet been treated in the available literature. However, the methods developed in [1] and [8] can be easily modified to yield the desired global existence result in the class of weak solutions specified above.

2.2 Weak–strong uniqueness

We are ready to state the first result of the present paper.

Theorem 2.2. *Let* $\Omega \subset R^N$, $N = 1, 2, 3$ *be a bounded Lipschitz domain. Let* $[\varrho, \mathbf{u}, c]$ *be a dissipative weak solution of the problem (1.2–1.6), (2.8) in* $(0,T) \times \Omega$ *in the sense specified in Definition 2.1. Suppose that the same problem admits a classical solution* $[r, \mathbf{U}, C]$, $r > 0$ *defined on the same time interval.*

Then

$$\varrho = r, \ \mathbf{u} = \mathbf{U}, \ c = C \ \text{in} \ (0,T) \times \Omega.$$

The rest of the paper is devoted to the proof of Theorem 2.2. Without loss of generality, replacing $c \approx c - c_b$, $C \approx C - c_b$, we suppose hereafter that

$$c|_{\partial\Omega} = C|_{\partial\Omega} = 0. \tag{2.13}$$

One can anticipate the existence of classical solution for the problem on a short time interval. This kind of results were obtained by Kotschote [12]. In space dimension one, the results of existence of strong solutions are global in time, see Chen and Guo [4].

2.3 Low Mach number

Our second result considers the incompressible limit. The target system is the same as in [11]:

$$\begin{aligned} \mathrm{div}_x \mathbf{U} &= 0, \\ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} + \nabla_x \Pi &= \nu \Delta_x \mathbf{U} - \mathrm{div}_x (\nabla_x C \otimes \nabla_x C), \\ \partial_t C + \mathbf{U} \cdot \nabla_x C &= \Delta_x C - F_c'(C), \end{aligned} \tag{2.14}$$

with the boundary conditions

$$\mathbf{U}|_{\partial\Omega} = 0, \ C|_{\partial\Omega} = 0. \tag{2.15}$$

To facilitate the analysis, we suppose that the initial data

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0, \ C(0, \cdot) = C_0$$

as well as the spatial domain are sufficiently regular for the problem to admit a smooth solution defined on a time interval $(0,T)$. The existence of global regular solutions for the incompressible Navier-Stokes-Allen-Cahn model can be seen as a special case of the model considered by Lin and Liu [14], while for the context of incompressible Navier-Stokes-Cahn-Hilliard model, we refer the interested reader to the work of Gal and Grasselli [9] and to the references within. The viscous limit of the incompressible Navier-Stokes-Allen-Cahn model towards the Euler-Allen-Cahn model is considered by Zhao, Guo and Huang in [17].

The primitive system is derived from (1.2), (1.3), (2.3), where the elastic pressure is scaled as $\frac{1}{\varepsilon^2}p_e(\varrho)$:

$$\begin{aligned}
&\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0,\\
&\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho)\,\mathrm{d}t = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) - \mathrm{div}_x\left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\right) + \nabla_x F_c(c),\\
&\partial_t c + \mathbf{u}\cdot\nabla_x\mathbf{u} = \Delta_x c - F_c'(c),
\end{aligned} \tag{2.16}$$

with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ c|_{\partial\Omega} = 0. \tag{2.17}$$

Accordingly, the weak formulation is specified in Definition 2.1 with the obvious modification applied to p_ε.

Our goal is to show the following result.

Theorem 2.3. *Let $\Omega\subset R^N$, $N = 2, 3$ be a bounded smooth domain. Suppose that the problem* (2.14), (2.15) *admits a smooth solution $[\mathbf{U}, C]$, with the initial data $[\mathbf{U}_0, C_0]$, on a time interval $[0, T]$. Let the initial data for the problem* (2.16), (2.17) *be well prepared, specifically,*

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = 1 + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \int_\Omega \varrho^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon}\to 0 \text{ in } L^\infty(\Omega),\ \mathbf{u}_{0,\varepsilon}\to\mathbf{U}_0 \text{ in } L^2(\Omega;R^N),$$

$$c(0,\cdot) = c_{0,\varepsilon}\in L^\infty\cap W^{1,2}_0(\Omega),\ \|c_{0,\varepsilon}\|_{L^\infty(\Omega)}\lesssim 1,\ c_{0,\varepsilon}\to C_0 \text{ in } W^{1,2}_0(\Omega)$$

as $\varepsilon\to 0$. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]_{\varepsilon>0}$ be dissipative weak solutions of the problem (2.16), (2.17) *emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, c_\varepsilon]$ such that*

$$\|c_\varepsilon\|_{L^\infty((0,T)\times\Omega)}\lesssim 1.$$

Then

$$\varrho_\varepsilon(t,\cdot)\to 1 \text{ in } L^1(\Omega),\ \mathbf{u}_\varepsilon(t,\cdot)\to\mathbf{U}(t,\cdot) \text{ in } L^2(\Omega;R^N),\ c_\varepsilon(t,\cdot)\to C(t,\cdot) \text{ in } W^{1,2}(\Omega)$$

uniformly for $t\in[0,T]$.

3 Relative energy inequality

Motivated by [6], we introduce a relative energy functional associated to (1.2–1.4) and derive a variant of the relative energy inequality. The origin of this method can be traced back to the pioneering work of Dafermos [5], see also Germain [10] or Leger and Vasseur [13] for more recent applications.

3.1 Relative energy

Let r, $r > 0$, $\mathbf{U}$, and C be arbitrary continuously differentiable functions. We define the *relative energy* as

$$\mathcal{E}\left(\varrho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{2}|\nabla_x c - \nabla_x C|^2 + F_e(\varrho) - F_e'(r)(\varrho - r) - F_e(r) \right] \ \mathrm{d}x.$$

Note that

$$\mathcal{E}\left(\varrho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) = \sum_{j=1}^{5} I_j,$$

where

$$I_1 = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + F_e(\varrho) + \frac{1}{2}|\nabla_x c|^2 \right] \ \mathrm{d}x,$$

$$I_2 = \int_\Omega \varrho \left[\frac{1}{2}|\mathbf{U}|^2 - F_e'(r) \right] \ \mathrm{d}x,$$

$$I_3 = -\int_\Omega \varrho\mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x, \ I_4 = -\int_\Omega \nabla_x c \cdot \nabla_x C \ \mathrm{d}x,$$

$$I_5 = \int_\Omega \left[\frac{1}{2}|\nabla_x C|^2 + p_e(r) \right] \ \mathrm{d}x, \ p_e(r) = F_e'(r)r - F_e(r)$$

We remark that in contrast with its counterpart in [6], the relative energy does not include all terms corresponding to the associated energy functional

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + F_e(\varrho) + F_c(c) \right] \ \mathrm{d}x.$$

On the other hand, by virtue of the Poincaré inequality,

$$\mathcal{E}\left(\varrho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) \geq c_p \int_\Omega |c - C|^2 \ \mathrm{d}x \text{ whenever } (c - C)|_{\partial\Omega} = 0. \tag{3.1}$$

3.2 Relative energy inequality

Our next goal is to compute the expression

$$\left[\mathcal{E}\left(\varrho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) \right]_{t=0}^{t=\tau},$$

provided $[\varrho, \mathbf{u}, c]$ is a weak solution of the system (1.2–1.4), and $[r, \mathbf{U}, C]$ are smooth functions, $r > 0$, $\mathbf{U}|_{\partial\Omega} = 0$, $(c - C)|_{\partial\Omega} = 0$.

3.2.1 Energy inequality

As the weak solution satisfies the energy inequality (2.12), we get

$$\begin{aligned}[I_1]_{t=0}^{t=\tau} &= \left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + F_e(\varrho) + F_c(c) + \frac{1}{2}|\nabla_x c|^2 \right] \ \mathrm{d}x \right]_{t=0}^{t=\tau} - \left[\int_\Omega F_c(c) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &\leq - \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \mu^2 \right] \ \mathrm{d}x - \left[\int_\Omega F_c(c) \ \mathrm{d}x \right]_{t=0}^{t=\tau}. \end{aligned} \tag{3.2}$$

Here and in what follows we set

$$\mu := \Delta_x c - F_c'(c)\,. \tag{3.3}$$

Moreover, we deduce from the Allen–Cahn equation that

$$\partial_t c + \mathbf{u} \cdot \nabla_x c = \mu, \ \ \partial_t F_c(c) = -\mathbf{u} \cdot \nabla_x F_c(c) + \mu F_c'(c)$$

Consequently, we get

$$- \left[\int_\Omega F_c(c) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = - \int_0^\tau \int_\Omega \partial_t F_c(c) \ \mathrm{d}x \, \mathrm{d}t = - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t$$

Thus, going back to (3.2), we may infer that

$$[I_1]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \mu^2 \right] \ \mathrm{d}x \, \mathrm{d}t \leq - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t. \tag{3.4}$$

3.2.2 Equation of continuity

Using the weak formulation (2.10) of the equation of continuity with

$$\varphi = \frac{1}{2}|\mathbf{U}|^2 - F_e'(r)$$

we obtain

$$\begin{aligned}[I_2]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[\varrho \Big(\mathbf{U} \cdot \partial_t \mathbf{U} - \partial_t F_e'(r) \Big) \right] \ \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \left[\varrho \mathbf{u} \cdot \Big(\mathbf{U} \cdot \nabla_x \mathbf{U} - \nabla_x F_e'(r) \Big) \right] \ \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.5}$$

Thus, summing up (3.4), (3.5), we deduce

$$\begin{aligned} &[I_1 + I_2]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \mu^2 \right] \ \mathrm{d}x \\ &\qquad \leq - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \int_0^\tau \int_\Omega [\varrho \mathbf{U} \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega [\varrho \partial_t F_e'(r) + \varrho \mathbf{u} \cdot \nabla_x F_e'(r)] \ \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.6}$$

3.2.3 Momentum equation

Plugging $\boldsymbol{\varphi} = \mathbf{U}$ in the weak formulation (2.11) of the momentum equation we get

$$
\begin{aligned}
[I_3]_{t=0}^{t=\tau} = & - \int_0^\tau \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \mathbf{u} \cdot \nabla_x \mathbf{U} + p_e(\varrho) \mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \ \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{t} \\
& - \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
[I_1 + I_2 + I_3]_{t=0}^{t=\tau} & + \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \right] \ \mathrm{d}x \, \mathrm{d}t \\
\leq & - \int_0^\tau \int_\Omega \left[F_c(c) \mathrm{div}_x \mathbf{u} + F_c'(c) \mu \right] \ \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^\tau \int_\Omega \left[\varrho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^\tau \int_\Omega \left[\varrho \partial_t F_e'(r) + \varrho \mathbf{u} \cdot \nabla_x F_e'(r) \right] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega p_e(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
& - \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{3.7}
$$

3.2.4 Allen–Cahn equation

Recalling our convention (2.13) we may use (2.3) to compute

$$
\begin{aligned}
[I_4]_{t=0}^{t=\tau} & = - \left[\int_\Omega \nabla_x c \cdot \nabla_x C \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \left[\int_\Omega c \Delta_x C \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\
& = \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \partial_t c \ \mathrm{d}x \, \mathrm{d}t \\
& = \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \left[\mu - \mathbf{u} \cdot \nabla_x c \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&[I_1+I_2+I_3+I_4]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big[\mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega [\varrho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega [\varrho \partial_t F_e'(r) + \varrho \mathbf{u} \cdot \nabla_x F_e'(r)] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega p_e(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \, [\mu - \mathbf{u} \cdot \nabla_x c] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{3.8}
$$

3.2.5 Conclusion

Adding the remaining integral I_5 to (3.8), we obtain the relative energy inequality in the form:

$$
\begin{aligned}
&\left[\mathcal{E} \left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U} \right) \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big[\mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\leq - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega [\varrho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega [\varrho \partial_t F_e'(r) + \varrho \mathbf{u} \cdot \nabla_x F_e'(r)] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega p_e(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \, [\mu - \mathbf{u} \cdot \nabla_x c] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \left(\frac{1}{2} |\nabla_x C|^2 + p_e(r) \right) \ \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{3.9}
$$

We recall that (3.9) holds for *any* trio of continuous differentiable test functions,

$$
r \in C^1([0,T] \times \overline{\Omega}), \ r > 0, \ \mathbf{U} \in C^1([0,T] \times \overline{\Omega}; R^N)), \ \mathbf{U}|_{\partial\Omega} = 0, \ C, \nabla_x C, \Delta_x C \in C^1([0,T] \times \overline{\Omega}).
$$

4 Weak–strong uniqueness

Our plan to prove Theorem 2.2 is to consider the strong solution $[r, \mathbf{U}, C]$ as test functions in (3.9) and to apply a Gronwall type argument. We proceed in several steps.

4.1 Convective term in the equation of continuity

We write

$$
\begin{aligned}
&\int_0^\tau \int_\Omega [\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U} + \varrho\mathbf{u}\cdot(\mathbf{U}-\mathbf{u})\cdot\nabla_x\mathbf{U}] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega [\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U} + \varrho(\mathbf{U}-\mathbf{u})\mathbf{U}\cdot\nabla_x\mathbf{U}] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\cdot\nabla_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_0^\tau \int_\Omega \left[\frac{\varrho}{r}(\mathbf{U}-\mathbf{u})\cdot\Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C\Delta_x C + \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\Big)\right] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad + c_1\int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \,\Big|\, r, C, \mathbf{U}\right) \ \mathrm{d}t,
\end{aligned}
$$

where the symbol c_i, $i = 1, \dots$ denotes a generic positive constant.

Consequently, relation (3.9) reduces to

$$
\begin{aligned}
&\left[\mathcal{E}\left(\varrho, c, \mathbf{u} \,\Big|\, r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x\mathbf{u}) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) + \mu^2\right] \ \mathrm{d}x \\
&\leq -\int_0^\tau \int_\Omega [F_c(c)\mathrm{div}_x\mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\frac{\varrho}{r}(\mathbf{U}-\mathbf{u})\cdot\Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C\Delta_x C + \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\Big)\right] \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega [\varrho\partial_t F_e'(r) + \varrho\mathbf{u}\cdot\nabla_x F_e'(r)] \ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega p_e(\varrho)\mathrm{div}_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right) : \nabla_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega F_c(c)\mathrm{div}_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t(\Delta_x C)c \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C\,[\mu - \mathbf{u}\cdot\nabla_x c] \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t\left(\frac{1}{2}|\nabla_x C|^2 + p_e(r)\right) \ \mathrm{d}x\,\mathrm{d}t + c_1\int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \,\Big|\, r, C, \mathbf{U}\right) \ \mathrm{d}t
\end{aligned} \tag{4.1}
$$

Now, we rewrite

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big[\frac{\varrho}{r}(\mathbf{U}-\mathbf{u})\cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\Big)\Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \Big[\Big(\frac{\varrho}{r}-1\Big)(\mathbf{U}-\mathbf{u})\cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\Big)\Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{U}-\mathbf{u})\cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C\Big) \ \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\Big) \ \mathrm{d}x \, \mathrm{d}t
\end{aligned}.$$

Moreover, similarly to [6], we show that

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big[\Big(\frac{\varrho}{r}-1\Big)(\mathbf{U}-\mathbf{u})\cdot \Big(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\Big)\Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq c_2 \int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \,\Big|\, r, C, \mathbf{U}\right) \ \mathrm{d}t + \frac{1}{2}\int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{4.2}$$

To see (4.2) we first record the Korn–Poincaré inequality (see e.g. [**?**])

$$\int_\Omega |\mathbf{u}-\mathbf{U}|^2 \ \mathrm{d}x \leq c_{kp} \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x. \tag{4.3}$$

Next, following [6], we introduce a cut–off function $\Psi \in C_c^\infty(0,\infty)$,

$$0 \leq \Psi \leq 1, \ \Psi \equiv 1 \text{ in } [\delta, \frac{1}{\delta}],$$

where δ is chosen so small that

$$r(t,x) \in [2\delta, \frac{1}{2\delta}] \text{ for all } (t,x) \in [0,T] \times \overline{\Omega}.$$

Moreover, for $h \in L^1((0,T)\times\Omega)$, we set

$$h = h_{\mathrm{ess}} + h_{\mathrm{res}}, \ h_{\mathrm{ess}} = \Psi(\varrho) h, \ h_{\mathrm{res}} = (1-\Psi(\varrho))h.$$

In view of (2.2) and (2.5), one has $F_e''(\varrho) = p_e'(\varrho)/\varrho$, the function is F_e is strictly convex and

$$F_e(\varrho) - F_e'(r)(\varrho - r) - F_e(r) \gtrsim (\varrho - r)^2_{\mathrm{ess}} + (1+\varrho)_{\mathrm{res}} \tag{4.4}$$

Consequently

$$\mathcal{E}\left(\varrho, \mathbf{u}, c \,\Big|\, r, \mathbf{U}, C\right) \gtrsim \int_\Omega \left([\mathbf{u}-\mathbf{U}]^2_{\mathrm{ess}} + [\varrho - r]^2_{\mathrm{ess}} + 1_{\mathrm{res}} + \varrho_{\mathrm{res}}\right) \ \mathrm{d}x. \tag{4.5}$$

Finally, we can write

$$\begin{aligned}&\int_0^\tau \int_\Omega \Big[\Big(\frac{\varrho}{r}-1\Big)(\mathbf{U}-\mathbf{u})\cdot\Big(\nabla_x F_c(C)-\nabla_x p_e(r)-\nabla_x C\Delta_x C+\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\Big)\Big] \ \mathrm{d}x\,\mathrm{d}t\\ &\quad\lesssim \int_0^\tau\int_\Omega |\varrho-r| \ |\mathbf{U}-\mathbf{u}| \ \mathrm{d}x\,\mathrm{d}t\\ &\quad\le \int_0^\tau\int_\Omega |[\varrho-r]_{\mathrm{ess}}| \ |\mathbf{U}-\mathbf{u}| \ \mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega |[\varrho-r]_{\mathrm{res}}| \ |\mathbf{U}-\mathbf{u}| \ \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Here, observing that

$$|[\varrho-r]_{\mathrm{ess}}| \ |\mathbf{U}-\mathbf{u}| = \sqrt{[\varrho-r]^2_{\mathrm{ess}}}\sqrt{|\mathbf{U}-\mathbf{u}|^2_{\mathrm{ess}}}\,, \tag{4.6}$$

in view of (4.5)

$$\int_\Omega |[\varrho-r]_{\mathrm{ess}}| \ |\mathbf{U}-\mathbf{u}| \ \mathrm{d}x \lesssim \mathcal{E}\left(\varrho,\mathbf{u},c \ \Big| \ r,\mathbf{U},C\right) .$$

On the other hand, since $\varrho_{\mathrm{ess}} \le \sqrt{\varrho_{\mathrm{ess}}}\sqrt{\varrho}$,

$$\begin{aligned}&\int_0^\tau\int_\Omega |[\varrho-r]_{\mathrm{res}}| \ |\mathbf{U}-\mathbf{u}| \ \mathrm{d}x\,\mathrm{d}t\\ &\quad\le c(\delta)\int_0^\tau\int_\Omega 1_{\mathrm{res}}+\varrho_{\mathrm{res}}+\varrho|\mathbf{u}-\mathbf{U}|^2 \ \mathrm{d}x\,\mathrm{d}t+\delta\int_0^\tau\int_\Omega |\mathbf{u}-\mathbf{U}|^2 \ \mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for any $\delta>0$. Combining (4.3) and (4.5) we obtain the desired estimate (4.2).

Summarizing the above we rewrite inequality (4.1) in the form

$$\begin{aligned}&\Big[\mathcal{E}\left(\varrho,c,\mathbf{u}\ \Big|r,C,\mathbf{U}\right)\Big]_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\Big[\frac{1}{2}\left(\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U})\right):(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})+\mu^2\Big] \ \mathrm{d}x\,\mathrm{d}t\\ &\le-\int_0^\tau\int_\Omega\left[F_c(c)\mathrm{div}_x\mathbf{u}+\mu F_c'(c)\right] \ \mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega(\mathbf{U}-\mathbf{u})\cdot\Big(\nabla_x F_c(C)-\nabla_x p_e(r)-\nabla_x C\Delta_x C\Big) \ \mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega\left[\varrho\partial_t F_e'(r)+\varrho\mathbf{u}\cdot\nabla_x F_e'(r)\right] \ \mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_\Omega p_e(\varrho)\mathrm{div}_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_\Omega\left(\nabla_x c\otimes\nabla_x c-\frac{1}{2}|\nabla_x c|^2\mathbb{I}\right):\nabla_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega F_c(c)\mathrm{div}_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega\partial_t(\Delta_x C)c \ \mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega\Delta_x C\left[\mu-\mathbf{u}\cdot\nabla_x c\right] \ \mathrm{d}x\,\mathrm{d}t\\ &+\int_0^\tau\int_\Omega\partial_t\left(\frac{1}{2}|\nabla_x C|^2+p_e(r)\right) \ \mathrm{d}x\,\mathrm{d}t+c_2\int_0^\tau\mathcal{E}\left(\varrho,c,\mathbf{u}\ \Big|r,C,\mathbf{U}\right) \ \mathrm{d}t\end{aligned}\tag{4.7}$$

4.2 Elastic pressure

Next, following step by step the arguments of [6] (Section 4.2), we deduce that:

$$
\begin{aligned}
&-\int_0^\tau \int_\Omega [\varrho \partial_t F_e'(r) + \varrho \mathbf{u} \cdot \nabla_x F_e'(r)] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega p_e(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
&\quad - \int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p_e(r) \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \partial_t p_e(r) \ \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq \frac{1}{2} \int_0^\tau \int_\Omega \Big[(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \Big] \ \mathrm{d}x \, \mathrm{d}t + c_3 \int_0^\tau \mathcal{E} \left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U} \right) \ \mathrm{d}t
\end{aligned}
$$

Accordingly, relation (4.7) reduces to

$$
\begin{aligned}
&\left[\mathcal{E} \left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U} \right) \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big[\frac{1}{2} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\leq - \int_0^\tau \int_\Omega [F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c)] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}) \cdot \Big(\nabla_x F_c(C) - \nabla_x C \Delta_x C \Big) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \, [\mu - \mathbf{u} \cdot \nabla_x c] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \frac{1}{2} |\nabla_x C|^2 \ \mathrm{d}x \, \mathrm{d}t + c_2 \int_0^\tau \mathcal{E} \left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U} \right) \ \mathrm{d}t
\end{aligned}
\tag{4.8}
$$

4.3 Terms containing the gradient of order parameter

We write

$$
\begin{aligned}
\int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \ \mathrm{d}x \, \mathrm{d}t &= \int_0^\tau \int_\Omega \partial_t C \Delta_x c \ \mathrm{d}x \, \mathrm{d}t \\
&= - \int_0^\tau \int_\Omega \mathbf{U} \cdot \nabla_x C \Delta_x c \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega [\Delta_x C - F_c'(C)] \, \Delta_x c \ \mathrm{d}x \, \mathrm{d}t \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \partial_t \frac{1}{2} |\nabla_x C|^2 \ \mathrm{d}x \, \mathrm{d}t = \int_0^\tau \int_\Omega \nabla_x C \cdot \partial_t \nabla_x C \ \mathrm{d}x \, \mathrm{d}t = - \int_0^\tau \int_\Omega \Delta_x C \partial_t C \ \mathrm{d}x \\
&\quad = \int_0^\tau \int_\Omega \mathbf{U} \cdot \nabla_x C \Delta_x C \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega [\Delta_x C - F_c'(C)] \, \Delta_x C \ \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

Regrouping terms containing the gradient of C we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega (\mathbf{u}-\mathbf{U})\cdot\nabla_x C\Delta_x C \ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega \left(\nabla_x c\otimes\nabla_x c - \frac{1}{2}|\nabla_x c|^2\mathbb{I}\right) : \nabla_x \mathbf{U} \ \mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \mathbf{U}\cdot\nabla_x C\Delta_x c \ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega \mathbf{u}\cdot\nabla_x c\Delta_x C \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathbf{U}\cdot\nabla_x C\Delta_x C \ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \mathbf{u}\cdot\nabla_x C\Delta_x C \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathbf{U}\cdot\nabla_x c\Delta_x c \ \mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \mathbf{U}\cdot\nabla_x C\Delta_x c \ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_\Omega \mathbf{u}\cdot\nabla_x c\Delta_x C \ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \Delta_x C(\nabla_x C-\nabla_x c)\cdot(\mathbf{u}-\mathbf{U}) \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathbf{U}\cdot(\nabla_x C-\nabla_x c)(\Delta_x C-\Delta_x c) \ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \Delta_x C(\nabla_x C-\nabla_x c)\cdot(\mathbf{u}-\mathbf{U}) \ \mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \nabla_x \mathbf{U} : \left[\nabla_x(C-c)\otimes\nabla_x(C-c) - \frac{1}{2}|\nabla_x(C-c)|^2\mathbb{I}\right] \ \mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Consequently, we deduce from (4.8) that

$$\begin{aligned}
&\left[\mathcal{E}\left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \left[\frac{1}{4}\left(\mathbb{S}(\nabla_x\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U})\right) : (\nabla_x\mathbf{u}-\nabla_x\mathbf{U}) + \mu^2\right] \ \mathrm{d}x\,\mathrm{d}t \\
&\leq -\int_0^\tau \int_\Omega \left[F_c(c)\mathrm{div}_x\mathbf{u} + \mu F_c'(c)\right] \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathrm{div}_x(\mathbf{u}-\mathbf{U})F_c(C) \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega F_c(c)\mathrm{div}_x\mathbf{U} \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\Delta_x c\Delta_x C - \Delta_x c F_c'(C)\right] \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C\mu \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\Delta_x C\Delta_x C - \Delta_x C F_c'(C)\right] \ \mathrm{d}x\,\mathrm{d}t + c_4\int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right) \ \mathrm{d}t
\end{aligned} \tag{4.9}$$

Finally, in view of (3.3),

$$\int_0^\tau \int_\Omega \mu^2 \ \mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_\Omega \left[|\Delta_x c|^2 - F_c'(c)\Delta_x c - \mu F_c'(c)\right] \ \mathrm{d}x\,\mathrm{d}t;$$

whence, after a simple manipulation,

$$
\begin{aligned}
&\left[\mathcal{E}\left(\varrho, c, \mathbf{u} \;\middle|\; r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \frac{1}{4}\int_0^\tau \int_\Omega \Big[(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega |\Delta_x c - \Delta_x C|^2 \ \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega \mathrm{div}_x \left(\mathbf{U} - \mathbf{u} \right) \left(F_c(c) - F_c(C) \right) \ \mathrm{d}x \, \mathrm{d}t \\
&\int_0^\tau \int_\Omega [\Delta_x (C - c)(F_c'(c) - F_c'(C))] \ \mathrm{d}x \, \mathrm{d}t + c_4 \int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \;\middle|\; r, C, \mathbf{U}\right) \ \mathrm{d}t
\end{aligned}
\tag{4.10}
$$

Thus, applying the Poincaré inequality (3.1), we obtain the desired conclusion

$$
\left[\mathcal{E}\left(\varrho, c, \mathbf{u} \;\middle|\; r, C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \lesssim \int_0^\tau \mathcal{E}\left(\varrho, c, \mathbf{u} \;\middle|\; r, C, \mathbf{U}\right) \ \mathrm{d}t;
$$

whence Gronwall's lemma completes the proof of Theorem 2.2.

5 Incompressible limit

Our ultimate goal is to prove Theorem 2.3. To begin, observe that $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]$ satisfy a modified version of the energy inequality, namely

$$
\begin{aligned}
&\left[\int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + \frac{1}{\varepsilon^2}\left(F_e(\varrho_\varepsilon) - F_e'(1)(\varrho_\varepsilon - 1) - F_e(1) \right) + F_c(c) \right] \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \left[\Delta_x c - F_c'(c) \right]^2 \ \mathrm{d}x \, \mathrm{d}t \lesssim \varepsilon,
\end{aligned}
\tag{5.1}
$$

Indeed thanks to our hypotheses concerning the initial data

$$
\int_\Omega \left(\varrho_\varepsilon(t, \cdot) - 1 \right) \ \mathrm{d}x = \int_\Omega \varrho^{0,\varepsilon} \ \mathrm{d}x = 0.
$$

In particular, the family $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]_{\varepsilon > 0}$ admits the energy bounds uniformly for $\varepsilon \to 0$.

Similarly to the proof of weak strong uniqueness, we use $r \equiv 1$, $\mathbf{U}$, and C as test functions in the relative energy inequality. Denoting

$$
\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u}, c \;\middle|\; \mathbf{U}, C\right) = \int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{2}|\nabla_x c - \nabla_x C|^2 + \frac{1}{\varepsilon^2}\left(F_e(\varrho) - F_e'(1)(\varrho - 1) - F_e(1) \right) \right] \ \mathrm{d}x
$$

we obtain

$$
\begin{aligned}
&\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U} \right)(\tau) + \int_0^\tau \int_\Omega \Big[\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2 \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\leq - \int_0^\tau \int_\Omega \left[F_c(c_\varepsilon) \mathrm{div}_x \mathbf{u}_\varepsilon + \mu_\varepsilon F_c'(c_\varepsilon) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\varrho_\varepsilon (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{U} + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega (\nabla_x c_\varepsilon \otimes \nabla_x c_\varepsilon) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c_\varepsilon \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \left[\mu_\varepsilon - \mathbf{u}_\varepsilon \cdot \nabla_x c_\varepsilon \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \frac{1}{2} |\nabla_x C|^2 \ \mathrm{d}x \, \mathrm{d}t + \mathcal{O}(\varepsilon),
\end{aligned}
\tag{5.2}
$$

where

$$
\mu_\varepsilon = \Delta_x c_\varepsilon - F_c'(c_\varepsilon).
$$

Now, similarly to Section 4.1 we may rewrite the convective term and use the fact that $\mathbf{U}$ satisfies (2.14) obtaining

$$
\begin{aligned}
&\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U} \right)(\tau) + \int_0^\tau \int_\Omega \Big[(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2 \Big] \ \mathrm{d}x \, \mathrm{d}t \\
&\leq - \int_0^\tau \int_\Omega \left[F_c(c_\varepsilon) \mathrm{div}_x \mathbf{u}_\varepsilon + \mu_\varepsilon F_c'(c_\varepsilon) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot (-\nabla_x \Pi - \mathrm{div}_x (\nabla_x C \otimes \nabla_x C) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega (\nabla_x c_\varepsilon \otimes \nabla_x c_\varepsilon) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c_\varepsilon \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \left[\mu_\varepsilon - \mathbf{u}_\varepsilon \cdot \nabla_x c_\varepsilon \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \frac{1}{2} |\nabla_x C|^2 \ \mathrm{d}x \, \mathrm{d}t + c_1 \int_0^\tau \mathcal{E}_\varepsilon \left(\varrho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U} \right) \mathrm{d}t + \mathcal{O}(\varepsilon).
\end{aligned}
\tag{5.3}
$$

Using strict convexity of the function F_e, we deduce from the energy inequality that

$$
\varrho_\varepsilon(t, \cdot) \to 1 \text{ in } L^1(\Omega) \text{ uniformly in } t \in [0, T],
$$

and

$$
\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega; R^N)).
$$

Similarly, writing

$$
\varrho_\varepsilon \mathbf{u}_\varepsilon = \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon,
$$

we deduce that $\varrho_\varepsilon \mathbf{u}_\varepsilon$ is an equi–integrable sequence converging weakly to $\mathbf{u}$. Passing to the limit in the equation of continuity, we get

$$\mathrm{div}_x \mathbf{u} = 0.$$

Consequently, relation (5.3) yields

$$\begin{aligned}
&\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, C, \mathbf{U} \right)(\tau) + \int_0^\tau \int_\Omega \Big[(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2 \Big] \ \mathrm{d}x \\
&\leq - \int_0^\tau \int_\Omega \mu_\varepsilon F_c'(c_\varepsilon) \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\mathbf{u}_\varepsilon - \mathbf{U}) \cdot \mathrm{div}_x (\nabla_x C \otimes \nabla_x C) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega (\nabla_x c_\varepsilon \otimes \nabla_x c_\varepsilon) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c_\varepsilon \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \Delta_x C \left[\mu_\varepsilon - \mathbf{u}_\varepsilon \cdot \nabla_x c_\varepsilon \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \partial_t \frac{1}{2} |\nabla_x C|^2 \ \mathrm{d}x \, \mathrm{d}t + c_1 \int_0^\tau \mathcal{E}_\varepsilon \left(\varrho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, C, \mathbf{U} \right) \mathrm{d}t + \mathcal{O}(\varepsilon).
\end{aligned} \tag{5.4}$$

At this stage, the rest of the proof is exactly the same as in Section 4.3.

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