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L^q-SOLUTION OF THE NEUMANN, ROBIN AND TRANSMISSION PROBLEM FOR THE SCALAR OSEEN EQUATION

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Abstract. We find necessary and sufficient conditions for the existence of an L^q-solution of the Neumann problem, the Robin problem and the transmission problem for the scalar Oseen equation in three-dimensional open sets. As a consequence we study solutions of the generalized jump problem.

1. Introduction

The Oseen equations represent a mathematical model describing the motion of a viscous incompressible fluid flow around an obstacle. They are obtained by linearizing the steady Navier-Stokes equations around a nonzero constant vector $\mathbf{u} = \mathbf{u}_\infty$, where $\mathbf{u}_\infty$ represents the velocity at infinity, and have the form

$$-\nu\Delta\mathbf{u} + \mathbf{u}_\infty \cdot \nabla\mathbf{u} + \nabla p = \mathbf{F}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^3$ denotes a bounded obstacle and $\mathbb{R}^3 \setminus \overline{\Omega}$ the domain containing the fluid. The velocity field $\mathbf{u}$ and the pressure function p are unknown, while the viscosity $\nu > 0$ and the external force density $\mathbf{F}$ acting on the fluid are given. Choosing $\mathbf{u}_\infty = (\lambda, 0, 0)$ and taking the divergence of the first equation in (1.1) we obtain the Poisson equation $\Delta p = \nabla \cdot \mathbf{F}$ for the pressure p, and each component u_j of the velocity satisfies the equation $-\nu\Delta u_j + \lambda\frac{\partial u_j}{\partial x_1} = F_j - \frac{\partial p}{\partial x_j}$. Thus we see that the Oseen equations (1.1) can be reduced to the scalar equation

$$-\nu\Delta u + \lambda\partial_1 u = f \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \tag{1.2}$$

with scalar functions u and $f = F_j - \dfrac{\partial\pi}{\partial x_j}$.

The system (1.1) introduced by C. W. Oseen [28] has mostly been studied in exterior domains with Dirichlet boundary conditions. Early works are due to Finn who considered these equations in two- and three-dimensional exterior domains using a weighted L^2-approach [10], [11]. Further important contributions are due to Farwig [7], Farwig, Sohr [8], and Kračmar, Novotný, Pokorný [16] in weighted Sobolev spaces. Galdi [12] considered the system in $W^{m,p}_{loc}$-spaces, and Enomoto, Shibata [5] and Kobayashi, Shibata [15] investigated the corresponding Oseen semigroup. Concerning the scalar equation (1.2), important results in weighted Sobolev spaces are given by Amrouche, Bouzit [1], [2] and Amrouche, Razafison [3]. All these results concern the exterior Dirichlet problem. Lately classical solutions of

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the Dirichlet problem, the Neumann problem and the Robin problem for the scalar Oseen equation has been studied by the integral equation method ([23], [24]).

In this paper we study so called L^q-solution of the Neumann problem, the Robin problem and the transmission problem for the scalar Oseen system. It means that we look for a solution such that the nontangential maximal function of u and ∇u are in $L^q(\partial\Omega)$ and boundary conditions are fulfilled in the sense of nontangential limits. We find necessary and sufficient conditions for the existence of an L^q-solution of the Neumann and Robin problem for bounded and unbounded domains with compact Lipschitz boundary. We solve also uniqueness of the problem and a continuous dependence on boundary conditions. Similar results we get also for the transmission problem. Remark that the essential difference between the Neumann problem for the Laplace equation and for the scalar Oseen equation is that the Neumann problem for the scalar Oseen equation $-\Delta u + \lambda \partial_1 u = 0$ with $\lambda \neq 0$ is uniquely solvable.

P. Krutitskii studied in [17], [18] classical solutions of the jump problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \qquad [u]_+ - [u]_- = f,\ [\partial u/\partial n]_+ - [\partial u/\partial n]_- = g \text{ on } \Gamma,$$

where Γ is a smooth open curve. Later he studied the generalized jump problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \qquad [u]_+ - [u]_- = f,\ [\partial u/\partial n]_+ - [\partial u/\partial n]_- + h[u]_+ = g \text{ on } \Gamma.$$

(See [19].) As a consequence of our result for the transmission problem we prove the existence of an L^q-solution of the generalized jump problem for the scalar Oseen equation corresponding to a crack $\Gamma \subset \mathbb{R}^3$. This result is new even for the Laplace equation.

2. Formulation of the problems

We shall look for so called L^q-solution of boundary value problems. For these reasons we need to define a nontangential limit and a nontangential maximal function.

Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary. If $x \in \partial\Omega$, $a > 0$ denote the nontangential approach regions of opening a at the point x by

$$\Gamma_a(x) = \{y \in \Omega; |x-y| < (1+a)\operatorname{dist}(y, \partial\Omega)\}.$$

If now v is a function defined in Ω we denote the nontangential maximal function of v on $\partial\Omega$ by

$$M_a(v)(x) = M_a^\Omega(v)(x) := \sup\{|v(y)|; y \in \Gamma_a(x)\}.$$

It is well known that there exists $c > 0$ such that for $a, b > c$ and $1 \leq q < \infty$ there exist $C_1, C_2 > 0$ such that

$$\|M_a v\|_{L^q(\partial\Omega)} \leq C_1 \|M_b v\|_{L^q(\partial\Omega)} \leq C_2 \|M_a v\|_{L^q(\partial\Omega)}$$

for any measurable function v in Ω. (See, e.g. [14] and [30, p. 62].) We shall suppose that $a > c$ and write $\Gamma(x)$ instead of $\Gamma_a(x)$. Next, define the nontangential limit of v at $x \in \partial\Omega$ by

$$v(x) = [v]_\Omega(x) = \lim_{\Gamma(x) \ni y \to x} v(y)$$

whenever the limit exists.

If $\Omega_+ \subset \mathbb{R}^3$ is an open set with compact Lipschitz boundary, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$, v is a function defined on $\Omega_+ \cup \Omega_-$, and $x \in \partial\Omega_+$, we denote by $[v(x)]_\pm$ the nontangential limit of v at x with respect to $\Omega_\pm$.

Let now $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $g \in L^q(\partial\Omega)$. *We say that u is an L^q-solution of the Robin problem for the scalar Oseen equation*

$$-\Delta u + \lambda \partial_1 u = 0 \ \text{in } \Omega, \quad \frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1 u + hu = g \ \text{on } \partial\Omega, \tag{2.1}$$

if $u \in \mathcal{C}^\infty(\Omega)$, $-\Delta u + \lambda\partial_1 u = 0$ in Ω, $M_a(u) + M_a(|\nabla u|) \in L^q(\partial\Omega)$, there exist nontangential limits of u and ∇u at almost all points of $\partial\Omega$, and the boundary condition $\partial u/\partial n - \lambda n_1 u/2 + hu = g$ is fulfilled in the sense of the nontangential limit at almost all points of $\partial\Omega$. If $h \equiv 0$ we say about the Neumann problem for the scalar Oseen equation. Here $n = n^\Omega$ is the unit exterior normal of Ω.

Let now $\Omega \subset \mathbb{R}^3$ be a bounded open set with compact Lipschitz boundary, $\lambda_+, \lambda_- \in \mathbb{R}$, $1 < q < \infty$, $h_+, h_- \in L^\infty(\partial\Omega)$, $g \in L^q(\partial\Omega)$, $f \in W^{1,q}(\partial\Omega)$. Let a_+, a_-, b_+, b_- be positive constants. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$, and by n denote the unit exterior normal of Ω. *We say that u is an L^q-solution of the transmission problem for the scalar Oseen equation*

$$\begin{aligned} &-\Delta u + \lambda_\pm \partial_1 u = 0 \ \text{in } \Omega_\pm, \\ &a_+[u]_+ - a_-[u]_- = f \ \text{on } \partial\Omega, \\ &b_+\left[\frac{\partial u}{\partial n} - \frac{\lambda_+}{2} n_1 u\right]_+ - b_-\left[\frac{\partial u}{\partial n} - \frac{\lambda_-}{2} n_1 u\right]_- + h_+[u]_+ + h_-[u]_- = g \ \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

if $u \in \mathcal{C}^\infty(\Omega_\pm)$, $-\Delta u + \lambda_\pm \partial_1 u = 0$ in $\Omega_\pm$, $M_a^{\Omega_+}(u) + M_a^{\Omega_+}(|\nabla u|) + M_a^{\Omega_-}(u) + M_a^{\Omega_-}(|\nabla u|) \in L^q(\partial\Omega)$, there exist nontangential limits of u and ∇u with respect to Ω_+ and Ω_- at almost all points of $\partial\Omega$, and the transmission conditions $a_+[u]_+ - a_-[u]_- = f$, $b_+[\partial u/\partial n - \lambda_+ n_1 u/2]_+ - b_-[\partial u/\partial n - \lambda_- n_1 u/2]_- + h_+[u]_+ + h_-[u]_- = g$ are fulfilled in the sense of the nontangential limit at almost all points of $\partial\Omega$.

3. Potentials for the scalar Oseen equation

We shall look for solutions of the Robin problem and the transmission problem for the scalar Oseen equation by the integral equation method. For that reason we need to define scalar Oseen boundary potentials and study their properties.

We say that E is a fundamental solution of the scalar Oseen equation

$$-\Delta u + \lambda \partial_1 u = 0 \tag{3.1}$$

if $-\Delta u + \lambda \partial_1 u = \delta_0$ in the sense of distribution. The fundamental solution of the scalar Oseen equation (3.1) is

$$E_\lambda(x) := \frac{1}{4\pi|x|}\, e^{-(|\lambda x| - \lambda x_1)/2}. \tag{3.2}$$

(Remark that $E_0(x)$ is a fundamental solution of the Laplace equation.) Clearly $E_\lambda(-x) = E_{-\lambda}(x)$. We have

$$|E_\lambda(x) - E_0(x)| = O(1), \ |\nabla E_\lambda(x) - \nabla E_0(x)| = O(|x|^{-1}) \quad \text{as } |x| \to 0 \tag{3.3}$$

by [24, Lemma 3.2]. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex and $\lambda \neq 0$ then

$$|\partial^\alpha E_0^\Omega \varphi(x)| = O(|x|^{-1-|\alpha|}) \qquad \text{as } |x| \to \infty, \tag{3.4}$$

$$|\partial^\alpha E_\lambda^\Omega \varphi(x)| = O(e^{-(|\lambda x| - \lambda x_1)/2}|x|^{-1}) \qquad \text{as } |x| \to \infty, \tag{3.5}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

If $\Omega \subset \mathbb{R}^3$ is an open set with compact Lipschitz boundary and $\varphi \in L^q(\partial\Omega)$ with $1 < q < \infty$ denote

$$E_\lambda^\Omega \varphi(x) = \int_{\partial\Omega} E_\lambda(x-y)\varphi(y)\, \mathrm{d}\sigma(y)$$

the scalar Oseen single layer potential with density φ. (If $\lambda = 0$ then $E_\lambda\varphi$ is a classical single layer potential for the Laplace equation.) Easy calculations yield that $E_\lambda^\Omega\varphi \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ and $-\Delta E_\lambda^\Omega \varphi + \lambda\partial_1 E_\lambda^\Omega\varphi = 0$ in $\mathbb{R}^3 \setminus \partial\Omega$.

Let now $y \in \partial\Omega$ be such that the unit exterior normal $n^\Omega(y)$ of Ω there exists at y. For $x \in \mathbb{R}^3 \setminus \{y\}$ define

$$K_\lambda^\Omega(x,y) = n^\Omega(y) \cdot \nabla E_\lambda(x-y) - \frac{\lambda}{2} n_1^\Omega(y) E_\Omega(x-y).$$

For $\varphi \in L^q(\partial\Omega)$ with $1 < q < \infty$ denote

$$D_\lambda^\Omega \varphi(x) = \int_{\partial\Omega} K_\lambda^\Omega(x,y)\varphi(y)\, \mathrm{d}\sigma(y)$$

the scalar Oseen double layer potential with density φ. (If $\lambda = 0$ then $D_\lambda\varphi$ is a classical double layer potential for the Laplace equation.) Since $K_\lambda^\Omega(\cdot, y)$ is a solution of the scalar Oseen equation (3.1) in $\mathbb{R}^3 \setminus \{y\}$, the double layer potential $D_\lambda^\Omega\varphi$ is a solution of the scalar Oseen equation (3.1) in $\mathbb{R}^3 \setminus \partial\Omega$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Then there exists a constant C such that*

$$\|M_a(D_\lambda^\Omega \varphi)\|_{L^q(\partial\Omega)} \le C\|\varphi\|_{L^q(\partial\Omega)}$$

for all $\varphi \in L^q(\partial\Omega)$. If $\varphi \in L^q(\partial\Omega)$ and $x \in \partial\Omega$, we define

$$K_\lambda^\Omega \varphi(x) = \lim_{r \downarrow 0} \int_{\partial\Omega \setminus B(x;r)} K_\lambda^\Omega(x,y)\varphi(y)\, \mathrm{d}\sigma(y)$$

whenever this integral makes sense. (Here $B(x;r) = \{y \in \mathbb{R}^3; |x-y| < r\}$.) Then K_λ^Ω is a bounded linear operator on $L^q(\partial\Omega)$. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$. If $\varphi \in L^q(\partial\Omega)$ then

$$[D_\lambda^\Omega \varphi(x)]_\pm = \pm\frac{1}{2}\varphi(x) + K_\lambda^\Omega\varphi(x)$$

for almost all $x \in \partial\Omega$.

(See [25], Theorem 2.9.)

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lispchitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Denote by $\mathcal{E}_\lambda^\Omega\varphi$ the restriction of $E_\lambda^\Omega\varphi$ onto $\partial\Omega$. If $\varphi \in L^q(\partial\Omega)$, then $\mathcal{E}_\lambda\varphi^\Omega(x)$ is the nontangential limit of $E_\lambda^\Omega\varphi$ at x for almost all $x \in \partial\Omega$, and*

$$\|M_a(E_\lambda^\Omega \varphi)\|_{L^q(\partial\Omega)} \le C\|\varphi\|_{L^q(\partial\Omega)} \tag{3.6}$$

with a constant C depending only on Ω, λ and q.

Proof. Since $|E_\lambda(x)| = O(|x|^{-1})$ as $|x| \to 0$, and $|E_\lambda(x)| \to 0$ as $|x| \to \infty$, the proposition follows from [22, Proposition 1]. $\square$

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. If $\varphi \in L^q(\partial\Omega)$, then there exists the nontangential limit of $\nabla E^\Omega_\lambda \varphi$ at x for almost all $x \in \partial\Omega$, and*

$$\|M_a(\nabla E^\Omega_\lambda \varphi)\|_{L^q(\partial\Omega)} \leq C\|\varphi\|_{L^q(\partial\Omega)}$$

with a constant C depending only on Ω, λ and q. Denote by $(K^\Omega_\lambda)'$ the adjoint operator of K^Ω_λ. Then $(K^\Omega_\lambda)'$ is a bounded linear operator on $L^q(\partial\Omega)$,

$$(K^\Omega_\lambda)'\varphi(x) = \lim_{r\downarrow 0} \int_{\partial\Omega\setminus B(x;r)} K^\Omega_\lambda(y,x)\varphi(y)\ \mathrm{d}\sigma(y) \tag{3.7}$$

for almost all $x \in \partial\Omega$. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$. If $\varphi \in L^q(\partial\Omega)$ then

$$\left[\left(\frac{\partial}{\partial n} - \frac{\lambda}{2} n_1\right) E^\Omega_\lambda \varphi(x)\right]_\pm = \pm\frac{1}{2}\varphi(x) - (K^\Omega_{-\lambda})'\varphi(x) \tag{3.8}$$

for almost all $x \in \partial\Omega$.

(See [25], Theorem 2.9.)

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Then $\mathcal{E}^\Omega_\lambda$, $K^\Omega_\lambda - K^\Omega_0$ and $(K^\Omega_\lambda)' - (K^\Omega_0)'$ are compact linear operators on $L^q(\partial\Omega)$.*

Proof. The operators $\mathcal{E}^\Omega_\lambda$, $K^\Omega_\lambda - K^\Omega_0$ and $(K^\Omega_\lambda)' - (K^\Omega_0)'$ are integral operators with weakly singular kernels by (3.3). So, these operators are compact in $L^q(\partial\Omega)$ by [9, §4.5.2, Satz 2]. □

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Then $\mathcal{E}^\Omega_\lambda : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is a bounded operator, $\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0 : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is a compact operator, $[\partial_j(E^\Omega_\lambda - E^\Omega_0)]_\Omega$ is a compact operator on $L^q(\partial\Omega)$.*

Proof. $\mathcal{E}^\Omega_\lambda$ maps $L^q(\partial\Omega)$ to $W^{1,q}(\partial\Omega)$ by Proposition 3.4 and Lemma 11.2 in Appendix.. Since $\mathcal{E}^\Omega_\lambda$ is a continuous operator on $L^q(\partial\Omega)$ by Proposition 3.3, the Closed graph theorem [29, Theorem 3.10] gives that $\mathcal{E}^\Omega_\lambda : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is a bounded operator.

Since $|\partial_j E_\lambda(x) - \partial_j E_0(x)| \leq C|x|^{-1}$, [22, Proposition 1] gives

$$[\partial_j E_\lambda g - \partial_j E_0 g]_\Omega(x) = \int_{\partial\Omega} [\partial_j E_\lambda(x-y) - \partial_j E_0(x-y)]g(y)\ \mathrm{d}\sigma(y)$$

for $g \in L^q(\partial\Omega)$. So, $g \mapsto [\partial_j E_\lambda g - \partial_j E_0 g]_\Omega$ is a compact operator on $L^q(\partial\Omega)$ by [9, §4.5.2, Satz 2].

$\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0$ is compact on $L^q(\partial\Omega)$ by Proposition 3.4. Proposition 3.3 and Lemma 11.2 in Appendix gives that

$$\partial_{\tau_{jk}}(\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0)g := (n_j\partial_k - n_k\partial_j)(\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0)g = n_j[\partial_k(\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0)g]_\Omega - n_k[\partial_j(\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0)g]_\Omega.$$

Since all tangential derivatives operators $\partial_{\tau_{jk}}(\mathcal{E}^\Omega_\lambda - \mathcal{E}^\Omega_0)$ are compact on $L^q(\partial\Omega)$, the operator $(E^\Omega_\lambda - E^\Omega_0) : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is compact. □

4. Behaviour at infinity

If Ω is an unbounded domain with compact Lipschitz boundary, then there exists $r > 0$ such that $\Gamma_a(x) \supset \{y \in \mathbb{R}^3; |y| > r\}$ for all $x \in \partial\Omega$. If u is an L^q-solution of the Robin problem in Ω, then $M_a^\Omega u \in L^q(\partial\Omega)$. This forces that u is bounded on the set $\{y \in \mathbb{R}^3; |y| > r\}$. In this section we shall study a behaviour of bounded solutions of the scalar Oseen equation. We need the following Liouville's theorem:

Theorem 4.1. *Let u be a tempered distribution in $\mathbb{R}^3$, $\lambda \in \mathbb{R}$. If $-\Delta u + \lambda \partial_1 u = 0$ in $\mathbb{R}^3$ in the sense of distributions, then u is a polynomial.*

(See [4], Chapter XI, §2, Theorem 1.)

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded open set with compact boundary, $\lambda \in \mathbb{R}$, $u \in \mathcal{C}^\infty(\Omega)$ be a bounded solution of the scalar Oseen equation (3.1) in Ω. Then there exists $u_\infty \in \mathbb{R}$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. If α is a multiindex then*

$$\begin{aligned} &\partial^\alpha[u(x) - u_\infty] = O(|x|^{-1-|\alpha|}) && \text{as } |x| \to \infty \quad \text{for } \lambda = 0, \\ &\partial^\alpha[u(x) - u_\infty] = O(e^{-(|\lambda x| - \lambda x_1)/2}|x|^{-1}) && \text{as } |x| \to \infty \quad \text{for } \lambda \neq 0. \end{aligned} \tag{4.1}$$

Proof. Fix $\varphi \in \mathcal{C}^\infty(\mathbb{R}^3$ with compact support such that $\varphi = 1$ on a neighbourhood of $\mathbb{R}^3 \setminus \Omega$. Define $v = u(1-\varphi)$ in Ω, $v = 0$ elsewhere. Then $v \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $g = -\Delta v + \lambda \partial_1 v$ has compact support. Define w by the convolution $w = g * E_\lambda$. Then $-\Delta w + \lambda \partial_1 w = g$. According to (3.4), (3.5), we have $\partial^\alpha w(x) = O(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for $\lambda = 0$, $\partial^\alpha w(x) = O(e^{-(|\lambda x| - \lambda x_1)/2}|x|^{-1})$ as $|x| \to \infty$ for $\lambda \neq 0$. Since $v - w$ is bounded, it is a tempered distribution. Since $-\Delta(v-w) + \lambda \partial_1 (v-w) = 0$ in $\mathbb{R}^3$, Theorem 4.1 gives that $v - w$ is a polynomial. Since $v - w$ is bounded, it is constant. Since $u = v$ in a neighbourhood of infinity, we obtain the proposition. □

5. Integral representation

In this section we prove a formula for an integral representation of solutions of the scalar Oseen equation.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Let u be a solution of the scalar Oseen equation (3.1) in Ω. Suppose that $M_a(u) + M_a(|\nabla u|) \in L^q(\partial\Omega)$, and there exist nontangential limits of u and ∇u at almost all points of $\partial\Omega$. If Ω is unbounded suppose that $u(x) \to 0$ as $|x| \to \infty$. Denote by g the Neumann condition*

$$g = \frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1^\Omega u \quad \text{on } \partial\Omega$$

(in the sense of a nontangential limit). Then

$$E_\lambda^\Omega g(x) + D_\lambda^\Omega u(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{5.1}$$

Proof. Suppose first that Ω is bounded and $u \in \mathcal{C}^2(\overline{\Omega})$. For $x \in \mathbb{R}^3$ denote $h_x(y) = E_\lambda(x-y)$. Then $\Delta h_x + \lambda \partial_1 h_x = 0$ in $\mathbb{R}^3 \setminus \{x\}$.

Let $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. According to Green's formula

$$E_\lambda^\Omega g(x) + D_\lambda^\Omega u(x) = \int_{\partial\Omega} \left[h\left(\frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1^\Omega u\right) + u\left(-\frac{\partial h}{\partial n} - \frac{\lambda}{2} n_1^\Omega h\right)\right] \, d\sigma$$

$$= \int_\Omega [h(\Delta u - \lambda \partial_1 u) - u(\Delta h + \lambda \partial_1 h)] \, \mathrm{d}y = 0.$$

Let now $x \in \Omega$. Using (5.1) for $\Omega(r) = \Omega \setminus \overline{B(x;r)}$

$$0 = \lim_{r \downarrow 0} \left[E_\lambda^{\Omega(r)} \left(\frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1^\Omega \right)(x) + D_\lambda^{\Omega(r)} u(x) \right] = E_\lambda^\Omega g(x) + D_\lambda^\Omega u(x)$$

$$- \lim_{r \downarrow 0} \left[E_\lambda^{B(x;r)} \left(\frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1^\Omega \right)(x) + \int_{\partial\Omega} (K_\lambda^{B(x;r)}(x,y) - K_0^{B(x;r)}(x,y)) u(y) \, \mathrm{d}\sigma(y) \right]$$

$$- \lim_{r \downarrow 0} \int_{\partial B(x;r)} \frac{u}{4\pi r^2} \, \mathrm{d}\sigma = E_\lambda^\Omega g(x) + D_\lambda^\Omega u(x) - u(x).$$

Let now Ω be bounded and u be general. Let Ω_j be the sequence of sets from Lemma 11.1 in Appendix. We have proved (5.1) for Ω_j. Letting $j \to \infty$ we obtain (5.1) for Ω by the Lebesgue lemma.

Let now Ω be unbounded. Fix $x \in \Omega$. Choose $r > 0$ such that $\partial\Omega \subset B(x;r)$. We have proved

$$u = E_\lambda^{\Omega \cap B(x;r)} (\partial u/\partial n - \lambda n_1 u/2) + D_\lambda^{\Omega \cap B(x;r)} u \qquad \text{in } \Omega \cap B(x;r). \tag{5.2}$$

Define

$$v(y) = \begin{cases} E_\lambda^\Omega g(y) + D_\lambda^\Omega u(y) - u(y), & y \in \Omega, \\ E_\lambda^{B(x;r)} (\partial u/\partial n^{B(x;r)} - \lambda n_1^{B(x;r)} u/2)(y) + D_\lambda^{B(x;r)} u(y), & y \notin \Omega. \end{cases}$$

According to (5.2) we have $E_\lambda^{B(x;r)} (\partial u/\partial n^{B(x;r)} - \lambda n_1^{B(x;r)} u/2) + D_\lambda^{B(x;r)} u = E_\lambda^\Omega g + D_\lambda^\Omega u - u$ in $\Omega \cap B(x;r)$. Thus v is a solution of the scalar Oseen equation (3.1) in $\mathbb{R}^3$. So, v is a polynomial by Theorem 4.1. Since $v(x) \to 0$ as $|x| \to \infty$, we deduce that $v \equiv 0$. By the definition of v we see that (5.1) holds for $x \in \Omega$. Let now $x \notin \overline{\Omega}$. Choose $r > 0$ such that $\overline{B(x;r)} \cap \overline{\Omega} = \emptyset$. Define $u = 0$ on $B(x;r)$. Then

$$0 = u(x) = E_\lambda^{\Omega \cup B(x;r)} (\partial u/\partial n^{\Omega \cup B(x;r)} - \lambda n_1^{\Omega \cup B(x;r)} u/2)(x) + D_\lambda^{\Omega \cup B(x;r)} u(x)$$
$$= E_\lambda^\Omega g(x) + D_\lambda^\Omega u(x).$$

□

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}$. Let u be a solution of the scalar Oseen equation (3.1) in $\Omega_+ \cup \Omega_-$. Suppose that $M_a^{\Omega_+}(u) + M_a^{\Omega_+}(|\nabla u|) + M_a^{\Omega_-}(u) + M_a^{\Omega_-}(|\nabla u|) \in L^q(\partial\Omega)$, and there exist nontangential limits of u and ∇u with respect to Ω_+ and with respect to Ω_- at almost all points of $\partial\Omega$. Suppose that $u(x) \to 0$ as $|x| \to \infty$. Denote*

$$f = [u]_+ - [u]_-, \quad g_\pm = \left[\frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1^\Omega u \right]_\pm, \quad g = g_+ - g_-,$$

where n is the unit outward normal of Ω. Then

$$u = E_\lambda^\Omega g + D_\lambda^\Omega f \quad \text{in } \Omega_+ \cup \Omega_-. \tag{5.3}$$

Proof. According to Proposition 5.1

$$\pm E_\lambda^\Omega g_\pm(x) \pm D_\lambda^\Omega [u]_\pm(x) = \begin{cases} u(x) & x \in \Omega_\pm, \\ 0 & x \in \Omega_\mp. \end{cases}$$

Adding we get (5.3). □

6. Derivatives of a double layer potential

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}^3$, $1 < q < \infty$, $f \in W^{1,q}(\partial\Omega)$. If $x \in \Omega$ then*

$$\partial_j D^\Omega_\lambda f(x) = \sum_{k=1}^3 \partial_k E^\Omega_\lambda(\partial_{\tau_{jk}} f)(x) - \lambda \partial_1 E^\Omega_\lambda(fn_j)(x) - \frac{\lambda}{2}\partial_j E^\Omega_\lambda(fn_1)(x), \tag{6.1}$$

where $\partial_{\tau_{jk}} = n^\Omega_j \partial_k - n^\Omega_k \partial_j$.

Proof. Fix $x \in \Omega$. Suppose first that $f \in \mathcal{C}^\infty(R^m)$. Choose $\varphi \in \mathcal{C}^\infty(\mathbb{R}^3)$ with compact support such that $\varphi = 1$ on $\partial\Omega$, $\varphi = 0$ on a neighbourhood of x and put $\tilde f = f\varphi$. By virtue of the Green formula

$$\sum_{k=1}^3 \partial_k E^\Omega_\lambda(\partial_{\tau_{jk}} f)(x) = \sum_{k=1}^3 \int_{\partial\Omega} [-\partial_k E_\lambda(x-\cdot)][n_j \partial_k \tilde f - n_k \partial_j \tilde f] \, \mathrm{d}\sigma$$

$$= \sum_{k=1}^3 \int_{\partial\Omega} [n_k \partial_j(\tilde f \partial_k E_\lambda(x-\cdot)) - n_j \partial_k(\tilde f \partial_k E_\lambda(x-\cdot))] \, \mathrm{d}\sigma$$

$$+ \int_{\partial\Omega} \tilde f \Big[n_j \Delta E_\lambda(x-\cdot) - \sum_{k=1}^3 n_k \partial_k \partial_j E_\lambda(x-\cdot) \Big] \, \mathrm{d}\sigma$$

$$= \int_\Omega \sum_{k=1}^3 [\partial_j \partial_k (\tilde f E_\lambda(x-\cdot)) - \partial_k \partial_j (\tilde f E_\lambda(x-\cdot))] \, \mathrm{d}y$$

$$- \int_{\partial\Omega} \tilde f n_j \lambda \partial_1 E^\Omega_\lambda(x-\cdot) \, \mathrm{d}\sigma + \partial_j D^\Omega_\lambda f(x) + \partial_j \int_{\partial\Omega} \tilde f \frac{\lambda}{2} n_1 E_\lambda(x-\cdot) \, \mathrm{d}\sigma$$

$$= \partial_j D^\Omega_\lambda f(x) + \lambda \partial_1 E^\Omega_\lambda(fn_j)(x) + \frac{\lambda}{2}\partial_j E^\Omega_\lambda(fn_1)(x).$$

Let now $f \in W^{1,q}(\partial\Omega)$. Choose $f_k \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $f_k \to f$ in $W^{1,q}(\partial\Omega)$. We have proved (6.1) for f_k. Letting $k \to \infty$ we obtain (6.1) for f. $\square$

Proposition 6.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. If $f \in W^{1,q}(\partial\Omega)$, then there exists a nontangential limit of $\nabla D^\Omega_\lambda f$ at almost all points of $\partial\Omega$ and*

$$\|M_a(\nabla D^\Omega_\lambda f)\|_{L^q(\partial\Omega)} \le C\|f\|_{L^q(\partial\Omega)} \tag{6.2}$$

with a constant C depending only on Ω, q and a. The operator

$$[\partial_j D^\Omega_\lambda]_\Omega - [\partial_j D^\Omega_0]_\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$$

is compact.

Proof. Let $f \in W^{1,q}(\partial\Omega)$. We have (6.1) by Lemma 6.1. According to Proposition 3.3 there exists a nontangential limit of $\nabla D^\Omega_\lambda f$ at almost all points of $\partial\Omega$ and (6.2) holds.

$g \mapsto [\partial_j E_\lambda g]_\Omega - [\partial_j E_0 g]_\Omega$ is a compact operator on $L^q(\partial\Omega)$ by Proposition 3.5. By virtue of (6.1) we deduce that $[\partial_j D^\Omega_\lambda]_\Omega - [\partial_j D^\Omega_0]_\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is compact. $\square$

Proposition 6.3. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Then K^Ω_λ is a bounded linear operator on $W^{1,q}(\partial\Omega)$ and $K^\Omega_\lambda - K^\Omega_0$ is a compact operator on $W^{1,q}(\partial\Omega)$.*

Proof. If $f \in W^{1,q}(\partial\Omega)$ then $\frac{1}{2}f + K_\lambda^\Omega f$ is the nontangential limit of $D_\lambda^\Omega f$ with respect to Ω. (See Proposition 3.1.) Proposition 6.2 and Lemma 11.2 in Appendix give that $\frac{1}{2}f + K_\lambda^\Omega f \in W^{1,q}(\partial\Omega)$. Hence $K_\lambda^\Omega f \in W^{1,q}(\partial\Omega)$. Since K_λ^Ω is a continuous operator in $L^q(\partial\Omega)$, Closed graph theorem ([29, Theorem 3.10]) gives that K_λ^Ω is a bounded linear operator on $W^{1,q}(\partial\Omega)$.

$K_\lambda^\Omega - K_0^\Omega$ is a compact operator on $L^q(\partial\Omega)$ by Proposition 3.4. For the tangential derivative

$$\partial_{\tau_{jk}}[K_\lambda^\Omega - K_0^\Omega]f := (n_j^\Omega \partial_k - n_k^\Omega \partial_j)[K_\lambda^\Omega - K_0^\Omega]f$$

one has by Lemma 11.2

$$\partial_{\tau_{jk}}[K_\lambda^\Omega - K_0^\Omega]f = n_j[\partial_k(D_\lambda^\Omega - D_0^\Omega)f]_\Omega - n_k[\partial_j(D_\lambda^\Omega - D_0^\Omega)f]_\Omega.$$

So, $\partial_{\tau_{jk}}[K_\lambda^\Omega - K_0^\Omega] : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is compact by Proposition 6.2. Hence $K_\lambda^\Omega - K_0^\Omega$ is a compact operator on $W^{1,q}(\partial\Omega)$. □

Proposition 6.4. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$.*

- *There exists a bounded linear operator $H_\lambda^\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ such that*

$$\left[\frac{\partial D_\lambda^\Omega f(x)}{\partial n} - \frac{\lambda}{2} n_1(x) D_\lambda^\Omega f(x)\right]_+ = \left[\frac{\partial D_\lambda^\Omega f(x)}{\partial n} - \frac{\lambda}{2} n_1(x) D_\lambda^\Omega f(x)\right]_- = H_\lambda^\Omega f(x)$$

  for almost all $x \in \partial\Omega$.
- *$H_\lambda^\Omega - H_0^\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is a compact operator.*

Proof. Let $f \in W^{1,q}(\partial\Omega)$. Define

$$H_\lambda^\Omega f = \left[\frac{\partial D_\lambda^\Omega f}{\partial n} - \frac{\lambda}{2} n_1 D_\lambda^\Omega f\right]_+ .$$

Then $H_\lambda^\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is a bounded linear operator by (6.2).

Define $u = D_\lambda^\Omega f$. Denote $g_\pm = [\partial u/\partial n - \lambda n_1 u/2]_\pm$, $g = g_+ - g_-$. Since $[u]_+ - [u]_- = f$ by Proposition 3.1, Corollary 5.2 gives

$$D_\lambda^\Omega f = u = E_\lambda^\Omega g + D_\lambda^\Omega f \qquad \text{in } \Omega_+ \cup \Omega_- .$$

Hence $E_\lambda^\Omega g = 0$ in $\Omega_+ \cup \Omega_-$. According to Proposition 3.3

$$0 = \left[\left(\frac{\partial}{\partial n} - \frac{\lambda}{2}\right) E_\lambda^\Omega g\right]_+ - \left[\left(\frac{\partial}{\partial n} - \frac{\lambda}{2}\right) E_\lambda^\Omega g\right]_- = g.$$

Thus

$$H_\lambda^\Omega f = g_+ = g_- = \left[\frac{\partial D_\lambda^\Omega f}{\partial n} - \frac{\lambda}{2} n_1 D_\lambda^\Omega f\right]_- .$$

(6.1), Proposition 6.2, Proposition 3.1 and Proposition 3.4 give that $H_\lambda^\Omega - H_0^\Omega : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is a compact operator. □

7. Regular L^q-solutions of the Dirichlet problem

In this auxiliary section we study regular L^q-solutions of the Dirichlet problem of the scalar Oseen equation.

Let now $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $1 < q < \infty$, $g \in W^{1,q}(\partial\Omega)$. *We say that u is a regular L^q-solution of the Dirichlet problem for the scalar Oseen equation*

$$-\Delta u + \lambda \partial_1 u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{7.1}$$

if $u \in \mathcal{C}^\infty(\Omega)$, $-\Delta u + \lambda\partial_1 u = 0$ *in* Ω, $M_a(u) + M_a(|\nabla u|) \in L^q(\partial\Omega)$, *there exist nontangential limits of* u *and* ∇u *at almost all points of* $\partial\Omega$, *and the boundary condition* $u = g$ *is fulfilled in the sense of the nontangential limit at almost all points of* $\partial\Omega$.

Lemma 7.1. *Let* $\Omega \subset \mathbb{R}^3$ *be an open set with compact Lipschitz boundary,* $\lambda \in \mathbb{R}$, $h \in L^\infty(\partial\Omega)$, $g \in L^2(\partial\Omega)$. *Let* u *be an* L^2*-solution of the Robin problem for the scalar Oseen equation* (2.1). *If* Ω *is unbounded suppose moreover that* $u(x) \to 0$ *as* $|x| \to \infty$. *Then*

$$\int_{\partial\Omega} gu \, \mathrm{d}\sigma = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_{\partial\Omega} hu^2 \, \mathrm{d}\sigma. \tag{7.2}$$

If $\lambda \neq 0$, $h \geq 0$ *and* $ug = 0$ *on* $\partial\Omega$ *then* $u \equiv 0$.

Proof. Suppose first that Ω is bounded. Let $\Omega(k)$ be a sequence of open sets from Lemma 11.1 in Appendix. According to the Gauss-Green theorem and the Lebesgue lemma

$$\int_{\partial\Omega} gu \, \mathrm{d}\sigma = \int_{\partial\Omega} hu^2 \, \mathrm{d}\sigma + \lim_{k\to\infty} \int_{\partial\Omega(k)} \left(u\frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1 u^2 \right) \, \mathrm{d}\sigma$$

$$= \int_{\partial\Omega} hu^2 \, \mathrm{d}\sigma + \lim_{k\to\infty} \int_{\Omega(k)} [|\nabla u|^2 + u(\Delta u - \lambda\partial_1 u)] \, \mathrm{d}x = \int_{\partial\Omega} hu^2 \, \mathrm{d}\sigma + \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

Let now Ω be unbounded. Put $h = 0$ outside $\partial\Omega$. Using (7.2) for $G(r) := \Omega \cap B(0;r)$

$$\int_{\partial G(r)} hu^2 \, \mathrm{d}\sigma + \int_{G(r)} |\nabla u|^2 \, \mathrm{d}x = \int_{\partial\Omega} gu \, \mathrm{d}\sigma + \int_{\partial B(0;1)} w_r(x) \, \mathrm{d}\sigma$$

where $w_r(x) = r^2 u(rx)[\partial u(rx)/\partial n - (\lambda/2) n_1 u(rx)]$. Proposition 4.2 gives $|w_r(x)| \leq C$ and $w_r(x) \to 0$ as $r \to \infty$ for $x \in \partial B(0;r)$. Letting $r \to \infty$ we obtain (7.2) by Lebesgue's lemma.

Let $\lambda \neq 0$, $h \geq 0$ and $ug = 0$ on $\partial\Omega$. We can suppose that Ω is connected. The relation (7.2) gives $\nabla u = 0$ in Ω, $hu^2 = 0$ on $\partial\Omega$. So, there exists a constant c such that $u \equiv c$. Suppose that $c \neq 0$. Then $0 = ug = cg$ forces $g \equiv 0$. Since $0 = hu^2 = c^2 h$ we infer that $h \equiv 0$. Thus $0 = g = \partial u/\partial n - (\lambda/2) n_1 u + hu = c(\lambda/2) n_1$ on $\partial\Omega$. Hence $n_1 = 0$ on $\partial\Omega$, what is impossible. Therefore $u \equiv 0$. $\square$

Proposition 7.2. *Let* $\Omega \subset \mathbb{R}^3$ *be an open set with compact Lipschitz boundary,* $1 < q \leq 2$, $\lambda \in \mathbb{R}$. *Then* $\mathcal{E}_\lambda^\Omega : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ *is an isomorphism.*

Proof. $\mathcal{E}_0^\Omega : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is a Fredholm operator with index 0 by [13, Theorem 2.2.22]. Since $\mathcal{E}_0^\Omega$ is injective (see [20, Chapter I, Theorem 1.15]), it is an isomorphism.

Let $\lambda \neq 0$. Proposition 3.5 gives that $\mathcal{E}_\lambda^\Omega - \mathcal{E}_0^\Omega : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is compact. So, $\mathcal{E}_\lambda^\Omega : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is a Fredholm operator with index 0 by [27, § 16, Theorem 16]. Let $f \in L^q(\partial\Omega)$, $\mathcal{E}_\lambda^\Omega f = 0$. Then $f \in L^2(\partial\Omega)$ by [26, Lemma 11.9.21]. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$. Then $E_\lambda^\Omega f$ is an L^2-solution of the Neumann problem

$$-\Delta u + \lambda\partial_1 u = 0 \quad \text{in } \Omega_\pm, \qquad \frac{\partial u}{\partial n} - \frac{\lambda}{2} n_1 u = g_\pm \quad \text{on } \partial\Omega_\pm$$

for some $g_\pm \in L^2(\partial\Omega)$. (See Proposition 3.2 and Proposition 3.3.) Lemma 7.1 gives that $E_\lambda^\Omega f = 0$ in $\Omega_\pm$. Thus

$$f = [f/2 - (K_{-\lambda}^\Omega)' f] - [-f/2 - (K_{-\lambda}^\Omega)' f] = 0$$

by Proposition 3.3. Therefore $\mathcal{E}^\Omega_\lambda : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is an isomorphism. □

Theorem 7.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $1 < q \le 2$, $\lambda \in \mathbb{R}^1$. If $g \in W^{1,q}(\partial\Omega)$ then $u = E^\Omega_\lambda (\mathcal{E}^\Omega_\lambda)^{-1} g$ is a unique regular L^q-solution of the Dirichlet problem (7.1). Moreover,*

$$\|M^\Omega_a(u)\|_{L^q(\partial\Omega)} + \|M^\Omega_a(\nabla u)\|_{L^q(\partial\Omega)} \le C\|g\|_{W^{1,q}(\partial\Omega)} \tag{7.3}$$

where C does not depend on g.

Proof. $\mathcal{E}^\Omega_\lambda : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is an isomorphism by Proposition 7.2. So, $u = E^\Omega_\lambda (\mathcal{E}^\Omega_\lambda)^{-1} g$ is a regular L^q-solution of the Dirichlet problem (7.1) by Proposition 3.2 and Proposition 3.3.

Let now u be a regular L^q-solution of the Dirichlet problem (7.1) with $g \equiv 0$. Denote $f = \partial u/\partial n - (\lambda/2) n_1 u$ on $\partial\Omega$. Then $f \in L^q(\partial\Omega)$. Proposition 5.1 gives that $u = E^\Omega_\lambda f + D^\Omega_\lambda g = E^\Omega_\lambda f$. Since $\mathcal{E}^\Omega_\lambda f = u = 0$ on $\partial\Omega$, Proposition 7.2 gives that $f \equiv 0$. Thus $u = E^\Omega_\lambda f \equiv 0$.

The estimate (7.3) is a consequence of Proposition 3.2 and Proposition 3.3. □

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded open set with Lipschitz boundary, $1 < q \le 2$, $\lambda \in \mathbb{R}^1$, $g \in W^{1,q}(\partial\Omega)$. If u is a regular L^q-solution of the Dirichlet problem (7.1) then there exists $u_\infty \in \mathbb{R}^1$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. On the other hand, if $u_\infty \in \mathbb{R}^1$ is given then $u = E^\Omega_\lambda (\mathcal{E}^\Omega_\lambda)^{-1}(g - u_\infty) + u_\infty$ is a unique regular L^q-solution of the Dirichlet problem (7.1) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M^\Omega_a(u)\|_{L^q(\partial\Omega)} + \|M^\Omega_a(\nabla u)\|_{L^q(\partial\Omega)} \le C\left[\|g\|_{W^{1,q}(\partial\Omega)} + u_\infty|\right] \tag{7.4}$$

where C does not depend on g and u_∞.

Proof. Let u be a regular L^q-solution of the Dirichlet problem (7.1). Since u is bounded in a neighbourhood of infinity, Proposition 4.2 gives that there exists $u_\infty \in \mathbb{R}^1$ such that $u(x) \to u_\infty$ as $|x| \to \infty$.

$\mathcal{E}^\Omega_\lambda : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is an isomorphism by Proposition 7.2. If u_∞ is given then $u = E^\Omega_\lambda (\mathcal{E}^\Omega_\lambda)^{-1}(g - u_\infty) + u_\infty$ is a regular L^q-solution of the Dirichlet problem (7.1) by Proposition 3.2 and Proposition 3.3.

Let now u be a regular L^q-solution of the Dirichlet problem (7.1) with $g \equiv 0$ such that $u(x) \to 0$ as $|x| \to \infty$. Denote $f = \partial u/\partial n - (\lambda/2) n_1 u$ on $\partial\Omega$. Then $f \in L^q(\partial\Omega)$. Proposition 5.1 gives that $u = E^\Omega_\lambda f + D^\Omega_\lambda g = E^\Omega_\lambda f$. Since $\mathcal{E}^\Omega_\lambda f = u = 0$ on $\partial\Omega$, Proposition 7.2 gives that $f \equiv 0$. Thus $u = E^\Omega_\lambda f \equiv 0$.

The estimate (7.4) is a consequence of Proposition 3.2 and Proposition 3.3. □

8. Neumann and Robin problem

We shall look for a particular solution of the Robin problem (2.1) in the form of a single layer potential. If $\varphi \in L^q(\partial\Omega)$ then $E^\Omega_\lambda \varphi$ is an L^q-solution of the problem (2.1) if $\frac{1}{2}\varphi - (K^\Omega_{-\lambda})'\varphi + h\mathcal{E}^\Omega_\lambda \varphi = g$. (See Proposition 3.2 and Proposition 3.3.)

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $\lambda \in \mathbb{R}$, $h \in L^\infty(\partial\Omega)$, $1 < q < \infty$. Suppose that $q \le 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Then $\frac{1}{2}I - (K^\Omega_{-\lambda})' + h\mathcal{E}^\Omega_\lambda$ is a Fredholm operator with index 0 in $L^q(\partial\Omega)$. If $\lambda \ne 0$ and $h \ge 0$ then $\frac{1}{2}I - (K^\Omega_{-\lambda})' + h\mathcal{E}^\Omega_\lambda$ is an isomorphism in $L^q(\partial\Omega)$.*

Proof. For $\lambda = 0$ and $h \equiv 0$ see [6, Theorem 1.2] and [13, Theorem 2.2.22]. The operator $(K_0^\Omega)' - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega$ is compact on $L^q(\partial\Omega)$ by Proposition 3.4. So, $T := \frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega$ is a Fredholm operator with index 0 in $L^q(\partial\Omega)$ by [29, Theorem 5.10].

Let now $\lambda \neq 0$ and $h \geq 0$. Let $\varphi \in L^q(\partial\Omega)$, $T\varphi = 0$. Since T is a Fredholm operator with index 0 in $L^q(\partial\Omega)$ and in $L^2(\partial\Omega)$, [21, Lemma 9] gives that $\varphi \in L^2(\partial\Omega)$. Thus $E_\lambda^\Omega\varphi$ is an L^2-solution of the problem (2.1) with $g \equiv 0$. Moreover, $E_\lambda^\Omega\varphi(x) \to 0$ as $|x| \to \infty$. Lemma 7.1 gives that $E_\lambda^\Omega\varphi = 0$ in Ω. The function $E_\lambda^\Omega\varphi$ is an L^2 solution of the problem $-\Delta u + \lambda\partial_1 u = 0$ in $R^3 \setminus \overline{\Omega}$, $\partial u/\partial n - (\lambda/2)n_1 u = g$ on $\partial(R^3 \setminus \overline{\Omega})$ for some $g \in L^2(\partial\Omega)$. Since $E_\lambda^\Omega\varphi = 0$ on $\partial(R^3 \setminus \overline{\Omega})$ by Proposition 3.2, Lemma 7.1 gives that $E_\lambda^\Omega\varphi = 0$ in $R^3 \setminus \overline{\Omega}$. According to Proposition 3.3

$$\varphi = \left[\frac{1}{2}\varphi - (K_{-\lambda}^\Omega)'\varphi\right] - \left[-\frac{1}{2}\varphi - (K_{-\lambda}^\Omega)'\varphi\right] = 0.$$

Thus T is an isomorphism on $L^q(\partial\Omega)$. □

Theorem 8.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $\lambda \in \mathbb{R} \setminus \{0\}$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $1 < q < \infty$. Suppose that $q \leq 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Fix $g \in L^q(\partial\Omega)$. Put $\varphi = [\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega]^{-1}g$. Then $E_\lambda^\Omega\varphi$ is a unique L^q-solution of the Robin problem (2.1). Moreover,*

$$\|M_a^\Omega(u)\|_{L^q(\partial\Omega)} + \|M_a^\Omega(\nabla u)\|_{L^q(\partial\Omega)} \leq C\|g\|_{L^q(\partial\Omega)} \tag{8.1}$$

where C does not depend on g.

Proof. $\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega$ is an isomorphism in $L^q(\partial\Omega)$ by Proposition 8.1. So, $E_\lambda^\Omega\varphi$ is an L^q-solution of the Robin problem (2.1).

Let now u be an L^q-solution of the Robin problem (2.1) with $g \equiv 0$. We can suppose that $q \leq 2$. Then u is a regular L^q-solution of the Dirichlet problem in Ω. According to Theorem 7.3 there exists $\psi \in L^q(\partial\Omega)$ such that $u = E_\lambda^\Omega\psi$. So, $[\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega]\psi = g \equiv 0$, what forces $\psi \equiv 0$. Thus $u = E_\lambda^\Omega\psi \equiv 0$.

The estimate (8.1) is a consequence of Proposition 3.2 and Proposition 3.3. □

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded open set with compact Lipschitz boundary, $\lambda \in \mathbb{R} \setminus \{0\}$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$, $1 < q < \infty$. Suppose that $q \leq 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Fix $g \in L^q(\partial\Omega)$. If u is an L^q-solution of the Robin problem (2.1) then there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Let u_∞ be given. Put $\varphi = [\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega]^{-1}[g - u_\infty(h - n_1\lambda/2)]$. Then $E_\lambda^\Omega\varphi + u_\infty$ is a unique L^q-solution of the Robin problem (2.1) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,*

$$\|M_a^\Omega(u)\|_{L^q(\partial\Omega)} + \|M_a^\Omega(\nabla u)\|_{L^q(\partial\Omega)} \leq C\left(\|g\|_{L^q(\partial\Omega)} + |u_\infty|\right) \tag{8.2}$$

where C does not depend on g.

Proof. $\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega$ is an isomorphism in $L^q(\partial\Omega)$ by Proposition 8.1. If u_∞ is given then $E_\lambda^\Omega\varphi + u_\infty$ is an L^q-solution of the Robin problem (2.1) such that $u(x) \to u_\infty$ as $|x| \to \infty$.

Let u be an L^q-solution of the Robin problem (2.1). Put $p = \min(q, 2)$. Then u is a regular L^p-solution of the Dirichlet problem in Ω. According to Theorem 7.4 there exists a constant u_∞ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Let now $u_\infty = 0$, $g \equiv 0$. According to Theorem 7.4 there exists $\psi \in L^q(\partial\Omega)$ such that $u = E_\lambda^\Omega\psi$. So, $[\frac{1}{2}I - (K_{-\lambda}^\Omega)' + h\mathcal{E}_\lambda^\Omega]\psi = g \equiv 0$, what forces $\psi \equiv 0$. Thus $u = E_\lambda^\Omega\psi \equiv 0$.

The estimate (8.2) is a consequence of Proposition 3.2 and Proposition 3.3. □

9. Transmission problem

Let now $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $\lambda_+, \lambda_- \in \mathbb{R}$, $1 < q < \infty$, $h_+, h_- \in L^\infty(\partial\Omega)$. Let a_+, a_-, b_+, b_- be positive constants. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$, and by n denote the unit exterior normal of Ω.

Lemma 9.1. *Let u be an L^2-solution of the transmission problem for the scalar Oseen equation (2.2) such that $u(x) \to 0$ as $|x| \to \infty$. If $h_\pm \geq 0$ and $f \equiv 0$, $g \equiv 0$ then $u \equiv 0$.*

Proof. Since $[u]_+ = (a_-/a_+)[u]_-$ on $\partial\Omega$ and $-n$ is the outward normal to Ω_-, Lemma 7.1 gives

$$0 = \int_{\partial\Omega} [u]_+ \left\{ b_+ \left[\frac{\partial u}{\partial n} - \frac{\lambda_+}{2} n_1 u \right]_+ - b_- \left[\frac{\partial u}{\partial n} - \frac{\lambda_-}{2} n_1 u \right]_- + h_+[u]_+ + h_-[u]_- \right\} \, \mathrm{d}\sigma$$

$$= b_+ \int_{\Omega_+} |\nabla u|^2 \, \mathrm{d}x + \frac{b_- a_-}{a_+} \int_{\Omega_-} |\nabla u|^2 \, \mathrm{d}x + \int_{\partial\Omega} \left[h_+[u]_+^2 + h_- \frac{a_-}{a_+} [u]_-^2 \right] \, \mathrm{d}\sigma.$$

Hence $\nabla u \equiv 0$ and u is constant on each component of $\mathbb{R}^3 \setminus \partial\Omega$. The condition $u(x) \to 0$ as $|x| \to \infty$ forces that $u = 0$ on the unbounded component of $\mathbb{R}^3 \setminus \partial\Omega$. Since $a_+[u]_+ = a_-[u]_-$ on $\partial\Omega$, we infer that $u \equiv 0$. □

Lemma 9.2. *Let $\psi, g \in L^q(\partial\Omega)$ and $\varphi, f \in W^{1,q}(\partial\Omega)$. Define*

$$T_1(\varphi, \psi) = a_+(\mathcal{E}^\Omega_{\lambda_+} \psi + \frac{1}{2}\varphi + K^\Omega_{\lambda_+}\varphi) - a_-(\mathcal{E}^\Omega_{\lambda_-}\psi - \frac{1}{2}\varphi + K^\Omega_{\lambda_-}\varphi), \tag{9.1}$$

$$\begin{aligned} T_2(\varphi, \psi) = \; & b_+[\tfrac{1}{2}\psi - (K^\Omega_{-\lambda_+})'\psi + H^\Omega_{\lambda_+}\varphi] - b_-[-\tfrac{1}{2}\psi - (K^\Omega_{-\lambda_-})'\psi + H^\Omega_{\lambda_-}\varphi] \\ & + h_+(\mathcal{E}^\Omega_{\lambda_+}\psi + \tfrac{1}{2}\varphi + K^\Omega_{\lambda_+}\varphi) + h_-(\mathcal{E}^\Omega_{\lambda_-}\psi - \tfrac{1}{2}\varphi + K^\Omega_{\lambda_-}\varphi), \end{aligned} \tag{9.2}$$

$T(\varphi, \psi) = [T_1(\varphi, \psi), T_2(\varphi, \psi)]$. Then

$$u = E^\Omega_{\lambda_\pm}\psi + D^\Omega_{\lambda_\pm}\varphi \qquad \text{in } \Omega_\pm \tag{9.3}$$

is an L^q-solution of the transmission problem for the scalar Oseen equation (2.2) if and only if $T(\varphi, \psi) = (f, g)$.

Proof. Lemma is a consequence of Proposition 3.1, Proposition 3.2, Proposition 3.3, Proposition 6.2 and Proposition 6.4. □

Proposition 9.3. *Let $a_+ = a_- = 1$, $h_+ \geq 0$, $h_- \geq 0$. Let T be an operator from Lemma 9.2. Suppose that one from the following conditions is satisfied:*

- *$b_+ = b_-$,*
- *$q = 2$,*
- *$\partial\Omega$ is of class $\mathcal{C}^1$.*

Then T is an isomorphism on $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$.

Proof. We prove that T is a Fredholm operator with index 0. Denote $\tilde{T}(\varphi, \psi) = [\tilde{T}_1(\varphi, \psi), \tilde{T}_2(\varphi, \psi)] = [\varphi, b_+[\frac{1}{2}\psi - (K^\Omega_0)'\psi + H^\Omega_0\varphi] - b_-[-\frac{1}{2}\psi - (K^\Omega_0)'\psi + H^\Omega_0\varphi]$, i.e the operator T for $\lambda_+ = \lambda_- = 0$, $h_+ \equiv h_- \equiv 0$. Clearly, $\tilde{T}$ is a Fredholm operator with index 0 on $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ if and only if $S\psi := b_+[\frac{1}{2}\psi - (K^\Omega_0)'\psi] - b_-[-\frac{1}{2}\psi - (K^\Omega_0)'\psi]$ is a Fredholm operator with index 0 on $L^q(\partial\Omega)$. If $b_+ = b_-$

then $S\psi = b_+\psi$. If $b_+ \neq b_-$ then S is a Fredholm operator with index 0 in $L^2(\partial\Omega)$ by [6, Theorem 2.5]. If $\partial\Omega$ is of class $\mathcal{C}^1$ then $(K_0^\Omega)'$ is a compact operator on $L^q(\partial\Omega)$ by [6, Theorem 1.8] and thus S is a Fredholm operator with index 0 on $L^q(\partial\Omega)$. Since $\tilde T$ is a Fredholm operator with index 0 in $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ and $T - \tilde T$ is compact by Proposition 3.4, Proposition 3.5, Proposition 6.3 and Proposition 6.4, the operator T is a Fredholm operator with index 0 in $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$.

Let now $(\varphi, \psi) \in W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$, $T(\varphi, \psi) = 0$. Since T is a Fredholm operator with index 0 in $W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$, [21, Lemma 9] gives that $(\varphi, \psi) \in W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$. Define

$$v_\pm = E^\Omega_{\lambda_\pm}\psi + D^\Omega_{\lambda_\pm}\varphi \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega, \qquad u = v_\pm \quad \text{in } \Omega_\pm.$$

Then u is an L^2-solution of the problem (2.2) with $g \equiv 0$, $f \equiv 0$ by Lemma 9.2. Lemma 9.1 gives that $v_\pm = 0$ in $\Omega_\pm$. According to Proposition 3.1 and Proposition 3.2

$$[v_-]_+ = \varphi + [v_-]_- = \varphi, \quad [v_+]_- = -\varphi + [v_+]_+ = -\varphi. \tag{9.4}$$

Proposition 3.3 and Proposition 6.4 force

$$[\partial v_-/\partial n - (\lambda_-/2) n_1 v_-]_+ = \psi + [\partial v_-/\partial n - (\lambda_-/2) n_1 v_-]_- = \psi, \tag{9.5}$$

$$[\partial v_+/\partial n - (\lambda_+/2) n_1 v_+]_- = -\psi + [\partial v_+/\partial n - (\lambda_+/2) n_1 v_+]_- = -\psi.$$

So, $\tilde u = v_-$ in Ω_+, $\tilde u = -v_+$ in Ω_- is an L^2-solution of the transmission problem

$$-\Delta \tilde u + \lambda_\mp \tilde u = 0 \quad \text{in } \Omega_\pm,$$

$$[\tilde u]_+ - [\tilde u]_- = 0, \quad \left[\frac{\partial \tilde u}{\partial n} - \frac{\lambda_-}{2} n_1 \tilde u\right]_+ - \left[\frac{\partial \tilde u}{\partial n} - \frac{\lambda_+}{2} n_1 \tilde u\right]_- = 0 \text{ on } \partial\Omega.$$

Lemma 9.1 gives that $v_\pm = 0$ in $\Omega_\mp$. According to (9.4) and (9.5) we obtain $\varphi = [v_-]_+ = 0$, $\psi = [\partial v_-/\partial n - (\lambda_-/2) n_1 v_-]_+ = 0$. Since T is a Fredholm operator with index 0 in $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$, it is an isomorphism. $\square$

Theorem 9.4. *Let $h_\pm \geq 0$. Suppose that one from the following conditions is satisfied:*

- $b_+/a_+ = b_-/a_-$,
- $q = 2$,
- *$\partial\Omega$ is of class $\mathcal{C}^1$.*

Define

$$\begin{aligned} S(\psi_+, \psi_-) = \; & [a_+ \mathcal{E}^\Omega_{\lambda_+}\psi_+ - a_- \mathcal{E}^\Omega_{\lambda_-}\psi_-, b_+(\tfrac12 \psi_+ - (K^\Omega_{-\lambda_+})'\psi_+) \\ & - b_-(-\tfrac12 \psi_- - (K^\Omega_{-\lambda_-})'\psi_-) + h_+ \mathcal{E}^\Omega_{\lambda_+}\psi_+ + h_- \mathcal{E}^\Omega_{\lambda_-}\psi_-]. \end{aligned}$$

Then $S : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is an isomorphism. If u is an L^q-solution of the transmission problem (2.2), then there exists $u_\infty \in \mathbb{R}^1$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Let now $u_\infty \in \mathbb{R}^1$, $f \in W^{1,q}(\partial\Omega)$, $g \in L^1(\partial\Omega)$ be given. Put $(\psi_+, \psi_-) = S^{-1}[f - a_+ u_\infty + a_- u_\infty, g - h_+ u_\infty - h_- u_\infty]$. Then $u = E^\Omega_{\lambda_\pm}\psi_\pm + u_\infty$ in $\Omega_\pm$ is a unique L^q-solution of the transmission problem (2.2) such that $u(x) \to u_\infty$ as $|x| \to \infty$. Moreover,

$$\|M_a^{\Omega_\pm}(u) + M_a^{\Omega_\pm}(\nabla u)\|_{L^q(\partial\Omega)} \leq C[\|f\|_{W^{1,q}(\partial\Omega)} + \|g\|_{L^q(\partial\Omega)} + |u_\infty|] \tag{9.6}$$

where C does not depend on f, g and u_∞.

Proof. If $(\psi_+, \psi_-) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$, then $u = \mathcal{E}^\Omega_{\lambda_\pm}\psi_\pm + u_\infty$ in $\Omega_\pm$ is an L^q-solution of the transmission problem (2.2) such that $u(x) \to u_\infty$ as $|x| \to \infty$ if and only if $S(\psi_+, \psi_-) = [f - a_+u_\infty + a_-u_\infty, g - h_+u_\infty - h_-u_\infty]$. (See Proposition 3.2 and Proposition 3.3.)

Let now u be an L^q-solution of the transmission problem (2.2). Then u is a regular L^q-solution of some Dirichlet problem in $\Omega_\pm$. According to Theorem 7.4 there exists $u_\infty \in \mathbb{R}^1$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. If $u_\infty = 0$ then there exists $(\psi_+, \psi_-) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$ such that $u = E^\Omega_{\lambda_\pm}\psi_\pm$ in $\Omega_\pm$. (See Theorem 7.3 and Theorem 7.4.)

Suppose now that $a_+ = a_- = 1$. Let T be the operator from Lemma 9.2. Then T is an isomorphism on $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ by Proposition 9.3. Let $(f, g) \in W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ be given. Put $(\varphi, \psi) = T^{-1}(f, g)$, $u = E^\Omega_{\lambda_\pm}\psi + D^\Omega_{\lambda_\pm}\varphi$ in $\Omega_\pm$. Then u is an L^q-solution of the transmission problem (2.2) such that $u(x) \to 0$ as $|x| \to \infty$. (See Lemma 9.2.) We have proved that there exists $(\psi_+, \psi_-) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$ such that $u = \mathcal{E}^\Omega_{\lambda_\pm}\psi_\pm$ in $\Omega_\pm$. Hence $S(\psi_+, \psi_-) = (f, g)$ and thus $S(L^q(\partial\Omega) \times L^q(\partial\Omega)) = W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$. Denote $p = \min(q, 2)$,

$$\tilde{S}(\psi_+, \psi_-) = [\mathcal{E}^\Omega_0\psi_+ - \mathcal{E}^\Omega_0\psi_-\ ,\ b_+(\psi_+/2 - (K^\Omega_0)'\psi_+) - b_-(-\psi_-/2 - (K^\Omega_0)'\psi_-)]$$

i.e. S for $\lambda_+ = \lambda_- = 0$, $h_+ \equiv h_- \equiv 0$. We have proved that $\tilde{S}(L^q(\partial\Omega) \times L^q(\partial\Omega)) = W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$. Let now $\psi_\pm \in L^q(\partial\Omega)$, $\tilde{S}(\psi_+, \psi_-) = 0$. Since $\mathcal{E}^\Omega_0\psi_+ = \mathcal{E}^\Omega_0\psi_-$ and $\mathcal{E}^\Omega_0 : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is an isomorphism by Proposition 7.2, we deduce that $\psi_- = \psi_+$. Denote by $\tilde{T}$ the operator T for $\lambda_+ = \lambda_- = 0$, $h_+ \equiv h_- \equiv 0$. Since $\psi_- = \psi_+$ and $\tilde{S}(\psi_+, \psi_-) = 0$, we deduce $\tilde{T}(0, \psi_+) = 0$. Since $\tilde{T}$ is an isomorphism on $W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ by Proposition 9.3, we infer that $\psi_- = \psi_+ = 0$. Therefore $\tilde{S} : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is an isomorphism. Since $\tilde{S} - S : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is a compact operator by Proposition 3.4 and Proposition 3.5, $S : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is a Fredholm operator with index 0. Since $S(L^q(\partial\Omega) \times L^q(\partial\Omega)) = W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$, the operator $S : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is an isomorphism. Let now u be an L^q-solution of the transmission problem (2.2) such that $u(x) \to 0$ as $|x| \to \infty$ and $f \equiv 0 \equiv g$. We have proved that there exists $(\psi_+, \psi_-) \in L^q(\partial\Omega) \times L^q(\partial\Omega)$ such that $u = E^\Omega_{\lambda_\pm}\psi_\pm$ in $\Omega_\pm$. Hence $S(\psi_+, \psi_-) = 0$ and thus $\psi_+ \equiv 0 \equiv \psi_-$. Therefore $u \equiv 0$.

Let now $a_\pm$ be arbitrary. Define $v = ua_\pm$ in $\Omega_\pm$. Then u is an L^q-solution of the transmission problem (2.2) such that $u(x) \to u_\infty$ as $|x| \to \infty$ if and only if v is an L^q-solution of the transmission problem

$$-\Delta v + \lambda_\pm \partial_1 v = 0 \quad \text{in } \Omega_\pm, \qquad [v]_+ - [v]_- = f \quad \text{on } \partial\Omega,$$

$$\frac{b_+}{a_+}\left[\frac{\partial v}{\partial n} - \frac{\lambda_+}{2}n_1 v\right]_+ - \frac{b_+}{a_+}\left[\frac{\partial v}{\partial n} - \frac{\lambda_+}{2}n_1 v\right]_+ + \frac{h_+}{a_+}[v]_+ + \frac{h_-}{a_-}[v]_- = g \text{ on } \partial\Omega$$

such that $v(x) \to a_-u_\infty$ as $|x| \to \infty$. We have proved that there exists a unique L^q-solution of this problem. Therefore there exists a unique L^q-solution of the transmission problem (2.2) such that $u(x) \to u_\infty$ as $|x| \to \infty$. If $u_\infty = 0$ then there exist $\psi_+, \psi_- \in L^q(\partial\Omega)$ such that $u = E^\Omega_{\lambda_\pm}\psi_\pm$ in $\Omega_\pm$. Since $S(\psi_+, \psi_-) = [f, g]$, we deduce that $S(L^q(\partial\Omega) \times L^q(\partial\Omega)) = W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$. If $\psi_+, \psi_- \in L^q(\partial\Omega)$ and $S(\psi_+, \psi_-) = 0$, then $u = E^\Omega_{\lambda_\pm}\psi_\pm$ is an L^q-solution of the transmission problem (2.2) with $f \equiv 0 \equiv g$ such that $u(x) \to 0$ as $|x| \to \infty$. Thus $u \equiv 0$. Proposition 3.2 gives $\mathcal{E}^\Omega_{\lambda_\pm}\psi_\pm = 0$ on $\partial\Omega$. Since $\mathcal{E}_{\lambda_\pm} : L^p(\partial\Omega) \to W^{1,p}(\partial\Omega)$ is an isomorphism for

$p = \min(2, q)$ by Proposition 7.2, we deduce that $\psi_+ \equiv 0 \equiv \psi_-$. Thus $S : L^q(\partial\Omega) \times L^q(\partial\Omega) \to W^{1,q}(\partial\Omega) \times L^q(\partial\Omega)$ is an isomorphism. If $u_\infty \in \mathbb{R}^1$, $f \in W^{1,q}(\partial\Omega)$, $g \in L^q(\partial\Omega)$ are given and $(\psi_+, \psi_-) = S^{-1}[f - a_+ u_\infty + a_- u_\infty, g - h_+ u_\infty - h_- u_\infty]$, then $u = E^\Omega_{\lambda_\pm} \psi_\pm + u_\infty$ in $\Omega_\pm$ is a unique L^q-solution of the transmission problem (2.2) such that $u(x) \to u_\infty$ as $|x| \to \infty$. The estimate (9.6) is a consequence of Proposition 3.2 and Proposition 3.3. □

10. Jump problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with compact Lipschitz boundary. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$, and by n denote the unit exterior normal of Ω. Let Λ be a closed subset of $\partial\Omega$. Let $\lambda \in \mathbb{R}$, $1 < q < \infty$, $h_+, h_- \in L^\infty(\partial\Omega)$, $g \in L^q(\partial\Omega)$, $f \in W^{1,q}(\partial\Omega)$ be such that $f = g = h_+ = h_- = 0$ on $\partial\Omega \setminus \Lambda$. *We say that u is an L^q-solution of the generalized jump problem for the scalar Oseen equation*

$$\begin{aligned} &-\Delta u + \lambda\partial_1 u = 0 \ \textit{in}\ \mathbb{R}^3 \setminus \Lambda, \quad [u]_+ - [u]_- = f \ \textit{on}\ \Lambda, \\ &\left[\tfrac{\partial u}{\partial n} - \tfrac{\lambda}{2} n_1 u\right]_+ - \left[\tfrac{\partial u}{\partial n} - \tfrac{\lambda}{2} n_1 u\right]_- + h_+[u]_+ + h_-[u]_- = g \ \textit{on}\ \Lambda, \end{aligned} \tag{10.1}$$

if $u \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \Lambda)$, $-\Delta u + \lambda\partial_1 u = 0$ in $\mathbb{R}^3 \setminus \Lambda$, $M^{\Omega+}_a(u) + M^{\Omega+}_a(|\nabla u|) + M^{\Omega-}_a(u) + M^{\Omega-}_a(|\nabla u|) \in L^q(\partial\Omega)$, there exist nontangential limits of u and ∇u with respect to Ω_+ and Ω_- at almost all points of $\partial\Omega$, and the generalized jump conditions $[u]_+ - [u]_- = f$, $[\partial u/\partial n - \lambda_+ n_1 u/2]_+ - [\partial u/\partial n - \lambda_- n_1 u/2]_- + h_+[u]_+ + h_-[u]_- = g$ are fulfilled in the sense of the nontangential limit at almost all points of Λ. If $h_+ \equiv 0 \equiv h_-$ we say about the jump problem.

Lemma 10.1. *Let $a_+ = a_- = b_+ = b_- = 1$, $\lambda_+ = \lambda_- = \lambda$.*

(1) If u is an L^q-solution of the generalized jump problem (10.1), then u is an L^q-solution of the transmission problem (2.2).

(2) Let u be an L^q-solution of the transmission problem (2.2). Define $u = [u]_+$ on $\partial\Omega \setminus \Lambda$. Then u is an L^q-solution of the generalized jump problem (10.1).

Proof. The first proposition is trivial.

Let now u be an L^q-solution of the transmission problem (2.2) and $u = [u]_+$ on $\partial\Omega \setminus \Lambda$. Then

$$u = E^\Omega_\lambda f + D^\Omega_\lambda (g - h_+[u]_+ - h_-[u]_-) \tag{10.2}$$

in $\mathbb{R}^3 \setminus \partial\Omega$ by Corollary 5.2. Since $f = 0$, $(g - h_+[u]_+ - h_-[u]_-) = 0$ on $\partial\Omega \setminus \Lambda$, the function u is given by (10.2) and it is a solution of the scalar Oseen equation in $\mathbb{R}^3 \setminus \Lambda$. Thus u is an L^q-solution of the generalized jump problem (10.1). □

Theorem 10.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with compact Lipschitz boundary. Denote $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}$, and by n denote the unit exterior normal of Ω. Let Λ be a closed subset of $\partial\Omega$. Let $\lambda \in \mathbb{R}$, $1 < q < \infty$, $h_+, h_- \in L^\infty(\partial\Omega)$, $h_\pm \geq 0$, $g \in L^q(\partial\Omega)$, $f \in W^{1,q}(\partial\Omega)$ be such that $f = g = h_+ = h_- = 0$ on $\partial\Omega \setminus \Lambda$. If u is an L^q-solution of the generalized jump problem (10.1) then there exists $u_\infty \in \mathbb{R}^1$ such that $u(x) \to u_\infty$. On the other hand, if $u_\infty \in \mathbb{R}^1$ is given then there exists a unique L^q-solution of the generalized jump problem (10.1) such that $u(x) \to u_\infty$. Moreover, the estimate (9.6) holds with a constant C that does not depend on f, g and u_∞.*

Proof. The theorem is an easy consequence of Lemma 10.1 and Theorem 9.4. □

11. Appendix

Lemma 11.1. *If $\Omega \subset R^m$ is an open set with compact Lipschitz boundary then there is a sequence of open sets Ω_j with compact boundaries of class C^∞ such that*

- $\overline{\Omega}_j \subset \Omega$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(y) \in \Gamma_a(y)$ for each j and each $y \in \partial\Omega$ and $\sup\{|y - \Lambda_j(y)|; y \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\Lambda_j(E))$, and so that $\omega_j \to 1$ pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$.*
- *The normal vectors to Ω_j, $n(\Lambda_j(y))$, converge pointwise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$, to $n(y)$.*

(See [31, Theorem 1.12].)

Lemma 11.2. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < q < \infty$, $u \in \mathcal{C}^2(\Omega)$. Suppose that $M_a(u), M_a(|\nabla u|) \in L^q(\partial\Omega)$ and there exist nontangential limits of u and ∇u at almost all points of $\partial\Omega$. Define the nontangential derivative $\partial_{\tau_{jk}} u$ in the sense of distributions by*

$$\langle \partial_{\tau_{jk}} u, \varphi \rangle = \int_{\partial\Omega} u(n_k^\Omega \partial_j \varphi - n_j^\Omega \partial_k \varphi) \, \mathrm{d}\sigma.$$

Then $u \in W^{1,q}(\partial\Omega)$ and $\partial_{\tau_{jk}} u$ in the sense of distributions coincides with the nontangential limit $n_j^\Omega \partial_k u - n_k^\Omega \partial_j u$.

Proof. Let $\Omega(i)$ be a sequence of sets from Lemma 11.1. If $\varphi \in \mathcal{C}^\infty(R^m)$ has compact support, then the Gauss-Green theorem and the Lebesgue lemma give

$$\langle \partial_{\tau_{jk}} u, \varphi \rangle = \lim_{i\to\infty} \int_{\partial\Omega(i)} u(n_k^\Omega \partial_j \varphi - n_j^\Omega \partial_k \varphi) \, \mathrm{d}\sigma = \lim_{i\to\infty} \int_{\Omega(i)} (\partial_k u \partial_j \varphi - \partial_j u \partial_k \varphi) \, \mathrm{d}x$$

$$= \lim_{i\to\infty} \int_{\partial\Omega(i)} \varphi(n_j^\Omega \partial_k u - n_k^\Omega \partial_j u) \, \mathrm{d}\sigma = \int_{\partial\Omega} \varphi(n_j^\Omega \partial_k u - n_k^\Omega \partial_j u) \, \mathrm{d}\sigma.$$

Since $u, \partial_{\tau_{jk}} u \in L^q(\partial\Omega)$ we infer that $u \in W^{1,q}(\partial\Omega)$. $\square$

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