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THE ROBIN PROBLEM FOR THE BRINKMAN SYSTEM AND FOR THE DARCY-FORCHHEIMER-BRINKMAN SYSTEM

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Abstract. In this paper we study the Neumann problem and the Robin problem for the Darcy-Forchheimer-Brinkman system in $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ for a bounded domain $\Omega\subset\mathbb{R}^m$ with Lipschitz boundary. First we study the Neumann problem and the Robin problem for the Brinkman system by the integral equation method. If $\Omega\subset\mathbb{R}^m$ is a bounded domain with Lipschitz boundary and $2\le m\le 3$, then we prove the unique solvability of the Neumann problem and the Robin problem for the Brinkman system in $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$, where $3/2<q<3$. Then we get results for the Darcy-Forchheimer-Brinkman system from the results for the Brinkman system using the fixed point theorem. If $\Omega\subset\mathbb{R}^m$ is a bounded domain with Lipschitz boundary, $2\le m\le 3$, $3/2<q<3$, then we prove the existence of a solution of the Neumann problem and the Robin problem for the Darcy-Forchheimer-Brinkman system in $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ for small given data.

1. Introduction

Boundary value problems for the Darcy-Forchheimer-Brinkman system

$$\nabla p-\Delta\mathbf{u}+\lambda\mathbf{u}+\alpha|\mathbf{u}|\mathbf{u}+\beta(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{f},\quad \nabla\cdot\mathbf{u}=0\quad\text{in }\Omega \tag{1.1}$$

have been extensively studied lately. This system describes flows through porous media saturated with viscous incompressible fluids, where the inertia of such fluid is not negligible. The constants $\lambda,\alpha,\beta>0$ are determined by the physical properties of such a porous medium. (For further details we refer the reader to the book [21, p. 17] and the references therein.)

T. Grosan, M. Kohr and W. L. Wendland studied in [7] the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\mathbf{f}\equiv 0$ in $W^{1,2}(\Omega,\mathbb{R}^m)\times L^2(\Omega)$, where $\Omega\subset\mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $m=2$ or $m=3$. R. Gutt and T. Grosan studied in [8] the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\mathbf{f}\equiv 0$ in $W^{s,2}(\Omega,\mathbb{R}^m)\times W^{s-1,2}(\Omega)$, where $1\le s<3/2$, $\Omega\subset\mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $m=2$ or $m=3$. M. Kohr, M. Lanza de Cristoforis, W. L. Wendland studied in [13] the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\mathbf{f}\equiv 0$, $\beta=0$ in $W^{s,2}(\Omega,\mathbb{R}^m)\times W^{s-1,2}(\Omega)$, where $1\le s<3/2$, $\Omega\subset\mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $2\le m\le 4$. The author studied in [18] a bounded solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\beta=0$ on a bounded domain $\Omega\subset\mathbb{R}^m$ with Ljapunov boundary. M. Kohr, M. Lanza de Cristoforis, W. L. Wendland studied in [12] the

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Robin problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\beta = 0$ in the space $H^s(\Omega, \mathbb{R}^m) \times H^{s-1}(\Omega)$, where $1 < s < 3/2$, $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary and $m \in \{2, 3\}$. M. Kohr, M. Lanza de Cristoforis, W. L. Wendland studied in [12] the mixed Dirichlet-Robin problem for the Darcy-Forchheimer-Brinkman system (1.1) with $\beta = 0$ in $H^{3/2}(\Omega, \mathbb{R}^3) \times H^{1/2}(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a bounded creased domain with connected Lipschitz boundary. M. Kohr, M. Lanza de Cristoforis, W. L. Wendland studied in [12] the problem of Navier's type for the Darcy-Forchheimer-Brinkman system (1.1) with $\beta = 0$ in $H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with connected Lipschitz boundary. M. Kohr, M. Lanza de Cristoforis, S. E. Mikhailov, W. L. Wendland studied in [10] the transmission problem, where the Darcy-Forchheimer-Brinkman system is given in a bounded domain $\Omega_+ \subset \mathbb{R}^3$ with connected Lipschitz boundary and the Stokes system is given on its complementary domain Ω_-. Solutions are from $\mathcal{H}^1(\Omega_\pm) \times L^2(\Omega_\pm)$, where $\mathcal{H}^1(\Omega) = \{\mathbf{u} \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^); \partial_j u_i \in L^2(\Omega), (1 + |\mathbf{x}|^2)^{-1/2} u_j(\mathbf{x}) \in L^2(\Omega)\}$.

In this paper we study the Neumann problem and the Robin problem for the Darcy-Forchheimer-Brinkman system in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^m$ with Lipschitz boundary. First we study the Neumann problem and the Robin problem for the Brinkman system by the integral equation method. If $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary and $2 \le m \le 3$, then we prove the unique solvability of the Neumann problem and the Robin problem for the Brinkman system in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$, where $3/2 < q < 3$. Then we get results for the Darcy-Forchheimer-Brinkman system from the results for the Brinkman system using the fixed point theorem. If $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary, $2 \le m \le 3$, $3/2 < q < 3$, then we prove the existence of a solution of the Neumann problem and the Robin problem for the Darcy-Forchheimer-Brinkman system in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ for small given data.

2. Function spaces

First we remember definitions of several function spaces.

Let $\Omega \subset \mathbb{R}^m$ be an open set. We denote by $\mathcal{C}_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. If $k \in \mathbb{N}_0$, $1 < q < \infty$ we define the Sobolev space $W^{k,q}(\Omega) := \{f \in L^q(\Omega); \partial^\alpha f \in L^q(\Omega)$ for $|\alpha| \le m\}$ endowed with the norm

$$\|u\|_{W^{k,q}(\Omega)} = \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^q(\Omega)}.$$

(Clearly $W^{0,q}(\Omega) = L^q(\Omega)$.) If $s = k + \lambda$, $0 < \lambda < 1$, denote $W^{s,q}(\Omega) = \{u \in W^{k,q}(\Omega); \|u\|_{W^{s,q}(\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,q}(\Omega)} = \left[\|u\|^q_{W^{k,q}(\Omega)} + \sum_{|\alpha| = k} \int_{\Omega \times \Omega} \frac{|\partial^\alpha u(\mathbf{x}) - \partial^\alpha u(\mathbf{y})|^q}{|\mathbf{x} - \mathbf{y}|^{m + q\lambda}} \, \mathrm{d}(\mathbf{x}, \mathbf{y}) \right]^{1/q}.$$

Denote by $\mathring{W}^{k,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

If X is a Banach space we denote by X' its dual space. If $0 < s < \infty$, denote $W^{-s,q}(\Omega) := [\mathring{W}^{s,q'}(\Omega)]'$, where $q' = q/(q-1)$.

Denote by $D^{-1,q}(\Omega)$ the set of distributions u on Ω such that $\partial_j \in W^{-1,q}(\Omega)$ for $j = 1, \ldots, m$.

If $\Omega \subset V \subset \overline{\Omega}$ then we denote by $L^q_{\rm loc}(V)$ the space of all measurable functions u on Ω such that $u \in L^q(\omega)$ for each bounded open set ω with $\overline{\omega} \subset V$.

If $\Omega \subset \mathbb{R}^m$ is an open set with compact Lipschitz boundary, $0 < s < 1$, $1 < q < \infty$, denote $W^{s,q}(\partial\Omega) = \{u \in L^q(\partial\Omega); \|u\|_{W^{s,q}(\partial\Omega)} < \infty\}$ where

$$\|u\|_{W^{s,q}(\partial\Omega)} = \left[\|u\|^q_{L^q(\partial\Omega)} + \int\limits_{\partial\Omega\times\partial\Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^q}{|\mathbf{x}-\mathbf{y}|^{m-1+qs}} \, \mathrm{d}(\mathbf{x},\mathbf{y})\right]^{1/q}.$$

Further, $W^{-s,q}(\partial\Omega) := [W^{s,q'}(\partial\Omega)]'$, where $q' = q/(q-1)$.

We denote $\mathcal{C}^\infty_c(\Omega;\mathbb{R}^m) := \{(v_1,\dots,v_m); v_j \in \mathcal{C}^\infty_c(\Omega)\}$. Similarly for other spaces of functions.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain, i.e. an open connected set. If $1 < q < \infty$ then $D^{-1,q}(\Omega) \subset L^q_{\rm loc}(\Omega)$. Choose a bounded non-empty domain ω such that $\overline{\omega} \subset \Omega$. Then $D^{-1,q}(\Omega)$ is a Banach space equipped with the norm*

$$\|u\|_{D^{-1,q}(\Omega)} := \|u\|_{L^q(\omega)} + \|\nabla u\|_{W^{-1,q}(\Omega)}. \tag{2.1}$$

Different choices of ω give equivalent norms.

Proof. Let $u \in D^{-1,q}(\Omega)$. According to [24, Proposition I.1.1]

$$\langle \nabla u, \mathbf{\Phi}\rangle = 0 \quad \forall \mathbf{\Phi} \in \mathcal{C}^\infty_c(\Omega,\mathbb{R}^m), \nabla\cdot\mathbf{\Phi} = 0. \tag{2.2}$$

Let $\omega \subset \Omega$ be a bounded domain with Lipschitz boundary such that $\overline{\omega} \subset \Omega$. Since u satisfies (2.2), [22, Lemma 2.1.1] gives that there exists $p \in L^q(\omega)$ such that $\nabla p = \nabla u$ in ω. Since $\nabla(u-p) = 0$ in ω, $u - p$ is constant in ω. Hence $u \in L^q_{\rm loc}(\Omega)$.

Let ω be a bounded non-empty domain such that $\overline{\omega} \subset \Omega$. Let u_n be a Cauchy sequence with respect to the norm (2.1). Then $(u_n, \nabla u_n)$ is a Cauchy sequence in $L^q(\omega) \times W^{-1,q}(\Omega) \times \dots \times W^{-1,q}(\Omega)$. So, $(u_n,\nabla u_n) \to (f_0, f_1,\dots,f_m)$ in $L^q(\omega) \times W^{-1,q}(\Omega) \times \dots \times W^{-1,q}(\Omega)$. Clearly, $\partial_j f_0 = f_j$ in ω in the sense of distributions for $j = 1,\dots,m$. Define $\mathbf{f} = (f_1,\dots,f_m)$. Since u_n satisfy (2.2), we get

$$\langle \mathbf{f}, \mathbf{\Phi}\rangle = 0 \quad \forall \mathbf{\Phi} \in \mathcal{C}^\infty_c(\Omega,\mathbb{R}^m), \nabla\cdot\mathbf{\Phi} = 0. \tag{2.3}$$

According to [24, Proposition I.1.1] there exists a distribution u in Ω such that $\nabla u = \mathbf{f}$. Since $\nabla(u - f_0) = 0$ in ω, $u - f_0$ is constant. We can suppose that $u = f_0$ in ω. Let G be a bounded domain with Lipschitz boundary such that $\omega \subset G \subset \overline{G} \subset \Omega$. Since $\mathbf{f}$ satisfies (2.3), [22, Lemma 2.1.1] gives that there exists $p \in L^q(G)$ such that $\nabla p = \nabla u$ in G. Since $\nabla(u-p) = 0$ in G, $u - p$ is constant in G. Thus $u \in D^{-1,q}(\Omega)$ and $u_n \to u$ in $D^{-1,q}(\Omega)$.

Let ω, G be bounded non-empty domains such that $G \subset \omega \subset \overline{\omega} \subset \Omega$. If $u \in D^{-1,q}(\Omega)$ then

$$\|u\|_{L^q(G)} + \|\nabla u\|_{W^{-1,q}(\Omega)} \le \|u\|_{L^q(\omega)} + \|\nabla u\|_{W^{-1,q}(\Omega)}.$$

[31, Chapter II, §5, Corollary] gives that there exist a positive constant C such that

$$\|u\|_{L^q(\omega)} + \|\nabla u\|_{W^{-1,q}(\Omega)} \le C\left[\|u\|_{L^q(G)} + \|\nabla u\|_{W^{-1,q}(\Omega)}\right] \qquad \forall u \in D^{-1,q}(\Omega).$$

□

3. Formulation of the problem

Suppose first that $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary, and $(\mathbf{u}, p) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^m) \times \mathcal{C}^1(\overline{\Omega})$ is a classical solution of the Robin problem for the Brinkman system

$$\nabla p - \Delta \mathbf{u} + \lambda \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.1a}$$

$$T(\mathbf{u}, p)\mathbf{n} + h\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega, \tag{3.1b}$$

where

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI, \quad \hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

and I is the identity matrix. If $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$, then the Green formula gives

$$\int_\Omega \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma = \int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi}) + \lambda \mathbf{\Phi} \cdot \mathbf{u}] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\mathbf{u} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma.$$

(Compare [29, p. 14].) This formula motivates definition of a weak solution of the Robin problem for the Brinkman system.

Let $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary, $h \in L^\infty(\partial\Omega)$, $1 < q < \infty$, $q' = q/(q-1)$, $\mathbf{F} \in [W^{1,q'}(\Omega, \mathbb{R}^m)]'$. *We say that* $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q_{\text{loc}}(\overline{\Omega})$ *is a weak solution of the Robin problem for the Brinkman system*

$$\nabla p - \Delta \mathbf{u} + \lambda \mathbf{u} = \mathbf{F} \quad \textit{in} \quad \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \textit{in } \Omega, \tag{3.2a}$$

$$T(\mathbf{u}, p)\mathbf{n} + h\mathbf{u} = \mathbf{F} \qquad \textit{on } \partial\Omega \tag{3.2b}$$

if $\nabla \cdot \mathbf{u} = 0$ *in* Ω *and*

$$\langle \mathbf{F}, \mathbf{\Phi} \rangle = \int_\Omega [2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p(\nabla \cdot \mathbf{\Phi}) + \lambda \mathbf{\Phi} \cdot \mathbf{u}] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\mathbf{u} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma \tag{3.3}$$

for all $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$. *If* $h \equiv 0$ *we say about the Neumann problem for the Brinkman system.*

If Ω is bounded then $p \in L^q(\Omega)$ and the density of $\mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$ in $W^{1,q'}(\mathbb{R}^m, \mathbb{R}^m)$ gives that (3.3) holds for all $\mathbf{\Phi} \in W^{1,q'}(\mathbb{R}^m, \mathbb{R}^m)$.

If $\mathbf{F}$ is supported on the boundary then $(\mathbf{u}, p)$ is a weak solution of the problem (3.1) with $\mathbf{f} \equiv 0$ and $\mathbf{g} = \mathbf{F}$.

If $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q_{\text{loc}}(\overline{\Omega})$ then (3.3) holds for all $\mathbf{\Phi} \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^m)$ if and only if $(\mathbf{u}, p)$ is a solution of (3.2a) in the sense of distributions.

Remark that if $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary and $(\mathbf{u}, p) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^m) \times \mathcal{C}^1(\overline{\Omega})$ is a classical solution of the problem (3.1), then $(\mathbf{u}, p)$ is a weak solution of the problem (3.2) with

$$\langle \mathbf{F}, \mathbf{\Phi} \rangle := \int_\Omega \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma.$$

4. Brinkman system in $\mathbb{R}^m$

Lemma 4.1. *For* $t \in (0, \infty)$ *define* $L_t\varphi(\mathbf{x}) := \varphi(t\mathbf{x})$ *for* $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$. *More generally, for a distribution* f *we define*

$$\langle L_t f, \varphi \rangle := \langle f, t^m L_{1/t}\varphi \rangle.$$

Suppose that

$$\nabla p - \Delta \mathbf{u} + \lambda \mathbf{u} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \qquad \textit{in } \mathbb{R}^m \tag{4.1}$$

in the sense of distributions. Define $\tilde{p} := t^{-1} L_t p$, $\tilde{\mathbf{u}} := t^{-2} L_t \mathbf{u}$, $\tilde{\mathbf{f}} := L_t \mathbf{f}$. *Then*

$$\nabla \tilde{p} - \Delta \tilde{\mathbf{u}} + t^2 \lambda \tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \quad \nabla \cdot \tilde{\mathbf{u}} = 0. \tag{4.2}$$

Proof. If $p \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, $\mathbf{u} \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$ then easy calculation yields (4.2). If $\mathbf{u}$, p are distributions we can choose $p_k \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, $\mathbf{u}_k \in \mathcal{C}_c^\infty(\mathbb{R}^m, \mathbb{R}^m)$ such that $p_k \to p$, $\mathbf{u}_k \to \mathbf{u}$ in the sense of distributions. Now we get (4.2) by the limit process. □

Proposition 4.2. *Let* $\lambda \in (0, \infty)$, $1 < q < \infty$, $\mathbf{f} \in W^{-1,q}(\mathbb{R}^m, \mathbb{R}^m)$. *Then there exists a solution* $(\mathbf{u}, p) \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ *of* (4.1). *A velocity* $\mathbf{u}$ *is unique, a pressure* p *is unique up to an additive constant. Moreover,* $p \in D^{-1,q}(\mathbb{R}^m)$ *and*

$$\|\mathbf{u}\|_{W^{1,q}(\mathbb{R}^m)} + \inf_{c \in \mathbb{R}^1} \|p + c\|_{D^{-1,q}(\mathbb{R}^m)} \le C \|\mathbf{f}\|_{W^{-1,q}(\mathbb{R}^m)} \tag{4.3}$$

where C *depends only on* m, λ, q *and a choice of* ω *in* (2.1).

Proof. If $\lambda = 1$ then there exists a solution $(\mathbf{u}, p) \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ of (4.1) by [30, Theorem 5.5] and [5, Lemma IV.1.1]. Lemma 4.1 gives that there exists a solution $(\mathbf{u}, p) \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ of (4.1) for arbitrary $\lambda \in (0, \infty)$. Let $(\tilde{\mathbf{u}}, \tilde{p}) \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ be another solution of (4.1). Then $u_j - \tilde{u}_j$, $p - \tilde{p}_j$ are polynomials by [18, Proposition 5.1]. Since $u_j - \tilde{u}_j \in W^{1,q}(\mathbb{R}^m)$, we infer that $u_j - \tilde{u}_j \equiv 0$. Thus $\nabla(p - \tilde{p}) \equiv 0$ by (4.1). This forces that $p - \tilde{p}$ is constant. Since $\partial_j p = \Delta u_j - \lambda u_j + f_j \in W^{-1,q}(\mathbb{R}^m)$, we infer $p \in D^{-1,q}(\mathbb{R}^m)$.

Define

$$Q(\mathbf{u}, p) := \int_\Omega p \, \mathrm{d}\mathbf{x}.$$

Then $(\mathbf{u}, p) \mapsto (\nabla p - \Delta \mathbf{u} + \lambda \mathbf{u}, Qp)$ is a bounded linear operator from the Banach space $W^{1,q}_\sigma(\mathbb{R}^m, \mathbb{R}^m) \times D^{-1,q}(\mathbb{R}^m)$ to $W^{-1,q}(\mathbb{R}^m, \mathbb{R}^m)$, where $W^{1,q}_\sigma(\mathbb{R}^m, \mathbb{R}^m) := \{\mathbf{u} \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m); \nabla \cdot \mathbf{u} \equiv 0\}$. Since it is one-to-one and onto, it is an isomorphism. This gives the estimate (4.3). □

5. Fundamental solution of the Brinkman system

Let $\lambda \ge 0$. Then there exists a unique fundamental solution $E^\lambda = (E^\lambda_{ij})$, $Q^\lambda = (Q^\lambda_j)$ of the Brinkman system

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + \nabla p = 0, \quad \nabla \mathbf{u} = 0 \tag{5.1}$$

in $\mathbb{R}^m$ such that $E^\lambda(x) = o(|x|)$, $Q^\lambda(x) = o(|x|)$ as $|x| \to \infty$. Remember that for $i, j \in \{1, \dots, m\}$ we have

$$-\Delta E^\lambda_{ij} + \lambda E^\lambda_{ij} + \partial_i Q^\lambda_j = \delta_{ij} \delta_0, \quad \partial_1 E^\lambda_{1j} + \dots \partial_m E^\lambda_{mj} = 0, \tag{5.2}$$

$$-\Delta E^\lambda_{i,m+1} + \lambda E^\lambda_{i,m+1} + \partial_i Q^\lambda_{m+1} = 0, \quad \partial_1 E^\lambda_{1,m+1} + \dots \partial_m E^\lambda_{m,m+1} = \delta_0. \tag{5.3}$$

Clearly,

$$E^\lambda(-\mathbf{x}) = E^\lambda(\mathbf{x}), \quad Q^\lambda(-\mathbf{x}) = -Q^\lambda(\mathbf{x}). \tag{5.4}$$

If $j \in \{1, \dots, m\}$ then

$$Q^\lambda_j(x) = E^\lambda_{j,m+1}(x) = \frac{1}{\omega_n} \frac{x_j}{|x|^m},$$

$$Q^\lambda_{m+1} = \begin{cases} \delta_0(x) + (\lambda/\omega_m) \ln |x|^{-1}, & m = 2, \\ \delta_0(x) + (\lambda/\omega_m)(m-2)^{-1} |x|^{2-m}, & m > 2, \end{cases}$$

where ω_m is the area of the unit sphere in $\mathbb{R}^m$. (See [29, p. 60].) The expressions of E^λ can be found in the book [29, Chapter 2]. We omit them for the sake of brevity.

For $\lambda = 0$ we obtain the fundamental solution of the Stokes system. If $i, j \in \{1, \dots, m\}$, the components of E^0 are given by

$$E^0_{ij}(x) = \frac{1}{2\omega_m}\left\{\frac{\delta_{ij}}{(m-2)|x|^{m-2}} + \frac{x_i x_j}{|x|^m}\right\}, \quad m \geq 3 \tag{5.5}$$

$$E^0_{ij}(x) = \frac{1}{4\pi}\left\{\delta_{ij}\ln\frac{1}{|x|} + \frac{x_j x_k}{|x|^2}\right\}, \quad m = 2, \tag{5.6}$$

(see, e.g., [29, p. 16]).

If $i, j \leq m$ then

$$E^\lambda_{ij} = E^\lambda_{ji}, \tag{5.7}$$

$$|E^\lambda_{ij}(x) - E^0_{ij}(x)| = O(1) \quad \text{as } |x| \to 0 \tag{5.8}$$

by [29, p. 66] and

$$|\nabla E^\lambda_{ij}(x) - \nabla E^0_{ij}(x)| = O(|x|^{2-m}) \quad \text{as } |x| \to 0 \tag{5.9}$$

by [18, Lemma 4.1].

If $i, j \leq m$ and $\lambda > 0$, then

$$\partial^\alpha E_{ij}(x) = O(|x|^{-m-|\alpha|}), \quad |x| \to \infty \tag{5.10}$$

for each multiindex α. (See [14, Lemma 3.1].)

6. Volume potential

We denote $Q(x) = (Q^0_1(x), \dots, Q^0_m(x)) = (Q^\lambda_1(x), \dots, Q^\lambda_m(x))$. By $\tilde{E}^\lambda$ we denote the matrix of the type $m \times m$, where $\tilde{E}^\lambda_{ij}(x) = E^\lambda_{ij}(x)$ for $i, j \leq m$.

Proposition 6.1. *Let $0 < \lambda < \infty$, $1 < q < \infty$, $s \in \mathbb{R}^1$. Then $\mathbf{f} \mapsto \tilde{E}^\lambda * \mathbf{f}$, $\mathbf{f} \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R}^m)$, can be extended by a unique way as a bounded linear operator from $W^{s,q}(\mathbb{R}^m, \mathbb{R}^m)$ to $W^{s+2,q}(\mathbb{R}^m, \mathbb{R}^m)$.*

Proof. $\mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R}^m)$ is a dense subset of $W^{s,q}(\mathbb{R}^m, \mathbb{R}^m)$ by [25, §2.3.3], [26, §2.12, Theorem] and [1, Theorem 4.2.2]. This gives a uniqueness.

Suppose first that $s = -1$. If $\mathbf{f} \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R}^m)$, then $\mathbf{u} := \tilde{E}^\lambda * \mathbf{f}$, $p := Q * \mathbf{f}$ is a solution of (4.1). According to Proposition 4.2 there exists a solution $(\tilde{\mathbf{u}}, \tilde{p}) \in W^{1,q}(\mathbb{R}^m, \mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ of (4.1) such that

$$\|\tilde{\mathbf{u}}\|_{W^{1,q}(\mathbb{R}^m)} \leq C_1 \|\mathbf{f}\|_{W^{-1,q}(\mathbb{R}^m)}$$

with C_1 independent of $\mathbf{f}$. [18, Proposition 5.1] gives that $u_j - \tilde{u}_j$ are polynomials. Since $\mathbf{u} \in L^q(\{|\mathbf{x}| > r\})$ for sufficiently large r by (5.10), we infer that $\mathbf{u} \equiv \tilde{\mathbf{u}}$. Therefore $B : \mathbf{f} \mapsto \tilde{E}^\lambda * \mathbf{f}$, $\mathbf{f} \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R}^m)$, can be extended as a bounded linear operator $B : W^{-1,q}(\mathbb{R}^m, \mathbb{R}^m) \to W^{1,q}(\mathbb{R}^m, \mathbb{R}^m)$.

Let now $k \in \mathbb{N}_0$. Then $W^{k,q}(\mathbb{R}^m) \hookrightarrow W^{-1,q}(\mathbb{R}^m)$ by [26, §2.3.3, Remark 4]. In particular, there exists a constant C_2 such that

$$\|\mathbf{f}\|_{W^{-1,q}(\mathbb{R}^m)} \leq C_2 \|\mathbf{f}\|_{L^q(\mathbb{R}^m)}. \tag{6.1}$$

If $\mathbf{f} \in W^{k,q}(\mathbb{R}^m, \mathbb{R}^m)$ and α is a multi-index with $|\alpha| \le k+1$, then $\partial^\alpha \tilde{E}^\lambda * \mathbf{f} = \tilde{E}^\lambda * \partial^\alpha \mathbf{f}$ and therefore

$$\|\partial^\alpha \tilde{E}^\lambda * \mathbf{f}\|_{W^{1,q}(\mathbb{R}^m)} \le C_1 \|\partial^\alpha \mathbf{f}\|_{W^{-1,q}(\mathbb{R}^m)}.$$

This, (6.1) and $\partial_j : L^q(\mathbb{R}^m) \to W^{-1,q}(\mathbb{R}^m)$ bounded yield that

$$B : W^{k,q}(\mathbb{R}^m, \mathbb{R}^m) \to W^{k+2,q}(\mathbb{R}^m, \mathbb{R}^m)$$

is bounded.

Let now $k \in \mathbb{N}_0$, $0 < \theta < 1$, $s = k - 1 + \theta$. Then

$$(W^{k-1,q}(\mathbb{R}^m), W^{k,q}(\mathbb{R}^m))_{\theta,q} = W^{s,q}(\mathbb{R}^m),$$

$$(W^{k+1,q}(\mathbb{R}^m), W^{k+2,q}(\mathbb{R}^m))_{\theta,q} = W^{s+2,q}(\mathbb{R}^m)$$

where $(\ ,\ )_{\theta,q}$ denotes the real interpolation. (See [3, Theorem 6.4.5].) Thus $B : W^{s,q}(\mathbb{R}^m, \mathbb{R}^m) \to W^{s+2,q}(\mathbb{R}^m, \mathbb{R}^m)$ is bounded by [23, Lemma 22.3].

Let now $s < -1$. Denote $q' = q/(q-1)$. Since $B : W^{-s-2,q'}(\mathbb{R}^m) \to W^{-s,q'}(\mathbb{R}^m)$ is bounded, we infer that $B' : W^{s,q}(\mathbb{R}^m, \mathbb{R}^m) \to W^{s+2,q}(\mathbb{R}^m, \mathbb{R}^m)$ is bounded. Since $\tilde{E}^\lambda(-\mathbf{x}) = \tilde{E}^\lambda(\mathbf{x})$ by (5.4) and $\tilde{E}_{ij} = \tilde{E}_{ji}$ by (5.7), Fubini's theorem yields that $B' = B$. □

7. Brinkman boundary layer potentials

Let now $\Omega \subset \mathbb{R}^m$ be an open set with compact Lipschitz boundary. If $1 < q < \infty$ and $\mathbf{g} \in L^q(\partial\Omega, \mathbb{R}^m)$ then the single-layer potential for the Brinkman system $E^\lambda_\Omega \mathbf{g}$ and its associated pressure potential $Q_\Omega \mathbf{g}$ are given by

$$E^\lambda_\Omega \mathbf{g}(x) := \int_{\partial\Omega} \tilde{E}^\lambda(x-y)\mathbf{g}(y)\, \mathrm{d}\sigma(y),$$

$$Q_\Omega \mathbf{g}(x) := \int_{\partial\Omega} Q(x-y)\mathbf{g}(y)\, \mathrm{d}\sigma(y).$$

More generally, if $\mathbf{g} = (g_1, \dots, g_m)$, where g_j are distributions supported on $\partial\Omega$ then we define

$$E^\lambda_\Omega \mathbf{g}(x) := \langle \mathbf{g}, \tilde{E}^\lambda(x - \cdot)\rangle, \quad Q_\Omega \mathbf{g}(x) := \langle \mathbf{g}, Q(x - \cdot)\rangle.$$

Remark that $(E^\lambda_\Omega \mathbf{g}, Q_\Omega \mathbf{g})$ is a solution of the Brinkman system (5.1) in the set $\mathbb{R}^m \setminus \partial\Omega$.

Denote

$$K^\lambda_\Omega(y,x) = -T_x(\tilde{E}^\lambda(x-y), Q(x-y))\mathbf{n}^\Omega(x),$$

where

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI, \quad \hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

is the stress tensor corresponding to a velocity $\mathbf{u}$ and a pressure p. Now we define a double layer potential. For $\mathbf{\Psi} \in L^q(\partial\Omega, \mathbb{R}^m)$ define in $\mathbb{R}^m \setminus \partial\Omega$

$$(D^\lambda_\Omega \mathbf{\Psi})(x) = \int_{\partial\Omega} K^\lambda_\Omega(x,y)\mathbf{\Psi}(y)\, \mathrm{d}\sigma(y), \tag{7.1}$$

and the corresponding pressure by

$$(\Pi^\lambda_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi^\lambda_\Omega(\mathbf{x}, \mathbf{y})\mathbf{\Psi}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}). \tag{7.2}$$

If $m > 2$ then

$$\Pi^\lambda_\Omega(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_m}\left\{-(\mathbf{y}-\mathbf{x})\frac{2m(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{m+2}} + \frac{2\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m} - \lambda\frac{|\mathbf{x}-\mathbf{y}|^{2-m}}{m-2}\mathbf{n}^\Omega(\mathbf{y})\right\},$$

where ω_m is the surface of the unit sphere. If $m = 2$ then

$$\Pi^\lambda_\Omega(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left\{-(\mathbf{y}-\mathbf{x})\frac{4(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{4}} + \frac{2\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m} - \lambda\left(\ln\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathbf{n}^\Omega(\mathbf{y})\right\}.$$

Remark that $D^\lambda_\Omega\mathbf{\Psi} \in \mathcal{C}^\infty(\mathbb{R}^m\setminus\partial\Omega,\mathbb{R}^m)$, $\Pi^\lambda_\Omega\mathbf{\Psi} \in \mathcal{C}^\infty(\mathbb{R}^m\setminus\partial\Omega,\mathbb{R}^1)$ and $\nabla\Pi^\lambda_\Omega\mathbf{\Psi} - \Delta D^\lambda_\Omega\mathbf{\Psi} + \lambda D^\lambda_\Omega\mathbf{\Psi} = 0$, $\nabla\cdot D^\lambda_\Omega\mathbf{\Psi} = 0$ in $\mathbb{R}^m\setminus\partial\Omega$.

Define

$$K_{\Omega,\lambda}\mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon\searrow 0}\int\limits_{\partial\Omega\setminus B(\mathbf{x};\epsilon)} K^\lambda_\Omega(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\sigma(\mathbf{y}),\quad \mathbf{x}\in\partial\Omega,$$

where $B(\mathbf{x};\epsilon) = \{\mathbf{y}\in\mathbb{R}^m; |\mathbf{x}-\mathbf{y}| < \epsilon\}$.

Lemma 7.1. *Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary, $\lambda \geq 0$, $1 < q < \infty$, $0 < s < 1$. Then $K_{\Omega,\lambda}$ is a bounded linear operator on $W^{s,q}(\partial\Omega,\mathbb{R}^m)$ and its adjoint operator $K'_{\Omega,\lambda}$ is a bounded linear operator on $W^{-s,q/(q-1)}(\partial\Omega,\mathbb{R}^m)$.*

(See [11, Lemma 3.1].)

Lemma 7.2. *Let $\Omega\subset\mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $\lambda\geq 0$, $1 < q < \infty$, $0 < s < 1$. If $s \neq 2$ suppose moreover that $s \neq 1 - 1/q$. Then $D^\lambda_\Omega : W^{s,q}(\partial\Omega,\mathbb{R}^m) \to W^{s+1/q,q}(\Omega,\mathbb{R}^m)$ is a bounded linear operator. If $\mathbf{\Phi}\in W^{s,q}(\partial\Omega,\mathbb{R}^m)$ then $\frac{1}{2}\mathbf{\Phi} + K_{\Omega,\lambda}\mathbf{\Phi}$ is the trace of $D^\lambda_\Omega\mathbf{\Phi}$.*

(See [11, Lemma 3.1].)

Proposition 7.3. *Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary, $\lambda > 0$, $1 < q < \infty$, $-1 < s < 0$. Then $E^\lambda_\Omega : W^{s,q}(\partial\Omega,\mathbb{R}^m) \to W^{s+1+1/q,q}(\mathbb{R}^m)$ is bounded.*

Proof. Put $q' = q/(q-1)$. Then

$$W^{s,q}(\partial\Omega,\mathbb{R}^m) \hookrightarrow [\mathring{W}^{1/q'-s,q'}(\mathbb{R}^m,\mathbb{R}^m)]' = W^{s-1/q',q}(\mathbb{R}^m,\mathbb{R}^m)$$

by [9, Chapter VI, Theorem 1] and [19, Theorem 3.18]. Since $s-1/q'+2 = s+1+1/q$, Proposition 6.1 gives the proposition. □

Lemma 7.4. *Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary, $1 < q < \infty$, $\mathbf{\Phi}\in W^{-1/q,q}(\partial\Omega,\mathbb{R}^m)$, $\lambda\geq 0$. Denote by $\mathcal{E}^\lambda_\Omega\mathbf{\Phi}$ the restricition of $E^\lambda_\Omega\mathbf{\Phi}$ onto $\partial\Omega$. Then $\mathcal{E}^\lambda_\Omega\mathbf{\Phi}$ is the trace of $E^\lambda_\Omega\mathbf{\Phi}$. If $h\in L^\infty(\partial\Omega)$, then $(\mathbf{u},p) := (E^\lambda_\Omega\mathbf{\Phi}, Q_\Omega\mathbf{\Phi})$ is a solution of the Robin problem (3.1) with $\mathbf{f}\equiv 0$ and $\mathbf{g} = \frac{1}{2}\mathbf{\Phi} - K'_{\Omega,\lambda}\mathbf{\Phi} + h\mathcal{E}^\lambda_\Omega\mathbf{\Phi}$. Moreover, $\mathcal{E}^\lambda_\Omega : W^{-1/q,q}(\partial\Omega,\mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega,\mathbb{R}^m)$ is a bounded operator.*

(See [11, Lemma 3.1].)

Proposition 7.5. *Let $\Omega\subset\mathbb{R}^m$ be an open set with compact Lipschitz boundary, $1 < q < \infty$, $\lambda\geq 0$, $0 < s < 1$. Suppose that one from the following conditions is fulfilled:*

(1) $q = 2$.
(2) $\partial\Omega$ *is of class* $\mathcal{C}^1$.
(3) $2 \leq m \leq 3$ *and* $3/2 \leq q \leq 3$.

Then $\frac{1}{2}I \pm K_{\Omega,\lambda}$ *are Fredholm operators with index* 0 *in* $W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ *and in* $W^{s,2}(\partial\Omega;\mathbb{R}^m)$*, and* $\frac{1}{2}I \pm K'_{\Omega,\lambda}$ *in* $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ *are Fredholm operators with index* 0*.*

Proof. Denote $q' = q/(q-1)$. If $\partial\Omega$ is of class $\mathcal{C}^1$ then $K_{\Omega,0}$ is a compact operator on $W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ and on $W^{1-1/q',q'}(\partial\Omega;\mathbb{R}^m)$ by [16, p. 232]. Therefore $K'_{\Omega,0}$ is a compact operator on $[W^{1-1/q',q'}(\partial\Omega;\mathbb{R}^m)]' = W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$. Hence $\frac{1}{2}I \pm K_{\Omega,0}$ are Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$, and $\frac{1}{2}I \pm K'_{\Omega,0}$ are Fredholm operators with index 0 in $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$.

$\frac{1}{2}I \pm K_{\Omega,0}$ are Fredholm operators with index 0 in $W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ and in $W^{s,2}(\partial\Omega;\mathbb{R}^m)$, and $\frac{1}{2}I \pm K'_{\Omega,0}$ are Fredholm operators with index 0 in the space $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ in the other cases by [20, Theorem 10.5.3].

$K_{\Omega,\lambda} - K_{\Omega,0}$ is a compact operator in $W^{1-1/q,q}(\partial\Omega;\mathbb{R}^m)$ and in $W^{s,2}(\partial\Omega;\mathbb{R}^m)$, $K'_{\Omega,\lambda} - K'_{\Omega,0}$ is a compact operator in $W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ by [11, Theorem 3.1]. This gives the proposition. □

8. Integral representation

The following lemma is well known for classical solutions of the Neumann problem for the Brinkman system.

Lemma 8.1. *Let* $\Omega \subset \mathbb{R}^m$ *be a bounded open set with Lipschitz boundary. Let* $\lambda \geq 0$*,* $1 < q < \infty$*,* $\mathbf{f} \equiv 0$*,* $\mathbf{g} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$*,* $h \equiv 0$*. If* $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ *is a solution of the Neumann problem* (3.1) *then*

$$D^\lambda_\Omega \mathbf{u}(\mathbf{x}) + E^\lambda_\Omega \mathbf{g}(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \notin \overline{\Omega}, \end{cases} \tag{8.1}$$

$$\Pi^\lambda_\Omega \mathbf{u}(\mathbf{x}) + Q_\Omega \mathbf{g}(\mathbf{x}) = \begin{cases} p(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \notin \overline{\Omega}. \end{cases} \tag{8.2}$$

Proof. If $\mathbf{x} \notin \overline{\Omega}$ then (8.1), (8.2) are an easy consequence of the Green formula. (See the proof of the lemma for classical solutions of the Robin problem in [29].)

Let now $\mathbf{x} \in \Omega$. Put $\omega := \Omega \setminus \overline{B(\mathbf{x}; r)}$. Define $\mathbf{g} = T(\mathbf{u}, p)\mathbf{n}^\omega$ on $\partial\omega \setminus \partial\Omega$. Then

$$D^\lambda_\omega \mathbf{u}(\mathbf{x}) + E^\lambda_\omega \mathbf{g}(\mathbf{x}) = 0, \quad \Pi^\lambda_\omega \mathbf{u}(\mathbf{x}) + Q_\omega \mathbf{g}(\mathbf{x}) = 0. \tag{8.3}$$

[29, p. 60] gives

$$D^\lambda_{B(\mathbf{x};r)} \mathbf{u}(\mathbf{x}) - E^\lambda_{B(\mathbf{x};r)} \mathbf{g}(\mathbf{x}) = \mathbf{u}(\mathbf{x}), \quad \Pi^\lambda_{B(\mathbf{x};r)} \mathbf{u}(\mathbf{x}) - Q_{B(\mathbf{x};r)} \mathbf{g}(\mathbf{x}) = p(\mathbf{x}). \tag{8.4}$$

Adding (8.3) and (8.4) we obtain (8.1), (8.2). □

9. Robin problem for the Brinkman system

First we study the problem (3.1) with $\mathbf{f} \equiv 0$ and $\mathbf{g} \in W^{1/q-1,q}(\partial\Omega, \mathbb{R}^m)$. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary. Let $G(1), \dots, G(k)$ be all bounded components of $\mathbb{R}^m \setminus \overline{\Omega}$. Fix open balls $B(j)$ such that $\overline{B}(j) \subset G(j)$. Choose $\mathbf{\Psi}_j \in W^{1,\infty}(\partial G(j), \mathbb{R}^m)$ such that

$$\int_{\partial G(j)} \mathbf{\Psi}_j \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma \neq 0. \tag{9.1}$$

Define $\mathbf{\Psi}_j = 0$ on $\partial\Omega \setminus \partial G(j)$. If $\mathbf{\Phi} \in W^{1/q-1,q}(\partial\Omega, \mathbb{R}^m)$ we define the modified Brinkman single layer potential by

$$\grave{E}^\lambda_\Omega \mathbf{\Phi} := E^\lambda_\Omega \mathbf{\Phi} + \sum_{j=1}^k \langle \mathbf{\Phi}, \mathbf{\Psi}_j \rangle D^\lambda_{B(j)} \mathbf{n}^{B(j)}, \tag{9.2}$$

$$\grave{Q}^\lambda_\Omega \mathbf{\Phi} := Q^\lambda_\Omega \mathbf{\Phi} + \sum_{j=1}^k \langle \mathbf{\Phi}, \mathbf{\Psi}_j \rangle \Pi^\lambda_{B(j)} \mathbf{n}^{B(j)}. \tag{9.3}$$

(If $\partial\Omega$ is connected then $\grave{E}^\lambda_\Omega \mathbf{\Phi} = E^\lambda_\Omega \mathbf{\Phi}$, $\grave{Q}^\lambda_\Omega \mathbf{\Phi} = Q^\lambda_\Omega \mathbf{\Phi}$.) Proposition 7.3 and Lemma 7.4 give that $(\grave{E}^\lambda_\Omega \mathbf{\Phi}, \grave{Q}^\lambda_\Omega \mathbf{\Phi})$ is a solution of the Robin problem (3.1) if and only if $\tau^\lambda_{\Omega,h} \mathbf{\Phi} = \mathbf{g}$ where

$$\tau^\lambda_{\Omega,h} \mathbf{\Phi} := \frac{1}{2} \mathbf{\Phi} - K'_{\Omega,\lambda} \mathbf{\Phi} + \sum_{j=1}^k \langle \mathbf{\Phi}, \mathbf{\Psi}_j \rangle T(D^\lambda_{B(j)} \mathbf{n}^{B(j)}, \Pi^\lambda_{B(j)} \mathbf{n}^{B(j)}) \mathbf{n}^\Omega + h \grave{E}^\lambda_\Omega \mathbf{\Phi}.$$

Lemma 9.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 < q < \infty$, $\lambda > 0$, $h \in L^\infty(\partial\Omega)$. Suppose that one from the following conditions is fulfilled:*

a) $q = 2$.
b) $\partial\Omega$ is of class $\mathcal{C}^1$.
c) $2 \le m \le 3$ and $3/2 \le q \le 3$.

Let $(\mathbf{u}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ be a weak solution of (3.2).

(1) If $q = 2$ then

$$\langle \mathbf{F}, \mathbf{u} \rangle = \int_\Omega [2|\hat{\nabla} \mathbf{u}|^2 + \lambda |\mathbf{u}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathbf{u}|^2 \mathrm{d}\sigma. \tag{9.4}$$

(2) If $h \ge 0$ and $\mathbf{F} \equiv 0$ then $\mathbf{u} \equiv 0$, $p \equiv 0$.

Proof. Suppose first that $q = 2$. The definition of the weak solution of the Robin problem and the density of $\mathcal{C}^\infty_c(\Omega, \mathbb{R}^m)$ in $W^{1,2}(\Omega, \mathbb{R}^m)$ give (9.4).

Let now $h \ge 0$ and $\mathbf{F} \equiv 0$. Since $T(\mathbf{u}, p)\mathbf{n}^\Omega = -h\mathbf{u}$, Lemma 8.1 gives

$$\mathbf{u} = D^\lambda_\Omega \mathbf{u} - E^\lambda_\Omega (h\mathbf{u}), \quad p = \Pi^\lambda_\Omega \mathbf{u} - Q_\Omega(h\mathbf{u}) \quad \text{in } \Omega. \tag{9.5}$$

For the trace of $\mathbf{u}$ we obtain from Lemma 7.2 and Lemma 7.4

$$\mathbf{u} = \frac{1}{2} \mathbf{u} + K_{\Omega,\lambda} \mathbf{u} - \mathcal{E}^\lambda_\Omega h \mathbf{u} \quad \text{on } \partial\Omega.$$

Hence $H\mathbf{u} = 0$, where $H\mathbf{v} = \frac{1}{2}\mathbf{v} - K_{\Omega,\lambda}\mathbf{v} + \mathcal{E}^\lambda_\Omega h \mathbf{v}$. The operator $\frac{1}{2} I - K_{\Omega,\lambda}$ is a Fredholm operator with index 0 in $W^{1-1/q,q}(\Omega, \mathbb{R}^m)$, in $W^{1-1/q,2}(\Omega, \mathbb{R}^m)$ and in $W^{1/2,2}(\Omega, \mathbb{R}^m)$ by Proposition 7.5. The operator $\mathbf{v} \mapsto \mathcal{E}^\lambda_\Omega h \mathbf{v}$ is a compact operator in $W^{1-1/q,q}(\Omega, \mathbb{R}^m)$, in $W^{1-1/q,2}(\Omega, \mathbb{R}^m)$ and in $W^{1/2,2}(\Omega, \mathbb{R}^m)$ by [11, Lemma 3.1]. So, H is a Fredholm operator with index 0 in $W^{1-1/q,q}(\Omega, \mathbb{R}^m)$, in $W^{1-1/q,2}(\Omega, \mathbb{R}^m)$ and in $W^{1/2,2}(\Omega, \mathbb{R}^m)$. [17, Lemma 9] gives that $\mathbf{u} \in W^{1/2,2}(\partial\Omega; \mathbb{R}^m)$. According to [11, Lemma 3.1] one has $D^\lambda_\Omega \mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^m)$, $\Pi^\lambda_\Omega \mathbf{u} \in L^2(\Omega)$. The representation (9.5), Proposition 7.3 and [11, Lemma 3.1] give that $(\mathbf{u}, p) \in W^{1,2}(\Omega, \mathbb{R}^m) \times L^2(\Omega)$. Thus

$$0 = \langle \mathbf{F}, \mathbf{u} \rangle = \int_\Omega [|\hat{\nabla} \mathbf{u}|^2 + \lambda |\mathbf{u}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathbf{u}|^2 \mathrm{d}\sigma.$$

Hence $\mathbf{u} \equiv 0$. Since $\nabla p = \Delta \mathbf{u} - \lambda \mathbf{u} \equiv 0$, there exists a constant c such that $p \equiv 0$. So, $(\mathbf{u}, p)$ is a classical solution of the Robin problem (3.1). So, $0 = T(\mathbf{u}, \mathbf{p})\mathbf{n}^\Omega + h\mathbf{u} = -c\mathbf{n}^\Omega$. Hence $c = 0$. □

Theorem 9.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 < q < \infty$, $\lambda > 0$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that one from the following conditions is fulfilled:*

(1) $q = 2$.
(2) $\partial\Omega$ is of class $\mathcal{C}^1$.
(3) $2 \leq m \leq 3$ and $3/2 \leq q \leq 3$.

Then $\tau^\lambda_{\Omega,h}$ is an isomorphism in $W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$. If $\mathbf{g} \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ then

$$(\mathbf{u}, p) := (\grave{E}^\lambda_\Omega (\tau^\lambda_{\Omega,h})^{-1}\mathbf{g}, \grave{Q}^\lambda_\Omega (\tau^\lambda_{\Omega,h})^{-1}\mathbf{g}) \tag{9.6}$$

is a unique solution in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of the Robin problem (3.1) with $\mathbf{f} \equiv 0$. Moreover,

$$\|\mathbf{u}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|p\|_{L^q(\Omega)} \leq C\|\mathbf{g}\|_{W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)} \tag{9.7}$$

where a constant C does not depend on $\mathbf{g}$.

Proof. $\mathcal{E}^\lambda_\Omega : W^{-1/q,q}(\partial\Omega, \mathbb{R}^m) \to W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m) \hookrightarrow L^q(\partial\Omega, \mathbb{R}^m)$ by Lemma 7.4. $L^q(\partial\Omega, \mathbb{R}^m) \hookrightarrow W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$ compactly by [28, Theorem 1.97], [27, §2.5.7, Proposition] and [25, §2.3.2, Proposition 2]. Thus $\tau^\lambda_{\Omega,h} - [\frac{1}{2}I - K'_{\Omega,\lambda}]$ is a compact operator in $W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$. Since $\frac{1}{2}I - K'_{\Omega,\lambda}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ by Proposition 7.5, we infer that $\tau^\lambda_{\Omega,h}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$.

The uniqueness of a solution of the problem (3.1) in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ follows from Lemma 9.1. Let $\mathbf{\Phi} \in W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$, $\tau^\lambda_{\Omega,h}\mathbf{\Phi} = 0$. Then $(\mathbf{u}, p) := (\grave{E}^\lambda_\Omega \mathbf{\Phi}, \grave{Q}^\lambda_\Omega \mathbf{\Phi})$ is a weak solution of the Robin problem (3.1) in $W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ with $\mathbf{f} \equiv 0$, $\mathbf{g} \equiv 0$. So, $\mathbf{u} = 0$ in Ω, $p = 0$ in Ω. The trace of $\mathbf{u}$ is equal to

$$\mathcal{E}^\lambda_\Omega \mathbf{\Phi} + \sum_{j=1}^k \langle \mathbf{\Phi}, \mathbf{\Psi}_j \rangle D^\lambda_{B(j)} \mathbf{n}^{B(j)} = 0 \tag{9.8}$$

by Lemma 7.4. Since $\nabla \cdot \mathcal{E}^\lambda_\Omega \mathbf{\Phi} = 0$, $\nabla \cdot D^\lambda_{B(j)} \mathbf{n}^{B(j)} = 0$ in $G(i)$ for $j \neq i$, Green's formula gives

$$\int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathcal{E}^\lambda_\Omega \mathbf{\Phi} \, d\sigma = 0, \quad \int_{\partial G(i)} \mathbf{n}^\Omega \cdot D^\lambda_{B(j)} \mathbf{n}^{B(j)} \, d\sigma = 0, \quad j \neq i.$$

This and (9.8) give

$$\langle \mathbf{\Phi}, \mathbf{\Psi}_i \rangle \int_{\partial G(i)} \mathbf{n}^\Omega \cdot D^\lambda_{B(i)} \mathbf{n}^{B(i)} \, d\sigma = 0. \tag{9.9}$$

Using [18, Proposition 7.2] on $B(i)$ and $G(i) \setminus \overline{B(i)}$

$$\int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[\frac{1}{2}\mathbf{n}^{B(i)} + K_{B(i),\lambda} \mathbf{n}^{B(i)} \right] \, d\sigma = 0,$$

$$\int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[\frac{1}{2}\mathbf{n}^{B(i)} - K_{B(i),\lambda} \mathbf{n}^{B(i)} \right] \, d\sigma + \int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^\lambda_{B(i)} \mathbf{n}^{B(i)} \, d\sigma = 0.$$

Adding

$$\int_{\partial G(i)} \mathbf{n}^{G(i)} \cdot D^{\lambda}_{B(i)} \mathbf{n}^{B(i)} \, \mathrm{d}\sigma = - \int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \mathbf{n}^{B(i)} \, \mathrm{d}\sigma \neq 0.$$

This and (9.9) give $\langle \mathbf{\Phi}, \mathbf{\Psi}_i \rangle = 0$. So,

$$0 = (\mathbf{u}, p) = (\grave{E}^{\lambda}_{\Omega} \mathbf{\Phi}, \grave{Q}^{\lambda}_{\Omega} \mathbf{\Phi}) = (E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi}) \quad \text{in } \Omega.$$

Hence $\mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi} = 0$ on $\partial\Omega$ by Lemma 7.4. Since $\tau^{\lambda}_{\Omega,h}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega; \mathbb{R}^m)$ and in $W^{-1/2,2}(\partial\Omega; \mathbb{R}^m)$, [17, Lemma 9] gives that $\mathbf{\Phi} \in W^{-1/2,2}(\partial\Omega; \mathbb{R}^m)$. Thus $E^{\lambda}_{\Omega} \mathbf{\Phi} \in W^{1,2}(\mathbb{R}^m; \mathbb{R}^m)$ by Proposition 7.3 and $Q_{\Omega} \mathbf{\Phi} \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ by [20, Theorem 10.5.1]. For a fixed $i \in \{1, \dots, k\}$ there exist $\mathbf{F} \in [W^{1,2}(G(i); \mathbb{R}^m)]'$ such that $(E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi})$ is a weak solution of the Robin problem (3.2) in $G(i)$. Since $(E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi})$ is a solution of the homogeneous Brinkman system in $G(i)$, we infer that $\mathbf{F}$ is supported on $\partial G(i)$. Since $\mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi} = 0$ on $\partial G(i)$, Lemma 9.1 gives

$$0 = \langle \mathbf{F}, \mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi} \rangle = \int_{G(i)} [2 |\hat{\nabla} E^{\lambda}_{\Omega} \mathbf{\Phi}|^2 + \lambda |E^{\lambda}_{\Omega} \mathbf{\Phi}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial G(i)} h |\mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi}|^2 \, \mathrm{d}\sigma.$$

Hence $E^{\lambda}_{\Omega} \mathbf{\Phi} = 0$ in $G(i)$. So,

$$\nabla Q_{\Omega} \mathbf{\Phi} = \Delta E^{\lambda}_{\Omega} \mathbf{\Phi} - \lambda E^{\lambda}_{\Omega} \mathbf{\Phi} = 0$$

in $G(i)$. Therefore there exists a constant c_i such that $Q_{\Omega} \mathbf{\Phi} = c_i$ on $G(i)$. Denote by $G(0)$ the unbounded component of $\mathbb{R}^m \setminus \overline{\Omega}$. Put $h = 0$ on $\mathbb{R}^m \setminus \partial\Omega$. For $r > 0$ denote $\omega(r) := G(0) \cap B(0; r)$. For a fixed $r > 0$ there exist $\mathbf{F} \in [W^{1,2}(\omega(r); \mathbb{R}^m)]'$ such that $(E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi})$ is a weak solution of the Robin problem (3.2) in $\omega(r)$. Since $(E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi})$ is a solution of the homogeneous Brinkman system in $\omega(r)$, we infer that $\mathbf{F}$ is supported on $\partial\omega(r)$. Lemma 9.1 gives

$$\langle \mathbf{F}, \mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi} \rangle = \int_{\omega(r)} [2 |\hat{\nabla} E^{\lambda}_{\Omega} \mathbf{\Phi}|^2 + \lambda |E^{\lambda}_{\Omega} \mathbf{\Phi}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial\omega(r)} h |E^{\lambda}_{\Omega} \mathbf{\Phi}|^2 \, \mathrm{d}\sigma.$$

Since $h = 0$ on $\partial\omega(r) \setminus \partial\Omega$ and $E^{\lambda}_{\Omega} \mathbf{\Phi} = 0$ on $\partial\Omega$, we obtain

$$\int_{\omega(r)} [2 |\hat{\nabla} E^{\lambda}_{\Omega} \mathbf{\Phi}|^2 + \lambda |E^{\lambda}_{\Omega} \mathbf{\Phi}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi}|^2 \, \mathrm{d}\sigma = \int_{\partial B(0;r)} (E^{\lambda}_{\Omega} \mathbf{\Phi}) T(E^{\lambda}_{\Omega} \mathbf{\Phi}, Q_{\Omega} \mathbf{\Phi}) \mathbf{n}.$$

Letting $r \to \infty$ we obtain by (5.10)

$$\int_{G(0)} [2 |\hat{\nabla} E^{\lambda}_{\Omega} \mathbf{\Phi}|^2 + \lambda |E^{\lambda}_{\Omega} \mathbf{\Phi}|^2] \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h |\mathcal{E}^{\lambda}_{\Omega} \mathbf{\Phi}|^2 \, \mathrm{d}\sigma = 0.$$

Hence $E^{\lambda}_{\Omega} \mathbf{\Phi} = 0$ in $G(0)$. So,

$$\nabla Q_{\Omega} \mathbf{\Phi} = \Delta E^{\lambda}_{\Omega} \mathbf{\Phi} - \lambda E^{\lambda}_{\Omega} \mathbf{\Phi} = 0$$

in $G(0)$. Therefore there exists a constant c_0 such that $Q_{\Omega} \mathbf{\Phi} = c_0$ on $G(0)$. Since $Q_{\Omega} \mathbf{\Phi}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, we infer that $c_0 = 0$. Using Lemma 7.4 on Ω and on $G(i)$ we infer that $(\frac{1}{2} - K'_{\Omega,\lambda}) \mathbf{\Phi} = 0$, $(\frac{1}{2} + K'_{\Omega,\lambda}) \mathbf{\Phi} = -c(i) \mathbf{n}^{\Omega}$ on $\partial G(i)$. So,

$$\mathbf{\Phi} = \left(\frac{1}{2} - K'_{\Omega,\lambda} \right) \mathbf{\Phi} + \left(\frac{1}{2} + K'_{\Omega,\lambda} \right) \mathbf{\Phi} = -c(i) \mathbf{n}^{\Omega} \qquad \text{on } \partial G(i).$$

We have proved for $i \in \{1, \dots, k\}$ that $\langle \mathbf{\Phi}, \mathbf{\Psi}_i \rangle = 0$. So, (9.1) gives that $c(i) = 0$. Hence $\mathbf{\Phi} \equiv 0$. Since $\tau^{\lambda}_{\Omega,h}$ is a Fredholm operator with index 0 in $W^{-1/q,q}(\partial\Omega, \mathbb{R}^m)$ and trivial kernel, it is an isomorphism.

If $\mathbf{g} \in W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ then $(\mathbf{u}, p)$ given by (9.6) is a unique solution in $W^{1,q}(\Omega,\mathbb{R}^m) \times L^q(\Omega)$ of the Robin problem (3.1) with $\mathbf{f} \equiv 0$. The estimate (9.7) is a consequence of Proposition 7.3. □

Theorem 9.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $1 < q < \infty$, $q' = q/(q-1)$, $\lambda > 0$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Suppose that one from the following conditions is fulfilled:*

(1) $q = 2$.
(2) $\partial\Omega$ is of class $\mathcal{C}^1$.
(3) $2 \leq m \leq 3$ and $3/2 \leq q \leq 3$.

If $\mathbf{F} \in [W^{1,q'}(\partial\Omega;\mathbb{R}^m)]$ then there exists a unique solution $(\mathbf{u}, p) \in W^{1,q}(\Omega,\mathbb{R}^m) \times L^q(\Omega)$ of the Robin problem (3.2). Moreover,

$$\|\mathbf{u}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|p\|_{L^q(\Omega)} \leq C\|\mathbf{F}\|_{[W^{1,q'}(\Omega;\mathbb{R}^m)]'} \tag{9.10}$$

where a constant C does not depend on $\mathbf{F}$.

Proof. Define $\langle\tilde{\mathbf{F}}, \boldsymbol{\Psi}\rangle := \langle\mathbf{F}, \boldsymbol{\Psi}\rangle$ for $\boldsymbol{\Psi} \in \mathring{W}^{1,q'}(\mathbb{R}^m,\mathbb{R}^m)$. Then $\tilde{\mathbf{F}} \in W^{-1,q}(\mathbb{R}^m,\mathbb{R}^m)$ and

$$\|\tilde{\mathbf{F}}\|_{W^{-1,q}(\mathbb{R}^m,\mathbb{R}^m)} \leq \|\mathbf{F}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}. \tag{9.11}$$

According to Proposition 4.2 there exists $(\tilde{\mathbf{u}}, \tilde{p}) \in W^{1,q}(\mathbb{R}^m,\mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ such that

$$\nabla\tilde{p} - \Delta\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} = \tilde{\mathbf{F}}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \mathbb{R}^m$$

and

$$\|\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|\tilde{p}\|_{L^q(\Omega)} \leq C_1\|\tilde{\mathbf{F}}\|_{W^{-1,q}(\mathbb{R}^m,\mathbb{R}^m)} \tag{9.12}$$

where C_1 does not depend on $\tilde{\mathbf{F}}$. Clearly, there exists $\mathbf{G} \in [W^{1,q'}(\Omega,\mathbb{R}^m)]'$ such that $(\tilde{\mathbf{u}}, p)$ is a solution of the Robin problem

$$\nabla\tilde{p} - \Delta\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} = \mathbf{G} \quad \text{in} \quad \Omega, \qquad \nabla\cdot\tilde{\mathbf{u}} = 0 \quad \text{in } \Omega,$$
$$T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} + h\tilde{\mathbf{u}} = \mathbf{G} \qquad \text{on } \partial\Omega.$$

Moreover,

$$\|\mathbf{G}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'} \leq C_2\left[\|\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|\tilde{p}\|_{L^q(\Omega)}\right] \tag{9.13}$$

where C_2 does not depend on $\tilde{\mathbf{u}}$ and $\tilde{p}$. Since $\tilde{\mathbf{F}} = \mathbf{F}$ in Ω, we infer that $\mathbf{F} - \mathbf{G}$ is supported on $\partial\Omega$. Using [6, Theorem 1.5.1.2] we deduce that $\mathbf{F} - \mathbf{G} \in [W^{1-1/q',q'}(\partial\Omega;\mathbb{R}^m)]' = W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)$ and

$$\|\mathbf{F} - \mathbf{G}\|_{W^{-1/q,q}(\partial\Omega;\mathbb{R}^m)} \leq C_3\|\mathbf{F} - \mathbf{G}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'} \tag{9.14}$$

where C_3 does not depend on $\mathbf{F}$ and $\mathbf{G}$. According to Theorem 9.2 there exists a solution $(\hat{\mathbf{u}}, \hat{p}) \in W^{1,q}(\mathbb{R}^m,\mathbb{R}^m) \times L^q_{\text{loc}}(\mathbb{R}^m)$ of the problem

$$\nabla\hat{p} - \Delta\hat{\mathbf{u}} + \lambda\hat{\mathbf{u}} = 0 \quad \text{in} \quad \Omega, \qquad \nabla\cdot\tilde{\mathbf{u}} = 0 \quad \text{in } \Omega,$$
$$T(\hat{\mathbf{u}}, \hat{p})\mathbf{n} + h\hat{\mathbf{u}} = \mathbf{F} - \mathbf{G} \qquad \text{on } \partial\Omega.$$

Moreover,

$$\|\hat{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|\hat{p}\|_{L^q(\Omega)} \leq C_4\|\mathbf{F} - \mathbf{G}\|_{W^{-1/q,q}(\partial\Omega,\mathbb{R}^m)} \tag{9.15}$$

where C_4 does not depend on $\mathbf{F}$ and $\mathbf{G}$. Put $\mathbf{u} := \tilde{\mathbf{u}} + \hat{\mathbf{u}}$, $p := \tilde{p} + \hat{p}$. Then $(\mathbf{u}, p) \in W^{1,q}(\Omega,\mathbb{R}^m) \times L^q(\Omega)$ is a solution of the Robin problem (3.2). This solution is unique by Theorem 9.2. The estimate (9.10) is a consequence of (9.11), (9.12), (9.13), (9.14) and (9.15). □

10. Robin problem for the Darcy-Forchheimer-Brinkman system

In this section we study the Robin problem for the Darcy-Forchheimer-Brinkman system

$$\nabla p - \Delta \mathbf{u} + \lambda \mathbf{u} + \alpha |\mathbf{u}|\mathbf{u} + \beta(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{G} \quad \text{in} \quad \Omega, \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{10.1a}$$

$$T(\mathbf{u},p)\mathbf{n} + h\mathbf{u} = \mathbf{G} \qquad \text{on } \partial\Omega \tag{10.1b}$$

in $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ for Ω bounded. Denote

$$L_{\alpha,\beta}\mathbf{u} := \alpha|\mathbf{u}|\mathbf{u} + \beta(\mathbf{u}\cdot\nabla)\mathbf{u}.$$

We restrict ourselves to such q for which $L_{\alpha,\beta}\mathbf{u} \in L^1(\Omega,\mathbb{R}^m)\cap[W^{1,q'}(\Omega,\mathbb{R}^m)]'$ for $\mathbf{u}\in W^{1,q'}(\Omega,\mathbb{R}^m)$ and $q' = q/(q-1)$. If $\alpha,\beta\in\mathbb{R}^1$, $h\in L^\infty(\Omega)$, $\mathbf{G}\in[W^{1,q'}(\Omega,\mathbb{R}^m)]'$ then $(\mathbf{u},p)\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ is a weak solution of the Robin problem for the Darcy-Forchheimer-Brinkman system (10.1) if $\nabla\cdot\mathbf{u}=0$ in Ω and

$$\langle \mathbf{G},\mathbf{\Phi}\rangle = \int_\Omega \{2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi} - p(\nabla\cdot\mathbf{\Phi}) + \mathbf{\Phi}\cdot[\lambda\mathbf{u}+\alpha|\mathbf{u}|\mathbf{u}+\beta(\mathbf{u}\cdot\nabla)\mathbf{u}]\}\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} h\mathbf{u}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma$$

for all $\mathbf{\Phi}\in\mathcal{C}_c^\infty(\mathbb{R}^m,\mathbb{R}^m)$ (or equivalently for all $\mathbf{\Phi}\in W^{1,q'}(\mathbb{R}^m,\mathbb{R}^m)$).

Theorem 10.1. *Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with Lipschitz boundary, $2\le m\le 3$. Let $1<q<\infty$, $q'=q/(q-1)$, $\lambda>0$, $\alpha,\beta\in\mathbb{R}^1$, $h\in L^\infty(\partial\Omega)$, $h\ge 0$. Suppose that one from the following conditions is fulfilled:*

(1) $3/2 < q \le 3$.
(2) $q = 3/2$ *and* $m = 2$.
(3) $q = 3/2$ *and* $\beta = 0$.
(4) $\partial\Omega$ *is of class* $\mathcal{C}^1$, $m=2$ *and* $\beta=0$.
(5) $\partial\Omega$ *is of class* $\mathcal{C}^1$, $m=3$, $\beta=0$ *and* $q>6/5$.
(6) $\partial\Omega$ *is of class* $\mathcal{C}^1$ *and* $\frac{6-m}{5-m}<q$.

Then the following hold:

- $L_{\alpha,\beta}\mathbf{u}\in L^1(\Omega,\mathbb{R}^m)\cap[W^{1,q'}(\Omega,\mathbb{R}^m)]'$ *for all* $\mathbf{u}\in W^{1,q'}(\Omega,\mathbb{R}^m)$.
- *There exist* $\delta,\epsilon,C\in(0,\infty)$ *such that the following holds: If*

$$\mathbf{G}\in[W^{1,q'}(\Omega,\mathbb{R}^m)]', \quad \|\mathbf{G}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}<\delta, \tag{10.2}$$

then there exists a unique weak solution $(\mathbf{u},p)\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ *of the Robin problem for the Darcy-Forchheimer-Brinkman system (10.1) such that*

$$\|\mathbf{u}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}<\epsilon. \tag{10.3}$$

If $\mathbf{G},\tilde{\mathbf{G}}\in[W^{1,q'}(\Omega,\mathbb{R}^m)]'$, $(\mathbf{u},p),(\tilde{\mathbf{u}},\tilde p)\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$, (10.3), (10.1), $\|\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}<\epsilon$,

$$\nabla\tilde p - \Delta\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} + \alpha|\tilde{\mathbf{u}}|\tilde{\mathbf{u}} + \beta(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} = \tilde{\mathbf{G}} \quad \textit{in} \quad \Omega, \qquad \nabla\cdot\tilde{\mathbf{u}}=0 \quad \textit{in } \Omega, \tag{10.4a}$$

$$T(\tilde{\mathbf{u}},\tilde p)\mathbf{n} + h\tilde{\mathbf{u}} = \tilde{\mathbf{G}} \qquad \textit{on } \partial\Omega, \tag{10.4b}$$

then

$$\|\mathbf{u}\|_{W^{1,q}(\Omega,\mathbb{R}^m)} + \|p\|_{L^q(\Omega)} \le C\|\mathbf{G}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}, \tag{10.5}$$

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}+\|p-\tilde{p}\|_{L^q(\Omega)}\leq C\|\mathbf{G}-\tilde{\mathbf{G}}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}. \tag{10.6}$$

Proof. According to Lemma 11.2 and Lemma 11.3 there exists a constant C_1 such that if $\mathbf{u},\tilde{\mathbf{u}}\in W^{1,q'}(\Omega,\mathbb{R}^m)$ then $L_{\alpha,\beta}\mathbf{u}\in L^1(\Omega,\mathbb{R}^m)\cap[W^{1,q'}(\Omega,\mathbb{R}^m)]'$ and

$$\|L_{\alpha,\beta}\mathbf{u}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}\leq C_1\|\mathbf{u}\|^2_{W^{1,q}(\Omega,\mathbb{R}^m)}, \tag{10.7}$$

$$\|L_{\alpha,\beta}\mathbf{u}-L_{\alpha,\beta}\tilde{\mathbf{u}}\|_{[W^{1,q'}(\Omega)]'}\leq C_1\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega)}[\|\mathbf{u}\|_{W^{1,q}(\Omega)}+\|\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega)}] \tag{10.8}$$

because

$$L_{\alpha,\beta}\mathbf{u}-L_{\alpha,\beta}\tilde{\mathbf{u}}=\alpha|\mathbf{u}|(\mathbf{u}-\tilde{\mathbf{u}})+\beta(\mathbf{u}\cdot\nabla)(\mathbf{u}-\tilde{\mathbf{u}})+\alpha(|\mathbf{u}|-|\tilde{\mathbf{u}}|)\tilde{\mathbf{u}}+\beta[(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla]\tilde{\mathbf{u}}.$$

According to Theorem 9.3 there exists a constant C_2 such that for each $\mathbf{F}\in[W^{1,q'}(\partial\Omega;\mathbb{R}^m)]$ there exists a unique solution $(\mathbf{u},p)\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ of the Robin problem (3.2) and

$$\|\mathbf{u}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}+\|p\|_{L^q(\Omega)}\leq C_2\|\mathbf{F}\|_{[W^{1,q'}(\Omega;\mathbb{R}^m)]'}. \tag{10.9}$$

Remark that $(\mathbf{u},p)$ is a solution of (10.1) if $(\mathbf{u},p)$ is a solution of (3.2) with $\mathbf{F}=\mathbf{G}-L_{\alpha,\beta}\mathbf{u}$. Put

$$\epsilon:=\frac{1}{4(C_1+1)(C_2+1)},\quad \delta:=\frac{\epsilon}{2(C_2+1)}.$$

If $(\mathbf{u},p),(\tilde{\mathbf{u}},\tilde{p})\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ are solution of (10.1) and (10.4) with (10.3) and $\|\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}<\epsilon$, then

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}+\|p-\tilde{p}\|_{L^q(\Omega)}\leq C_2[\|\mathbf{G}-\tilde{\mathbf{G}}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}$$

$$+\|L_{\alpha,\beta}\mathbf{u}-L_{\alpha,\beta}\tilde{\mathbf{u}}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}]\leq C_2[\|\mathbf{G}-\tilde{\mathbf{G}}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}+2\epsilon C_1\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}].$$

Since $2C_1C_2\epsilon<1/2$ we get subtracting $2\epsilon C_1C_2\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}$ from the both sides

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}+\|p-\tilde{p}\|_{L^q(\Omega)}\leq 2C_2\|\mathbf{G}-\tilde{\mathbf{G}}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}.$$

Therefore a solution of (10.1) satisfying (10.3) is unique. Putting $\tilde{p}\equiv 0$, $\tilde{\mathbf{u}}\equiv 0$, $\tilde{\mathbf{G}}\equiv 0$ we obtain (10.5) with $C=2C_2$.

Put $X:=\{\mathbf{v}\in W^{1,q}(\Omega,\mathbb{R}^m);\|\mathbf{v}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}\leq\epsilon\}$. Fix $\mathbf{G}$ satisfying (10.2). For $\mathbf{v}\in X$ there exists a unique solution $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})\in W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ of (3.2) with $\mathbf{F}=\mathbf{G}-L_{\alpha,\beta}\mathbf{v}$. Remember that $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})$ is a solution of (10.1) if and only if $\mathbf{u}^{\mathbf{v}}=\mathbf{v}$. According to (10.9), (10.7)

$$\|\mathbf{u}^{\mathbf{v}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}\leq C_2\left[\|\mathbf{G}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}+\|L_{\alpha,\beta}\mathbf{v}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}\right]\leq C_2\delta+C_2C_1\epsilon^2.$$

Since $C_2\delta+C_2C_1\epsilon^2<\epsilon$, we infer $\mathbf{u}^{\mathbf{v}}\in X$. If $\mathbf{w}\in X$ then

$$\|\mathbf{u}^{\mathbf{v}}-\mathbf{u}^{\mathbf{w}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}\leq C_2\|L_{\alpha,\beta}\mathbf{v}-L_{\alpha,\beta}\mathbf{w}\|_{[W^{1,q'}(\Omega,\mathbb{R}^m)]'}\leq 2\epsilon\|\mathbf{u}^{\mathbf{v}}-\mathbf{u}^{\mathbf{w}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}$$

by (10.8). Since $2\epsilon<1$, the Fixed point theorem ([4, Satz 1.24]) gives that there exists $\mathbf{v}\in X$ such that $\mathbf{u}^{\mathbf{v}}=\mathbf{v}$. So, $(\mathbf{u}^{\mathbf{v}},p^{\mathbf{v}})$ is a solution of (10.1) in $W^{1,q}(\Omega,\mathbb{R}^m)\times L^q(\Omega)$ satisfying $\|\mathbf{u}^{\mathbf{v}}\|_{W^{1,q}(\Omega,\mathbb{R}^m)}\leq\epsilon$. □

11. Appendix

Lemma 11.1. *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary. Let $s(i) \geq s \in N_0$, $1 \leq p, p(1), p(2) < \infty$, $s(i) - s \geq m[1/p(i) - 1/p]$, $s(1) + s(2) - s > m[1/p(1) + 1/p(2) - 1/p] \geq 0$. Then there exists a constant C such that the following holds: If $u \in W^{s(1),p(1)}(\Omega)$, $v \in W^{s(2),p(2)}(\Omega)$ then $uv \in W^{s,p}(\Omega)$ and*

$$\|uv\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s(1),p(1)}(\Omega)}\|v\|_{W^{s(2),p(2)}(\Omega)}.$$

(See [2, Corollary 6.3].)

Lemma 11.2. *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \in \{2, 3\}$. Let $1 < q < \infty$, $q' = q/(q-1)$. If $m = 3$ suppose moreover $q > 6/5$. Then there exists a constant C such that if $u, v \in W^{1,q}(\Omega)$, $\mathbf{w}, \tilde{\mathbf{w}} \in W^{1,q}(\Omega, \mathbb{R}^m)$ then $uv, |\mathbf{w}|v \in L^1(\Omega) \cap [W^{1,q'}(\Omega)]'$ and*

$$\|uv\|_{[W^{1,q'}(\Omega)]'} \leq C\|u\|_{W^{1,q}(\Omega)}\|v\|_{W^{1,q}(\Omega)}, \tag{11.1}$$

$$\|\ |\mathbf{w}|v\|_{[W^{1,q'}(\Omega)]'} \leq C\|\mathbf{w}\|_{W^{1,q}(\Omega;\mathbb{R}^m)}\|v\|_{W^{1,q}(\Omega)}, \tag{11.2}$$

$$\|\ |\mathbf{w}|v - |\tilde{\mathbf{w}}|v\|_{[W^{1,q'}(\Omega)]'} \leq C\|\mathbf{w} - \tilde{\mathbf{w}}\|_{W^{1,q}(\Omega;\mathbb{R}^m)}\|v\|_{W^{1,q}(\Omega)}. \tag{11.3}$$

Proof. Suppose first that $m = 2$. Since $1 - 0 > 0 = 2(1/q - 1/q)$, $1 + 1 - 0 > 2/q = 2(1/q + 1/q - 1/q)$, Lemma 11.1 gives that $uv \in L^q(\Omega)$ and there exists a constant C_1 such that

$$\|uv\|_{L^q(\Omega)} \leq C_1\|u\|_{W^{1,q}(\Omega)}\|v\|_{W^{1,q}(\Omega)}. \tag{11.4}$$

Thus $uv \in [W^{1,q'}(\Omega)]'$ and Hölder's inequality forces (11.1). [32, Corollary 2.1.8] gives $|w_j| \in W^{1,q}(\Omega)$ for $j = 1, \dots, m$ and

$$\|\ |w_j|\ \|_{W^{1,q}(\Omega)} = \|w_j\|_{W^{1,q}(\Omega)}$$

Thus

$$\|\ |\mathbf{w}|v\|_{L^q(\Omega)} \leq \sum_{j=1}^{m} \|\ |w_j|v\|_{L^q(\Omega)} \leq mC_1\|\mathbf{w}\|_{W^{1,q}(\Omega;\mathbb{R}^m)}\|v\|_{W^{1,q}(\Omega)}.$$

So, $|\mathbf{w}|v \in [W^{1,q'}(\Omega)]'$ and Hölder's inequality forces (11.2). Since

$$\|\ |\mathbf{w}|v - |\tilde{\mathbf{w}}|v\|_{L^q(\Omega)} \leq \|\ |\mathbf{w} - \tilde{\mathbf{w}}|v\|_{L^q(\Omega)} \leq mC_1\|\mathbf{w} - \tilde{\mathbf{w}}\|_{W^{1,q}(\Omega;\mathbb{R}^m)}\|v\|_{W^{1,q}(\Omega)}$$

we obtain (11.3) by Hölder's inequality.

Let now $m = 3$. Suppose first that $q > 3/2$. Since $1 - 0 > 0 = 3(1/q - 1/q)$, $1 + 1 - 0 > 3/q = 3(1/q + 1/q - 1/q)$, Lemma 11.1 gives that $uv \in L^q(\Omega)$ and there exists a constant C_1 such that (11.4) holds. Thus $uv \in [W^{1,q'}(\Omega)]'$ and Hölder's inequality gives (11.1). Let now $6/5 < q \leq 3/2$. Then there exists $r \in (1, q)$ such that $1 + 1 - 0 > 3(1/q + 1/q - 1/r) \geq 0$. Since $1 - 0 > 0 > 3(1/q - 1/r)$, Lemma 11.1 gives that $uv \in L^r(\Omega)$ and there exists a constant C_1 such that

$$\|uv\|_{L^r(\Omega)} \leq C_1\|u\|_{W^{1,q}(\Omega)}\|v\|_{W^{1,q}(\Omega)}.$$

Put $r' = r/(r-1)$. Since $q' \geq 3$, [15, Theorem 5.7.7, Theorem 5.7.8] give that $W^{1,q'}(\Omega) \hookrightarrow L^{r'}(\Omega)$. Hölder's inequality gives (11.1). The relations (11.2), (11.3) we deduce by the same way as in the case $m = 2$. □

Lemma 11.3. *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m \in \{2,3\}$. Let $\frac{6-m}{5-m} < q < \infty$, $q' = q/(q-1)$. Then there exists a constant C such that if $u \in W^{1,q}(\Omega)$, $v \in L^q(\Omega)$ then $uv \in L^1(\Omega) \cap [W^{1,q'}(\Omega)]'$ and*

$$\|uv\|_{[W^{1,q'}(\Omega)]'} \le C\|u\|_{W^{1,q}(\Omega)}\|v\|_{L^q(\Omega)}. \tag{11.5}$$

Proof. Suppose first that $q > m$. Since $\min(1-0, 0-0) = 0 = m(1/q - 1/q)$, $1 + 0 - 0 > m/q = m(1/q + 1/q - 1/q)$, Lemma 11.1 gives that $uv \in L^q(\Omega)$ and there exists a constant C_1 such that

$$\|uv\|_{L^q(\Omega)} \le C_1\|u\|_{W^{1,q}(\Omega)}\|v\|_{L^q(\Omega)}.$$

Thus $uv \in [W^{1,q'}(\Omega)]'$ and Hölder's inequality forces (11.5).

Let now $q \le m$. Suppose first that $m = 2$. Then there exists $r \in (1,q)$ such that $1 + 0 - 0 > 2(1/q + 1/q - 1/r) \ge 0$. Since $\min(1-0, 0-0) = 0 > 2(1/q - 1/r)$, Lemma 11.1 gives that $uv \in L^r(\Omega)$ and there exists a constant C_1 such that

$$\|uv\|_{L^r(\Omega)} \le C_1\|u\|_{W^{1,q}(\Omega)}\|v\|_{L^q(\Omega)}.$$

Put $r' = r/(r-1)$. Since $q' \ge 2$, [15, Theorem 5.7.7, Theorem 5.7.8] give that $W^{1,q'}(\Omega) \hookrightarrow L^{r'}(\Omega)$. Hölder's inequality gives (11.5).

Suppose now that $m = 3$. Since $3/2 \le q' < 3$, [15, Theorem 5.7.7, Theorem 5.7.8] and [32, Corollary 2.1.8] give that there exists a constant C_1 such that

$$\|u\|_{L^3(\Omega)} \le C_1\|u\|_{W^{1,q}(\Omega)} = C_1\|\ |u|\ \|_{W^{1,q}(\Omega)},$$

$$\|\varphi\|_{L^{3q'/(3-q')}(\Omega)} \le C_1\|\varphi\|_{W^{1,q'}(\Omega)} = C_1\|\ |\varphi|\ \|_{W^{1,q'}(\Omega)} \quad \forall \varphi \in W^{1,q'}(\Omega).$$

Since

$$\frac{1}{q} + \frac{3-q'}{3q'} + \frac{1}{3} = \frac{1}{q} + \frac{1}{q'} = 1$$

Hölder's inequality yields

$$\left|\int_\Omega uv\varphi\ \mathrm{d}\mathbf{x}\right| \le \int_\Omega |u||v||\varphi|\ \mathrm{d}\mathbf{x} \le \|u\|_{L^3(\Omega)}\|v\|_{L^q(\Omega)}\|\varphi\|_{L^{3q'/(3-q')}(\Omega)}$$
$$\le C_1^2\|u\|_{W^{1,q}(\Omega)}\|v\|_{L^q(\Omega)}\|\varphi\|_{W^{1,q'}(\Omega)}.$$

(In particular for $\varphi \equiv 1$ we obtain $uv \in L^1(\Omega)$.) Thus $uv \in [W^{1,q'}(\Omega)]'$ and (11.5) holds. ∎

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