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**Free sequences in  $\mathcal{P}(\omega)/fin$**

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# FREE SEQUENCES IN $\mathcal{P}(\omega)/\text{fin}$

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**ABSTRACT.** We investigate maximal free sequences in the Boolean algebra  $\mathcal{P}(\omega)/\text{fin}$ , as defined by D. Monk in [Mon11]. We provide some information on the general structure of these objects and we are particularly interested in the minimal cardinality of a free sequence, a cardinal characteristic of the continuum denoted  $\mathfrak{f}$ . Answering a question of Monk, we demonstrate the consistency of  $\omega_1 = \mathfrak{f} < \mathfrak{u} = \omega_2$ . In fact, this consistency is demonstrated in the model of S. Shelah for  $\mathfrak{i} < \mathfrak{u}$  [She92]. Our paper provides a streamlined and mostly self contained presentation of this construction.

## 1. INTRODUCTION

The paper uses the following convention: For an element  $a$  of a Boolean algebra  $B$  we denote  $a^0$  the complement of  $a$ , occasionally we also use  $a^1$  to denote  $a$ . This convention is used even for subsets of  $\omega$  which are considered as elements of the Boolean algebra  $\mathcal{P}(\omega)$ .

Free sequences in Boolean algebras were explicitly defined by Donald Monk in [Mon11].

**Definition 1.** Sequence  $A = \langle a_\alpha \mid \alpha \in \gamma \rangle$  of elements of a Boolean algebra of ordinal length  $\gamma$  is a *free sequence* if the family  $\{a_\alpha^1 \mid \alpha < \beta\} \cup \{a_\alpha^0 \mid \beta \leq \alpha < \gamma\}$  is centered for each  $\beta \leq \gamma$ .

The concept of free sequences comes from an analogous notion in topological spaces. A sequence of points  $\langle x_\alpha \mid \alpha < \gamma \rangle$  in a topological space is a free sequence if the topological closure of  $\langle x_\alpha \mid \alpha < \beta \rangle$  is disjoint from the topological closure of  $\langle x_\alpha \mid \beta \leq \alpha < \gamma \rangle$  for each  $\beta \leq \gamma$ . These objects were first considered by A. Arhangel'skiĭ in [Arh69] who introduced this concept in order to solve a famous problem of Alexandroff and Urysohn about the bound on cardinality of first countable compact spaces. In the topological context, the most important consideration seems to be the maximal possible size of a free sequence, this gives rise to a cardinal invariant of a topological space closely related to the tightness of the

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space, see e.g. [Bel86, Tod93], and existence of  $\pi$ -bases with additional combinatorial properties [Gor10].

S. Todorćević defined an algebraic version of the topological notion of a free sequence in [Tod90] and demonstrated that the algebraic formulation is often more convenient than the original topological concept (see also [Tod99]). For compact zero-dimensional topological spaces the algebraic definition of Todorćević coincides via the Stone duality with the notion of a free sequence in a Boolean algebra as defined by Monk. Nevertheless, the notion of a free sequence in a Boolean algebra is not precisely dual to the notion of a free sequence of points in a topological space, see the discussion in [Mon11].

A free sequence  $\langle a_\alpha \mid \alpha \in \gamma \rangle$  is *maximal* if it is maximal with respect to end-extension, i.e. there exist no  $a_\gamma$  such that  $\langle a_\alpha \mid \alpha \in \gamma \rangle \frown \langle a_\gamma \rangle$  is also a free sequence. Monk was primarily interested in the spectrum of possible cardinalities of maximal free sequences in Boolean algebras. Most notably, for a Boolean algebra  $B$  he defined  $f(B)$  to be the cardinal  $\min\{|A| \mid A \text{ is a maximal free sequence in } B\}$ . Monk investigated the relation of this cardinal with other cardinal characteristics of Boolean algebras. Let us remark at this point that the relation of the cardinal spectrum of possible cardinalities of maximal free sequences of a given Boolean algebra with the ordinal spectrum of the actual ordinal lengths of maximal free sequences is quite unclear. Even the question whether  $f(B)$  is realized by a maximal free sequence of ordinal length exactly  $f(B)$  is in general quite non-trivial.

One of the main problems stated in [Mon11] was the relation of  $f(B)$  and the ultrafilter number  $u(B)$ ; the minimal size of an ultrafilter base in  $B$ . One of the instances of this problem was solved by K. Selker [Sel15] who used forcing to demonstrate that the existence of a Boolean algebra  $B$  such that  $\omega = f(B) < u(B) = \omega_1$  is consistent with ZFC+CH.

The present paper is solely interested in free sequences in the Boolean algebra  $\mathcal{P}(\omega)/\text{fin}$ . We make several observations on free sequences and the relation of the free sequence number with other cardinal characteristics of the continuum. Most notably, we prove that the free sequence number is strictly smaller than the ultrafilter number  $u$  in the model for  $i < u$  of Shelah [She92]. As the paper of Shelah is considered to be somewhat cryptic, we opted for providing a streamlined, complete and mostly self contained presentation of the forcing construction from [She92]. All the core ingredients of this construction are originally due to Shelah. Our contribution, apart of the presentation, is the argument concerning free sequences and the free sequence number  $f$ . Reader interested only in Shelah's construction may skip Section 2 and other parts of this paper which are concerned with free sequences.

## 2. BASIC CONSIDERATIONS

We will start with exploring basic facts about possible incarnations of maximal free sequences in  $\mathcal{P}(\omega)/\text{fin}$ .<sup>1</sup> We define the *free sequence number*  $\mathfrak{f}$  to be the minimal cardinality of a maximal free sequence in  $\mathcal{P}(\omega)/\text{fin}$ , i.e.  $\mathfrak{f} = \mathfrak{f}(\mathcal{P}(\omega)/\text{fin})$ . For a given free sequence  $A = \langle a_\alpha \in \mathcal{P}(\omega) \mid \alpha < \gamma \rangle$  we denote the set of admissible intersections as

$$\text{comb}(A) = \left\{ \bigcap_{\alpha \in \Gamma} a_\alpha \cap \bigcap_{\alpha \in \Delta} a_\alpha^0 \mid \Gamma, \Delta \in [\gamma]^{<\omega}, \Gamma < \Delta \right\}.$$

We will also consider the filter generated by a free sequence, this is just the filter the free sequence generates as a centered subset of  $\mathcal{P}(\omega)/\text{fin}$ .

The free sequence number is closely related to other well known cardinal characteristics of the continuum. Let us give a brief overview of the relevant definitions.

Let  $\mathcal{U}$  be a non-principal ultrafilter on  $\omega$ . The character  $\chi(\mathcal{U})$  of  $\mathcal{U}$  is the minimal cardinality of a base of  $\mathcal{U}$ , the  $\pi$ -character  $\pi\chi(\mathcal{U})$  is the minimal cardinality of a  $\pi$ -base<sup>2</sup> of  $\mathcal{U}$ . The ultrafilter number  $\mathfrak{u}$  is the cardinal  $\min\{\chi(\mathcal{U}) \mid \mathcal{U} \text{ is a non-principal ultrafilter on } \omega\}$ , the reaping number  $\mathfrak{r}$  is the cardinal  $\min\{\pi\chi(\mathcal{U}) \mid \mathcal{U} \text{ is a non-principal ultrafilter on } \omega\}$ . We opted for a nonstandard definition of the reaping number as it is more suitable for our purposes.

**Theorem 2** ([BS91]). *The reaping number  $\mathfrak{r}$  as defined above is equal to the minimal cardinality of a family  $\mathcal{R} \subset [\omega]^\omega$  such that for each  $x \subset \omega$  there is  $r \in \mathcal{R}$  such that  $r \subset^* x$  or  $r \cap x =^* \emptyset$ .*

We also need a variant of the ultrafilter number, let  $\mathfrak{u}^*$  be the cardinal  $\min\{\chi(\mathcal{U}) \mid \mathcal{U} \text{ is a non-principal ultrafilter such that } \chi(\mathcal{U}) = \pi\chi(\mathcal{U})\}$ . The existence of an ultrafilter satisfying  $\chi(\mathcal{U}) = \pi\chi(\mathcal{U})$  is unclear in general, if no such ultrafilter exists, we declare  $\mathfrak{u}^*$  to be the continuum  $\mathfrak{c}$ . Bell and Kunen [BK81] proved that there is always an ultrafilter  $\mathcal{U}$  such that  $\pi\chi(\mathcal{U}) = \text{cof } \mathfrak{c}$ , therefore the following question is open only in case the continuum is a singular cardinal.

**Question 3.** Does ZFC imply the existence of an ultrafilter  $\mathcal{U}$  such that  $\chi(\mathcal{U}) = \pi\chi(\mathcal{U})$ ?

**Observation 4.**  $\mathfrak{r} \leq \mathfrak{u} \leq \mathfrak{u}^*$ . If  $\mathfrak{r} = \mathfrak{u}$ , then  $\mathfrak{u}^* = \mathfrak{u}$ .

We say that  $\mathcal{X} \subset [\omega]^\omega$  is an independent system if for every function  $f: \mathcal{X} \rightarrow 2$  is the family  $\{a^{f(a)} \mid a \in \mathcal{X}\}$  centered. An independent system is maximal if it is maximal with respect to inclusion. The independence

<sup>1</sup>We will not formally distinguish between the elements of the Boolean algebra  $\mathcal{P}(\omega)/\text{fin}$  and their representatives in  $\mathcal{P}(\omega)$ . We write  $a \subset^* b$  when  $b \setminus a$  is finite.

<sup>2</sup> $\mathcal{B} \subset [\omega]^\omega$  is a  $\pi$ -base of  $\mathcal{U}$  if there exists some  $B \in \mathcal{B}$ ,  $B \subset^* U$  for each  $U \in \mathcal{U}$ . A  $\pi$ -base  $\mathcal{B}$  is a base of  $\mathcal{U}$  if moreover  $\mathcal{B} \subset \mathcal{U}$ .

number  $i$  is the minimal cardinality of a maximal independent system. Although the definitions of a maximal independent system and a maximal free sequence are somewhat similar, we know very little about the relations between these objects and the relation between the cardinal characteristics  $i$  and  $f$ .

A strictly  $\subset^*$ -decreasing sequence in  $[\omega]^\omega$  is always a free sequence. Maximal such decreasing sequences (with respect to end-extension) are called towers, the smallest cardinality of a tower is the tower number  $t$ . A tower does not need to be a maximal free sequence. On the other hand if a free sequence generates an ultrafilter, then it is maximal. This observation allows us to deduce that there are maximal free sequences of ordinal length  $\omega_1$  in the Miller model as it contains such towers which generate ultrafilters [Mil84]. In particular, the Miller model demonstrates the consistency of  $\omega_1 = u = f < i = c = \omega_2$ .

**Question 5.** Is  $i < f$  consistent with ZFC?

The first part of the following proposition is already in [Mon11].

**Proposition 6.**  $\tau \leq f \leq u^*$

*Proof.* First assume that  $A$  is a free sequence of size smaller than  $\tau$ . Let  $\mathcal{U}$  be a non-principal ultrafilter extending  $A$ ,  $\text{comb}(A)$  is not a  $\pi$ -base of  $\mathcal{U}$  as it is of size  $< \tau$ . Choose  $a \in \mathcal{U}$  such that  $a^0 \cap c$  is infinite for each  $c \in \text{comb}(A)$ . Now  $A \frown \langle a \rangle$  is a free sequence and the first inequality is proved.

Assuming  $u^* < c$ , let  $\{u_\alpha \mid \alpha < u^*\}$  be a base of an ultrafilter  $\mathcal{U}$  such that  $\pi\chi(\mathcal{U}) = \chi(\mathcal{U})$ . Using induction on  $\alpha$  we can define a free sequence  $\langle a_\alpha \mid \alpha < u^* \rangle$ . Start with  $a_0 = u_0$ . If  $A_\beta = \langle a_\alpha \mid \alpha < \beta \rangle$  is defined, use  $|\text{comb}(A_\beta)| < \pi\chi(\mathcal{U})$  to find  $b_\beta \in \mathcal{U}$  such that  $b_\beta^0 \cap c$  is infinite for each  $c \in \text{comb}(A_\beta)$ . Let  $a_\beta = b_\beta \cap u_\beta$ , notice that  $A_\beta \frown \langle a_\beta \rangle$  is a free sequence. Finally, the constructed free sequence is a base of the ultrafilter  $\mathcal{U}$  and hence it is maximal.  $\square$

**Corollary 7.** If  $\tau = u$ , then  $f = u = \tau$ .

**Question 8.** Is  $\tau < f$  consistent with ZFC? What about  $u < f$ ?

The natural candidate for a model satisfying  $\tau < f$  is the model constructed in [GS90]. Corollary 7 presents a substantial obstacle when constructing a model where  $u < f$ . In such model would necessarily  $\tau < u < f$  hold, and this cannot be achieved using the usual technique of countable support forcing iteration.

The next proposition generalizes a property of decreasing sequences to arbitrary free sequences.

**Proposition 9.** Let  $A = \langle a_\alpha \mid \alpha < \gamma \rangle$  be a free sequence and  $\text{cf}\gamma = \omega$ . Then the free sequence  $A$  does not generate an ultrafilter.

*Proof.* Let  $\langle \gamma_i \mid i \in \omega \rangle$  be a sequence of ordinals cofinal in  $\gamma$ . For  $i \in \omega$  choose an ultrafilter  $\mathcal{U}_i$  extending the centered family  $\{a_\alpha \mid \alpha < \gamma_i\} \cup \{a_\alpha^0 \mid \gamma_i \leq \alpha < \gamma\}$ . If  $\mathcal{U}$  is an ultrafilter extending  $A$ , then  $\langle \mathcal{U}_i \mid i \in \omega \rangle$  is a sequence in the Stone space converging to  $\mathcal{U}$ , a contradiction.  $\square$

In fact, the same argument can be used to prove to following, presumably well known, fact.

**Observation 10.** Let  $\mathcal{X}$  be an independent system and  $f : \mathcal{X} \rightarrow 2$  any function. Then  $\mathcal{X}_f = \{a^{f(a)} \mid a \in \mathcal{X}\}$  does not generate an ultrafilter.

*Proof.* If  $\mathcal{X}$  is infinite, then  $\mathcal{X}_f$  can be ordered with an order type of cofinality  $\omega$ , and then use Proposition 9.  $\square$

The maximal free sequences constructed so far generate ultrafilters. The next proposition shows an elementary example demonstrating that this does not need to be case for a general free sequence.

**Proposition 11.** *For any given maximal free sequence there exists a maximal free sequence of the same cardinality which does not generate an ultrafilter.*

*Proof.* We can assume that  $\omega = X \cup Y$  for  $X, Y$  infinite disjoint, and there are maximal free sequences  $A = \langle a_\alpha \subset X \mid \alpha \in \gamma \rangle$  and  $B = \langle b_\alpha \subset Y \mid \alpha \in \gamma \rangle$  in  $\mathcal{P}(X)$  and  $\mathcal{P}(Y)$  respectively. For  $\langle \alpha, i \rangle \in \gamma \times 2$  let  $c_{\alpha, i} = a_\alpha \cup b_{\alpha+i}$ . Considering the lexicographical order on  $\gamma \times 2$  we get a sequence  $C = \langle c_{\alpha, i} \mid \langle \alpha, i \rangle \in \gamma \times 2 \rangle$ . This sequence does not generate an ultrafilter as both  $X$  and  $Y$  are positive with respect to the filter the sequence generates. We claim that  $C$  is a maximal free sequence on  $\omega$ . Checking that  $C$  is a free sequence is straightforward. To verify the maximality, take any  $z \subset \omega$ . If  $z$  is not positive with respect to both the filters generated by  $A$  and  $B$ , then  $C \frown \langle z \rangle$  is not centered. Assume  $z$  is positive with respect to the filter generated by  $A$ . As  $A$  is maximal, there are  $\Gamma < \Delta \in [\gamma]^\omega$  such that  $\{a_\alpha \mid \alpha \in \Gamma\} \cup \{a_\alpha^0 \mid \alpha \in \Delta\} \cup \{z^0 \cap X\}$  has only finite intersection. We may moreover suppose that there is  $\alpha \in \Gamma$  such that  $\alpha + 1 \in \Delta$ . As the intersection of  $\{c_\alpha \mid \alpha \in \Gamma \times 2\} \cup \{c_\alpha^0 \mid \alpha \in \Delta \times 2\}$  is a subset of  $X$ , it has only finite intersection with  $z^0$  and  $C$  cannot be end-extended by  $z$ . The reasoning when  $z$  is positive with respect to  $B$  is analogous.  $\square$

Regarding the proof Proposition 11, if the free sequences  $A$  and  $B$  generate ultrafilters, we can use similar construction, defining a free sequence  $C = \langle c_\alpha = a_\alpha \cup b_\alpha \mid \alpha \in \gamma \rangle \frown \langle X \rangle$ . This way we get an example of a maximal free sequence such that the order type of  $C$  is not a limit ordinal.

### 3. TOWARDS $\mathfrak{i} = \mathfrak{f} < \mathfrak{u}$

The rest of the paper is focused on proving that  $\mathfrak{f} < \mathfrak{u}$  is consistent with ZFC. The model where this holds is the model for  $\mathfrak{i} < \mathfrak{u}$  due to

Shelah [She92]. As the original paper is not easy to digest, we opted to include the proof. Our original contribution here is only the proof that  $i = j$  in this model.

Let us start with reviewing some basic terminology and folklore knowledge. An ideal on  $\omega$  is a set  $\mathcal{I} \subset \mathcal{P}(\omega)$ , such that if  $I, J \in \mathcal{I}$  and  $A \subset I$ , then  $A \in \mathcal{I}$  and  $I \cup J \in \mathcal{I}$ . The ideal  $\mathcal{I}$  is proper if  $\omega \notin \mathcal{I}$ . All ideals considered here will be proper ideals on  $\omega$  containing all finite subsets of  $\omega$ . A filter will generally be a dual of such ideal. for an ideal  $\mathcal{I}$  we denote the dual filter as  $\mathcal{I}^*$ . We say that  $\mathcal{K}$  is a *co-filter* if  $\mathcal{P}(\omega) \setminus \mathcal{K}$  is a filter.

For a filter base  $\mathcal{H} \subset \mathcal{P}(\omega)$  we denote  $\langle \mathcal{H} \rangle$  the filter generated by  $\mathcal{H}$ , i.e.  $F \in \langle \mathcal{H} \rangle$  iff  $H \subset^* F$  for some  $H \in \mathcal{H}$ . We use the same notation for co-filters generated by a co-filter base, the intended meaning of the notation should be always apparent from the context. We will need a folklore classification of filters. For  $A \subseteq \omega$  we denote  $\epsilon_A: \omega \rightarrow A$  the unique increasing surjection, and  $\bar{\epsilon}_F \in \omega^\omega$  the function  $\bar{\epsilon}_F: n \mapsto \epsilon_F(n+1) - \epsilon_F(n)$ . Filter  $\mathcal{F}$  is non-meager if the family  $\{\epsilon_F \mid F \in \mathcal{F}\}$  is unbounded in  $(\omega^\omega, <^*)$ . Filter  $\mathcal{F}$  is rare<sup>3</sup> if the family  $\{\bar{\epsilon}_F \mid F \in \mathcal{F}\}$  is dominating. Filter  $\mathcal{F}$  is a P-filter if for each  $C \in [\mathcal{F}]^\omega$  there exists  $F \in \mathcal{F}$  such that  $F \subset^* X$  for each  $X \in C$ .

We will use the following standard *diagonal* properties of these filters.

**Fact 12.** Filter  $\mathcal{F}$  is a non-meager P-filter if and only if for each sequence  $\{F_n \in \mathcal{F} \mid n \in \omega\}$  there exist  $F \in \mathcal{F}$  such that  $F \setminus n \subset F_n$  for infinitely many  $n \in \omega$ .

Notice that the condition in the preceding fact can be equivalently formulated as “ $F \setminus (n+1) \subset F_n$  for infinitely many  $n \in F$ .”

**Fact 13.** Filter  $\mathcal{F}$  is a rare P-filter iff  $\mathcal{F}$  has the *diagonal property*, i.e. for each  $\{F_n \in \mathcal{F} \mid n \in \omega\}$  there exists  $F \in \mathcal{F}$  such that  $F \setminus (n+1) \subseteq F_n$  for each  $n \in F$ .

Forcing notion  $P$  is bounding if for every generic extension  $V[G]$  and each  $f \in \omega^\omega \cap V[G]$  there is  $g \in \omega^\omega \cap V[G]$  such that  $f \leq g$ . Forcing  $P$  has the Sacks property if for each  $f \in \omega^\omega \cap V[G]$  there exists a sequence  $\{G_n \mid |G_n| \leq 2^n, n \in \omega\} \in V$  such that  $f(n) \in G_n$  for each  $n \in \omega$ . Every forcing with the Sacks property is bounding. Every rare or non-meager filter generates a filter with the same property in every generic extension via a bounding forcing. Every P-filter generates a P-filter in a generic extension via a proper forcing.

We will use the standard notation for the Cohen poset, the set  $\mathbf{C}_\kappa = \{h: \kappa \rightarrow 2 \mid |h| < \omega\}$  ordered by reverse inclusion. If  $\kappa = \omega$ , we write just  $\mathbf{C}$ . Set  $D \subset \mathbf{C}_\kappa$  is dense if for each  $h \in \mathbf{C}_\kappa$  exists  $g \in D$ ,  $g \supset h$ . For dense sets  $C, D \subset \mathbf{C}_\kappa$  we say that  $C$  *refines*  $D$  if for each  $h \in C$  exists  $g \in D$

<sup>3</sup>Rare filters are also called Q-filters. We opted for the original terminology of Choquet.



such that  $g \subseteq h$ . If  $W$  is an extension of a model set theory  $V$ , we say that  $W$  is *Cohen-preserving* if for each dense  $D \subset \mathbf{C}$ ,  $D \in W$  exists  $C \in V$  which refines  $D$ . We say that a forcing is Cohen-preserving if every generic extension via this forcing is Cohen-preserving. Although this property of forcing notions is considered in the literature, e.g. [BJ95, 6.3.C], there does not seem to be a unified terminology.

The following proposition is implicitly proved in [Mil81]. We learned both the proposition and the proof from O. Guzmán. We reproduce the proof for the sake of completeness.

**Proposition 14.** *If a forcing notion has the Sacks property, then it is Cohen-preserving.*

*Proof.* Suppose that  $V[G]$  is a generic extension via a forcing which has the Sacks property, let  $D \in V[G]$  be an open dense subset of  $\mathbf{C}$ . We will without loss of generality work with  $2^{<\omega}$  instead of  $\mathbf{C}$ . As the extension is bounding, there is  $f: \omega \rightarrow \omega$  in  $V$  such that for each  $t \in 2^n$  there is  $s \in 2^{f(n)}$  such that  $t \hat{\ } s \in D$ . Fix a dense subset  $\{t_n \mid n \in \omega\}$  of  $2^{<\omega}$  in  $V$  such that  $|t_n| = n$ .

In  $V[G]$  define a function  $h: \omega \rightarrow [2^{<\omega}]^{<\omega}$  such that  $|h(n)| = n + 1$  for each  $n \in \omega$ . The function is defined in the following way. Given  $n \in \omega$  let  $r_n(0) = n$ . When  $r_n(i)$  for  $i \leq n + 1$  is defined, choose  $s^i \in 2^{f(r_n(i))}$  such that  $x \hat{\ } s^i \in D$  for each  $x \in 2^{r_n(i)}$  and let  $r_n(i + 1) = r_n(i) + f(r_n(i))$ . Finally let  $h(n) = \langle s^i \mid i \leq n \rangle$ . As the extension has the Sacks property, there is a sequence  $\langle H(n) \subset [2^{<\omega}]^{n+1} \mid n \in \omega \rangle$  in  $V$ , such that  $|H(n)| = n + 1$  and  $h(n) \in H(n)$  for each  $n \in \omega$ . We denote  $H(n) = \langle S_k(n) \mid k \leq n \rangle$  and  $S_k(n) = \langle s_k^i(n) \mid i \leq n \rangle$ . We may assume that  $|s_k^i(n)| = f(r_n(i))$  for each  $k, i \leq n, n \in \omega$ .

Finally let  $z_n = t_n \hat{\ } s_0^0(n) \hat{\ } s_1^1(n) \hat{\ } \dots \hat{\ } s_n^n(n)$ . The set  $C = \{z_n \mid n \in \omega\} \in V$  is obviously dense, and  $C \subset D$  because for each  $n \in \omega$  there is  $k$  such that  $s_k^k(n) \in h(n)$ .  $\square$

Since the posets  $\mathbf{C}_\kappa$  are c.c.c., being Cohen-preserving already guarantees an analogous property for these posets as well.

**Lemma 15.** *Let  $P$  be a proper Cohen-preserving forcing,  $G$  a generic filter on  $P$ . For each  $\kappa$  and each dense  $D \subset \mathbf{C}_\kappa$  in  $V[G]$ , there exists  $C \in V$  refining  $D$ .*

*Proof.* Since  $\mathbf{C}_\kappa$  is c.c.c. there is a countable set  $a \in V[G]$ ,  $a \subset \kappa$ , and a countable dense  $D' \subseteq D$ ,  $D' \subset \mathbf{C}_a$ . Since  $P$  is proper, there exist a countable  $b \in V$  such that  $a \subset b$ , i.e.  $D' \subset \mathbf{C}_b$ . As  $P$  is Cohen-preserving, there exists  $C \in \mathbf{C}_b \cap V$  refining  $D'$ , hence also refining  $D$ .  $\square$

## 4. THE FORCING NOTION

We will say that  $E = \{e_k \subset \omega \mid k \in \omega\}$  is a partition if  $e_k \cap e_j = \emptyset$  for  $k \neq j$ . We will usually deal with infinite partitions and we always assume  $\min e_k < \min e_j$  for  $k < j$ . We denote  $\text{dom } E = \bigcup E$ . Partition  $D = \{d_k \mid k \in \omega\}$  is coarser than  $E$  if each element of  $D$  is a union of elements of  $E$ . If  $\mathcal{I}$  is an ideal on  $\omega$ , we say that  $E$  is an  $\mathcal{I}$ -partition if  $e_k \in \mathcal{I}$  for each  $k \in \omega$  and  $\text{dom } E \in \mathcal{I}^*$ .

For the purpose of this paper a tree  $T$  is an initial subtree of the tree of finite 0-1 sequences  $(2^{<\omega}, \subseteq)$  with no maximal elements (leaves). For  $t \in T$  we denote  $T[t]$  the subtree consisting of all nodes of  $T$  compatible with  $t$ . For  $n \in \omega$  we denote by  $T^{(n)}$  the set of all nodes  $t \in T$  such that  $|t| = n$  (i.e. the nodes from the  $n$ -th level). A node  $t \in T$  is a *branching node* of  $T$  if both  $t \hat{\ } 0 \in T$  and  $t \hat{\ } 1 \in T$ . We say that the  $n$ -th level is a *branching level* if each element of  $T^{(n)}$  is a branching node. We say that a tree is *uniformly branching* if each branching node is an element of a branching level.

Given a tree  $T$  we say that the level  $m$  *depends* on a level  $n$  if for  $n \leq m$ ,  $n$  is a branching level, and for each  $s, t \in T^{(m+1)}$  is  $s(m) + s(n) = t(m) + t(n) \pmod{2}$ . We call such levels *dependent* levels, levels which are not dependent are *independent*. Note that for a given dependent level  $m$  there is a unique  $n$  such that  $m$  depends on  $n$ , and each branching level depends on itself. We say that a level is independent if it does not depend on any level. To each uniformly branching tree  $T$  we assign a partition of  $\omega$  denoted  $E^T = \{e_k^T \mid k \in \omega\}$  such that if  $m$  and  $n$  are dependent levels, then  $m$  and  $n$  are in the same element of  $E_T$  iff  $m$  and  $n$  depend on the same level, and  $\text{dom } E^T$  is exactly the set of all dependent levels. The superscripts will occasionally be omitted if clear from the context. Let  $\mathcal{I}$  be an ideal on  $\omega$ , we say that a uniformly branching tree  $T$  is  $\mathcal{I}$ -suitable if  $E^T$  is an  $\mathcal{I}$ -partition. The poset of  $\mathcal{I}$ -suitable trees ordered by inclusion will be denoted  $\mathbf{Q}_{\mathcal{I}}$ . Note that for  $S < T \in \mathbf{Q}_{\mathcal{I}}$  dependent levels of  $T$  can in general be independent levels of  $S$ , and independent levels of  $T$  can become dependent levels in  $S$ . Thus  $S < T$  does not necessarily imply that  $E^S$  is coarser than  $E^T$ , on the other hand  $E^S \upharpoonright \text{dom } E^T = \{e_k^S \cap \text{dom } E^T \mid k \in \omega\}$  is coarser than  $E^T$ .

This poset is sometimes called the *party forcing*.<sup>4</sup> This version of the forcing is slightly different than the one used in [She92], the conditions of the poset used by Shelah did explicitly remember the partitions  $E^T$ . Nevertheless, our version of the poset works in the same way. This type of forcing was also recently used by Guzmán [GG] to prove that the homogeneity number  $\mathfrak{hm}$  can be consistently smaller than  $\mathfrak{u}$ .

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<sup>4</sup>Organizing a party in the Hilbert hotel is a difficult task, guests may or may not like their lesser colleagues.

For  $T \in \mathbf{Q}_{\mathcal{I}}$  and a partial function  $f : \omega \rightarrow 2$  we denote by  $T_f$  the largest subtree of  $T$  with the property that if  $k \in \text{dom } f$ ,  $n \in e_k^T$ ,  $n$  is a branching level of  $T$  (i.e.  $n = \min e_k^T$ ), and  $t \in T^{(n)}$ , then  $t \frown i \in T_f$  only if  $f(k) = i$ . Note that  $f$  being finite is a sufficient condition guaranteeing  $T_f \in \mathbf{Q}_{\mathcal{I}}$ .

The forcing will be used to destroy a given ultrafilter, when we use the dual ideal as a parameter, the generic real will witness that the ultrafilter does not generate an ultrafilter in the generic extension.

**Lemma 16.** *Let  $\mathcal{I}$  be a proper ideal on  $\omega$  and let  $G$  be a  $\mathbf{Q}_{\mathcal{I}}$ -generic filter. Then  $r = \bigcup \bigcap G \in 2^\omega$  and  $r \notin \langle \mathcal{I} \rangle \cup \langle \mathcal{I}^* \rangle$ .*

*Proof.* The first part of the lemma is immediate since  $\mathcal{I}$  extends the Fréchet ideal. Let  $T \in \mathbf{Q}_{\mathcal{I}}$  be a condition and  $I \in \mathcal{I}$ . Pick any integer  $n \in \text{dom } E^T \setminus I$ , hence  $n \in e_k^T$  for some  $k \in \omega$ . Put  $f_i : \{k\} \rightarrow 2$ ,  $f_i : k \mapsto i$  for  $i \in 2$ . For both  $i \in 2$  the conditions  $T_{f_i} \in \mathbf{Q}_{\mathcal{I}}$  decide whether  $n \in r$ , and they do so in opposite ways. That is at least one of them forces that  $r \notin I$ . The argument for  $r \notin \langle \mathcal{I}^* \rangle$  is analogous.  $\square$

Let  $a \subset \omega$ . Suppose that  $S < T$  are conditions in  $\mathbf{Q}_{\mathcal{I}}$  such that for each  $k \in a$ , if  $n$  is the splitting level of  $T$  in  $e_k^T$ , then  $n$  is also a splitting level of  $S$  (i.e.  $a$ -th splitting levels are preserved). We will denote this relation by  $S <_a T$ .

**Lemma 17.** *Let  $T \in \mathbf{Q}_{\mathcal{I}}$  be a condition,  $x$  a name for an element of  $V$ , and  $n \in \omega$ . There exists a condition  $S <_n T$  such that for each  $f \in {}^n 2$  the condition  $S_f$  decides the value of  $x$ .*

*Proof.* Fix enumeration  ${}^n 2 = \{f_i \mid i \in 2^n\}$ , denote  $T^0 = T$ , and for  $i \in 2^n$  repeat the following procedure.

Suppose that  $T^i$  is defined. Find a condition  $S^i < T_{f_i}^i$  and  $y_i \in V$  such that  $S^i \Vdash x = y_i$ . Then let  $T^{i+1}$  be the largest subtree of  $T$  in  $\mathbf{Q}_{\mathcal{I}}$  such that  $T_{f_i}^{i+1} = S^i$ . Note that  $T^{i+1} <_n T^i$ .

Finally let  $S = T^{2^n}$ . Then  $S <_n T$ , and every  $R < S$  is compatible with some  $S^i$ . Thus  $S$  and the set  $Y = \{y_i \mid i \in 2^n\}$  are as required.  $\square$

Before proving the properness of the forcing  $\mathbf{Q}_{\mathcal{I}}$  we introduce a game with  $\mathcal{I}$ -partitions  $\text{PG}(\mathcal{I})$ . Player I starts the game with choosing an  $\mathcal{I}$ -partition  $E^0$  and then players I and II alternate in building a sequence of  $\mathcal{I}$ -partitions. In round  $n$  player II plays an  $\mathcal{I}$ -partition  $D^n$  coarser than  $E_n$ , puts  $\Delta_n = \text{dom } E^n \setminus \text{dom } D^n$ , and in the next round player I replies with an  $\mathcal{I}$ -partition  $E_{n+1}$  coarser than  $D_n$ . After  $\omega$  many rounds player I wins iff  $r = \bigcup \{\Delta_n \mid n \in \omega\} \in \mathcal{I}$ .

**Lemma 18.** *Player I has no winning strategy in the game  $\text{PG}(\mathcal{I})$ .*

*Proof.* If player I has a winning strategy, then he also has a winning strategy, such that moreover  $\bigcap \{\text{dom } E^n \mid n \in \omega\} = \emptyset$  (where  $\langle E^n \rangle$  is the

sequence of moves of player I). Assuming player I uses this strategy, player II will play simultaneously two matches of the game  $\text{PG}(\mathcal{I})$ . He passes his first move in the first match and then he always imitates the moves of player I in the other game. This produces results  $r, r'$  of the two matches such that  $r \cup r' = \text{dom } E^0 \in \mathcal{I}^*$ . Thus in at least one of the two matches player II won.  $\square$

**Proposition 19.** *Let  $\mathcal{I}$  be a maximal ideal. The forcing  $\mathbf{Q}_{\mathcal{I}}$  is proper and has the Sacks property.*

*Proof.* We will prove both statements simultaneously. Let  $T \in \mathbf{Q}_{\mathcal{I}}$  be a condition and  $g$  a name for a function in  $\omega^\omega$ . Let  $\theta$  be large enough and fix a countable elementary submodel  $M \prec H(\theta)$  such that  $\mathbf{Q}_{\mathcal{I}}, T, g \in M$ . Enumerate all  $\mathbf{Q}_{\mathcal{I}}$ -names for ordinals in  $M$  as  $\{\sigma_n \mid n \in \omega\}$ . We will construct a condition  $Q < T$  such that for each  $f \in {}^{n+1}2$ ,  $n \in \omega$  the condition  $Q_f$  decides the value of  $g(n)$ , and forces  $\sigma_n$  to be some element of  $M$ . This will prove the proposition.

Two players will play the game  $\text{PG}(\mathcal{I})$  in the model  $M$ , player I will attempt to construct the desired condition during the course of the game. Player I starts by finding a condition  $T_0 < T$ ,  $T_0 \in M$  which decides  $g(0)$  and  $\sigma_0$ . His first move in the game is  $E^0 = E^{T_0}$ , and the reply of player II is an  $\mathcal{I}$ -partition  $D^0$ .

Suppose that in the  $n$ -th round of the game, condition  $T_n$  was defined and player II played an  $\mathcal{I}$ -partition  $D^n$  coarser than  $E^n = \{e_k^{T_n} \mid n \leq k \in \omega\} \upharpoonright \text{dom } D^{n-1}$ . In the next round player I first picks some condition  $T'_n <_n T_n$  in  $M$  such that  $e_k^{T'_n} = e_k^{T_n}$  for  $k < n-1$ ,  $e_{n-1}^{T'_n} = e_{n-1}^{T_n} \cup \Delta_n$ , and  $\{e_k^{T'_n} \mid k \geq n\} = D^n$ . Then using Lemma 17 he finds a condition  $T_{n+1} <_n T'_n$  in model  $M$  such that;

- $(T_{n+1})_f$  decides  $g(n+1)$  for each  $f \in {}^{n+1}2$ , and
- $(T_{n+1})_f$  decides  $\sigma_{n+1}$  for each  $f \in {}^{n+1}2$ .

Finally player I passes the  $\mathcal{I}$ -partition  $E^{n+1} = \{e_k^{T_{n+1}} \mid n+1 \leq k \in \omega\} \upharpoonright \text{dom } D^n$  to player II and awaits his response.

This strategy is not winning for player I, so we can assume that the game is played so that player II wins, i.e.  $r = \bigcup \{\Delta_n \mid n \in \omega\} \in \mathcal{I}^*$  (the ideal  $\mathcal{I}$  is maximal).

Once the game is over, define  $Q = \bigcap \{T_n \mid n \in \omega\}$ . Notice that for  $e_n^Q \in E^Q$  is  $e_n^Q \cap r \subset \Delta_n$  and  $r \subset \text{dom } E^Q$ , thus  $\text{dom } E^Q \in \mathcal{I}^*$ ,  $E^Q$  is an  $\mathcal{I}$ -partition and  $Q \in \mathbf{Q}_{\mathcal{I}}$ . Since  $Q <_n T_n$  for each  $n \in \omega$ ,  $Q$  is the desired condition.  $\square$

**Corollary 20.** *The poset  $\mathbf{Q}_{\mathcal{I}}$  is a Cohen-preserving forcing notion.*

The proof of Proposition 19 gives us in fact the following.

**Corollary 21.** *Let  $T \in \mathbf{Q}_{\mathcal{A}}$  be a condition and  $X$  be a name for a subset of  $\omega$ . There is a condition  $S < T$  such that for each  $n \in \omega$  and  $f \in 2^{n+1}$ ,  $S_f$  forces either  $n \in X$  or  $n \notin X$ .*

## 5. DENSE INDEPENDENT SYSTEMS

Let  $\mathcal{A} \subset \mathcal{P}(\omega)$  be an independent system. Remember that the set of finite partial functions  $\{h: \mathcal{A} \rightarrow 2\}$  is denoted  $\mathbf{C}_{\mathcal{A}}$ , and it carries the usual inclusion order. For each  $h \in \mathbf{C}_{\mathcal{A}}$  we put  $\mathcal{A}^h = \bigcap \{A^{h(A)} \mid A \in \text{dom } h\} \in [\omega]^\omega$ . For  $X \subseteq \omega$  we will say that  $h \in \mathbf{C}_{\mathcal{A}}$  *reaps*  $X$  if either  $\mathcal{A}^h \subset^* X$  or  $\mathcal{A}^h \cap X =^* \emptyset$ . If the first option  $\mathcal{A}^h \subset^* X$  occurs, we say that  $h$  *hits*  $X$ . The independent system  $\mathcal{A}$  is maximal iff the set  $\{h \mid h \text{ reaps } X\}$  is nonempty for each  $X \subseteq \omega$ .

We say that the independent system  $\mathcal{A}$  is *dense* if the set  $\{h \mid h \text{ reaps } X\}$  is dense in  $\mathbf{C}_{\mathcal{A}}$  for each  $X \subseteq \omega$ . It is easy to see that every dense independent system is maximal. Dense independent systems were originally introduced in [GS90] and recently studied in [FM18]. For each maximal independent system  $\mathcal{A}$  there exists  $h \in \mathbf{C}_{\mathcal{A}}$  such that  $\mathcal{A} \upharpoonright \mathcal{A}^h$  is a dense independent system, see [GS90, Lemma 6.6, 6.7].

Denote by  $\mathcal{D}$  the collection of dense subsets of  $\mathbf{C}_{\mathcal{A}}$ . The filter on  $\omega$  generated by sets of form  $F(D) = \bigcup \{\mathcal{A}^h \mid h \in D\}$  for some  $D \in \mathcal{D}$  will be denoted  $\mathcal{F}_{\mathcal{A}}$ . Notice that  $X \in \mathcal{F}_{\mathcal{A}}$  iff  $\{h \mid h \text{ hits } X\}$  is dense in  $\mathbf{C}_{\mathcal{A}}$ . We will denote  $\mathcal{C}_{\mathcal{A}} = \{\omega \setminus \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}}\}$ . The following observation will be crucial for the preservation of maximality of a given independent system.

**Lemma 22.** *An independent system  $\mathcal{A}$  is dense if and only if the co-filter  $\mathcal{P}(\omega) \setminus \mathcal{F}_{\mathcal{A}}$  is generated by the set  $\mathcal{C}_{\mathcal{A}}$ .*

*Proof.* Suppose that  $\mathcal{A}$  is dense and  $X \subseteq \omega$ . If  $\{h \mid h \text{ hits } X\}$  is dense in  $\mathbf{C}_{\mathcal{A}}$ , then  $X \in \mathcal{F}_{\mathcal{A}}$ . Otherwise there is  $h \in \mathbf{C}_{\mathcal{A}}$  such that  $\mathcal{A}^h \cap X =^* \emptyset$  and  $X \in \mathcal{C}_{\mathcal{A}}$ .

To verify the other implication let  $X \subseteq \omega$  and  $h \in \mathbf{C}_{\mathcal{A}}$  be given, let  $X' = (X \cap \mathcal{A}^h) \cup (\omega \setminus \mathcal{A}^h)$ . If  $X' \in \mathcal{F}_{\mathcal{A}}$ , then there is  $h' \supset h$  such that  $\mathcal{A}^{h'} \subset^* X'$ , and hence  $\mathcal{A}^{h'} \subset^* X$ . Otherwise  $X' \in \mathcal{C}_{\mathcal{A}}$ , there is  $h'$  such that  $\mathcal{A}^{h'} \cap X' =^* \emptyset$ . Thus  $h \subseteq h'$  and  $\mathcal{A}^{h'} \cap X =^* \emptyset$ .  $\square$

The definition of  $\mathcal{C}_{\mathcal{A}}$  is absolute for all models of set theory. The definition of  $\mathcal{F}_{\mathcal{A}}$  behaves well when considering Cohen-preserving extension.

**Lemma 23.** *Let  $\mathcal{A} \in V$  be an independent system and let  $W$  be a Cohen-preserving extension of  $V$ . The filter  $\mathcal{F}_{\mathcal{A}}^W$  is generated by  $\mathcal{F}_{\mathcal{A}}^V$ .*

*Proof.* Follows immediately from Lemma 15.  $\square$

**Remark 24.** Lemmas 22 and 23 imply that to prove that a dense independent system  $\mathcal{A} \in V$  remains dense in a Cohen-preserving extension  $W$ , it is sufficient to demonstrate that in  $W$  is  $\mathcal{P}(\omega) = \langle \mathcal{F}_{\mathcal{A}}^V \rangle \cup \langle \mathcal{C}_{\mathcal{A}} \rangle$ .

**Proposition 25.** *Assume CH. There exists an independent system  $\mathcal{A}$  with the following properties:*

- (1)  $\mathcal{A}$  is dense,
- (2)  $\mathcal{F}_{\mathcal{A}}$  is a rare P-filter.

We call an independent system satisfying properties (1) and (2) *selective*.

*Proof.* Enumerate the functions in  $\omega^\omega$  as  $\{f_\alpha \mid \alpha \in \omega_1, \alpha \text{ limit}\}$ , enumerate maximal antichains in  $\mathbf{C}_{\omega_1}$  as  $\{H_\alpha \mid \alpha \in \omega_1, 0 < \alpha \text{ limit}\}$  so that  $H_\alpha \subset \mathbf{C}_\alpha$ , and enumerate all elements of  $\mathcal{P}(\omega) \times \mathbf{C}_{\omega_1}$  as  $\{\langle X_\alpha, g_\alpha \rangle \mid \alpha \in \omega_1\}$  so that  $g_\alpha \in \mathbf{C}_\alpha$ .

We proceed by induction, for  $\alpha < \omega_1$  we will define  $\langle A_\alpha, B_\alpha \mid \alpha < \omega_1 \rangle$  such that  $A_\alpha \subset B_\alpha \subset^* B_\beta \subset \omega$  for  $\alpha < \beta$ , and  $\vec{\mathcal{A}}_\alpha = \langle A_\beta \cap B_\alpha \mid \beta < \alpha \rangle$  is an independent system. We write  $\mathcal{A}_\alpha = \langle A_\beta \mid \beta < \alpha \rangle$ .

Start with  $B_0$  such that  $f_0 < \bar{\epsilon}_{B_0}$ . If  $\langle A_\alpha, B_\alpha \mid \alpha < \beta \rangle$  and  $B_\beta$  are defined, let  $B_{\beta+1} = B_\beta$  and choose any  $A_\beta \subset B_\beta$  such that  $\vec{\mathcal{A}}_{\beta+1}$  is an independent system, this is possible since  $\vec{\mathcal{A}}_\alpha$  is countable and hence not maximal. Moreover, letting  $Z_\beta = \vec{\mathcal{A}}_\alpha^{g_\alpha}$ , if it is possible to choose  $A_\beta$  such that  $A_\beta \cap Z_\beta = X_\beta \cap Z_\beta$ , do so.<sup>5</sup>

Suppose  $\langle A_\alpha, B_\alpha \mid \alpha < \omega_1 \rangle$  is defined for all  $\alpha < \beta$ ,  $\beta$  limit.

**Claim.** *There is  $B_\beta \subset \omega$  such that  $B_\beta \subset^* B_\alpha$  for  $\alpha < \beta$ ,  $f_\beta < \bar{\epsilon}_{B_\beta}$ ,  $B_\beta \subset \bigcup \{ \mathcal{A}^h \mid h \in H_\alpha \}$ , and  $\vec{\mathcal{A}}_\beta$  is an independent system.*

Fix a sequence  $\alpha(n)$  converging to  $\beta$  and an enumeration  $\{g_i \mid i \in \omega\}$  of all functions in  $\mathbf{C}_\beta$  which extend some element of  $H_\alpha$ , with infinite repetitions, and so that  $\text{dom } h_i \subset \alpha(i)$  for each  $i \in \omega$ .

Since the sets  $C_i = \mathcal{A}_\beta^{g_i} \cap \bigcap \{ B_{\alpha(j)} \mid j \leq i \}$  are infinite for all  $i \in \omega$ , it is possible to choose infinite  $B_\beta$  such that  $\epsilon_{B_\beta}(i) \in C_i$  and  $f_\beta < \bar{\epsilon}_{B_\beta}$ . This is as required since  $\vec{\mathcal{A}}_\beta$  is an independent system.  $\blacksquare$

This completes the inductive construction. We constructed an independent system  $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ . To check that it is dense take any  $\langle X_\beta, g_\beta \rangle \in \mathcal{P}(\omega) \times \mathbf{C}_{\omega_1}$ . If  $A_\beta$  was chosen so that  $A_\beta \cap Z_\beta = X_\beta \cap Z_\beta$ , we are done. If  $A_\beta$  was not chosen with this property, there is some  $g \in \mathbf{C}_\beta$ ,  $g_\beta \subset g$  such that  $\mathcal{A}^g$  reaps  $X_\beta \cap B_\beta$  and we are also done, as we can extend  $g$  by declaring  $g: \beta \mapsto 1$  to achieve  $\mathcal{A}^g \subset B_\beta$ .

The inductive construction ensures that the filter generated by the decreasing tower  $\mathcal{T} = \{B_\alpha \mid \alpha \in \omega_1\}$  is a rare P-filter.

**Claim.** *The filter  $\mathcal{F}_{\mathcal{A}}$  is the filter generated by  $\mathcal{T}$ .*

For  $\alpha \in \omega_1$  let  $D = \{h \in \mathbf{C}_{\mathcal{A}} \mid \text{there is } k \in \omega \text{ such that } A_{\alpha+k} \in \text{dom } h\}$ . The set  $D$  is dense in  $\mathbf{C}_{\mathcal{A}}$  and  $F(D) \subset B_\alpha$  is witnessing  $B_\alpha \in \mathcal{F}_{\mathcal{A}}$ .

<sup>5</sup>We will be slightly abusing the notation, identifying  $\mathbf{C}_{\vec{\mathcal{A}}_\alpha}$  with  $\mathbf{C}_\alpha$  etc.

On the other hand take any dense  $D \subset \mathbf{C}_{\mathcal{A}}$ . There is some  $\beta \in \omega_1$  such that  $H_\beta \subset D$ . Since each element of  $D$  is compatible with some element of  $H_\beta$ , we have that  $F(H_\beta) \subset F(D)$ . The set  $B_\beta$  was chosen so that  $B_\beta \subset F(H_\beta)$ .  $\square$

**Theorem 26.** *Let  $\mathcal{A}$  be a selective independent system and let  $\mathcal{G}$  be a maximal ideal. If  $G$  is a  $\mathbf{Q}_{\mathcal{G}}$ -generic filter, then  $\mathcal{A}$  is a selective independent system in  $V[G]$ .*

*Proof.* The system  $\mathcal{A}$  remains independent in any extension. Since  $\mathbf{Q}_{\mathcal{G}}$  is Cohen-preserving, Lemma 23 states that in  $V[G]$  the filter  $\mathcal{F}_{\mathcal{A}}^{V[G]}$  is generated by  $\mathcal{F}_{\mathcal{A}}^V$ . Thus the filter  $\mathcal{F}_{\mathcal{A}}^{V[G]}$  is a P-filter since  $\mathbf{Q}_{\mathcal{G}}$  is proper, and it is rare since  $\mathbf{Q}_{\mathcal{G}}$  has the Sacks property. To show that  $\mathcal{A}$  remains dense in the extension we will use Remark 24.

Let  $T \in \mathbf{Q}_{\mathcal{G}}$  be a condition and  $X$  a name for a subset of  $\omega$ . Suppose that no stronger condition forces that  $X \in \langle \mathcal{C}_{\mathcal{A}} \rangle$ , i.e. for each  $S < T$  is  $X_S = \{n \in \omega \mid S \Vdash n \notin X\} \in \mathcal{F}_{\mathcal{A}}$ . We will show that such  $T$  forces that  $X \in \langle \mathcal{F}_{\mathcal{A}} \rangle$ . In particular, for given  $h \in \mathbf{C}_{\mathcal{A}}$  we find  $g \supset h$  and  $Q < T$  such that  $Q \Vdash \mathcal{A}^g \subset^* X$ .

We may assume that for each  $n \in \omega$  and  $f \in 2^n$  the condition  $T_f$  decides  $X \cap n$  (use Corollary 21). For  $n \in \omega$  put  $X_n = \bigcap \{X_{T_f} \mid f \in 2^n\} \in \mathcal{F}_{\mathcal{A}}$ . Note that for each  $n < k$ ,  $k \in X_n$  there is a condition  $T_n(k) <_{\omega \setminus [n,k]} T$  such that  $T_n(k) \Vdash k \in X$ . The filter  $\mathcal{F}_{\mathcal{A}}$  has the diagonal property, i.e. there is  $F \in \mathcal{F}_{\mathcal{A}}$  such that  $F \setminus (n+1) \subseteq X_n$  for each  $n \in F$ . Let  $\{k_n \mid n \in \omega\} = F$  be the increasing enumeration. The choice of  $F$  ensures that for each  $n \in \omega$  the condition  $T_{k_n}(k_{n+1})$  is defined.

Since  $\mathcal{A}$  is dense, there are  $g_0, g_1 \supset h$  such that  $\mathcal{A}^{g_0} \cup \mathcal{A}^{g_1} \subset^* F$ , and  $\mathcal{A}^{g_0} \cap \mathcal{A}^{g_1} = \emptyset$ . For  $i \in 2$  put  $Q_i = \bigcap \{T_{k_n}(k_{n+1}) \mid k_{n+1} \in \mathcal{A}^{g_i}\}$ . The sets  $d_i = \bigcup \{[k_n, k_{n+1}) \mid k_{n+1} \in \mathcal{A}^{g_i}\}$  are disjoint for  $i \in 2$ , therefore for at least one  $i \in 2$  is  $\bar{d}_i = \bigcup \{e_k^T \mid k \in d_i\} \in \mathcal{G}$ . For this  $i$  is  $Q_i \in \mathbf{Q}_{\mathcal{G}}$ . To check this, notice that for  $k \in \omega$ ,  $e_k^{Q_i} \in E^{Q_i}$  is  $e_k^{Q_i} \subseteq e' \cup \bar{d}_i \cup (\omega \setminus \text{dom } E^T)$  for some  $e' \in E^T$ , and also  $\text{dom } E^T \subset \text{dom } E^{Q_i} \cup \bar{d}_i$ , thus  $\text{dom } E^{Q_i} \in \mathcal{G}^*$ . Since  $Q_i < T_{k_n}(k_{n+1})$  for each  $k_{n+1} \in \mathcal{A}^{g_i}$ , and all but finitely many elements of  $\mathcal{A}^{g_i}$  are of the form  $k_{n+1}$ , we have that  $Q_i \Vdash \mathcal{A}^{g_i} \subset^* X$ .  $\square$

Let  $\mathcal{A}$  be a dense independent system and let  $B$  be a free sequence. We say that  $B$  is a *free sequence associated with  $\mathcal{A}$*  if  $B$  is a maximal free sequence and  $B$  generates the filter  $\mathcal{F}_{\mathcal{A}}$ .

**Theorem 27.** *Let  $B$  be a maximal free sequence associated with a dense independent system  $\mathcal{A}$  in a model of set theory  $V$ . Let  $W$  be a Cohen-preserving extension of  $V$  such that  $\mathcal{A}$  remains dense in  $W$ . Then  $B$  remains to be a maximal free sequence associated with  $\mathcal{A}$  in  $W$ .*

*Proof.* Lemma 23 states that  $\mathcal{F}_{\mathcal{A}} \cap V$  generates  $\mathcal{F}_{\mathcal{A}}$  in  $W$  so it remains to show that  $B$  is a maximal free sequence in  $W$ . Take  $X \subset \omega$  in  $W$ , we need

to show that  $B$  cannot be end-extended by  $X$ . If  $X \in \mathcal{F}_{\mathcal{A}}^*$ , we are done so suppose this is not the case. Since  $\mathcal{A}$  is dense in  $W$ , there is  $h \in \mathbf{C}_{\mathcal{A}}$  such that  $\mathcal{A}^h \subset^* X$ . As  $\mathcal{A}^h \in V$ ,  $\mathcal{A}^h \notin \mathcal{F}_{\mathcal{A}}^*$ , and  $B$  cannot be end-extended by  $\mathcal{A}^h$ , there is  $b \in \text{comb}(B)$  such that  $b \subset^* \mathcal{A}^h$ . Now  $X^0 \cap b =^* \emptyset$  witnesses that  $B$  cannot be end-extended by  $X$ .  $\square$

**Proposition 28.** *Assume  $\mathfrak{t} = \mathfrak{c}$  and let  $\mathcal{T}$  be tower. There is a maximal decreasing free sequence  $\{a_\alpha \mid \alpha \in \mathfrak{c}\}$  which is cofinal with  $\mathcal{T}$ .*

*Proof.* Let  $\mathcal{F}$  be the filter generated by  $\mathcal{T}$ . If  $\mathcal{F}$  is an ultrafilter, we are done. If this is not the case, fix an enumeration  $\{X_\alpha \mid \alpha \in \mathfrak{c}, \alpha \text{ even}\}$  of  $\mathcal{P}(\omega) \setminus (\mathcal{F} \cup \mathcal{F}^*)$ . We construct the tower  $\{a_\alpha \mid \alpha \in \mathfrak{c}\}$  cofinal in  $\mathcal{T}$  by induction. If  $\beta < \mathfrak{c}$  is even and  $a_\alpha$  is defined for each  $\alpha < \beta$ , find  $t \in \mathcal{T}$  such that  $t \subset^* a_\alpha$  and  $a_\alpha \setminus t$  is infinite for each  $\alpha < \beta$ , and let  $a_\beta = t$  (choose  $a_0 \in \mathcal{T}$  arbitrary). Then find  $s \in \mathcal{T}$  such that  $(t \setminus s) \cap X_\beta$  is infinite (use the assumptions on  $\mathcal{T}$  and  $X_\beta$ ) and let  $a_{\beta+1} = s \cup (t \setminus X_\beta)$ . Notice that  $a_\beta \setminus a_{\beta+1}$  is an infinite subset of  $X_\beta$ . Now it is easy to check that the sequence we defined is a maximal free sequence.  $\square$

**Corollary 29.** *Assume CH. For every selective independent system  $\mathcal{A}$  there exists free sequence  $B$  associated with  $\mathcal{A}$ .*

**Theorem 30.** *It is consistent that  $\omega_1 = \mathfrak{i} = \mathfrak{f} < \mathfrak{u} = \mathfrak{c} = \omega_2$ .*

*Proof.* Start in a model of CH and run a countable support iteration of length  $\omega_2$  of posets of form  $\mathbf{Q}_{\mathcal{A}}$  with the parameter  $\mathcal{A}$  ranging over all maximal ideals on  $\omega$  in all intermediate models. Lemma 16 together with the usual reflection argument implies that the final generic extension does not contain any ultrafilter base of size  $\omega_1$ , i.e.  $\mathfrak{u} = \mathfrak{c} = \omega_2$ .

Use Proposition 25 to find a selective independent system in the ground-model. Theorem 26 states that the independent system remains selective in all successor stages of the iteration and Theorem 32 together with Remark 24 ensure that it remains selective also in limit stages of the iteration. Thus the ground-model independent system remains selective and in particular maximal in the final extension,  $\mathfrak{i} = \omega_1$ . Finally use Corollary 29 in the ground-model to find a free sequence associated with a selective independent system. Theorem 27 states that this free sequence is still maximal in the final generic extension,  $\mathfrak{f} = \omega_1$ .  $\square$

It is worth noting that in the resulting model all the usually considered cardinal characteristics of the continuum, except of  $\mathfrak{u}$ , are equal to  $\omega_1$ . For a this was proved by Guzmán [GG].

#### APPENDIX: PRESERVATION THEOREM FOR THE ITERATION

The forcing iteration argument in Section 5 uses a typical preservation theorem for countable support forcing iteration, in this instance



the preservation of a filter–co-filter pair. This theorem follows the usual pattern described in [She98, Gol93]. However, as specific instances of preservation theorems are sometimes difficult to derive from the general statements given in these sources, we decided to provide the proof of the relevant preservation theorem in this appendix, making the paper more self-contained.

Let  $\mathcal{F}$  be a filter on  $\omega$ . We will use the following game  $G(\mathcal{F})$ . Players I and II alternate for  $\omega$  many rounds. In the  $n$ -th round player I plays a set  $F_n \in \mathcal{F}$ , and player II responds with  $a_n \in F_n$ . Player II wins if  $\{a_n \mid n \in \omega\} \in \mathcal{F}$ . The following is well known.

**Fact 31.** Player I does not have a winning strategy in the game  $G(\mathcal{F})$  iff  $\mathcal{F}$  is a rare P-filter.

**Theorem 32.** Let  $\mathcal{F}$  be a P-filter on  $\omega$ , denote  $\mathcal{K} = \mathcal{P}(\omega) \setminus \mathcal{F}$ . For  $\delta$  limit let  $P_\delta = \langle P_\alpha, Q_\alpha \mid \alpha < \delta \rangle$  be a countable support iteration of proper forcing notions such that for each  $\alpha < \delta$

$$P_\alpha \Vdash \mathcal{F} \text{ is a rare filter and } \langle \mathcal{F} \rangle \cup \langle \mathcal{K} \rangle = \mathcal{P}(\omega).$$

Then also  $P_\delta \Vdash \langle \mathcal{F} \rangle \cup \langle \mathcal{K} \rangle = \mathcal{P}(\omega)$ .

By  $\langle \mathcal{F} \rangle$  and  $\langle \mathcal{K} \rangle$  we denote the upwards, respectively downwards closure of  $\mathcal{F}$  and  $\mathcal{K}$  in the appropriate models. The assumption for  $\alpha = 0$  states that  $\mathcal{F}$  is a rare P-filter in the ground model  $V$ . Standard arguments shows that  $\langle \mathcal{F} \rangle$  is a P-filter in any generic extension via a proper forcing, and  $\langle \mathcal{F} \rangle$  is rare in any generic extension via a bounding forcing.

*Proof.* If the cofinality of  $\delta$  is uncountable, no new reals are added at stage  $\delta$  of the iteration, and the conclusion of the theorem holds true. Therefore we will assume that the cofinality of  $\delta$  is countable, and by passing to a cofinal sequence of  $\delta$ , it is sufficient to prove the theorem in case  $\delta = \omega$ . In the following  $G_\alpha$  denotes exclusively generic filters on  $P_\alpha$ . We use  $P$  to denote posets  $P_\delta/G_\alpha$  in the intermediate generic extensions  $V[G_\alpha]$ . Let  $X$  be a  $P$ -name for a subset of  $\omega$ . For  $r \in P$  denote  $X_r = \{n \in \omega \mid r \Vdash n \in X\}$ .

**Lemma 33.** Let  $\mathcal{H}$  be a rare P-filter and  $p \in P$  a condition. If  $X_r \in \mathcal{H}$  for each  $r < p$ , then there exists  $H \in \mathcal{H}$  and a sequence  $\langle r_i \in P \mid i \in \omega \rangle$ ,  $r_0 = p$ ,  $r_{i+1} < r_i$  such that  $r_i \Vdash H \cap i \subset X$  for each  $i \in \omega$ .

*Proof.* Put  $p_0 = p$  and let play the game  $G(\mathcal{H})$  as follows. In the  $n$ -th round player I plays the set  $X_{p_n} \in \mathcal{H}$ , player II responds with some  $a_n \in X_{p_n}$ . Player I then chooses  $p_{n+1} \in P$ ,  $p_{n+1} < p_n$  such that  $p_{n+1} \Vdash a_n \in X$  and proceeds to the next round. Since  $\mathcal{H}$  is a rare P-filter, this strategy is not winning for player I. Thus there is a sequence of moves of player II and conditions  $\langle p_n \mid n \in \omega \rangle$  such that player II wins the game, i.e.  $H = \{a_n \mid n \in \omega\} \in \mathcal{H}$ . A sequence of conditions  $\langle r_i \mid i \in \omega \rangle$  such that  $r_i = p_{a_n}$  if  $a_n < i \leq a_{n+1}$  is as required in the lemma.  $\square$

Let  $p$  be a condition in  $P_\omega$ . The goal is to find a stronger condition which forces either  $X \in \langle \mathcal{F} \rangle$  or  $X \in \langle \mathcal{H} \rangle$ . In case there exists an intermediate extension  $V[G_\alpha]$ ,  $p \in G_\alpha$  and  $r \in P/G_\alpha$ ,  $r < p/G_\alpha$  such that  $X_r \notin \langle \mathcal{F} \rangle$  (in  $V[G_\alpha]$ ), then  $r \Vdash X \in \langle \mathcal{H} \rangle$  due to the assumption of the theorem, and there exists a condition in  $P_\omega$  stronger than  $p$  forcing the same statement. Therefore we will assume in the rest of the proof that this is not the case.

For a sufficiently large  $\theta$  fix a countable elementary submodel  $N \prec H(\theta)$  such that  $X, p, \mathcal{F}, P_\omega \in N$ . Use Lemma 33 in  $N$  for  $\mathcal{H} = \mathcal{F}$  and  $P = P_\omega$  to get  $H \in \mathcal{F} \cap N$  and a sequence  $\langle r_n^0 \in P_\omega \mid n \in \omega \rangle \in N$ . Since  $\mathcal{F}$  is a P-filter, there exists  $A^* \in \mathcal{F}$  such that  $A^* \subset H$ , and  $A^* \subset^* F$  for each  $F \in \mathcal{F} \cap N$ .

**Lemma 34.** *Let  $q$  be a  $(P_i, N)$ -master condition, and let  $\langle F_n \mid n \in \omega \rangle \in N[G_i]$  be a sequence of elements of  $\mathcal{F}$ . Then*

$q \Vdash$  *There are infinitely many  $n \in \omega$  such that  $A^* \setminus n \subset F_n$ .*

*Proof.* Since  $N[G_i] \prec H(\theta)[G_i]$  and  $\mathcal{F}$  generates a non-meager filter in  $H(\theta)[G_i]$ , there is  $F \in \mathcal{F} \cap N[G_i]$  such that  $F \setminus n \subset F^n$  for infinitely many  $n$  (Fact 12). Now  $q \Vdash F \in N$  and we can use that  $A^* \subset^* F$ .  $\square$

We will inductively construct a condition  $q < p$  such that  $q \Vdash A^* \subset X$ . Specifically, we construct two sequences of conditions  $p_i, q_i$  for  $i \in \omega$  with the following properties;

- (1)
  - $p_i \in P_\omega$ ,
  - $p_{i+1} < p_i$ ,
  - $p_{i+1} \upharpoonright i = p_i \upharpoonright i$ ,
  - $q_i \in P_i$ ,
  - $q_{i+1} \upharpoonright i = q_i$ ,
  - $q_i < p_i \upharpoonright i$ ,
  - $q_i$  is a  $(N, P_i)$ -master condition;
- (2)  $q_i \Vdash (p_i/G_i \Vdash A^* \cap i \subset X)$ ,
- (3)  $q_i \Vdash$  (There is a sequence  $\langle r_n^i \in P_\omega/G_i \mid n \in \omega \rangle \in N[G_i]$ ,  $r_n^i < p_i/G_i$  such that  $r_n^i \Vdash A^* \cap n \subset X$ ).

The construction starts with putting  $p_0 = p$  and let  $q_0$  be a trivial condition (in the trivial forcing  $P_0$ ). Existence of the sequence  $\langle r_n^0 \in P_\omega \mid n \in \omega \rangle$  follows from the choice of  $A^*$ .

Suppose that  $p_i, q_i$  are defined, work in  $N[G_i]$  assuming  $q_i \in G_i$ . For each  $n \in \omega$  consider a model  $N[G_{i+1}]$  such that  $r_n^i \upharpoonright (i+1) \in G_{i+1}/G_i$ . Use Lemma 33 in  $N[G_{i+1}]$  for  $\langle \mathcal{F} \rangle$  and  $r_n^i/G_{i+1}$  to get  $H_n \in \langle \mathcal{F} \rangle \cap N[G_{i+1}]$  and a sequence  $\langle s_k^n \in P_\omega/G_{i+1} \mid k \in \omega \rangle \in N[G_{i+1}]$  as in the lemma. We can assume that  $H_n \in \mathcal{F} \cap N[G_{i+1}]$ , and by strengthening  $r_n^i \upharpoonright \{i\}$  to  $t_n^i \upharpoonright \{i\} \in N[G_i]$  we can decide  $H_n$  to be some  $F_n \in \mathcal{F} \cap N[G_i]$ . Since  $q_i$  is  $(N, P_i)$ -master, Lemma 34 implies that there is  $m > i$  such that  $A^* \setminus m \subset F_m$ .

Define  $p_{i+1} = p_i \upharpoonright i \hat{\ } t_m^i$ , and let  $q_{i+1} < p_{i+1} \upharpoonright i + 1$  be any  $(N, P_{i+1})$ -master condition such that  $q_{i+1} \upharpoonright i = q_i$ . Property (1) is obviously satisfied. Property (2) follows from  $m > i$ , the inductive hypothesis for  $r_m^i$ , and from  $q_{i+1} \hat{\ } (p_{i+1}/G_{i+1}) < q_i \hat{\ } r_m^i$ . To justify (3) notice that  $q_{i+1}$  forces that the sequence  $\langle s_k^m \mid k \in \omega \rangle$  satisfies the condition required for  $\langle r_n^i \mid n \in \omega \rangle$ ; for  $y \in A^* \cap m$  this follows from the inductive hypothesis on  $r_m^i$ , and for  $y \in A^*, x \geq m$  from the choice of  $\langle s_k^m \mid k \in \omega \rangle$  and  $A^* \setminus m \subset F_m$ .

Once the inductive construction is done, the condition  $q = \bigcup \{q_i \mid i \in \omega\}$  forces that  $A^* \subset X$ . The inclusion  $A^* \cap i \subset X$  is guaranteed by property (2) and  $q < q_i \hat{\ } (p_i/G_i)$ .  $\square$

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