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Universal Proof Theory: Semi-analytic Rules and Uniform Interpolation

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Abstract

In [7] and [8], Iemhoff introduced a connection between the existence of a terminating sequent calculi of a certain kind and the uniform interpolation property of the super-intuitionistic logic that the calculus captures. In this paper, we will generalize this relationship to also cover the substructural setting on the one hand and a much more powerful class of rules on the other. The resulted relationship then provides a uniform method to establish uniform interpolation property for the logics \mathbf{FL}_e , \mathbf{FL}_{ew} , \mathbf{CFL}_e , \mathbf{CFL}_{ew} , \mathbf{IPC} , \mathbf{CPC} and their \mathbf{K} and \mathbf{KD} -type modal extensions. More interestingly though, on the negative side, we will show that no extension of \mathbf{FL}_e can enjoy a certain natural type of terminating sequent calculus unless it has the uniform interpolation property. It excludes almost all super-intuitionistic logics and the logics $\mathbf{K4}$ and $\mathbf{S4}$ from having such a reasonable calculus.

1 Introduction

Proof systems are and always have been the main tool in any investigation of the behavior of the mathematical theories from searching for the consistency proofs and finding the possible decision procedures to capturing the admissible rules and extracting the actual programs from given proofs. Following this huge effectiveness, a technical approach has emerged to first design and

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then study the appropriate proof systems tailored for their sole use in proving the properties of a given interesting theory. In this respect, proof systems have been treated as the second rank citizens contrary to the independent interesting mathematical objects that they could have been. Fortunately, in the recent years, alongside this instrumentalist approach, another approach has been also emerged; an approach that is more interested in the general behavior of the proof systems than their possible technical use in proof theory, although it happens to bring its own fruits in the latter aspect, as well (see [7], [8], [3]). This general approach widens the proof theoretic horizon with its own structural problems including *the existence problem* (when does a theory have a certain type of proof system?), *the equivalence problem* (when are two proof systems equivalent?) and *the characterization problem* (is there any characterization of the proof systems relative to a natural equivalence relation?). Imitating the term universal algebra for the generic study of the algebraic structures, we will call this approach *the universal proof theory*¹ which focuses on the model theoretic style investigation of the different possible proof systems in their most general form.

As the first step in this so-called universal proof theory and following the spirit of [7] and [8], we begin with the most basic problem of the kind, the *existence problem*, addressing the existence of the natural sequent style proof systems for a given propositional or modal logic. For this purpose, we have to develop some strong relationships between the existence of some sort of proof systems and some regularity conditions of the logic. One loose example of such a relationship is the relationship between the existence of a terminating calculus for a logic and its decidability. Why these relationships are important? Because they reduce the existence problem partially or completely to the regularity conditions of the logic that are calculus-independent and probably more amenable to our tools. Again using our loose example, we know that if a logic is not decidable, it can not have a terminating calculus; a fact which solves the existence problem negatively.

This paper is devoted to one of this kind of relationships and to explain how, we have to browse the history a little bit, first. The story begins with Pitts' seminal work, [9], in which he introduced a proof theoretic method to prove the uniform interpolation property for the propositional intuitionistic logic. His technique is built on the following two main ideas: First he extended the notion of uniform interpolation from a logic to its sequent calculus in a way that the uniform p -interpolants for a sequent are roughly the best

¹We are grateful to Masoud Memarzadeh for this elegant terminological suggestion.

left and right p -free formulas that if we add them to the left or right side of the sequent, they make the sequent provable. This reduces the task of proving uniform interpolation for the logic, to the task of finding these new uniform interpolants for all sequents. For the latter, he assigned two sets of p -free formulas to any sequent using the structure of the formulas occurred in the sequent itself. To define these sets, though, he needed the second crucial tool of the game namely the terminating calculus for **IPC**, introduced in [4] by Dyckhoff. The terminating calculus provides a well-founded order on sequents on which we can define the sets that we have mentioned before, recursively.

Later, as witnessed in [8], Iemhoff recognized that the main point in the first part of Pitts' argument is flexible enough to apply on any rule with a certain general form. This observation then lets her to lift the technique from the intuitionistic logic to any extension of the intuitionistic logic presented with a generic terminating calculus consisting of that certain sort of rules that she calls centered rules. These rules are very natural rules to consider and they are roughly the rules with one main formula in their consequence such that the rule respects both the side of this main formula and the occurrence of atoms in it, i.e. if the main formula occurred in the left-side (right-side) of the consequence, all non-contextual formulas in the premises should also occur in the left-side (right-side) and any occurrence of any atom in these formulas should also occur in the main formula. The usual conjunction and disjunction rules are the prototype examples of these rules while the implication rules are the non-examples since they clearly do not respect the side of the main formula.

As we explained, the investigations in [8] lead to an exciting relationship between the existence of a terminating calculus consisting only of the centered rules for a logic and the uniform interpolation property of the logic. Iemhoff used this relationship first in a positive manner to prove the uniform interpolation for some well-known super-intuitionistic and super-intuitionistic modal logics including **IPC**, **CPC**, **K** and **KD** and their intuitionistic versions. And then she switched to the negative part to show that no centered extension of the intuitionistic logic can have a terminating centered calculus unless it has the uniform interpolation property. Since uniform interpolation is a rare property for a logic, it excludes almost all logical systems, including all super-intuitionistic logics except the seven logics with the uniform interpolation property from having a terminating centered calculus.

Now we are ready to explain what we will pursue in this paper. Our ap-

proach is a generalization of the mentioned relationship between the existence of a terminating calculus consisting of certain sort of rules and the uniform interpolation property. Our results are the generalization of the results in [7] and [8], in the following two aspects. First we use a much more general class of rules that we will call semi-analytic and context-sharing semi-analytic rules. These rules can be defined roughly as the centered rules relaxing the side preserving condition. Therefore, they cover a vast variety of rules including centered rules, implication rules, non-context sharing rules in substructural logics and so many others. Second, we lower the base logic from the intuitionistic logic to the basic substructural logic \mathbf{FL}_e . It helps to provide a uniform method to establish the uniform interpolation property which is applicable simultaneously for \mathbf{FL}_e , \mathbf{FL}_{ew} , \mathbf{CFL}_e , \mathbf{CFL}_{ew} and their \mathbf{K} and \mathbf{KD} modal extensions on the one hand and the intuitionistic and classical logics and their modal extensions on the other. (For the classical modal case see [2], for the sub-structural logics see [1] and for intuitionistic and intuitionistic modal logics see [9] and [8].) It also sets the scene to provide the same characterization for any semi-analytic extensions of \mathbf{FL}_e if we first provide a terminating calculus for them.

While it is very appealing to develop a general method to prove uniform interpolation, the main application of our investigation belongs to the negative side of the relationship. Applying our result negatively, we can also push the result in [8] further to show that the logics without uniform interpolation property can not even have our more general type of terminating calculi. For instance, using the well-known result on the characterization of all super-intuitionistic logics, [6], we know that except \mathbf{IPC} , \mathbf{LC} , \mathbf{KC} , \mathbf{Bd}_2 , \mathbf{Sm} , \mathbf{GSc} and \mathbf{CPC} , non of the super-intuitionistic logics have a terminating calculus consisting of semi-analytic and context-sharing semi-analytic rules together with the centered axioms. The same also goes for the modal logics $\mathbf{K4}$ and $\mathbf{S4}$.

2 Preliminaries

In this section we will cover some of the preliminaries needed for the following sections. The definitions are similar to the same concepts in [8], but they have been changed whenever it is needed.

First, note that all of the finite objects that we will use here can be represented by a fixed reasonable binary string code. Therefore, by the length of any object O including formulas, proofs, etc. we mean the length of this

string code and we will denote it by $|O|$.

Definition 2.1. Let \mathcal{L} and \mathcal{L}' be two languages. By a *translation* $t : \mathcal{L} \rightarrow \mathcal{L}'$, we mean an assignment which assigns a formula $\phi_C(\bar{p}) \in \mathcal{L}'$ to any logical connective $C(\bar{p}) \in \mathcal{L}$ such that any p_i has at most one occurrence in $\phi_C(\bar{p})$. It is possible to extend a translation from the basic connectives of the language to all of its formulas in an obvious compositional way. We will denote the translation of a formula ϕ by ϕ^t and the translation of a multiset Γ , by $\Gamma^t = \{\phi^t \mid \phi \in \Gamma\}$.

Note that for any translation t we have $|\psi^t| \leq O(1)|\psi|$ which shows that all translations are polynomially bounded.

In this paper, we will work with a fixed but arbitrary language \mathcal{L} that is augmented by a translation $t : \{\wedge, \vee, \rightarrow, *, 0, 1\} \cup \mathcal{L} \rightarrow \mathcal{L}$ that fixes all logical connectives in \mathcal{L} . For this reason and w.l.o.g, we will assume that the language already includes the connectives $\{\wedge, \vee, \rightarrow, *, 0, 1\}$. In addition, whenever we investigate the multi-conclusion systems we always assume that the translation expands to include $+$.

Example 2.2. The usual language of classical propositional logic is a valid language in our setting. In this case, there is a *canonical translation* that sends fusion, addition, 1 and 0 to conjunction, disjunction, \top and \perp , respectively. In this paper, whenever we pick this language, we assume that we are working with this canonical translation.

By a sequent, we mean an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulas in the language, and it is interpreted as $*\Gamma \rightarrow \dagger\Delta$. By a single-conclusion sequent $\Gamma \Rightarrow \Delta$ we mean a sequent that $|\Delta| \leq 1$, and we call it multi-conclusion otherwise. We denote multisets by capital Greek letters such as $\Sigma, \Gamma, \Pi, \Delta$ and Λ . However, sometimes we use the bar notation for multisets to make everything simpler. For instance, by $\bar{\phi}$, we mean a multiset consisting of formulas ϕ_i .

Meta-language is the language with which we define the sequent calculi. It extends our given language with the formula symbols (variables) such as ϕ and ψ . A meta-formula is defined as the following: Atomic formulas and formula symbols are meta-formulas and if $\bar{\phi}$ is a set of meta-formulas, then $C(\bar{\phi})$ is also a meta-formula, where $C \in \mathcal{L}$ is a logical connective of the language. Moreover, we have infinitely many variables for meta-multisets and we use capital Greek letters again for them, whenever it is clear from the context whether it is a multiset or a meta-multiset variable. A meta-multiset is a

multiset of meta-formulas and meta-multiset variables. By a meta-sequent we mean a sequent where the antecedent and the succedent are both meta-multisets. We use meta-multiset variable and context, interchangeably.

For a meta-formula ϕ , by $V(\phi)$ we mean the meta-formula variables and atomic constants in ϕ . A meta-formula ϕ is called p -free, for an atomic formula or meta-formula variable p , when $p \notin V(\phi)$.

Let us recall some of the notions related to sequent calculi and some of the important systems that we will use throughout the paper.

For a sequent $S = (\Gamma \Rightarrow \Delta)$, by S^a we mean the antecedent of the sequent, which is Γ , and by S^s we mean the succedent of the sequent, which is Δ . And, the multiplication of two sequents S and T is defined as $S \cdot T = (S^a \cup T^a \Rightarrow T^s \cup S^s)$.

By a rule we mean an expression of the form

$$\frac{S_1, \dots, S_n}{S_0}$$

where S_i 's are meta-sequents. By an instance of a rule, we mean substituting multisets of formulas for its contexts and substituting formulas for its meta-formula variables. A rule is backward applicable to a sequent S , when the conclusion of the rule is S .

By a sequent calculus G , we mean a set of rules. A sequent S is derivable in G , denoted by $G \vdash S$, if there exists a tree with sequents as labels of the nodes such that the label of the root is S and in each node the set of the labels of the children of the node together with the label of the node itself, constitute an instance of a rule in the system. This tree is called the proof of S in G which is sometimes called a tree-like proof to emphasize its tree-like form.

Now consider the following set of rules:

Identity:

$$\overline{\phi \Rightarrow \phi}$$

Contextual Axioms:

$$\overline{\Gamma \Rightarrow \top, \Delta} \quad \overline{\Gamma, \perp \Rightarrow \Delta}$$

Context-free Axioms:

$$\overline{\Rightarrow 1} \quad \overline{0 \Rightarrow}$$

Rules for 0 and 1:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta} L1 \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta} R0$$

Conjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} L\wedge \quad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} R\wedge$$

Disjunction Rules:

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} L\vee \quad \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee \quad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} R\vee$$

Fusion Rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi * \psi \Rightarrow \Delta} L* \quad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \phi * \psi, \Delta, \Lambda} R*$$

Implication Rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \rightarrow \psi \Rightarrow \Delta, \Lambda} L\rightarrow \quad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta} R\rightarrow$$

The system \mathbf{FL}_e consists of the single-conclusion version of all of these rules. In the multi-conclusion case define \mathbf{CFL}_e with the same rules as \mathbf{FL}_e , this time in their full multi-conclusion version and add $+$ to the language and the following rules to the systems:

Rules for $+$:

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi + \psi \Rightarrow \Delta, \Lambda} L+ \quad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi + \psi, \Delta} R+$$

Moreover, we have the following additional rules that we will use later:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} Lw \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} Rw$$

Note that in the single-conclusion cases, in the rule (*Rw*), Δ is empty.

Contraction rules:

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} Lc \quad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \phi, \Delta} Rc$$

The rule (*Rc*) is only allowed in multi-conclusion systems.

If we consider the logic \mathbf{FL}_e and add the weakening rules (contraction rules), the resulted system is called \mathbf{FL}_{ew} (\mathbf{FL}_{ec}). The same also goes for \mathbf{CFL}_{ew} and \mathbf{CFL}_{ec} .

We also have the following rule:

Context-sharing left implication:

$$\frac{\Gamma \Rightarrow \phi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \rightarrow \psi \Rightarrow \Delta}$$

Finally, note that Γ and Δ are multisets everywhere, therefore the exchange rule is built in and hence admissible in our system. Moreover, note that the calculi defined in this section are written in the given language which can be any extension of the language of the system itself. For instance, \mathbf{FL}_e is the calculus with the mentioned rules on our fixed language that can have more connectives than $\{\wedge, \vee, *, \rightarrow, \top, \perp, 1, 0\}$.

Definition 2.3. We will define the sequent calculus for intuitionistic logic, which was first introduced by Dyckhoff in [4].

$$\begin{array}{c} \Gamma, p \Rightarrow p \quad At \quad , \quad \Gamma, \perp \Rightarrow \Delta \quad L\perp \\ \\ \frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} L\wedge \quad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta} R\wedge \\ \\ \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} L\vee \quad \frac{\Gamma \Rightarrow \phi, \psi}{\Gamma \Rightarrow \phi \vee \psi} R\vee \end{array}$$

$$\begin{array}{c}
\frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \phi \rightarrow \psi} R \rightarrow \\
\frac{\Gamma, p, \psi \Rightarrow \Delta}{\Gamma, p, p \rightarrow \psi \Rightarrow \Delta} L_1 \rightarrow \frac{\Gamma, \phi \rightarrow (\psi \rightarrow \gamma) \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \rightarrow \gamma \Rightarrow \Delta} L_2 \rightarrow \\
\frac{\Gamma, \phi \rightarrow \gamma, \psi \rightarrow \gamma \Rightarrow \Delta}{\Gamma, \phi \vee \psi \rightarrow \gamma \Rightarrow \Delta} L_3 \rightarrow \frac{\Gamma, \psi \rightarrow \gamma \Rightarrow \phi \rightarrow \psi \quad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \rightarrow \psi) \rightarrow \gamma \Rightarrow \Delta} L_4 \rightarrow
\end{array}$$

where p is an atom. Structural rules and the cut rule are admissible in the system and we have $|\Delta| \leq 1$.

Definition 2.4. A calculus is *terminating* if for any sequent S , the number of rules which are backward applicable to S are finite. Moreover, there is a well-founded order on the sequents such that the order of the following are less than the order of S :

- the premises of a rule whose conclusion is S ;
- subsequents of S , and
- any sequent S' of the form $(\Gamma, \Pi \Rightarrow \Delta, \Lambda)$, where S is of the form $(\Gamma, \Box\Pi \Rightarrow \Delta, \Box\Lambda)$. Note that $\Pi \cup \Lambda$ must be non-empty.

Definition 2.5. Let L and L' be two logics such that $\mathcal{L}_L \subseteq \mathcal{L}_{L'}$. We say L' is an *extension* of L if $L \vdash A$ implies $L' \vdash A$.

Definition 2.6. Let G and H be two sequent calculi such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. We say H is an *extension* of G if $G \vdash \Gamma \Rightarrow \Delta$ implies $H \vdash \Gamma \Rightarrow \Delta$. It is called an *axiomatic extension*, if the provable sequents in G are considered as axioms of H , to which H adds some rules.

Definition 2.7. Let G be a sequent calculus and L be a logic with the same language as G 's. We say G is a *sequent calculus for the logic L* when:

$$G \vdash \Gamma \Rightarrow \Delta \quad \text{if and only if} \quad L \vdash (*\Gamma \rightarrow +\Delta).$$

Note that if the calculus is single-conclusion, by $+\Delta$, we mean Δ if Δ is a singleton, and 0 if Δ is empty. Therefore, in this case we do not need the $+$ operator.

Theorem 2.8. *Let L be a logic and G a single-conclusion (multi-conclusion) sequent calculus for L . If L extends \mathbf{FL}_e (\mathbf{CFL}_e), then cut is admissible in G .*

Proof. Assume that $G \vdash \Gamma \Rightarrow A, \Delta$ and $G \vdash \Gamma', A \Rightarrow \Delta'$. Hence $L \vdash * \Gamma \rightarrow A + (\perp \Delta)$ and $L \vdash (* \Gamma') * A \rightarrow (\perp \Delta')$. Since L extends \mathbf{FL}_e (\mathbf{CFL}_e) and in this theory the formula

$$[* \Gamma \rightarrow A + (\perp \Delta)] * [(* \Gamma') * A \rightarrow (\perp \Delta')]$$

implies the formula

$$[(* \Gamma) * (* \Gamma') \rightarrow (\perp \Delta) + (\perp \Delta')]$$

the last formula is provable in L which implies $G \vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. \square

Definition 2.9. We say a logic L has Craig interpolation property if for any formulas ϕ and ψ if $L \vdash \phi \rightarrow \psi$, then there exists formula θ such that $L \vdash \phi \rightarrow \theta$ and $L \vdash \theta \rightarrow \psi$ and $V(\theta) \subseteq V(\phi) \cap V(\psi)$.

Definition 2.10. We say a logic L has the uniform interpolation property if for any formulas ϕ and any atomic formula p , there are two p -free formulas, the p -pre-interpolant, $\forall p\phi$ and the p -post-interpolant $\exists p\phi$, such that

- (i) $L \vdash \forall p\phi \rightarrow \phi$,
- (ii) For any p -free formula ψ if $L \vdash \psi \rightarrow \phi$ then $L \vdash \psi \rightarrow \forall p\phi$,
- (iii) $L \vdash \phi \rightarrow \exists p\phi$, and
- (iv) For any p -free formula ψ if $L \vdash \phi \rightarrow \psi$ then $L \vdash \exists p\phi \rightarrow \psi$.

3 Semi-analytic Rules

In this section we will introduce a class of rules which we will investigate in the rest of this paper. First let us begin with the single-conclusion case in which all sequents have at most one succedent.

Definition 3.1. A rule is called a *left semi-analytic rule* if it is of the form

$$\frac{\langle \langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle \langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi_1, \dots, \Pi_m, \Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

where Π_j , Γ_i and Δ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{j,s} V(\bar{\psi}_{js}) \cup \bigcup_{j,s} V(\bar{\theta}_{js}) \subseteq V(\phi)$$

and it is called a *right semi-analytic rule* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \phi}$$

where Γ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,r} V(\bar{\psi}_{ir}) \subseteq V(\phi)$$

Moreover, a rule is called a *context-sharing semi-analytic* rule if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

where Γ_i and Δ_i 's are meta-multiset variables and

$$\bigcup_{i,r} V(\bar{\phi}_{ir}) \cup \bigcup_{i,s} V(\bar{\psi}_{is}) \cup \bigcup_{i,s} V(\bar{\theta}_{is}) \subseteq V(\phi)$$

We will call the conditions for the variables in all the semi-analytic rules, the *occurrence preserving* conditions. Note that in the left rule, for each i we have $|\Delta_i| \leq 1$, and since the size of the succedent of the conclusion of the rule must be at most 1, it means that at most one of Δ_i 's can be non-empty.

For the multi-conclusion case, we define a rule to be *left multi-conclusion semi-analytic* if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n}$$

with the same occurrence preserving condition as above and the same condition that Γ_i 's and Δ_i 's are meta-multiset variables. A rule is defined to be a *right multi-conclusion semi-analytic* rule if it is of the form

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i\rangle_r\rangle_i}{\Gamma_1, \dots, \Gamma_n \Rightarrow \phi, \Delta_1, \dots, \Delta_n}$$

again with the similar occurrence preserving condition and the same condition that Γ_i 's and Δ_i 's are meta-multiset variables. Whenever it is clear from the context, we will omit the phrase “multi-conclusion”.

A rule is called *modal semi-analytic* if it has one of the following forms:

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi} K \quad \frac{\Gamma \Rightarrow}{\Box\Gamma \Rightarrow} D$$

where Γ is a meta-multiset variable. Note that we always have the condition that whenever the rule (D) is present, the rule (K) must be present, as well. In the case of the modal rules, we use the convention that $\Box\emptyset = \emptyset$.

By the notation $\langle\langle\cdot\rangle_r\rangle_i$ we mean first considering the sequents ranging over r and then ranging over i . For instance, $\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i$ is short for the following set of sequents where $1 \leq r \leq m_i$ and $1 \leq i \leq n$:

$$\begin{aligned} \Gamma_1, \bar{\phi}_{11} \Rightarrow \bar{\psi}_{11}, \dots, \Gamma_1, \bar{\phi}_{1m_1} \Rightarrow \bar{\psi}_{1m_1}, \\ \Gamma_2, \bar{\phi}_{21} \Rightarrow \bar{\psi}_{21}, \dots, \Gamma_2, \bar{\phi}_{2m_2} \Rightarrow \bar{\psi}_{2m_2}, \\ \vdots \\ \Gamma_n, \bar{\phi}_{n1} \Rightarrow \bar{\psi}_{n1}, \dots, \Gamma_n, \bar{\phi}_{nm_n} \Rightarrow \bar{\psi}_{nm_n}. \end{aligned}$$

$\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i$ and $\langle\langle\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j$ are defined similarly.

Both in the single-conclusion and multi-conclusion case, a rule is called *semi-analytic*, if it is either a left semi-analytic rule, a right semi-analytic rule or it is of the form of a semi-analytic modal rule. In all the semi-analytic rules, the meta-variables and atomic constants occurring in the meta-formulas of the premises of the rule, should also occur in the meta-formulas in the consequence. Because of this condition, we call these rules semi-analytic. This occurrence preserving condition is a weaker version of the analyticity property in the analytic rules, which demands the formulas in the premises to be sub-formulas of the formulas in the consequence.

Example 3.2. A generic example of a left semi-analytic rule is the following:

$$\frac{\Gamma, \phi_1, \phi_2 \Rightarrow \psi \quad \Gamma, \theta \Rightarrow \eta \quad \Pi, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \Pi, \alpha \Rightarrow \Delta}$$

where

$$V(\phi_1, \phi_2, \psi, \theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

and a generic example of a context-sharing left semi-analytic rule is:

$$\frac{\Gamma, \theta \Rightarrow \eta \quad \Gamma, \mu_1, \mu_2, \mu_3 \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$$

where

$$V(\theta, \eta, \mu_1, \mu_2, \mu_3) \subseteq V(\alpha)$$

Moreover, for a generic example of a right semi-analytic rule we can have

$$\frac{\Gamma, \phi \Rightarrow \psi \quad \Gamma, \theta_1, \theta_2 \Rightarrow \eta \quad \Pi, \mu_1, \mu_2, \Rightarrow \nu}{\Gamma, \Pi \Rightarrow \alpha}$$

where

$$V(\phi, \psi, \theta_1, \theta_2, \eta, \mu_1, \mu_2, \nu) \subseteq V(\alpha)$$

Here are some remarks. First note that in any left semi-analytic rule there are two types of premises; the type whose right hand-side includes meta-multi variables and the type whose right hand-side includes meta-formulas. This is a crucial point to consider. Any left semi-analytic rule allows any kind of combination of sharing/combining contexts in any type. However, between two types, we can only combine the contexts. The case in which we can share the contexts of the two types is called context-sharing semi-analytic rule. This should explain why our second example is called context-sharing left semi-analytic while the first is not. The reason is the fact that the two types share the same context in the second rule while in the first one this situation happens in just one type. The second point is the presence of contexts. This is very crucial for almost all the arguments in this paper, that any sequent present in a semi-analytic rule should have meta-multiset variables as left contexts and in the case of left rules, at least one meta-multiset variable for the right hand-side must be present.

Example 3.3. Now for more concrete examples, note that all the usual conjunction, disjunction and implication rules for **IPC** are semi-analytic. The same also goes for all the rules in sub-structural logic **FL_e**, the weakening and the contraction rules and some of the well known restricted versions of them including the following rules for exponentials in linear logic:

$$\frac{\Gamma, !\phi, !\phi \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !\phi \Rightarrow \Delta}$$

For a context-sharing semi-analytic rule, consider the following rule in the Dyckhoff calculus for **IPC** (see [4]):

$$\frac{\Gamma, \psi \rightarrow \gamma \Rightarrow \phi \rightarrow \psi \quad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\phi \rightarrow \psi) \rightarrow \gamma \Rightarrow \Delta}$$

Example 3.4. For a concrete non-example consider the cut rule; it is not semi-analytic because it does not preserve the variable occurrence condition. Moreover, the following rule in the calculus of **KC**:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}$$

in which Δ should consist of negation formulas is not a multi-conclusion semi-analytic rule, simply because the context is not free for all possible substitutions. The rule of thumb is that any rule in which we have *side conditions on the contexts* is not semi-analytic.

Definition 3.5. A sequent is called a *centered axiom* if it has the following form:

- (1) Identity axiom: $(\phi \Rightarrow \phi)$
- (2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$
- (3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$
- (4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$
- (5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where in 2-5, the variables in any pair of elements in $\bar{\alpha}$, $\bar{\beta}$, $\bar{\phi}$ are equal and Γ and Δ are meta-multiset variables. A sequent is called context-free centered axiom if it has the form (1), (2) or (3).

Example 3.6. It is easy to see that the axioms given in the preliminaries are examples of centered axioms. Here are some more examples:

$$\begin{aligned} \neg 1 \Rightarrow \quad , \quad \Rightarrow \neg 0 \\ \phi, \neg \phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg \phi \\ \Gamma, \neg \top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg \perp \end{aligned}$$

where the first four are context-free while the last two are contextual.

4 Uniform Interpolation

In this section we will generalize the investigations of [8] to also cover the sub-structural setting and semi-analytic rules. We will show that any extension of a sequent calculus by semi-analytic rules preserves uniform interpolation if the resulted system turns out to be terminating. Our method here is similar to the method used in [8].

As a first step, let us generalize the notion of uniform interpolation from logics to sequent calculi. The following definition offers three versions of such a generalization, each of which suitable for different forms of rules.

Definition 4.1. Let G and H be two sequent calculi. G has H -uniform interpolation if for any sequent S and T where $T^s = \emptyset$ and any atom p , there exist p -free formulas $I(S)$ and $J(T)$ such that

- (i) $S \cdot (I(S) \Rightarrow)$ is derivable in H .
- (ii) For any p -free multiset Γ , if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $\Gamma \Rightarrow I(S)$ is derivable in H .
- (iii) $T \cdot (\Rightarrow J(T))$ is derivable in H .
- (iv) For any p -free multisets Γ and Δ , if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $J(T), \Gamma \Rightarrow \Delta$ is derivable in H .

Similarly, we say G has *weak H -uniform interpolation* if instead of (ii) we have

- (ii') For any p -free multiset Γ , if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in H where $\tilde{S} = (S^a \Rightarrow)$.

We say G has *strong H -uniform interpolation* if instead of (ii) we have

- (ii'') For any p -free multisets Γ and Δ , if $S \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $\Gamma \Rightarrow I(S), \Delta$ is derivable in H .

Note that in the case of the strong uniform interpolation, T^s can be non-empty, and we have multi-conclusion rules.

We call $I(S)$ a left p -interpolant (weak p -interpolant, strong p -interpolant) of S and $J(T)$ a right p -interpolant (weak right p -interpolant, strong right p -interpolant) of T in G relative to H . The system H has uniform interpolation property (weak uniform interpolation property, strong uniform interpolation property) if it has H -uniform interpolation (weak H -uniform interpolation, strong H -uniform interpolation).

Theorem 4.2. *If G is a sequent calculus with (weak/strong) uniform interpolation and complete for a logic L extending $(\mathbf{FL}_e/\mathbf{CFL}_e)$ \mathbf{FL}_e , L has the uniform interpolation property.*

Proof. First note that since G is complete for L , $L \vdash \phi \rightarrow \psi$ iff $G \vdash \phi \Rightarrow \psi$. Hence we can rewrite the definition of the uniform interpolation using the sequent system G . Now pick $S = (\Rightarrow A)$. By uniform interpolation property of G , there is a p -free formula $I(S)$ such that $S \cdot (I(S) \Rightarrow)$ and for any p -free Σ if $S \cdot (\Sigma \Rightarrow)$, then $\Sigma \Rightarrow I(S)$. It is clear that $I(S)$ works as the p -pre-interpolant of A , because firstly $I(S) \Rightarrow A$ and secondly if $B \Rightarrow A$

then $B \Rightarrow I(S)$ for any p -free B . The same argument also works for the p -post-interpolant. In the case of weak uniform interpolation, first note that by definition if $T = (\Rightarrow)$ then $(\Rightarrow J(T))$. Secondly, note that since G is complete for L , the calculus should admit the cut rule by Theorem 2.8. Now we claim that $I(S)$ works again. The reason now is that if $B \Rightarrow A$ for a p -free B , then $J(\tilde{S}), B \Rightarrow I(S)$. Since $\tilde{S} = T$ and we have the cut rule, $B \Rightarrow A$. The case for strong uniform interpolation is similar to the interpolation case. \square

In the following theorem, we will check the uniform interpolation property for a set of centered axioms. It can also be considered as an example to show how this notion works in practice.

Theorem 4.3. *Let G and H be two sequent calculi such that every provable sequent in G is also provable in H and G consists only of finite centered axioms. Then:*

- (i) *If H extends \mathbf{FL}_e , then G has H -uniform interpolation.*
- (ii) *If H extends \mathbf{FL}_e and has the left weakening rule, then G has weak H -uniform interpolation.*
- (iii) *If both G and H are multi-conclusion and H extends \mathbf{CFL}_e , then G has strong H -uniform interpolation.*

Proof. To prove part (i) of the theorem, we have to find $I(S)$ and $J(T)$ for given sequents $S = (\Sigma \Rightarrow \Lambda)$ and $T = (\Pi \Rightarrow)$ such that the four conditions in the Definition 4.1 hold. We will denote our $I(S)$ and $J(T)$ by $\forall pS$ and $\exists pT$, respectively.

First, we will prove (i) and we will investigate the case $\exists pT$, first. For that purpose, define $\exists pT$ as the following

$$[(\ast \Pi_p) \ast \top] \wedge 0 \wedge \perp$$

where Π_p is the subset of Π consisting of all p -free formulas and by $\ast \Pi_p$ we mean $\phi_1 \ast \dots \ast \phi_k$, where $\{\phi_1, \dots, \phi_k\} = \Pi_p$. Note that \top appears in the first conjunct only when $\Pi - \Pi_p$ is non-empty. Moreover, 0 only appears as a conjunct when T is of the form axiom 3 (which is $\bar{\beta} \Rightarrow$) and $\bar{\beta} = \Pi$, and \perp only appears as a conjunction when T is of the form of axiom 4 (which is $\Sigma, \bar{\phi} \Rightarrow \Lambda$) and we have $\bar{\phi} \subseteq \Pi$.

First, we have to show that $\Pi \Rightarrow \exists pT$ holds in H . Note that Π is of the form $\Pi_p \cup (\Pi - \Pi_p)$. By definition, for every $\psi \in \Pi_p$ we have $\psi \Rightarrow \psi$

and hence using the rule (R^*) we have $\Pi_p \Rightarrow * \Pi_p$ holds in H (note that since H extends \mathbf{FL}_e , it has the rule (R^*)). On the other hand, using the axiom for \top we have $\Pi - \Pi_p \Rightarrow \top$ and then using the rule (R^*) we have $\Pi_p, \Pi - \Pi_p \Rightarrow (* \Pi_p) * \top$, which is $\Pi \Rightarrow (* \Pi_p) * \top$.

The formula 0 appears as a conjunct when T is of the form axiom 3 and $\bar{\beta} = \Pi$, which means that in this case $\Pi \Rightarrow$ is an instance of axiom 3 and it holds in H . Hence, using the rule $(R0)$ we have $\Pi \Rightarrow 0$.

The formula \perp appears as a conjunct when T is of the form axiom 4 and $\bar{\phi} \subseteq \Pi$. Hence, $\Pi \Rightarrow \perp$ is an instance of axiom 4 when we let Δ to be \perp .

Now, we have to show that if for p -free sequents \bar{C} and \bar{D} if $\Pi, \bar{C} \Rightarrow \bar{D}$ is provable in G , then $\exists pT, \bar{C} \Rightarrow \bar{D}$ is provable in H . Therefore, $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of one of the centered axioms and we have five cases to consider:

- (1) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\phi \Rightarrow \phi$. Then, since $\bar{D} = \phi$, it means that ϕ is p -free. There are two cases; first, if $\Pi = \phi$ and $\bar{C} = \emptyset$, then $* \Pi_p = \phi$ and since $\Pi - \Pi_p = \emptyset$, we do not have \top in the conjunct. Hence, $\Pi \Rightarrow \phi$ and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow \bar{D}$. Second, if $\Pi = \emptyset$ and $\bar{C} = \phi$, then $* \Pi_p = 1$ and since $\Pi - \Pi_p = \emptyset$, then \top does not appear in the first conjunct in the definition of $\exists pT$. Hence, since $\bar{C} \Rightarrow \bar{D}$ is equal to $\phi \Rightarrow \phi$ and this is of the form of the axiom 1, using the rule $(L1)$ we have $1, \phi \Rightarrow \phi$ and using $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$.
- (2) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Rightarrow \bar{\alpha}$. Then, since $\bar{D} = \bar{\alpha}$, it means that $\bar{\alpha}$ is p -free and $\Pi = \bar{C} = \emptyset$. Hence, like the above case $* \Pi_p = 1$ and we do not have \top in the definition, either. Again, using the rule $(L1)$ we have $1 \Rightarrow \bar{\alpha}$ and by $(L\wedge)$ we have $\exists pT \Rightarrow \bar{\alpha}$.
- (3) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $(\bar{\beta} \Rightarrow)$. Then there are two cases; first if $\bar{\beta} = \Pi$, then we must have 0 as one of the conjuncts in the definition of $\exists pT$. We have $\bar{C} = \bar{D} = \emptyset$ and $0 \Rightarrow$ is an axiom in H and using the rule $(L\wedge)$ we have $\exists pT \Rightarrow$. Second, if $\Pi \subsetneq \bar{\beta}$, since we have $\bar{\beta} = \Pi, \bar{C}$ and \bar{C} is p -free, and we have this condition that for any two formulas in $\bar{\beta}$ they have the same variables, we have Π is p -free, as well, which means every formula in Π is p -free and $\Pi = \Pi_p$ and \top does not appear in the definition of $\exists pT$. Hence, using the rule (L^*) on $\Pi, \bar{C} \Rightarrow$, we have $* \Pi_p, \bar{C} \Rightarrow$ and by the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow$.
- (4) If $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$, then there are two cases; first if $\bar{\phi} \subseteq \Pi$, then by definition of $\exists pT$, \perp is one of the

conjuncts. Therefore, since $\perp, \bar{C} \Rightarrow \bar{D}$ is an instance of an axiom in H , using the rule $(L\wedge)$ we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is derivable in H . Second, if $\bar{\phi} \not\subseteq \Pi$, then at least one of the elements in $\bar{\phi}$ is in \bar{C} and hence it is p -free. Therefore, by the condition that for any two formulas in $\bar{\phi}$ they have the same variables, $\bar{\phi}$ is p -free. Hence, there can not be any element of $\bar{\phi}$ present in $\Pi - \Pi_p$ and hence $\bar{\phi} \subseteq \Pi_p, \bar{C}$. Therefore, we have $\Pi_p, \bar{C} \Rightarrow \bar{D}$ because it is of the form of the axiom $\Gamma, \bar{\phi} \Rightarrow \Delta$ of G and hence it is provable in H . Therefore, using the axiom $(L*)$ we have $(*\Pi_p) * \top, \bar{C} \Rightarrow \bar{D}$ and by $(L\wedge)$, $\exists pT, \bar{C} \Rightarrow \bar{D}$. (Note that it is possible that $\Pi - \Pi_p$ is empty. It is easy to show that in this case the claim also holds. It is enough to drop \top in the last part of the proof.)

- (5) Consider the case where $\Pi, \bar{C} \Rightarrow \bar{D}$ is of the form of the axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$. Then, since $\bar{\phi} \subseteq \bar{D}$, we have $\exists pT, \bar{C} \Rightarrow \bar{D}$ is an instance of the same axiom $\Gamma \Rightarrow \bar{\phi}, \Delta$ when we substitute Γ by $\exists pT, \bar{C}$.

Now, we will investigate the case $\forall pS$ for S of the form $\Sigma \Rightarrow \Lambda$. Define $\forall pS$ as the following

$$[(*\Sigma_p \rightarrow \perp)] \vee [*(\bar{\beta} - \Sigma)] \vee \phi \vee 1 \vee \top$$

where in the first disjunct, Σ_p means the p -free part of Σ , the second disjunct appears whenever there exists an instance of an axiom of the form (3) in G where $\Sigma \subseteq \bar{\beta}$, $\Lambda = \emptyset$ and $\bar{\beta}$ is p -free. The third disjunct appears if $\Sigma = \emptyset$ and $\Lambda = \phi$ where ϕ is p -free. The fourth disjunct appears if $\Sigma \Rightarrow \Lambda$ equals to one of the instances of the axiom (1), (2), or (3) in G . And finally, the fifth disjunct appears when $\bar{\phi} \subseteq \Sigma$ for an instance of $\bar{\phi}$ in axiom (4) in G or $\bar{\phi} \subseteq \Lambda$ for an instance of $\bar{\phi}$ in axiom (5) in G .

First we have to show that $\Sigma, \forall pS \Rightarrow \Lambda$. For this purpose, we have to prove that for any possible disjunct X , we have $\Sigma, X \Rightarrow \Lambda$. For the first disjunct note that $\Sigma_p \Rightarrow *\Sigma_p$ and $\Sigma - \Sigma_p, \perp \Rightarrow \Lambda$. Hence, $\Sigma, (*\Sigma_p \rightarrow \perp) \Rightarrow \Lambda$.

For the second disjunct, we have $\Sigma \subseteq \bar{\beta}$ and $\Lambda = \emptyset$. Therefore

$$\Sigma, *(\bar{\beta} - \Sigma) \Rightarrow \Lambda$$

by the axiom (3) itself. For the third disjunct, note that $\Sigma = \emptyset$ and $\Lambda = \phi$ where ϕ is p -free. Hence $\Sigma, \phi \Rightarrow \Lambda$ by axiom (1). For the fourth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an axiom itself and hence $\Sigma, 1 \Rightarrow \Lambda$. Finally, for the fifth disjunct, note that $\Sigma \Rightarrow \Lambda$ is an instance of the axioms (4) or (5) which means if we also add \top to the left hand-side of the sequent, it remains provable.

Now we have to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda$ then $\bar{C} \Rightarrow \forall pS$. For this purpose, we will check all possible axiomatic forms for $\Sigma, \bar{C} \Rightarrow \Lambda$.

- (1) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (1), there are two possible cases. First if $\Sigma = \emptyset$ and $\bar{C} = \phi$ and $\Lambda = \phi$. Then ϕ will be p -free and hence ϕ appears in $\forall pS$ as a disjunct. Since $\bar{C} \Rightarrow \phi$, we have $\bar{C} \Rightarrow \forall pS$. For the second case, if $\Sigma = \phi$ and $\bar{C} = \emptyset$ then $\Sigma \Rightarrow \Lambda$ is an instance of the axiom (1) which means that 1 is a disjunct in $\forall pS$. Since $(\Rightarrow 1)$ and $\bar{C} = \emptyset$ we have $\bar{C} \Rightarrow \forall pS$.
- (2) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \emptyset$ and $\Lambda = \bar{\alpha}$. Therefore, 1 is a disjunct in $\forall pS$ and since $\bar{C} = \emptyset$ we have $\bar{C} \Rightarrow \forall pS$.
- (3) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (3). Then there are two cases to consider. First if $\Sigma = \bar{\beta}$. Then $\bar{C} = \emptyset$ and $\Lambda = \emptyset$. By definition, 1 is a disjunct in $\forall pS$ and again like the previous cases $\bar{C} \Rightarrow \forall pS$. Second if $\Sigma \subsetneq \bar{\beta}$. Then $\bar{\beta} \cap \bar{C}$ is non-empty. Pick $\psi \in \bar{\beta} \cap \bar{C}$. ψ is p -free, since any pair of the elements in $\bar{\beta}$ have the same variables, $\bar{\beta}$ is p -free. Now by definition, $\ast(\bar{\beta} - \Sigma)$ is a disjunct in $\forall pS$. Since $\bar{C} = \bar{\beta} - \Sigma$, we have $\bar{C} \Rightarrow \forall pS$.
- (4) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (4). Similar to the previous case, there are two cases. If $\bar{\phi} \subseteq \Sigma$, then by definition \top is a disjunct in $\forall pS$ and there is nothing to prove. In the second case, at least one the elements of ϕ is in \bar{C} and hence p -free. Since any pair of the elements in $\bar{\phi}$ have the same variables, $\bar{\phi}$ is p -free. We can partition Σ, \bar{C} to $\Sigma_p, \bar{C}, (\Sigma - \Sigma_p)$. Since every element of $(\Sigma - \Sigma_p)$ has p , and $\bar{\phi}$ is p -free, the whole ϕ should belong to Σ_p, \bar{C} . Therefore, by the axiom (4) itself, $\Sigma_p, \bar{C} \Rightarrow \perp$ which implies $\bar{C} \Rightarrow (\ast \Sigma_p \rightarrow \perp)$. By definition $(\ast \Sigma_p) \rightarrow \perp$ is a disjunct in $\forall pS$ and hence $\bar{C} \Rightarrow \forall pS$.
- (5) If $\Sigma, \bar{C} \Rightarrow \Lambda$ is an instance of the axiom (5). Then $\bar{\phi} \subseteq \Lambda$. By definition \top is a disjunct in $\forall pS$ and therefore, there is nothing to prove.

For (ii), note that using the part (i) we have formulas $\exists pT$ and $\forall pS$ for any sequents S and T ($T^s = \emptyset$) with the conditions of H -uniform interpolation. The conditions for the weak H -uniform interpolation is the same except for the second part of the left weak p -interpolant which demands that if $\Sigma, \bar{C} \Rightarrow \Lambda$, then $\exists p\bar{S}, \bar{C} \Rightarrow \forall pS$. If we use the same uniform interpolants, we satisfy all the conditions of weak H -uniform interpolation. The reason is that except the mentioned condition, all of the others are the same as the

conditions for H -interpolation and for the other condition, we can argue as follows: By $\Sigma, \bar{C} \Rightarrow \Lambda$, we have $\bar{C} \Rightarrow \forall pS$ and by the left weakening rule we will have $\exists p\tilde{S}, \bar{C} \Rightarrow \forall pS$.

For (iii), first note that proving the existence of the right interpolants is enough. It is sufficient to define $\forall pS = \neg\exists pS$ and using the assumption that \mathbf{CFL}_e is admissible in H to reduce the conditions of $\forall pS$ to $\exists pS$. Now define $\exists pS$ for any $S = (\Sigma \Rightarrow \Lambda)$ as:

$$[(\ast \Sigma_p) \ast \top] \wedge [\neg(\perp + (\dashv \Lambda_p))] \wedge 0 \wedge \perp$$

where by $\ast \Sigma_p$ we mean $\psi_1 \ast \dots \ast \psi_r$, where $\{\psi_1, \dots, \psi_r\} = \Pi_p$ and $\dashv \Lambda_p$ is defined similarly. Note that in $[(\ast \Sigma_p) \ast \top]$ the formula \top appears iff $\Sigma \neq \Sigma_p$, and \perp appears in the second conjunct iff $\Lambda \neq \Lambda_p$. The third conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (1), (2) and (3) in G and the fourth conjunct appears if $\Sigma \Rightarrow \Lambda$ is an instance of an axiom of the forms (4), (5) in G .

First, we have to show that $\Sigma \Rightarrow \exists pS, \Lambda$. For that purpose, we have to check that for any conjunct X we have $\Sigma \Rightarrow X, \Lambda$. For the first conjunct, if $\Sigma \neq \Sigma_p$ then note that $\Sigma_p \Rightarrow \ast \Sigma_p$ and $\Sigma - \Sigma_p \Rightarrow \top, \Lambda$ therefore

$$\Sigma \Rightarrow [(\ast \Sigma_p) \ast \top], \Lambda$$

If $\Sigma = \Sigma_p$, then there is no need for \top and the claim is clear by $\Sigma \Rightarrow \ast \Sigma_p$. For the second conjunct, if $\Lambda \neq \Lambda_p$ note that $\dashv \Lambda_p \Rightarrow \Lambda_p$ and $\Sigma, \perp \Rightarrow \Lambda - \Lambda_p$, hence

$$\Sigma, [\perp + (\dashv \Lambda_p)] \Rightarrow \Lambda$$

hence

$$\Sigma \Rightarrow [\neg(\perp + (\dashv \Lambda_p))], \Lambda$$

If $\Lambda = \Lambda_p$, similar to the case before, there is no need for \perp .

The cases for the third and the fourth conjuncts are similar to the similar cases in the proof of (i).

Now we want to prove that if $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$, then $\exists pS, \bar{C} \Rightarrow \bar{D}$. For this purpose, we will check all the cases one by one:

- (1) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (1), we have four cases to check.

- If $\phi \in \bar{C}$ and $\phi \in \bar{D}$, then $\Sigma = \Lambda = \emptyset$ and $\bar{C} = \bar{D} = \phi$. Hence $*\Sigma_p = 1$. Therefore, since $1, \bar{C} \Rightarrow \bar{D}$ we have $\exists pS, \bar{C} \Rightarrow \bar{D}$.
 - If $\phi \in \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \emptyset$ and $\Lambda = \phi$. Therefore, ϕ is p -free and hence $\Lambda_p = \phi$. Since $\bar{D} = \emptyset$ and $\Lambda = \phi$, we have $, \neg\phi, \bar{C} \Rightarrow \bar{D}$. Therefore, $\neg(+\Lambda_p), \bar{C} \Rightarrow \bar{D}$.
 - If $\phi \notin \bar{C}$ and $\phi \in \bar{D}$. This case is similar to the previous case.
 - If $\phi \notin \bar{C}$ and $\phi \notin \bar{D}$ then $\Sigma = \Lambda = \phi$ and $\bar{C} = \bar{D} = \emptyset$. Hence, by definition, we have 0 as a conjunct in $\exists pS$. Since $0 \Rightarrow$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$.
- (2) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (2). Then $\Sigma = \bar{C} = \emptyset$. There are two cases to consider. If $\Lambda = \bar{\alpha}$. Then by definition 0 appears in $\exists pS$. Since $\bar{D} = \emptyset$ and $(0 \Rightarrow)$ we have $\bar{C}, \exists pS \Rightarrow \bar{D}$. If $\Lambda \subsetneq \bar{\alpha}$, then $\bar{D} \cap \bar{\alpha}$ is non empty. Therefore, there exists a p -free formula in $\bar{\alpha}$. Since the variables of any pair in $\bar{\alpha}$ are equal, $\bar{\alpha}$ is p -free. Therefore, $\Lambda \subseteq \bar{\alpha}$ is p -free, hence $\Lambda = \Lambda_p$ (and \perp does not appear in the second conjunct). Since $(\Rightarrow \Lambda, \bar{D})$, we have $(\Rightarrow +\Lambda, \bar{D})$ therefore $(\neg(+\Lambda_p) \Rightarrow \bar{D})$ which implies $(\exists pS \Rightarrow \bar{D})$.
- (3) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (3). This case is similar to the previous case (2).
- (4) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (4). There are two cases to consider. If $\bar{\phi} \subseteq \Sigma$. Then by definition \perp is a conjunct in $\exists pS$ and therefore there is nothing to prove. For the second case, if $\bar{\phi} \not\subseteq \Sigma$, then $\bar{\phi} \cap \bar{C}$ is non-empty. Hence, $\bar{\phi}$ has a p -free element. Since the variables of any pair in $\bar{\phi}$ are equal, $\bar{\phi}$ is p -free. Since $\bar{\phi} \subseteq \Sigma_p, \bar{C}, \Sigma - \Sigma_p$ and $\bar{\phi}$ is p -free, we should have $\bar{\phi} \subseteq \Sigma_p, \bar{C}$. Therefore, if $\Sigma \neq \Sigma_p$, by the axiom (4) itself, $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$. Since $(*\Sigma_p) * \top$ is a conjunct in $\exists pS$, we will have $\exists pS, \bar{C} \Rightarrow \bar{D}$. Note that if $\Sigma = \Sigma_p$, then we will use $\Sigma_p, \bar{C} \Rightarrow \bar{D}$ instead of $\top, \Sigma_p, \bar{C} \Rightarrow \bar{D}$.
- (5) If $\Sigma, \bar{C} \Rightarrow \Lambda, \bar{D}$ is an instance of the axiom (5). This case is similar to the previous case 4.

□

4.1 The Single-conclusion Case

In this section, we assume that for any sequent $S = \Gamma \Rightarrow \Delta$ we have $|\Delta| \leq 1$. We will show how the single-conclusion semi-analytic and context-sharing

semi-analytic rules preserve the uniform interpolation property. For this purpose, we will investigate these two kinds of rules separately. First we will study the semi-analytic rules and then we will show in the presence of weakening and context-sharing implication rules, we can also handle the context-sharing semi-analytic rules.

4.1.1 Semi-analytic Case

Let us begin right away with the following theorem which is one of the main theorems of this paper.

Theorem 4.4. *Let G and H be two sequent calculi and H extends \mathbf{FL}_e . If H is a terminating sequent calculus axiomatically extending G with only semi-analytic rules, then if G has H -uniform interpolation property, then so does H .*

Proof. For any sequent U and V where $V^s = \emptyset$ and any atom p , we define two p -free formulas, denoted by $\forall pU$ and $\exists pV$ and we will prove that they meet the conditions for the left and the right p -interpolants of U and V , respectively. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus H .

If V is the empty sequent we define $\exists pV$ as 1 and otherwise, we define $\exists pV$ as the following

$$\left(\bigwedge_{par} \ast_i \exists pS_i \right) \wedge \left(\bigwedge_{LR} [\left(\ast_j \bigwedge_s \forall pT_{js} \right) \ast_{i \neq 1} \left(\ast_r \bigwedge_r \forall pS_{ir} \right) \rightarrow \bigvee_r \exists pS_{1r}] \right) \wedge (\Box \exists pV') \wedge (\exists^G pV).$$

In the first conjunct, the conjunction is over all non-trivial partitions of $V = S_1 \cdot \dots \cdot S_n$ and i ranges over the number of S_i 's, in this case $1 \leq i \leq n$. In the second conjunct, the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{js} \rangle_s \rangle_j$, $\langle \langle S_{ir} \rangle_r \rangle_{i \neq 1}$ and $\langle S_{1r} \rangle$ and the conclusion is V , where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir} 's are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Note that picking the block $\langle S_{1r} \rangle$ is arbitrary and we include all conjuncts related to any choice of $\langle S_{1r} \rangle$.) The conjunct $\Box \exists pV'$ appears in the definition whenever V is of the form $(\Box \Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since G has the H -uniform interpolation property, by definition there exists $J(V)$ as right p -interpolant of V . We choose one such $J(V)$ and

denote it as $\exists^G pV$ and include it in the definition.

If U is the empty sequent define $\forall pU$ as 0. Otherwise, define $\forall pU$ as the following

$$\begin{aligned} & \left(\bigvee_{par} \left(\bigstar_{i \neq 1} \exists pS_i \rightarrow \forall pS_1 \right) \right) \vee \left(\bigvee_{LR} \left[\left(\bigstar_j \bigwedge_s \forall pT_{js} \right) * \left(\bigstar_i \bigwedge_r \forall pS_{ir} \right) \right] \right) \\ & \vee \left(\bigvee_{RR} \left(\bigstar_i \bigwedge_r \forall pS_{ir} \right) \right) \vee \left(\Box \forall pU' \right) \vee \left(\forall^G pU \right). \end{aligned}$$

In the first disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \dots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \emptyset$ and $S_1 \neq U$. (Note that in this case, if $S^s = \emptyset$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists pU \rightarrow \forall pS_1$ where $\forall pS_1 = 0$. But this does not make any problem, since the definition of $\exists pU$ is prior to the definition of $\forall pU$.) In the second disjunct, the big disjunction is over all left semi-analytic rules that are backward applicable to U in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{js} \rangle_s \rangle_j$ and $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is U . In the third disjunct, the big disjunction is over all right semi-analytic rules backward applicable to U in H . The premise of the rule is $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is U . The fourth disjunct is on all semi-analytic modal rules with the result U and the premise U' . And finally, since G has the H -uniform interpolation property, by definition there exists $I(U)$ as left p -interpolant of U . We choose one such $I(U)$ and denote it as $\forall^G pU$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents and we prove both cases $\forall pU$ and $\exists pV$ simultaneously. First we have to show that

(i) $V \cdot (\Rightarrow \exists pV)$ is derivable in H .

(ii) $U \cdot (\forall pU \Rightarrow)$ is derivable in H .

We show them using induction on the order of the sequents U and V . When proving (i), we assume that (i) holds for sequents whose succedents are empty and with order less than the order of V and (ii) holds for any sequent with order less than the order of V . We have the same condition for U when proving (ii).

To prove (i), note that if V is the empty sequent, then by definition $\exists pV = 1$ and hence (i) holds. For the rest, we have to show that $V \cdot (\Rightarrow X)$

is derivable in H for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule $(R\wedge)$ it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show $\Gamma \Rightarrow X$ is derivable in H .

- In the case that the conjunct is $(\bigwedge_{par\ i} * \exists pS_i)$, we have to show that for any non-trivial partition $S_1 \cdot \dots \cdot S_n$ of V we have $\Gamma \Rightarrow * \exists pS_i$ is derivable in H . Since the order of each S_i is less than the order of V and $S_i^s = (\Gamma_i \Rightarrow)$ for $1 \leq i \leq n$ where $\bigcup_{i=1}^n \Gamma_i = \Gamma$, we can use the induction hypothesis and we have $\Gamma_i \Rightarrow \exists pS_i$. Using the right rule for $(*)$ we have $\Gamma_1, \dots, \Gamma_n \Rightarrow * \exists pS_i$ which is $\Gamma \Rightarrow * \exists pS_i$.
- For the second conjunct in the definition of $\exists pV$, we have to check that for every left semi-analytic rule we have

$$V \cdot (\Rightarrow [(* \bigwedge_j \forall pT_{js}) * (* \bigwedge_{i \neq 1} \forall pS_{ir}) \rightarrow \bigvee_r \exists pS_{1r}]).$$

is derivable in H . Therefore, V is the conclusion of a left semi-analytic rule such that the premises are $\langle\langle T_{js} \rangle_s \rangle_j$, $\langle\langle S_{ir} \rangle_r \rangle_i$ and $\langle S_{1r} \rangle_r$ and hence the order of all of them are less than the order of V . We can easily see that the claim holds since by induction hypothesis we can add $\forall pT_{js}$ and $\forall pS_{ir}$ to the left side of the sequents T_{js} and S_{ir} for $i \neq 1$. And again by induction hypothesis we can add $\exists pS_{1r}$ to the right side of the sequents S_{1r} . Then using the rules $L\wedge$, $L*$ and $R\vee$ the claim follows. What we have said so far can be seen precisely in the following:

Note that $\langle\langle T_{js} \rangle_s \rangle_j$ is of the form $\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j$ and $\langle\langle S_{ir} \rangle_r \rangle_i$ is of the form $\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_i$ and V is of the form

$$\Pi_1, \dots, \Pi_m, \Gamma_1, \dots, \Gamma_n, \phi \Rightarrow$$

Using induction hypothesis we have for every $1 \leq j \leq m$

$$(\Pi_j, \forall pT_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \dots, (\Pi_j, \forall pT_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \dots$$

for every $1 < i \leq n$ we have

$$(\Gamma_i, \forall pS_{i1}, \bar{\phi}_{i1} \Rightarrow), \dots, (\Gamma_i, \forall pS_{ir}, \bar{\phi}_{ir} \Rightarrow), \dots$$

and for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \exists pS_{11}), \dots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists pS_{1r}), \dots$$

Hence, using the rule $(L\wedge)$, for every $1 \leq j \leq m$ we have

$$(\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \dots, (\Pi_j, \bigwedge_s \forall p T_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \dots$$

and for every $1 < i \leq n$ we have

$$(\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{i1} \Rightarrow), \dots, (\Gamma_i, \bigwedge_r \forall p S_{ir}, \bar{\phi}_{ir} \Rightarrow), \dots$$

and using the rule $(R\vee)$, for $i = 1$ we have

$$(\Gamma_1, \bar{\phi}_{11} \Rightarrow \bigvee_r \exists p S_{1r}), \dots, (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}) \dots$$

Substituting all these three in the original left semi-analytic rule (we can do this, since in the original rule, there are contexts, Δ_i 's in the right hand side of the sequents $S'_{ir}s$), we conclude

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall p T_{js} \rangle_j, \langle \bigwedge_r \forall p S_{ir} \rangle_{i \neq 1} \Rightarrow \bigvee_r \exists p S_{1r}.$$

where $\Pi = \Pi_1, \dots, \Pi_m$, $\Gamma = \Gamma_1, \dots, \Gamma_n$, $\langle \bigwedge_s \forall p T_{js} \rangle_j = \bigwedge_s \forall p T_{1s}, \dots, \bigwedge_s \forall p T_{ms}$ and $\langle \bigwedge_r \forall p S_{ir} \rangle_{i \neq 1} = \bigwedge_r \forall p S_{2r}, \dots, \bigwedge_r \forall p S_{nr}$.

Now, using the rule (L^*) we have

$$\Pi, \Gamma, \phi, (* \bigwedge_j \bigwedge_s \forall p T_{js}) * (* \bigwedge_{i \neq 1} \bigwedge_r \forall p S_{ir}) \Rightarrow \bigvee_r \exists p S_{1r}.$$

And finally, using the rule $R \rightarrow$ we conclude

$$\Pi, \Gamma, \phi \Rightarrow [(* \bigwedge_j \bigwedge_s \forall p T_{js}) * (* \bigwedge_{i \neq 1} \bigwedge_r \forall p S_{ir}) \rightarrow \bigvee_r \exists p S_{1r}].$$

- Consider the conjunct $\Box \exists p T'$. In this case, T must have been of the form $(\Box \Gamma \Rightarrow)$ and T' of the form $(\Gamma \Rightarrow)$. By definition, the order of T' is less than the order of T . Hence, by induction hypothesis we have $T' \cdot (\Rightarrow \exists p T')$ or in other words $\Gamma \Rightarrow \exists p T'$. Now, we use the rule K and we have $\Box \Gamma \Rightarrow \Box \exists p T'$ which means $T \cdot (\Rightarrow \Box \exists p T')$.
- The last case is $\exists^G p V$. We have to show $V \cdot (\Rightarrow \exists^G p V)$ is provable in H which is the case since G has H -uniform interpolation property and by Definition 4.1 part (iii) there exists p -free formula J such that $V \cdot (\Rightarrow J)$ is derivable in H . We chose one such J and call it $\exists^G p V$, hence $V \cdot (\Rightarrow \exists^G p V)$ in H by definition.

To prove (ii), note that if U is the empty sequent, then by definition $\forall pU = 0$ and hence (ii) holds. For the rest, we have to show that $U \cdot (X \Rightarrow)$ is derivable in H for any X that is one of the disjuncts in the definition of $\forall pU$. Then, using the rule ($L\vee$) it follows that $U \cdot (\forall pU \Rightarrow)$. Since U is of the form $\Gamma \Rightarrow \Delta$, we have to show $\Gamma, X \Rightarrow \Delta$ is derivable in H .

- In the case that the disjunct is $(\bigvee_{\substack{par \\ i \neq 1}} (* \exists pS_i \rightarrow \forall pS_1))$ we have to prove that for any partitions of $U = S_1 \cdot \dots \cdot S_n$ such that $S_i^s = \emptyset$ for each $i \neq 1$ and $S_1 \neq U$, we have $U \cdot ((* \exists pS_i \rightarrow \forall pS_1) \Rightarrow)$. First, consider the case that non of S_i 's are equal to U (or in other words, $S^s \neq \emptyset$); then the order of each S_i is less than the order of S and we can use the induction hypothesis. Since for $i \neq 1$ the succedent of each S_i is empty, we have $S_i = (\Gamma_i \Rightarrow)$ and $(\Gamma_i \Rightarrow \exists pS_i)$ and using the rule $R*$ we have $(\Gamma_2, \dots, \Gamma_n \Rightarrow * \exists pS_i)$. And for $S_1 = \Gamma_1 \Rightarrow \Delta$ we have $\Gamma_1, \forall pS_1 \Rightarrow \Delta$. Hence using the rule $L \rightarrow$ we conclude

$$\Gamma_1, \dots, \Gamma_n, * \exists pS_i \rightarrow \forall pS_1 \Rightarrow \Delta$$

and the claim follows.

In the case that $U^s = \emptyset$, it is possible that for $i \neq 1$, one of S_i 's is equal to U . In this case what appears in the definition of $\forall pU$ is $\exists pU \rightarrow \forall pS_1$ which is equivalent to $\exists pU \rightarrow 0$. But, we can do this, since we defined $\exists pU$ prior to the definition of $\forall pU$ and we have proved $U \cdot (\Rightarrow \exists pU)$ prior to the case that we are checking now.

- In the case that the disjunct is $(\bigvee_{L\mathcal{R}} [(* \bigwedge_{j \ s} \forall pT_{js}) * (* \bigwedge_{i \ r} \forall pS_{ir})])$, we have to prove that for any left semi-analytic rule that is backward applicable to U in H we have $U \cdot ((* \bigwedge_{j \ s} \forall pT_{js}) * (* \bigwedge_{i \ r} \forall pS_{ir}) \Rightarrow)$. The premises of the rule are $\langle \langle T_{js} \rangle_s \rangle_j$ and $\langle \langle S_{ir} \rangle_r \rangle_i$ and the conclusion is U . Since the orders of all T_{js} 's and S_{ir} 's are less than the order of U we can use the induction hypothesis and have $T_{js} \cdot (\forall pT_{js} \Rightarrow)$ and $S_{ir} \cdot (\forall pS_{ir} \Rightarrow)$. Using the rule ($L\wedge$) for context sharing sequents (when j is fixed and i is fixed we have context sharing sequents) and then using the rule ($L*$) for non context sharing sequents (when s and r are fixed and we are ranging over j and i) and then applying the same left rule we can prove the claim. The proof is similar to the second case of (i) and precisely it goes as the following: Using induction hypothesis we have for every $1 \leq j \leq m$

$$(\Pi_j, \forall pT_{j1}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \dots, (\Pi_j, \forall pT_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \dots$$

and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \forall pS_{i1}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \dots, (\Gamma_i, \forall pS_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \dots$$

Hence, using the rule $(L\wedge)$, for every $1 \leq j \leq m$ we have

$$(\Pi_j, \bigwedge_s \forall pT_{js}, \bar{\psi}_{j1} \Rightarrow \bar{\theta}_{j1}), \dots, (\Pi_j, \bigwedge_s \forall pT_{js}, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}), \dots$$

and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \bigwedge_r \forall pS_{ir}, \bar{\phi}_{i1} \Rightarrow \Delta_i), \dots, (\Gamma_i, \bigwedge_r \forall pS_{ir}, \bar{\phi}_{ir} \Rightarrow \Delta_i), \dots$$

Substituting these two in the original left semi-analytic rule, we conclude

$$\Pi, \Gamma, \phi, \langle \bigwedge_s \forall pT_{js} \rangle_j, \langle \bigwedge_r \forall pS_{ir} \rangle_i \Rightarrow \Delta,$$

and using the rule $(L*)$ we have

$$\Pi, \Gamma, \phi, (* \bigwedge_j \bigwedge_s \forall pT_{js}) * (* \bigwedge_i \bigwedge_r \forall pS_{ir}) \Rightarrow \Delta.$$

- In the case that the disjunct is $(\bigvee_{RR} (* \bigwedge_i \bigwedge_r \forall pS_{ir}))$, we have to prove that for any right semi-analytic rule backward applicable to U in H , we have $U \cdot (* \bigwedge_i \bigwedge_r \forall pS_{ir} \Rightarrow)$. In this case the premises of the rule are $\langle \langle S_{ir} \rangle_r \rangle_i$, where $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ and the conclusion is $U = (\Gamma_1, \dots, \Gamma_n \Rightarrow \phi)$. Since the order of each S_{ir} is less than the order of S , we can use the induction hypothesis and for every $1 \leq i \leq n$ we have

$$(\Gamma_i, \forall pS_{i1}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \dots, (\Gamma_i, \forall pS_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \dots$$

Using the rule $L\wedge$ we have

$$(\Gamma_i, \bigwedge_r \forall pS_{ir}, \bar{\phi}_{i1} \Rightarrow \bar{\psi}_{i1}), \dots, (\Gamma_i, \bigwedge_r \forall pS_{ir}, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}), \dots$$

and substituting it in the original right rule, we conclude

$$\Gamma, \langle \bigwedge_r \forall pS_{ir} \rangle_i \Rightarrow \phi,$$

and using the rule $(L*)$ we have

$$\Gamma, * \bigwedge_i \bigwedge_r \forall pS_{ir} \Rightarrow \phi.$$

- For the case that the disjunct is $\Box\forall pU'$ we have that U is the conclusion of a semi-analytic modal rule and the premise is U' . Hence, U is of the form $(\Box\Gamma \Rightarrow \Box\Delta)$ and U' is of the form $(\Gamma \Rightarrow \Delta)$. Since the order of U' is less than the order of U , we can use the induction hypothesis and we have $(\Gamma, \forall pU' \Rightarrow \Delta)$. Now, using the rule K we can conclude $(\Box\Gamma, \Box\forall pU' \Rightarrow \Box\Delta)$ which is equivalent to $U \cdot (\Box\forall pU' \Rightarrow)$.
- And finally, for the case that the disjunct is $\forall^G pU$ we have to show that $U \cdot (\forall^G pU \Rightarrow)$ holds in H , which does since G has H -uniform interpolation property and by Definition 4.1 part (i) there exists p -free formula I such that $U \cdot (I \Rightarrow)$ is derivable in H . We choose one such I and call it $\forall^G pU$ and hence we have $U \cdot (\forall^G pU \Rightarrow)$ in H by definition.

So far we have proved (i) and (ii). We want to show that H has H -uniform interpolation. Therefore, based on the Definition 4.1, we have to prove the following, as well:

- (iii) For any p -free multisets \bar{C} and \bar{D} , if $V \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in H , where $\bar{C} = C_1, \dots, C_k$ and $|\bar{D}| \leq 1$.
- (iv) For any p -free multiset \bar{C} , if $U \cdot (\bar{C} \Rightarrow)$ is derivable in G then $\bar{C} \Rightarrow \forall pU$ is derivable in H , where $\bar{C} = C_1, \dots, C_k$.

Recall that V is of the form $(\Gamma \Rightarrow)$ and U is of the form $(\Gamma \Rightarrow \Delta)$. We will prove (iii) and (iv) simultaneously using induction on the length of the proof and induction on the order of U and V . More precisely, first by induction on the order of U and V and then inside it, by induction on n , we will show:

- For any p -free multisets \bar{C} and \bar{D} , if $V \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in G with length less than or equal to n , then $\exists pV, \bar{C} \Rightarrow \bar{D}$ is derivable in H .
- For any p -free multiset \bar{C} , if $U \cdot (\bar{C} \Rightarrow)$ has a proof in G with length less than or equal to n , then $\bar{C} \Rightarrow \forall pU$ is derivable in H .

Where by the length we mean counting just the new rules that H adds to G .

First note that for the empty sequent and for (iii), we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in G , then $\bar{C}, 1 \Rightarrow \bar{D}$ is valid in H , which is trivial by the rule (L1). Similarly, for (iv), if $\bar{C} \Rightarrow$ is valid in G , then $\bar{C} \Rightarrow 0$ is valid in H , which is trivial by the rule (R0).

For the base of the other induction, note that if $n = 0$, for (iii) it means that $\Gamma, \bar{C} \Rightarrow \bar{D}$ is valid in G . By Definition 4.1 part (iv), $\exists^G pV, \bar{C} \Rightarrow \bar{D}$ and

hence $\exists pV, \bar{C} \Rightarrow \bar{D}$ is provable in H . For (iv), it means that $\Gamma, \bar{C} \Rightarrow \Delta$ is valid in G . Therefore, again by Definition 4.1, $\bar{C} \Rightarrow \forall^G pU$ and hence $\bar{C} \Rightarrow \forall pU$ is provable in H .

For $n \neq 0$, to prove (iii), we have to consider the following cases:

- The case that the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule, ϕ , is one of C_i 's). Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y}, \phi)$ and we want to prove $(\exists pV, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$. Consider $T_{js} = (\Pi_j \Rightarrow)$ and $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js} 's do not depend on the suffix s , we have $T_{j1} = \dots = T_{js}$ and we denote it by T_j . And, since S_{ir} 's do not depend on r , we have $S_{i1} = \dots = S_{ir}$ and we denote it by S_i . Therefore, $T_1, \dots, T_m, S_1, \dots, S_n$ is a partition of V . First, consider the case that it is a non-trivial partition. Then the order of all of them are less than the order of V and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p -free. Hence, we can use the induction hypothesis to get:

$$\exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i$$

If we let $\{\exists pT_j, \bar{X}_j\}$ and $\{\exists pS_i, \bar{Y}_i\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle \exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i \rangle_r \rangle_i}{\exists pT_1, \dots, \exists pT_m, \exists pS_1, \dots, \exists pS_n, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Using the rule (L^*) we have

$$(* \exists pT_j) * (* \exists pS_i), \bar{X}, \bar{Y}, \phi \Rightarrow \Delta.$$

Therefore using the rule $(L \wedge)$, we have $(\exists pV, \bar{C} \Rightarrow \bar{D})$.

If $T_1, \dots, T_m, S_1, \dots, S_n$ is a trivial partition of V , it means that one of them equals V and all the others are empty sequents. W.l.o.g. suppose $T_1 = V = (\Sigma \Rightarrow)$ and the others are empty. Then we must have had the following instance of the rule:

$$\frac{\langle\langle\Sigma, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i\rangle_r\rangle_i}{\Sigma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $V \cdot (\bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js})$ for every j and s are premises of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $V \cdot (\bar{C} \Rightarrow \bar{D})$, and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{js}$ and $\bar{\theta}_{js}$ are also p -free. Hence, for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $\exists pV, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}$. Substituting $\{\exists pV, \bar{X}_j\}$, $\{\bar{X}_j\}$, $\{\bar{Y}_i\}$ and $\{\Delta\}$ as the contexts of the premises in the original left rule we have

$$\frac{\langle\langle\exists pV, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \Delta_i\rangle_r\rangle_i}{\exists pV, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

which is $(\exists pV, \bar{C} \Rightarrow \bar{D})$.

- Consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore,

$$V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$$

is the conclusion of a left semi-analytic rule and V is of the form $(\Pi, \Gamma, \phi \Rightarrow)$ and $\bar{C} = (\bar{X}, \bar{Y})$ and we want to prove $(\exists pV, \bar{X}, \bar{Y} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle\Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\Gamma_1, \bar{Y}_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta} \quad (\dagger)$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$ and $\bigcup_i \bar{Y}_i = \bar{Y}$.

Since, \bar{X}_j 's and \bar{Y}_i 's are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take \bar{X}_j 's and \bar{Y}_i 's as contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \rangle_r \rangle_{i \neq 1} \quad \langle \Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta \rangle_r}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is back ward applicable to V and T_{js} 's and S_{ir} 's are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis for them. Note that we have $V \cdot (\bar{C} \Rightarrow \bar{D})$ is provable in H and from (\dagger) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and for $i \neq 1$, $S_{ir} \cdot (\bar{Y}_i \Rightarrow \Delta)$ and $S_{1r} \cdot (\bar{Y}_1 \Rightarrow)$ are also provable in H . Using the induction hypothesis we get

$$(\bar{X}_j \Rightarrow \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \forall p S_{ir})_{i \neq 1} \quad , \quad (\bar{Y}_1, \exists p S_{1r} \Rightarrow \Delta)$$

Note that we were allowed to use the induction hypothesis because for $i \neq 1$ we have $\Delta_i = \emptyset$ and Δ is p -free and T_{js} 's and S_{ir} 's meet the conditions of (iii) and (iv) in the induction step. Now, using the rules $(R\wedge)$ and $(L\vee)$ we have

$$(\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js}) \quad , \quad (\bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir})_{i \neq 1} \quad , \quad (\bar{Y}_1, \bigvee_r \exists p S_{1r} \Rightarrow \Delta)$$

Denote $(\bigwedge_s \forall p T_{js})$ as A_j and $(\bigwedge_r \forall p S_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee_r \exists p S_{1r})$ as C . We have

$$\frac{\frac{\langle \bar{X}_j \Rightarrow A_j \rangle_j}{\bar{X} \Rightarrow * A_j} \text{R*} \quad \frac{\langle \bar{Y}_i \Rightarrow B_i \rangle_{i \neq 1}}{Y_2, \dots, Y_n \Rightarrow * B_i} \text{R*}}{\bar{X}, Y_2, \dots, Y_n \Rightarrow (* A_j) * (* B_i) \text{R*}} \quad \bar{Y}_1, C \Rightarrow \Delta}{\bar{X}, \bar{Y}, (* A_j) * (* B_i) \rightarrow C \Rightarrow \Delta} \text{L}\rightarrow$$

Note that $(* A_j) * (* B_i) \rightarrow C$ is defined as the second conjunct in the definition of $\exists p V$ and hence using the rule $(L\wedge)$ we have $(\exists p V, \bar{C} \Rightarrow \Delta)$.

- Consider the case when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a right semi-analytic rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and V is of the form $(\Gamma \Rightarrow)$ and $\bar{D} = \phi$ and we want to prove $(\exists p V, \bar{C} \Rightarrow \phi)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_i}{\Gamma, \bar{C} \Rightarrow \phi}$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$. Denote $(\Gamma_i \Rightarrow)$ as S_i . Then we have that S_1, \dots, S_n is a partition of V . First consider the case where it is a non-trivial partition of V . Therefore, the order of any S_i is less than the order of V and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{ir}$ and $\bar{\phi}_{ir}$ are also p -free, we can use the induction hypothesis on the order, and get

$$\exists p S_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}$$

Now, substituting $\{\exists p S_i, \bar{C}_i\}$ as the context in the original rule, we get

$$\exists p S_1, \dots, \exists p S_n, \bar{C}_1, \dots, \bar{C}_n \Rightarrow \phi$$

then using the rule (L^*) we have

$$* \exists p S_i, \bar{C} \Rightarrow \phi$$

and since $* \exists p S_i$ appears as the first conjunct in the definition of $\exists p V$, using the rule $(L \wedge)$ we have $(\exists p V, \bar{C} \Rightarrow \phi)$.

It remains to investigate the case where S_1, \dots, S_n is a trivial partition of V . W.l.o.g. suppose $S_1 = V$ and all the others are the empty sequents. Hence, we must have had the following instance of the rule

$$\frac{\langle \Gamma, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r} \rangle_r \quad \langle \langle \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_{i \neq 1}}{\Gamma, \bar{C} \Rightarrow \phi}$$

We have, for all r , $V \cdot (\bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r})$ are the premises of $V \cdot (\bar{C} \Rightarrow \phi)$. Hence the length of tree proofs of all of them are less than the length of proof of $V \cdot (\bar{C} \Rightarrow \phi)$ and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{1r}$ and $\bar{\phi}_{1r}$ are also p -free, we can use the induction hypothesis (induction on the length of proof) and get $\exists p V, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r}$. Substituting $\{\exists p V, \bar{C}_1\}$ as the context in the original semi-analytic rule we get

$$\frac{\langle \exists p V, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \bar{\psi}_{1r} \rangle_r \quad \langle \langle \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir} \rangle_r \rangle_{i \neq 1}}{\exists p V, \bar{C} \Rightarrow \phi}$$

which is what we wanted.

- And the final case is when the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta})$ is the conclusion of a semi-analytic modal rule and V is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box\Delta}$, where $|\overline{\Box\Delta}| \leq 1$ and $V' = (\Gamma \Rightarrow)$. We want to prove $(\exists pV, \bar{C} \Rightarrow \bar{D})$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C}' \Rightarrow \bar{\Delta}}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box\Delta}}$$

Since the order of V' is less than the order of V , and C' and Δ are p -free, we can use the induction hypothesis and get

$$\exists pV', \bar{C}' \Rightarrow \bar{\Delta}$$

Using the rule K or D (depending on the cardinality of $\overline{\Box\Delta}$) we have $\Box\exists pV', \overline{\Box C'} \Rightarrow \overline{\Box\Delta}$ and since we have $\Box\exists pV'$ as one of the conjuncts in the definition of $\exists pV$, we conclude $\exists pV, \bar{C} \Rightarrow \bar{D}$ using the rule $(L\wedge)$.

Now, we have to prove (iv) . Similar to the proof of part (iii) , there are several cases to consider.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \in \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}, \phi$ and we want to prove $\bar{X}, \bar{Y}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$.

Consider $T_{js} = (\Pi_j \Rightarrow)$, $S_{1r} = \Gamma_1 \Rightarrow \Delta_1$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{js} 's do not depend on the suffix s , we have $T_{j1} = \dots = T_{js}$ and we denote it by T_j . And, since S_{ir} 's do not depend on r for $i \neq 1$, we have $S_{21} = \dots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1 . Therefore, $T_1, \dots, T_m, S_1, \dots, S_n$ is a partition of U . First, consider the case that S_1 does not equal U . Then the order of all of them are less than the order of U (or in some cases that the others can be equal to U , the length of their proof in the premises is lower) and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{js}$, $\bar{\theta}_{js}$ and $\bar{\phi}_{ir}$ are also p -free, we can use the induction hypothesis to get (for $i \neq 1$):

$$\exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow \quad , \quad \bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall pS_{1r}$$

If we let $\{\exists pT_j, \bar{X}_j\}$ and $\{\exists pS_i, \bar{Y}_i\}$ and $\{\bar{Y}_1\}$ and $\{\forall pS_{1r}\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle\exists pT_j, \bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\exists pS_i, \bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\bar{\phi}_{1r}, \bar{Y}_1 \Rightarrow \forall pS_{1r}\rangle_r}{\exists pT_1, \dots, \exists pT_m, \exists pS_2, \dots, \exists pS_n, \bar{X}, \bar{Y}, \phi \Rightarrow \forall pS_1}$$

Using the rule (L^*) we have

$$(* \exists pT_j) * (* \exists pS_i), \bar{X}, \bar{Y}, \phi \Rightarrow \forall pS_1.$$

Therefore using the rule $(R \rightarrow)$, we have

$$\bar{X}, \bar{Y}, \phi \Rightarrow (* \exists pT_j) * (* \exists pS_i) \rightarrow \forall pS_1.$$

Since the right side of the sequent is a disjunct in the definition of $\forall pU$, using the rule $(R \vee)$ we have $\bar{C}, \phi \Rightarrow \forall pU$.

In the case that $T_1, \dots, T_m, S_1, \dots, S_n$ is a trivial partition of U , it means that either $S_1 = U$ or $U^s = \emptyset$ and one of the others is equal to U .

If $S_1 = U = \Gamma \Rightarrow \Delta$, then all the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle\langle\bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\Gamma, \phi_{1r}, \bar{Y}_1 \Rightarrow \Delta\rangle_r}{\Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta}$$

Therefore, $U \cdot (\phi_{1r}, \bar{Y}_1 \Rightarrow)$ for every r are premises of $U \cdot (\bar{C} \Rightarrow)$, and hence the length of their trees are smaller than the length of the proof tree of $U \cdot (\bar{C} \Rightarrow)$ and since the rule is semi-analytic and ϕ is p -free then $\bar{\phi}_{1r}$ are also p -free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{Y}_1 \Rightarrow \forall pU)$. Substituting $\{\forall pU\}$, $\{\bar{X}_i\}$ and $\{\bar{Y}_i\}$ as the contexts of the premises in the original left rule and letting all the other contexts in the original left rule to be empty we have

$$\frac{\langle\langle\bar{\psi}_{js}, \bar{X}_j \Rightarrow \bar{\theta}_{js}\rangle_s\rangle_j \quad \langle\langle\bar{\phi}_{ir}, \bar{Y}_i \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\phi_{1r}, \bar{Y}_1 \Rightarrow \forall pU\rangle_r}{\bar{X}, \bar{Y}, \phi \Rightarrow \forall pU}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a left semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta)$ is the conclusion of a left semi-analytic rule and U is of the form $\Pi, \Gamma, \phi \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \bar{Y}$ and we want to prove $\bar{X}, \bar{Y} \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Pi_j, \bar{X}_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \Gamma_i, \bar{Y}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi, \Gamma, \bar{X}, \bar{Y}, \phi \Rightarrow \Delta} \quad (\ddagger)$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, $\bigcup_i \bar{Y}_i = \bar{Y}$ and $\bigcup_i \Delta_i = \Delta$.

Since, \bar{X}_j 's and \bar{Y}_i 's are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take \bar{X}_j 's and \bar{Y}_i 's in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js} \rangle_s \rangle_j \quad \langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r \rangle_i}{\Pi, \Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$, we can claim that this rule is backward applicable to U and T_{js} 's and S_{ir} 's are the premises of the rule. Hence, their orders are less than the order of U and we can use the induction hypothesis for them. Note that we have $U \cdot (\bar{C} \Rightarrow)$ is provable in H and from (\ddagger) we have that $T_{js} \cdot (\bar{X}_j \Rightarrow)$ and $S_{ir} \cdot (\bar{Y}_i \Rightarrow)$ are also provable in H . Using the induction hypothesis we get

$$\bar{X}_j \Rightarrow \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \forall p S_{ir}$$

Using the rule $(R \wedge)$ we get

$$\bar{X}_j \Rightarrow \bigwedge_s \forall p T_{js} \quad , \quad \bar{Y}_i \Rightarrow \bigwedge_r \forall p S_{ir}$$

and using the rule (R^*) we get

$$\bar{X}, \bar{Y} \Rightarrow (* \bigwedge_j \bigwedge_s \forall p T_{js}) * (* \bigwedge_r \bigwedge_r \forall p S_{ir}).$$

Since the right side of the sequent is appeared as the second disjunct in the definition of $\forall pU$, using the rule $(R \vee)$ we have $\bar{C} \Rightarrow \forall pU$.

- Consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a right semi-analytic rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Gamma, \bar{C} \Rightarrow \phi)$ is the conclusion of a right semi-analytic rule and U is of the form $\Gamma \Rightarrow \phi$ and we want to prove $\bar{C} \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle\Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma, \bar{C} \Rightarrow \phi} \quad (\star)$$

where $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$.

With the similar reasoning as in the previous case, since \bar{C}_i 's are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take \bar{C}_i 's in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}\rangle_r\rangle_i}{\Gamma \Rightarrow \phi}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir})$ we can claim that this rule is backward applicable to U and S_{ir} 's are the premises of the rule. Hence, their orders are less than the order of U and since the rule is semi-analytic and ϕ is p -free then $\bar{\psi}_{ir}$ and $\bar{\phi}_{ir}$ are also p -free, we can use the induction hypothesis for them. Note that we have $U \cdot (\bar{C} \Rightarrow)$ is provable in H and from (\star) we have that $S_{ir} \cdot (\bar{C}_i \Rightarrow)$ is also provable in H . Using the induction hypothesis we get for every i and r ,

$$\bar{C}_i \Rightarrow \forall pS_{ir}.$$

Using the rule $(R\wedge)$ we get $\bar{C}_i \Rightarrow \bigwedge_r \forall pS_{ir}$ and then using the rule $(R*)$ we get $\bar{C}_i \Rightarrow * \bigwedge_{i,r} \forall pS_{ir}$. And since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall pU$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall pU$.

- And the final case is when the last rule used in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a semi-analytic modal rule. Therefore, $U \cdot (\bar{C} \Rightarrow) = (\Box\Gamma, \Box\bar{C}' \Rightarrow \Box\Delta)$ is the conclusion of a semi-analytic modal rule and U is of the form $(\Box\Gamma \Rightarrow \Box\Delta)$ and $\bar{C} = \Box\bar{C}'$, where $|\Box\Delta| \leq 1$ and $U' = (\Gamma \Rightarrow \Delta)$. We want to prove $(\bar{C} \Rightarrow \forall pU)$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C}' \Rightarrow \bar{\Delta}}{\Box\Gamma, \Box\bar{C}' \Rightarrow \Box\bar{\Delta}}$$

Since the order of U' is less than the order of U and C' is p -free, we can use the induction hypothesis and get

$$\bar{C}' \Rightarrow \forall p U'$$

Using the rule K or D (depending on the cardinality of $\Box\bar{\Delta}$) we have $\Box\bar{C}' \Rightarrow \Box\forall p U'$ and since we have $\Box\forall p U'$ as one of the disjuncts in the definition of $\forall p U$, we conclude $\bar{C} \Rightarrow \forall p U$ using the rule $(R\vee)$.

□

Theorem 4.5. *Any terminating sequent calculus H that extends \mathbf{FL}_e and consists of centered axioms and semi-analytic rules, has H -uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.3 and Theorem 4.4. □

Corollary 4.6. *If $\mathbf{FL}_e \subseteq L$ and L has a terminating sequent calculus consisting of centered axioms and semi-analytic rules, then L has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.5 and Theorem 4.2. □

In the following application, we will use the Corollary 4.6 to generalize the result of [1] to also cover the modal cases:

Corollary 4.7. *The logics \mathbf{FL}_e , \mathbf{FL}_{ew} and their \mathbf{K} and \mathbf{KD} versions have uniform interpolation.*

Proof. Since all the rules of the usual calculi of these logics are semi-analytic and their axioms are centered and since in the absence of the contraction rule the calculi are clearly terminating, by Corollary 4.6, we can prove the claim. □

4.1.2 Context-Sharing Semi-analytic Case

In this subsection we will modify the investigations of the last subsection to also cover the context-sharing semi-analytic rules.

Theorem 4.8. *Let G and H be two sequent calculi with the property that the right and left weakening rules and the context-sharing ($L \rightarrow$) rule are admissible in H and H extends \mathbf{FL}_e . Then if H is a terminating sequent calculus axiomatically extending G with semi-analytic rules and context-sharing semi-analytic rules and G has weak H -uniform interpolation property, so does H .*

Proof. The proof is similar to the proof of Theorem 4.4. For any sequent U and V where $V^s = \emptyset$ and any atom p , we define two p -free formulas, denoted by $\forall pU$ and $\exists pV$ and we will prove that they meet the conditions in the definition of weak H -uniform interpolation. We define them simultaneously and the definition uses recursion on the rank of sequents which is specified by the terminating condition of the sequent calculus H .

If V is the empty sequent we define $\exists pV$ as 1 and otherwise, we define $\exists pV$ as the following

$$\begin{aligned} & \bigwedge_{LR_{csa}} \left(*_{i \neq 1} \left[\left(\bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) \right) \wedge \left(\bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is}) \right) \right] * \left(\bigwedge_s (\exists p \tilde{T}_{1s} \rightarrow \forall p T_{1s}) \rightarrow \bigvee_r \exists p S_{1r} \right) \right. \\ & \quad \wedge \bigwedge_{LR_{sa}} \left(*_{i \neq 1} \left(\bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) * \left(*_{j \neq 1} \left(\bigwedge_s (\exists p \tilde{T}_{js} \rightarrow \forall p T_{js}) \rightarrow \bigvee_r \exists p S_{1r} \right) \right) \right. \right. \\ & \quad \quad \left. \left. \wedge \left(\bigwedge_{par} *_{i \neq 1} \exists p S_i \right) \wedge (\Box \exists p V') \wedge (\exists^G p V) \right) \right. \end{aligned}$$

where for any sequent R , by \tilde{R} we mean $R^a \Rightarrow$. In the first conjunct (the first line), the first big conjunction is over all context semi-analytic rules that are backward applicable to V in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{is} \rangle_s \rangle_i$, $\langle \langle S_{ir} \rangle_r \rangle_{i \neq 1}$ and $\langle S_{1r} \rangle$ and the conclusion is V , where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir} 's are those who have context in the right side of the sequents (Δ_i) in the premises of the context-sharing semi-analytic rule. (Note that picking the block $\langle S_{1r} \rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r} \rangle$, we add one conjunct to the definition.)

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle \langle T_{js} \rangle_s \rangle_j$, $\langle \langle S_{ir} \rangle_r \rangle_{i \neq 1}$ and $\langle S_{1r} \rangle$ and the conclusion is V , where

$T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$ which means that S_{ir} 's are those who have context in the right side of the sequents (Δ_i) in the premises of the left semi-analytic rule. (Again note that picking the block $\langle S_{1r} \rangle$ between the S_{ir} blocks is arbitrary and for any choice of $\langle S_{1r} \rangle$, we add one conjunct to the definition.)

In the third conjunct (first one in the third line), the conjunction is over all non-trivial partitions of $V = S_1 \cdot \dots \cdot S_n$ and i ranges over the number of S_i 's, in this case $1 \leq i \leq n$.

The conjunct $\Box \exists p V'$ appears in the definition whenever V is of the form $(\Box \Gamma \Rightarrow)$ and we consider V' to be $(\Gamma \Rightarrow)$. And finally, since G has weak H -uniform interpolation property, by definition there exist $J(V)$ as weak right p -interpolant of V . We choose one such $J(V)$ and denote it as $\exists^G p V$ and include it in the definition.

If U is the empty sequent define $\forall p U$ as 0. Otherwise, define $\forall p U$ as the following

$$\begin{aligned} & \bigvee_{LR_{csa}} (* \bigwedge_i [\bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is})]) \\ \vee & \bigvee_{LR_{sa}} ([* \bigwedge_i (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir})] * [* \bigwedge_j \bigwedge_s (\exists p \tilde{T}_{js} \rightarrow \forall p T_{js})]) \\ & \vee (\bigvee_{RR} (* \bigwedge_i \bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}))) \\ \vee & \bigvee_{par} (* (\exists p S_i) \rightarrow \forall p S_1) \vee (\Box (\exists p \tilde{U}' \rightarrow \forall p U')) \vee (\forall^G p U). \end{aligned}$$

In the first conjunct (the first line), the first big conjunction is over all context sharing semi-analytic rules that are backward applicable to V in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{is} \rangle_s \rangle_i$, $\langle\langle S_{ir} \rangle_r \rangle_i$ and the conclusion is V , where $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the second conjunct (the second line), the first big conjunction is over all left semi-analytic rules that are backward applicable to V in H . Since H is terminating, there are finitely many of such rules. The premises of the rule are $\langle\langle T_{js} \rangle_s \rangle_j$, $\langle\langle S_{ir} \rangle_r \rangle_i$ and the conclusion is V , where $T_{js} = (\Pi_j, \bar{\psi}_{js} \Rightarrow \bar{\theta}_{js})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i)$.

In the third disjunct (the third line), the big disjunction is over all right semi-analytic rules backward applicable to U in H . The premise of the rule is $\langle\langle S_{ir}\rangle_r\rangle_i$ and the conclusion is U .

In the fourth disjunct, the big disjunction is over all partitions of $U = S_1 \cdot \dots \cdot S_n$ such that for each $i \neq 1$ we have $S_i^s = \emptyset$ and $S_1 \neq U$. (Note that in this case, if $S^s = \emptyset$ it may be possible that for one $i \neq 1$ we have $S_i = U$. Then the first disjunct of the definition must be $\exists pU \rightarrow \forall pS_1$ where $\forall pS_1 = 0$. But this does not make any problem, since the definition of $\exists pU$ is prior to the definition of $\forall pU$.)

The fifth disjunct is on all semi-analytic modal rules with the result U and the premise U' . And finally, since G has weak H -uniform interpolation property, by definition there exist $I(U)$ as left weak p -interpolant of U . We choose one such $I(U)$ and denote it as $\forall^G pU$ and include it in the definition.

To prove the theorem we use induction on the order of the sequents to prove both cases $\forall pU$ and $\exists pV$ simultaneously. First we have to show that:

- (i) $V \cdot (\Rightarrow \exists pV)$ is derivable in H .
- (ii) $U \cdot (\forall pU \Rightarrow)$ is derivable in H .

The proof is similar to the proof of the Theorem 4.4. Therefore, we will prove two of the cases, one for (i) and one for (ii), where there is a notable difference.

- In proving (i), we have to show that $V \cdot (\Rightarrow X)$ is derivable in H for any X that is one of the conjuncts in the definition of $\exists pV$. Then, using the rule (R_\wedge) it follows that $V \cdot (\Rightarrow \exists pV)$. Since V is of the form $\Gamma \Rightarrow$, we have to show $\Gamma \Rightarrow X$ is derivable in H .

Consider the case where X is the first conjunct in the definition of $\exists pV$. In this case, we have to prove that for any context-sharing semi-analytic rules that is backward applicable to V in H , we have $V \cdot (\Rightarrow Y)$ in H , where $X = \bigwedge_{LR_{sh}} Y$. Therefore, V is the conclusion of a context-sharing

semi-analytic rule and is of the form $(\Gamma, \phi \Rightarrow)$ such that the premises are $\langle\langle T_{is}\rangle_s\rangle_i$ and $\langle\langle S_{ir}\rangle_r\rangle_i$, where T_{is} is of the form $(\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ and we have $\{\Gamma_1, \dots, \Gamma_n\} = \Gamma$. Therefore, their orders are less than the order of V . Moreover, since $\tilde{T}_{is} = (T_{is}^a \Rightarrow)$ and $\tilde{S}_{ir} = (T_{ir}^a \Rightarrow)$ and they are subsequents of T_{is} and S_{ir} , their orders are less than or equal to the orders of T_{is} and S_{ir} . Hence, we can use

the induction hypothesis for all of them.

Using the induction hypothesis for T_{is} , \tilde{T}_{is} , S_{ir} and \tilde{S}_{ir} , for $i \neq 1$, we have the following

$$\Gamma_i, \bar{\psi}_{is}, \forall p T_{is} \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\psi}_{is} \Rightarrow \exists p \tilde{T}_{is},$$

$$\Gamma_i, \bar{\phi}_{ir}, \forall p S_{ir} \Rightarrow \quad , \quad \Gamma_i, \bar{\phi}_{ir} \Rightarrow \exists p \tilde{S}_{ir}.$$

And using the induction hypothesis for S_{1r} , T_{1s} and \tilde{T}_{1s} we have

$$\Gamma_1, \bar{\phi}_{1r} \Rightarrow \exists p S_{1r} \quad , \quad \Gamma_1, \bar{\psi}_{1s}, \forall p T_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\psi}_{1s} \Rightarrow \exists p \tilde{T}_{1s}.$$

Now, using the left context-sharing implication rule, we have

$$\Gamma_i, \bar{\psi}_{is}, \exists p \tilde{T}_{is} \rightarrow \forall p T_{is} \Rightarrow \bar{\theta}_{is}$$

$$\Gamma_i, \bar{\phi}_{ir}, \exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir} \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \exists p \tilde{T}_{1s} \rightarrow \forall p T_{1s} \Rightarrow \bar{\theta}_{1s}$$

Now, first using the rules $(L\wedge)$ and $(R\vee)$, we have

$$\Gamma_i, \bar{\psi}_{is}, \bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, \bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s (\exists p \tilde{T}_{1s} \rightarrow \forall p T_{1s}) \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r} \Rightarrow \bigvee_r \exists p S_{1r}.$$

For simplicity, denote $(\exists p \tilde{T}_{is} \rightarrow \forall p T_{is})$ as A_{is} and $(\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir})$ as B_{ir} . If we use the rule $(L\wedge)$ again, and the rule left weakening only for S_{1r} , and not changing the rule for T_{1r} , we have

$$\Gamma_i, \bar{\psi}_{is}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow \bar{\theta}_{is} \quad , \quad \Gamma_i, \bar{\phi}_{ir}, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) \Rightarrow$$

$$\Gamma_1, \bar{\psi}_{1s}, \bigwedge_s A_{1s} \Rightarrow \bar{\theta}_{1s} \quad , \quad \Gamma_1, \bar{\phi}_{1r}, \bigwedge_s A_{1s} \Rightarrow \bigvee_r \exists p S_{1r}.$$

Now, it is easy to see that the contexts are sharing and we can substitute the above sequents in the original rule. More precisely, in the original context-sharing semi-analytic rule consider $(\Gamma_i, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}))$ as the context of the premises (as Γ_i 's in definition of a context-sharing semi-analytic rule 3.1) for $i \neq 1$ and consider $(\Gamma_1, \bigwedge_s A_{1s})$ as the context of the premises for $i = 1$ (as Γ_1 's in definition of a context-sharing semi-analytic rule 3.1). Therefore, after substituting the above sequents in the original context-sharing semi-analytic rule, we conclude

$$\Gamma_1, \bigwedge_s A_{1s}, \Gamma_2, \dots, \Gamma_n, (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir})_{i \neq 1}, \phi \Rightarrow \bigvee_r \exists p S_{1r}$$

And finally, using the rule L^* and $R \rightarrow$ we get

$$\Gamma, \phi \Rightarrow (*_{i \neq 1} (\bigwedge_s A_{is} \wedge \bigwedge_r B_{ir}) * (\bigwedge_s A_{1s}) \rightarrow \bigvee_r \exists p S_{1r})$$

and this is what we wanted.

- To prove (ii), we have to show that $U \cdot (X \Rightarrow)$ is derivable in H for any X that is one of the disjuncts in the definition of $\forall p U$. Then, using the rule $(L\vee)$ it follows that $U \cdot (\forall p U \Rightarrow)$. Since U is of the form $(\Gamma \Rightarrow \Delta)$, we have to show $(\Gamma, X \Rightarrow \Delta)$ is derivable in H .

In the case that the disjunct is

$$\bigvee_{L\mathcal{R}_{csa}} (*_i [\bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is})]),$$

we have to prove that for any context-sharing semi-analytic rule that is backward applicable to U in H we have

$$U \cdot (*_i [\bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}) \wedge \bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is})] \Rightarrow).$$

The proof goes exactly as in the previous case (in proof of (i) for context-sharing semi-analytic rules), except that this time the succedents of S_{ir} 's and U are not empty and Δ_i 's and Δ appear in their positions everywhere. And, we do not separate the cases T_{1s} and S_{1r} and we proceed with the proof considering the induction hypothesis for every i , in a uniform manner.

Note that these two cases were the cases for the only rule that is not considered in the proof of 4.4. For the proof of (i) for the other conjuncts and (ii) for the other disjuncts, we proceed with the proof of the corresponding cases

as in the proof of 4.4, this time substituting $(\exists p\tilde{T}_{js} \rightarrow \forall pT_{js})$ for $\forall pT_{js}$ and $(\exists p\tilde{S}_{ir} \rightarrow \forall pS_{ir})$ for $\forall pS_{ir}$ wherever it is needed. One can easily see that the proof essentially goes as before, considering this minor change.

Secondly, we have to prove the following, as well.

- (iii) For any p -free multisets Γ and Δ , if $T \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $J(T), \Gamma \Rightarrow \Delta$ is derivable in H .
- (iv) For any p -free multiset Γ , if $S \cdot (\Gamma \Rightarrow)$ is derivable in G then $J(\tilde{S}), \Gamma \Rightarrow I(S)$ is derivable in H .

Again, since the spirit of the proof is the same as the proof of Theorem 4.4, we will prove two cases for the context-sharing semi-analytic rule, which were not present in the Theorem 4.4. We will prove (iii) and (iv) simultaneously using induction on the length of the proof and induction on the order of U and V as in the Theorem 4.4.

- o To prove (iii), consider the case where the last rule used in the proof of $V \cdot (\bar{C} \Rightarrow \bar{D})$ is a context-sharing semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $V \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \Delta)$ is the conclusion of a context-sharing semi-analytic rule and V is of the form $(\Gamma, \phi \Rightarrow)$ and we want to prove $(\exists pV, \bar{C} \Rightarrow \Delta)$. Hence, we must have had the following instance of the rule

$$\frac{\langle\langle\Gamma_i, \bar{C}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\Gamma_1, \bar{C}_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Gamma, \bar{C}, \phi \Rightarrow \Delta}$$

where $\bigcup_j \Pi_j = \Pi$, $\bigcup_i \Gamma_i = \Gamma$ and $\bigcup_i \bar{C}_i = \bar{C}$.

Since, \bar{C}_i 's are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take \bar{C}_i 's as contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma_i, \bar{\phi}_{ir} \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta\rangle_r}{\Gamma, \phi \Rightarrow \Delta}$$

If we let $T_{is} = (\Gamma_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is})$ and $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow)$ for $i \neq 1$ and $S_{1r} = (\Gamma_1, \bar{\phi}_{1r} \Rightarrow \Delta)$, we can claim that this rule is backward applicable to V and T_{is} 's and S_{ir} 's are the premises of the rule. Hence, their orders are less than the order of V and we can use the induction hypothesis

for them. Furthermore, since $\tilde{T}_{is} = (T_{is}^a \Rightarrow)$ and $\tilde{S}_{ir} = (S_{ir}^a \Rightarrow)$, their orders are smaller than or equal to the orders of T_{is} and S_{ir} and we can use the induction hypothesis for them, as well. Using the induction hypothesis (informally speaking, for the first two premises, use the induction hypothesis of \forall , and for the last premise use the induction hypothesis of \exists) we get

$$(\bar{C}_i, \exists p \tilde{T}_{is} \Rightarrow \forall p T_{is}) \quad , \quad (\bar{C}_i, \exists p \tilde{S}_{ir} \Rightarrow \forall p S_{ir})_{i \neq 1} \quad , \quad (\bar{C}_1, \exists p S_{1r} \Rightarrow \Delta)$$

Now, first using the rules $(R \rightarrow)$ and then using the rule $(R \wedge)$ and $(L \vee)$ we have

$$\begin{aligned} & (\bar{C}_i \Rightarrow \bigwedge_s (\exists p \tilde{T}_{is} \rightarrow \forall p T_{is})) \\ & (\bar{C}_i \Rightarrow \bigwedge_r (\exists p \tilde{S}_{ir} \rightarrow \forall p S_{ir}))_{i \neq 1} \\ & (\bar{C}_1, \bigvee_r \exists p S_{1r} \Rightarrow \Delta) \end{aligned}$$

Denote $(\bigwedge_s \forall p T_{js})$ as A_j and $(\bigwedge_r \forall p S_{ir})$ as B_i (for $i \neq 1$) and $(\bigvee_r \exists p S_{1r})$ as D . We have for $i \neq 1$

$$\bar{C}_i \Rightarrow A_i \quad , \quad \bar{C}_i \Rightarrow B_i$$

and for $i = 1$ we have

$$\bar{C}_1 \Rightarrow A_1 \quad , \quad \bar{C}_1, D \Rightarrow \Delta.$$

Now, and using the rule $(R \wedge)$ for $i \neq 1$ we get $\bar{C}_i \Rightarrow A_i \wedge B_i$. Together with $\bar{C}_1 \Rightarrow A_1$ and using the rule (R^*) we get

$$\bar{C}_1, \bar{C}_2, \dots, \bar{C}_n \Rightarrow \ast_i (A_i \wedge B_i) \ast A_1.$$

Consider the sequent $\bar{C}_1, D \Rightarrow \Delta$ and use the left weakening rule to get

$$\bar{C}_1, \bar{C}_2, \dots, \bar{C}_n, D \Rightarrow \Delta.$$

Now, use the rule left context-sharing implication to reach

$$\bar{C}, (\ast_i (A_i \wedge B_i) \ast A_1) \rightarrow D \Rightarrow \Delta.$$

And, we are done.

- For the proof of (iv) , consider the case where the last rule in the proof of $U \cdot (\bar{C} \Rightarrow)$ is a context-sharing semi-analytic rule and $\phi \in \bar{C}$. Therefore,

$$U \cdot (\bar{C} \Rightarrow) = \Gamma, \bar{X}, \phi \Rightarrow \Delta$$

is the conclusion of a context-sharing semi-analytic rule and U is of the form $\Gamma \Rightarrow \Delta$ and $\bar{C} = \bar{X}, \phi$ and we want to prove $\exists p\tilde{U}, \bar{X}, \phi \Rightarrow \forall pU$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle\Gamma_i, \bar{X}_i, \bar{\psi}_{is} \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \Delta_i\rangle_r\rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \Delta}$$

where $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_j \bar{X}_j = \bar{X}$, and $\bigcup_i \Delta_i = \Delta$. Consider $T_{is} = (\Gamma_i \Rightarrow)$, $S_{1r} = (\Gamma_1 \Rightarrow \Delta_1)$, and for $i \neq 1$ let $S_{ir} = (\Gamma_i \Rightarrow)$. Since T_{is} 's do not depend on the suffix s , we have $T_{i1} = \dots = T_{is}$ and we denote it by T_i . And, since S_{ir} 's do not depend on r for $i \neq 1$, we have $S_{21} = \dots = S_{ir}$ and we denote it by S_i and with the same line of reasoning we denote S_{1r} by S_1 . Therefore, S_1, \dots, S_n is a partition of U . First, consider the case that $S_1 \neq U$. Then the order of all of them are less than the order of U (or in some cases that one of the others equals to U , the length of the proof is lower) and since the rule is context sharing semi-analytic and ϕ is p -free then $\bar{\psi}_{is}$ and $\bar{\phi}_{ir}$ are also p -free, we can use the induction hypothesis to get (for $i \neq 1$):

$$\exists pT_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is} \quad , \quad \exists pS_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \quad , \quad \exists p\tilde{S}_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall pS_1$$

Note that for every $i \neq 1$ we have $T_i = S_i$ and for $i = 1$ we have $T_1 = \tilde{S}_1$ and we can rewrite the above sequents according to this new information. Hence, if we let $\{\exists pT_i, \bar{X}_i\}$ and $\{\forall pS_1\}$ be the contexts in the original left semi-analytic rule, we have the following

$$\frac{\langle\langle\exists pT_i, \bar{\psi}_{is}, \bar{X}_i \Rightarrow \bar{\theta}_{is}\rangle_s\rangle_i \quad \langle\langle\exists pT_i, \bar{\phi}_{ir}, \bar{X}_i \Rightarrow\rangle_r\rangle_{i \neq 1} \quad \langle\exists pT_1, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall pS_1\rangle_r}{\exists pT_1, \dots, \exists pT_n, \bar{X}, \phi \Rightarrow \forall pS_1}$$

Using first the rule (L^*) and second the rule $R \rightarrow$ we get

$$\exists pT_1, \bar{X}, \phi \Rightarrow \underset{i \neq 1}{*} \exists pT_i \rightarrow \forall pS_1$$

Since T_2, \dots, T_n, S_1 is a partition of U , the right hand side of the above sequent is appeared as one of the disjuncts in the definition of $\forall pU$. And since $T_1 \subseteq \tilde{U}$, we have

$$\exists p\tilde{U}, \bar{C} \Rightarrow \forall pU$$

and we are done.

We have to investigate the case when $S_1 = U$, as well. However, the line of reasoning is as above and as in the case of $\forall pU$, and $\phi \in \bar{C}$ in the proof of the Theorem 4.4. The important thing is that in the case where $S_1 = U$, with similar reasoning as above, at the end we get $\exists p\tilde{S}_1, \bar{C} \Rightarrow \forall pS_1$ which solves the problem. Note that this case is one of the main reasons that we have changed uniform interpolation to weak uniform interpolation.

And finally, to prove (iii) and (iv) for the other cases, use similar reasonings as in the proof of Theorem 4.4, this time substituting $(\exists p\tilde{T}_{js} \rightarrow \forall pT_{js})$ for $\forall pT_{js}$ and $(\exists p\tilde{S}_{ir} \rightarrow \forall pS_{ir})$ for $\forall pS_{ir}$ wherever it is needed, then the proof easily follows. \square

Theorem 4.9. *Any terminating sequent calculus H that extends **IPC** and consists of centered axioms, semi-analytic and context-sharing semi-analytic rules, has weak H -uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.3 and the Theorem 4.8. \square

Corollary 4.10. *If $\mathbf{IPC} \subseteq L$ and L has a terminating sequent calculus consisting of centered axioms, semi-analytic rules and context-sharing semi-analytic rules, then L has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.9 and the Theorem 4.2. \square

The clear application of this theorem is the uniform interpolation property for the logic **IPC**.

Corollary 4.11. [9] *The logic **IPC** has uniform interpolation.*

Proof. Use the Dyckhoff terminating calculus for **IPC** introduced in the Preliminaries section. Note that all the rules in this calculus, except the rule (L4) are semi-analytic, while (L4) is context-sharing semi-analytic and all the axioms are centered. Since this calculus admits weakening and context-sharing implication rules, by Theorem 4.10 we can prove the claim. \square

4.2 The Multi-conclusion Case

Finally we will move to the multi-conclusion case to handle the more general form of semi-analytic rules.

Theorem 4.12. *Let G and H be two sequent calculi and H extends \mathbf{CFL}_e . Then if H is a terminating sequent calculus axiomatically extending G with multi-conclusion semi-analytic rules and G has strong H -uniform interpolation property, so does H .*

Proof. For a given sequent $S = (\Gamma \Rightarrow \Delta)$ and an atom p , we define a p -free formula, denoted by $\forall pS$ and we will prove that it meets the conditions for the strong left and right p -interpolants of S , respectively.

If S is the empty sequent define $\forall pS$ as 0. Otherwise, define $\forall pS$ as

$$\bigvee_{\mathcal{R}} (* \bigwedge_i \bigwedge_r \forall pS_{ir}) \vee \bigvee_{par} (\bigoplus_i \forall pS_i) \vee (\Box \forall pS') \vee (\neg \Box \neg \forall pS'') \vee (\forall^G pS)$$

where the first disjunction is over all multi-conclusion semi-analytic rules backward applicable to S in H , which means the result is S and the premises are S_{ir} . Since H is terminating, there are finitely many of such rules. The second disjunction is over all non-trivial partitions of S . The third disjunction is over all semi-analytic modal rules with the result S and the premise S' . Moreover, If S is of the form $\Box \Gamma \Rightarrow$, then we consider S'' to be $\Gamma \Rightarrow$ and $\neg \Box \neg \forall pS''$ must be appeared in the definition of $\forall pS$. And finally $\forall^G pS$ is the strong left p -interpolant of a sequent S in G relative to H .

We define the strong right p -interpolant of S as $\neg \forall pS$ and we denote it by $\exists pS$. Note that if we prove $\forall pS$ is the strong left p -interpolant, it is easy to show that $\exists pS$ meets the conditions for the strong right p -interpolant. The reason is the following: First we have to show that $\Gamma \Rightarrow \Delta, \exists pS$ is provable in H . But we have $\Gamma, \forall pS \Rightarrow \Delta$ is provable in H and using the rule $(R0)$, we have $\Gamma, \forall pS \Rightarrow \Delta, 0$ which means $\Gamma \Rightarrow \Delta, \neg \forall pS$ is provable in H . Secondly, we have to show that if for p -free multisets Σ and Λ , if $\Gamma, \Sigma \Rightarrow \Lambda, \Delta$ is derivable in G , then $\exists pS, \Sigma \Rightarrow \Lambda$ is derivable in H . However, we have $\Sigma \Rightarrow \Lambda, \forall pS$ is derivable in H and using the axiom $0 \Rightarrow$ we can use the rule $(L \rightarrow)$ to get $\Sigma, \neg \forall pS \Rightarrow \Lambda$ in H .

Now let us prove that $\forall pS$ meets all the conditions of a strong left p -interpolant. The proof is similar to the proofs of the Theorems 4.4 and 4.8. To prove the theorem we use induction on the order of the sequents. First, we have to show that

(i) $S \cdot (\forall pS \Rightarrow)$ is provable in H .

We have to show that $\Gamma, X \Rightarrow \Delta$ is derivable in H for every disjunct X in the definition of $\forall pS$.

- In the case that the disjunct is $\bigvee_{\mathcal{R}} (* \bigwedge_i \bigwedge_r \forall pS_{ir})$, we have to show that for any multi-conclusion semi-analytic rule \mathcal{R} with the premises S_{ir} we have

$$S \cdot (* \bigwedge_i \bigwedge_r \forall pS_{ir} \Rightarrow)$$

where S is of the form $(\Gamma_1, \dots, \Gamma_n, \phi \Rightarrow \Delta_1, \dots, \Delta_n)$ and S_{ir} is of the form $(\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$. Note that since S'_{ir} s are the premises of the rule, the order of all of them are less than the order of S and we can use the induction hypothesis for them. We have for every i and r

$$\Gamma_i, \bar{\phi}_{ir}, \forall pS_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$

Using the rule $(L \wedge)$ we have for every i

$$\Gamma_i, \bar{\phi}_{ir}, \bigwedge_r \forall pS_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i$$

Using $\Gamma_i, \bigwedge_r \forall pS_{ir}$ as the left context in the original rule (we can do this, since $\bigwedge_r \forall pS_{ir}$ does not depend on r and it only ranges over i), we have

$$\Gamma_1, \dots, \Gamma_n, \langle \bigwedge_r \forall pS_{ir} \rangle_i, \phi \Rightarrow \Delta_1, \dots, \Delta_n$$

and then using the rule (L^*) , we have

$$\Gamma_1, \dots, \Gamma_n, (* \bigwedge_i \bigwedge_r \forall pS_{ir}), \phi \Rightarrow \Delta_1, \dots, \Delta_n.$$

- In the case that the disjunct is $\bigvee_{par} \bigoplus_i \forall pS_i$, we have to show that for any non-trivial partition S_1, \dots, S_n of S we have $S \cdot (\bigoplus_i \forall pS_i \Rightarrow)$ is derivable in H . Since the order of each S_i is less than the order of S , we can use the induction hypothesis for them and get $(\Gamma_i, \forall pS_i \Rightarrow \Delta_i)$. Using the rule $(L+)$ we get $\Gamma_1, \dots, \Gamma_n, (\bigoplus_i \forall pS_i) \Rightarrow \Delta_1, \dots, \Delta_n$.
- The proof of case that the disjunct is $\square \forall pS'$ is exactly the same as the similar case in the proof of the Theorem 4.4.

- In the case that the disjunct is $\neg\Box\neg\forall pS''$, the sequent S must have been of the form $(\Box\Gamma \Rightarrow)$ and S'' is of the form $(\Gamma \Rightarrow)$. Since the order of S'' is less than the order of S , we can use the induction hypothesis and get $(\Gamma, \forall pS'' \Rightarrow)$ is derivable in H . Using the rule $(R0)$ and then the rule $(R \rightarrow)$ we have $(\Gamma \Rightarrow \neg\forall pS'')$. Using the rule (K) we have $(\Box\Gamma \Rightarrow \Box\neg\forall pS'')$ and together with the axiom $(0 \Rightarrow)$ we can use the rule $(L \rightarrow)$ and we have $(\Box\Gamma, \neg\Box\neg\forall pS'' \Rightarrow)$ is derivable in H .
- The case for $\forall^G pS$, holds trivially by definition.

Second, we have to show that

- (ii) For any p -free multisets \bar{C} and \bar{D} , if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\bar{C} \Rightarrow \forall pS, \bar{D}$ is derivable in H .

We will prove it using induction on the length of the proof and induction on the order of S . More precisely, first by induction on the order of S and then inside it, by induction on n , we will show:

- For any p -free multisets \bar{C} and \bar{D} , if $S \cdot (\bar{C} \Rightarrow \bar{D})$ has a proof in G with length less than or equal to n , then $\bar{C} \Rightarrow \forall pS, \bar{D}$ is derivable in H .

First note that for the empty sequent, we have to show that if $\bar{C} \Rightarrow \bar{D}$ is valid in G , then $\bar{C} \Rightarrow 0, \bar{D}$ is valid in H , which is trivial by the rule $(R0)$.

For the base of the other induction, note that if $n = 0$, it means that $\Gamma, \bar{C} \Rightarrow \bar{D}, \Delta$ is valid in G . Therefore, by Definition 4.1, $\bar{C} \Rightarrow \forall^G pS, \bar{D}$ and hence $\bar{C} \Rightarrow \forall pS, \bar{D}$ is valid in H .

For $n \neq 0$ we have to consider the following cases:

- Consider the case that the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \in \bar{C}$ (which means that the main formula of the rule, ϕ , is one of C_i 's). Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $(\Gamma \Rightarrow \Delta)$ and $\bar{C} = (\bar{X}, \phi)$ and we want to prove $(\bar{X}, \phi \Rightarrow \forall pS, \bar{D})$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle\Gamma_i, \bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i\rangle_r\rangle_i}{\Gamma, \bar{X}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_i \bar{X}_i = \bar{X}$, $\bigcup_i \bar{D}_i = \bar{D}$ and $\bigcup_i \Delta_i = \Delta$. Consider $S_{ir} = (\Gamma_i \Rightarrow \Delta_i)$. Since S_{ir} 's do not depend on the suffix r , all of them

are equal and we denote it by S_i . Therefore, S_1, \dots, S_n is a partition of S . First, consider that it is a non-trivial partition of S . Then the order of all of them are less than the order of S and since the rule is semi-analytic and ϕ is p -free then $\bar{\phi}_{ir}$ and $\bar{\psi}_{ir}$ are also p -free, we can use the induction hypothesis to get for every i and r :

$$\bar{X}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \forall p S_i$$

If we let \bar{X}_i and $\bar{D}_i, \forall p S_i$ be the contexts in the left side and right side in the original rule, respectively, we have the following

$$\bar{X}, \phi \Rightarrow \bar{D}, \forall p S_1, \dots, \forall p S_n$$

Using the rule $(R+)$ we have

$$\bar{X}, \phi \Rightarrow \bar{D}, \bigvee_i \forall p S_i$$

Since the right side of the sequent is a disjunct in the definition of $\forall p U$, using the rule $(R\vee)$ we have $\bar{C}, \phi \Rightarrow \forall p S, \bar{D}$.

In the case that S_1, \dots, S_n is a trivial partition of S , it means that one of them equals S . W.l.o.g. suppose $S_1 = S$ and all of the others are the empty sequents. Then we must have had the following instance of the rule:

$$\frac{\langle \langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i \rangle_r \rangle_{i \neq 1} \quad \langle \Gamma, \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1, \Delta \rangle_r}{\Gamma, \phi, \bar{X} \Rightarrow \bar{D}, \Delta}$$

Therefore, $S \cdot (\phi_{1r}, \bar{X}_1 \Rightarrow \bar{\psi}_{1r}, \bar{D}_1)$ for every r are premises of $S \cdot (\bar{C} \Rightarrow \bar{D})$, and hence the length of their trees are smaller than the length of the proof tree of $S \cdot (\bar{C} \Rightarrow \bar{D})$ and since the rule is semi-analytic and ϕ is p -free then $\bar{\phi}_{1r}$ and $\bar{\psi}_{1r}$ are also p -free, which means that for all of them we can use the induction hypothesis (induction on the length of the proof), and we have $(\phi_{1r}, \bar{X}_1 \Rightarrow \forall p S, \bar{\psi}_{1r}, \bar{D}_1)$. Substituting $\{\bar{X}_j\}$ and $\{\forall p S, \bar{D}_1\}$ as the contexts of the premises in the original rule we have

$$\frac{\langle \langle \bar{\phi}_{ir}, \bar{X}_i \Rightarrow \bar{\psi}_{ir}, \bar{D}_i \rangle_r \rangle_{i \neq 1} \quad \langle \bar{\phi}_{1r}, \bar{X}_1 \Rightarrow \forall p S, \bar{\psi}_{1r}, \bar{D}_1 \rangle_r}{\bar{X}, \phi \Rightarrow \forall p S, \bar{D}}$$

which is what we wanted.

- Consider the case where the last rule in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a left multi-conclusion semi-analytic rule and $\phi \notin \bar{C}$. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta)$ is the conclusion of the rule and S is of the form $\Gamma, \phi \Rightarrow \Delta$ and we want to prove $\bar{C} \Rightarrow \forall pS, \bar{D}$. Hence, we must have had the following instance of the rule:

$$\frac{\langle\langle \Gamma_i, \bar{C}_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \bar{D}_i, \Delta_i \rangle_r \rangle_i}{\Gamma, \bar{C}, \phi \Rightarrow \bar{D}, \Delta}$$

where $\bigcup_i \Gamma_i = \Gamma$, $\bigcup_i \bar{C}_i = \bar{C}$, $\bigcup_i \bar{D}_i = \bar{D}$ and $\bigcup_i \Delta_i = \Delta$.

Since, \bar{C}_i 's and \bar{D}_i 's are in the context positions in the original rule, we can consider the same substitution of meta-sequents and meta-formulas as above in the original rule, except that we do not take \bar{C}_i 's and \bar{D}_i 's in the contexts. More precisely, we reach the following instance of the original rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma, \phi \Rightarrow \Delta}$$

If we let $S_{ir} = (\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bar{\psi}_{ir}, \Delta_i)$, we can claim that this rule is backward applicable to S and S_{ir} 's are the premises of the rule. Hence, their orders are less than the order of S and we can use the induction hypothesis for them. Using the induction hypothesis we get for every i and r

$$\bar{C}_i \Rightarrow \forall pS_{ir}, \bar{D}_i$$

Using the rule $(R\wedge)$ we get for every i

$$\bar{C}_i \Rightarrow \bigwedge_r \forall pS_{ir}, \bar{D}_i$$

and using the rule $(R*)$ we get

$$\bar{C} \Rightarrow * \bigwedge_i \bigwedge_r \forall pS_{ir}, \bar{D}.$$

Since the right side of the sequent is appeared as one of the disjuncts in the definition of $\forall pS$, using the rule $(R\vee)$ we have $\bar{C} \Rightarrow \forall pS, \bar{D}$.

- Consider the case when the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is a semi-analytic modal rule. Therefore, $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Box\Gamma, \Box\bar{C}' \Rightarrow \Box\bar{D}')$ is the conclusion of a semi-analytic modal rule. Hence, there are two cases to consider.

The first one is the case where S is of the form $(\Box\Gamma \Rightarrow)$ and $\bar{C} = \overline{\Box C'}$ and $\bar{D} = \overline{\Box D'}$, where $|\overline{\Box D'}| \leq 1$ and $S'' = (\Gamma \Rightarrow)$. We want to prove $(\bar{C} \Rightarrow \forall p S, \bar{D})$. We must have had the following instance of the rule

$$\frac{\Gamma, \bar{C}' \Rightarrow \bar{D}'}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

Since the order of S'' is less than the order of S and C' and D' are p -free, we can use the induction hypothesis and get

$$\bar{C}' \Rightarrow \forall p S'', \bar{D}'$$

Using the axiom $(0 \Rightarrow)$ and the rule $(L \rightarrow)$ we have

$$\bar{C}', \neg\forall p S'' \Rightarrow \bar{D}'$$

Now, using the rule K or D (depending on the cardinality of \bar{D}') we have

$$\overline{\Box C'}, \Box\neg\forall p S'' \Rightarrow \overline{\Box D'}$$

and using the rule $(R0)$ and $(R \rightarrow)$ we get

$$\overline{\Box C'} \Rightarrow \neg\Box\neg\forall p S'', \overline{\Box D'}$$

since we have $\neg\Box\neg\forall p S''$ as one of the disjuncts in the definition of $\forall p S$, we conclude $\bar{C} \Rightarrow \forall p S, \bar{D}$ using the rule $(R\vee)$.

The second case is when S is of the form $\Box\Gamma \Rightarrow \Box D'$, where D' is a p -free formula and S' is of the form $\Gamma \Rightarrow D$. We want to prove $\bar{C} \Rightarrow \forall p S$. Then we must have had the following instance of the rule

$$\frac{\Gamma, \bar{C}' \Rightarrow \bar{D}'}{\Box\Gamma, \overline{\Box C'} \Rightarrow \overline{\Box D'}}$$

Since \bar{C}' is in the context position of the original rule, we can consider the same substitution of meta-sequents as above in the original rule, except that we do not take \bar{C}' in the context. More precisely, we reach the following instance of the original rule:

$$\frac{\Gamma \Rightarrow \bar{D}'}{\Box\Gamma \Rightarrow \overline{\Box D'}}$$

Therefore, this rule is backward applicable to S and the order of the premise, S' , is less than the order of S and we can use the induction hypothesis for that to reach $C' \Rightarrow \forall pS'$. Then we can use the rule K and we get $\overline{\Box C'} \Rightarrow \Box \forall pS'$, which is a disjunct in the definition of $\forall pS$ and we have $\overline{C'} \Rightarrow \forall pS$.

- The case for the right multi-conclusion semi-analytic rules is similar to the cases for the left ones discussed in this proof, and the proof of other two cases are similar to the proof of the same cases in the Theorem 4.4.

□

Theorem 4.13. *Any terminating multi-conclusion sequent calculus H that extends \mathbf{CFL}_e and consists of centered axioms and multi-conclusion semi-analytic rules, has strong H -uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.3 and Theorem 4.12. □

Corollary 4.14. *If $\mathbf{CFL}_e \subseteq L$ and L has a terminating sequent calculus consisting of centered axioms and multi-conclusion semi-analytic rules, then L has uniform interpolation.*

Proof. The proof is a result of the combination of the Theorem 4.13 and Theorem 4.2. □

Using the Theorem 4.14, we can extend the results of [1] and [2] to:

Corollary 4.15. *The logics \mathbf{CFL}_e , \mathbf{CFL}_{ew} and \mathbf{CPC} and their \mathbf{K} and \mathbf{KD} modal versions have uniform interpolation property.*

Proof. For \mathbf{CFL}_e , \mathbf{CFL}_{ew} , since all the rules of the usual calculus of these logics are semi-analytic and their axioms are centered and since in the absence of the contraction rule the calculus is clearly terminating, by Theorem 4.14, we can prove the claim. For \mathbf{CPC} use the contraction-free calculus for which the proof goes as the other cases. □

In the negative side we use the negative results in [2], [5] and [6] to ensure that the following logics do not have uniform interpolation. Then we will use the Theorems 4.6, 4.10 and 4.14 to the non-existence of terminating calculus consisting only of semi-analytic and context-sharing semi-analytic rules together with centered axioms.

Corollary 4.16. *The logics $\mathbf{K4}$ and $\mathbf{S4}$ do not have a terminating sequent calculus consisting only of single conclusion (multi-conclusion) semi-analytic and context-sharing semi-analytic rules plus some centered axioms.*

Corollary 4.17. *Except the logics IPC, LC, KC, Bd₂, Sm, GSc and CPC, non of the super-intuitionistic logics have a terminating sequent calculus consisting only of single conclusion semi-analytic rules and context-sharing semi-analytic rules plus some centered axioms.*

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