

Equations of compressible fluids with random perturbations

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Field equations

Phase variables

mass density $\varrho = \varrho(t, \mathbf{x})$
bulk velocity $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^N$

Mass conservation

$$d\varrho + \operatorname{div}_x(\varrho \mathbf{u}) dt = 0$$

Balance of momentum

$$d(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) dt + \nabla_x p(\varrho) dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) dt + \boxed{\varrho \mathbf{g}(\varrho, \mathbf{u}) dW}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Boundary and initial conditions

spatially periodic boundary conditions $Q = ([-1, 1] |_{\{-1, 1\}})^N$

$$\varrho(0, \cdot) = \varrho_0, \quad \varrho \mathbf{u}(0, \cdot) = (\varrho \mathbf{u})_0$$

Stochastic forcing

Stochastic basis

$$\{\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathcal{P}\}$$

Random driving force in the momentum equation

$$\mathbf{g}(\varrho, \mathbf{u})dW \equiv \sum_{k=1}^{\infty} \mathbf{g}_k(\varrho, \mathbf{u})dW_k$$

$$\mathbf{g}_k \in W^{1,\infty}(R^{N+1}, R^N), \|\mathbf{g}_k\|_{W^{1,\infty}} \leq \alpha_k \text{ where } \sum_{k=1}^{\infty} \alpha_k^2 < \infty$$

Deterministic initial conditions

$$\varrho(0, \cdot) = \varrho_0, (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0 \mathcal{P} - \text{a.s.}$$

Weak martingale solution

Probability basis, forcing term

$$\left\{ \Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathcal{P} \right\}, \{W_k\}_{k=1}^{\infty}$$

Field equations

$$d \int_Q \varrho \varphi \, dx = \int_Q \varrho \mathbf{u} \cdot \nabla_x \varphi \, dx dt, \quad \int_Q \varrho(0, \cdot) \varphi \, dx = \int_Q \varrho_0 \varphi \, dx$$

$$d \int_Q \varrho \mathbf{u} \cdot \varphi \, dx = \int_Q [\varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \operatorname{div}_x \varphi] \, dx dt$$

$$- \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, dx dt + \sum_{k=1}^{\infty} \left(\int_Q \varrho \mathbf{g}_k(\varrho, \mathbf{u}) \cdot \varphi \, dx \right) dW_k$$

$$\int_Q \varrho \mathbf{u}(0, \cdot) \cdot \varphi \, dx = \int_Q (\varrho \mathbf{u})_0 \cdot \varphi \, dx$$

Dissipative martingale solutions

Energy inequality

$$\begin{aligned} & \left[\psi \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] dx \right]_{t=0}^{t=\tau} + \int_0^\tau \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx dt \\ & \leq \int_0^\tau \partial_t \psi \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] dx dt \\ & + \int_0^\tau \psi \int_Q \frac{1}{2} \sum_{k=1}^{\infty} \varrho |\mathbf{g}_k(\varrho, \mathbf{u})|^2 dx dt \int_0^\tau \psi \sum_{k=1}^{\infty} \left(\int_Q \varrho \mathbf{u} \cdot \mathbf{g}_k(\varrho, \mathbf{u}) dx \right) dW_k \end{aligned}$$

Pressure potential

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} dz$$

Available results

Existence for finite energy data

- Global existence for any finite energy data, $p(\varrho) \approx \varrho^\gamma$, $\gamma > \frac{N}{2}$ [Breit, Hofmanová 2015]
- Local (up to a stopping time) existence of unique strong solutions [Breit, EF, Hofmanová 2017], [Kim 2011]

Weak-strong uniqueness

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law [Breit, EF, Hofmanová 2016]

Augmented system

Mass conservation

$$d\rho + \operatorname{div}_x(\rho \mathbf{u}) dt = 0$$

Balance of momentum

$$d(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) dt + \nabla_x p(\rho) dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) dt + \boxed{\rho \mathbf{g}(\rho, \mathbf{u}) dW}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Velocity “potential”

$$d\mathbf{v} = \mathbf{u} dt, \quad \mathbf{v}(0) = \mathbf{v}_0$$

Canonical (trajectory) space

Label space

$$X = \cup_{M>0} X_M$$

$$X_M \equiv \left\{ [r, \mathbf{m}, \mathbf{w}] \mid r \geq 0, \int_Q \frac{1}{2} \frac{|\mathbf{m}|^2}{r} + P(r) \, dx < M, \mathbf{w} \in W^{1,2}(Q; \mathbb{R}^N) \right\}$$

Trajectory space

$$\Omega^{[0,\infty)} = \left\{ \xi = [\xi_1, \xi_2, \xi_3] \mid \xi \in C_{\text{loc}}([0, \infty); W^{-k,2}(Q; \mathbb{R}^{2N+1})) \right\}$$

Solution space

$U(x)$ – the law of a dissipative martingale solution $[\varrho, \mathbf{u}, \mathbf{v}]$ on $\Omega^{[0,\infty)}$

with the initial data $\delta_{[\xi_1, \xi_2, \xi_3]=x}$

$$x = [\varrho_0, (\varrho \mathbf{u})_0, \mathbf{v}_0] \in X$$

$U(x)$ – non-empty, convex, compact for all $x \in X$

Markovian family – conditional probability

Markovian family

$$x \in X \mapsto U(x) \in \text{prob} \left[\Omega^{[0, \infty)} \right] \text{ Borel measurable}$$

Conditional probability

$$\begin{aligned} U(x) \{ \xi|_{[0, T]} \in A \wedge \xi|_{[T, \infty)} \in B \} \\ = \int_{\xi \in A} S(T) \circ U([\xi(T)]) \{ B \} dU(x)[\xi] \end{aligned}$$

A Borel subset of $\Omega^{[0, T]}$, B Borel subset of $\Omega^{[T, \infty)}$

for a.a. $T > 0$, where the exceptional set may depend on $U(x)$

$$\Omega^I = \left\{ \xi \mid \xi \in C_{\text{loc}}(I; W^{-k, 2}(Q; R^{2N+1})) \right\}$$

Pre-Markovian family - disintegration

Families of measures

$$\mathcal{U}(x) = \left\{ U(x) \mid U(x) \text{ a law of a solution with the initial value } \delta_x \right\}$$

Disintegration

$$\begin{aligned} U(x) \{ \xi|_{[0,T]} \in A \wedge \xi|_{[T,\infty)} \in B \} \\ = \int_{\xi \in A} V_T[\xi]\{B\} dU(x)[\xi] \end{aligned}$$

\Rightarrow

$$S(-T)V_T[\xi] \in \mathcal{U}([\xi(T)) U(x) \text{ a.s.}$$

for a.a. $T > 0$, where the exceptional set may depend on $U(x)$

Pre-Markovian family – reconstruction

Conditional probability

$$V \in \mathcal{U}(x)$$

$$T > 0$$

$$y \in X \mapsto \tilde{U}(y) \in \mathcal{U}(y)$$

\Rightarrow

there exists $U(x) \in \mathcal{U}(x)$, $U|_{[0,T]} = V$

$$\begin{aligned} U(x) \{ \xi|_{[0,T]} \in A \wedge \xi|_{[T,\infty)} \in B \} \\ = \int_{\xi \in A} S(T) \circ \tilde{U}(\xi(T)) \{ B \} dU(x)[\xi] \end{aligned}$$