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Abstract

We consider a singular limit for the compressible Euler system in the low Mach number regime driven by a large external force. We show that any dissipative measure-valued solution approaches a solution of the lake equation in the asymptotic regime of low Mach and Froude numbers. The result holds for the ill-prepared initial data creating rapidly oscillating acoustic waves. We use dispersive estimates of Strichartz type to eliminate the effect of the acoustic waves in the asymptotic limit.

Key words: compressible Euler equations, singular limit, low Mach number, low Froude number, dissipative measure-valued solutions.

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1 Introduction

Singular limits of stratified fluids play an important role in the real world applications of fluid mechanics, notably in meteorology, see the survey by Klein [17]. We consider the following scaled Euler equations in the whole space $\mathbb{R}^2$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{Ma^2} \nabla_x p(\rho) = \frac{1}{Fr^2} \rho \nabla_x F, \end{cases} \tag{1.1}$$

where the unknown fields $\rho = \rho(t,x)$ and $\mathbf{u} = \mathbf{u}(t,x)$ represent the density and the velocity of an inviscid compressible fluid driven by an external potential force $\nabla_x F$. The Mach number Ma, proportional to the characteristic velocity divided by the sound speed, and the Froude number Fr, defined as the ratio of the flow inertia to the external force, play the role of singular (small)

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parameters. The symbol $p = p(\rho)$ denotes the barotropic pressure. The system is supplemented by the far field conditions

$$\mathbf{u} \to 0, \quad \rho \to \overline{\rho}, \text{ as } |x| \to \infty, \text{ where } \overline{\rho} > 0. \tag{1.2}$$

For $Ma = Fr = \epsilon$ and $\rho = \rho_\epsilon$, $\mathbf{u} = \mathbf{u}_\epsilon$, the system (1.1) takes the form

$$\begin{cases} \partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \\ \partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \frac{1}{\epsilon^2} \nabla_x p(\rho_\epsilon) = \frac{1}{\epsilon^2} \rho_\epsilon \nabla_x F. \end{cases} \tag{1.3}$$

Our goal is to study the singular limit $\epsilon \to 0$. Obviously, the density will approach an asymptotic profile $\widetilde{\rho}$,

$$\nabla_x p(\widetilde{\rho}) = \widetilde{\rho} \nabla_x F, \quad \widetilde{\rho} \to \overline{\rho}, \text{ as } |x| \to \infty. \tag{1.4}$$

Moreover, similarly to [4, 11], the limit velocity $\mathbf{v}$ solves the so-called lake equation

$$\begin{aligned} &\operatorname{div}(\widetilde{\rho} \mathbf{v}) = 0, \\ &\partial_t \mathbf{v} + \mathbf{v} \nabla_x \mathbf{v} + \nabla_x \Pi = 0. \end{aligned}$$

The results of the present paper can be seen as a natural extension of those in [11], where a similar problem is considered for the compressible Navier–Stokes equations with low viscosity. In contrast with [11], smooth solutions of the Euler system (1.3) will exhibit blow-up in a finite time no matter how smooth or small the initial data are. It seems therefore more appropriate to consider a suitable class of admissible weak solutions to (1.3). By admissible we mean that solutions will satisfy some form of the energy balance. Unfortunately, although the method of convex integration gave rise to several rather general existence results for the compressible Euler system, see e.g. Chiodaroli [5], De Lellis and Székelyhidi [7], the existence of global-in-time *admissible* weak solutions for arbitrary initial data remains an outstanding open problem.

The need for global admissible solutions of the Euler system leads to the concept of more general *dissipative measure–valued (DMV) solutions* introduced in the context of the full Euler system in [2, 3]. The reader may consult [13, 14, 16, 21] for applications of the theory of (DMV) solutions in fluid mechanics or their counterparts [6, 20] in other areas of mathematical physics. Compared with weak solutions, the advantage of (DMV) solutions is the following:

1 DMV solutions to the compressible Euler system exist globally in time.

2 Convergence to the limit system holds for any ill-prepared initial data.

To the best of our knowledge, there are only a few results concerning singular limits in the context of measure–valued solutions. The low Mach number limit was studied in [15], where it is shown that (DMV) solutions approach the smooth solutions of incompressible Euler system both for well-prepared and ill-prepared data. Bruell and Feireisl [1] identified the singular limit of the full compressible Euler system in the low Mach and strong stratification regime for the well-prepared data. Our goal is to consider the asymptotic limit of (DMV) solutions to the compressible Euler equations with ill-prepared data in the case of strong stratification. It seems interesting to compare the results of the present paper with those obtained in [11], where the same scaling was considered for the Navier–Stokes system with vanishing viscosity. The analysis in [11] leans, among other things, on the estimates on the pressure term based on the presence of viscosity that are *not* available for the Euler system. The extension of the results of [11] to the Euler system is therefore not straightforward.

The paper is organized as follows. In Section 2, we introduce the dissipative measure solutions, relative energy and the other necessary material. In Section 3, we state our main theorem. Section 4 is devoted to deriving uniform bounds of the Euler system independent of ϵ. In Section 5, we perform the necessary analysis of the acoustic waves. The proof of the main theorem is completed in Section 6.

2 Preliminaries, measure-valued solutions, relative energy

First observe that it is more convenient to rewrite the Euler system in terms of the conservative variables ρ, $\mathbf{m} = \rho\mathbf{u}$. Let $\mathcal{Q} = \{[\rho, \mathbf{m}] | \rho \in [0, \infty), \mathbf{m} \in \mathbb{R}^2\}$ be the natural phase space associated to solutions $[\rho, \mathbf{m}] = [\rho, \rho\mathbf{u}]$.

2.1 Dissipative measure–valued solutions

A *dissipative measure-valued (DMV) solution* of the Euler system (1.1) is a parameterized family of probability measures

$$\{Y_{t,x}\}_{t\in[0,T],x\in\Omega}, \quad (t,x) \mapsto Y_{t,x} \in L^\infty_{weak-(\star)}((0,T)\times\Omega;\mathcal{P}(\mathcal{Q})), \tag{2.1}$$

satisfying

• the continuity equation

$$\int_0^T \int_{\mathbb{R}^2} [\langle Y_{t,x};\rho\rangle\partial_t\varphi + \langle Y_{t,x};\mathbf{m}\rangle\nabla_x\varphi]dxdt = -\int_{\mathbb{R}^2} \langle Y_{0,x};\rho\rangle\varphi(0)dx, \tag{2.2}$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2)$;

• the momentum equation

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2} [\langle Y_{t,x};\mathbf{m}\rangle\partial_t\varphi + \langle Y_{t,x};\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\rangle : \nabla_x\varphi]dxdt + \int_0^T \int_{\mathbb{R}^2} \langle Y_{t,x};p(\rho)\rangle \mathrm{div}\varphi dxdt \\ &= -\int_{\mathbb{R}^2} \langle Y_{0,x};\mathbf{m}\rangle\varphi(0)dx - \int_0^T \int_{\mathbb{R}^2} \langle Y_{t,x};\rho\rangle\nabla_x F\cdot\varphi dxdt - \int_0^T \int_{\mathbb{R}^2} \nabla_x\varphi : d\mu_c, \end{aligned} \tag{2.3}$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2;\mathbb{R}^2)$, where $\mu_c \in \mathcal{M}([0,T]\times\mathbb{R}^2;\mathbb{R}^2\times\mathbb{R}^2)$ is the so–called momentum concentration measure;

• the energy inequality

$$\begin{aligned} &\int_{\mathbb{R}^2} [\langle Y_{\tau,x};\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \big(P(\rho) - P'(\widetilde{\rho})(\rho-\widetilde{\rho}) - P(\widetilde{\rho})\big)\rangle dx + \mathcal{D}(\tau) \\ &\quad \le \int_{\mathbb{R}^2} \langle Y_{0,x};\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \big(P(\rho) - P'(\widetilde{\rho})(\rho-\widetilde{\rho}) - P(\widetilde{\rho})\big)\rangle dx + \int_{\mathbb{R}^2} \langle Y_{t,x};\mathbf{m}\rangle\nabla_x F dx, \end{aligned} \tag{2.4}$$

for a.a $\tau \in (0,T)$, where

$$P(\rho) = \rho\int_{\overline{\rho}}^{\rho} \frac{p(z)}{z^2}dz, \tag{2.5}$$

and $\mathcal{D}$ is a non-negative function $\mathcal{D} \in L^\infty(0,T)$, satisfying the compatibility condition

$$\int_0^\tau \int_{\mathbb{R}^2} |\mu_c| dxdt \leq C \int_0^\tau \xi(t)\mathcal{D}(t)dt, \text{ for some } \xi \in L^1(0,T). \tag{2.6}$$

Remark 2.1. *The measure $Y_{0,x}$ plays the role of* initial conditions.

Remark 2.2. *We easily observe*

$$P''(\rho) = \frac{p'(\rho)}{\rho}; \text{ whence } P''(\widetilde{\rho})\nabla_x \widetilde{\rho} = \nabla_x F. \tag{2.7}$$

Remark 2.3. *We need to define the function*

$$[\rho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\rho}$$

on the vacuum set as

$$[\rho, \mathbf{m}] \to \frac{|\mathbf{m}|^2}{\rho} = \begin{cases} \infty, & \text{if } \rho = 0 \text{ and } \mathbf{m} \neq 0, \\ \frac{|\mathbf{m}|^2}{\rho}, & \text{if } \rho > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

Accordingly, it follows from the energy inequality (2.4) that

$$\mathrm{Supp}[Y_{t,x}] \cap \{[\rho, \mathbf{m}] \in \mathcal{Q} | \rho = 0, \mathbf{m} \neq 0]\} = \emptyset \text{ for a.a. } (t,x). \tag{2.9}$$

2.2 Relative energy

Motivated by [9, 3], we introduce the relative energy

$$\mathcal{E}(\rho, \mathbf{m}|r, \mathbf{U}) = \int_{\mathbb{R}^2} \langle Y_{t,x}; \frac{1}{2}\rho|\frac{\mathbf{m}}{\rho} - \mathbf{U}(t,x)|^2 + (P(\rho) - P'(r)(\rho - r) - P(r))\rangle dx, \tag{2.10}$$

where $r > 0$, $\mathbf{U}$ are smooth "test" functions, $r - \overline{\rho}$, U compactly supported in $\mathbb{R}^2$.

As shown in [3], any (DMV) solution of (1.1) satisfies the relative energy inequality

$$\begin{aligned} \mathcal{E}(\rho, \mathbf{m}|r, \mathbf{U})|_{t=0}^{t=\tau} + \mathcal{D}(\tau) \leq & \int_0^\tau \int_{\mathbb{R}^2} \langle Y_{t,x}; (\partial_t \mathbf{U} + \frac{\mathbf{m}}{\rho}\nabla_x U)(\rho \mathbf{U} - \mathbf{m})\rangle dxdt \\ + & \int_0^\tau \int_{\mathbb{R}^2} \langle Y_{t,x}; (r-\rho)\partial_t P'(r) + (r\mathbf{U} - \mathbf{m})\nabla_x P'(r)\rangle dxdt \\ - & \int_0^\tau \int_{\mathbb{R}^2} \langle Y_{t,x}; p(\rho) - p(r)\rangle \mathrm{div}\mathbf{U} dxdt + \int_0^\tau \int_{\mathbb{R}^2} \langle Y_{t,x}; \mathbf{m} - \rho \mathbf{U}\rangle \nabla_x F dxdt + \int_0^\tau \int_{\mathbb{R}^2} \nabla_x \mathbf{U} : d\mu_c. \end{aligned} \tag{2.11}$$

for a.a. $\tau \in [0,T]$, and any $r, \mathbf{U} \in C^1([0,T] \times \mathbb{R}^2)$, $r - \overline{\rho}$, $\mathbf{U}$ compactly supported in $\mathbb{R}^2$.

3 Main result

Before stating our main result, we collect several mostly technical hypotheses and known facts concerning the limit system.

3.1 Pressure and the static density profile

We suppose the pressure p is a continuously differentiable function of the density such that for some $\gamma > 1$,

$$p \in C^1[0,\infty) \cap C^\infty(0,\infty),\ p(0) = 0,\ p'(\rho) > 0 \text{ for all } \rho > 0,\ \lim_{\rho\to\infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = p_\infty > 0. \tag{3.1}$$

Furthermore, to facilitate the analysis, we consider

$$F \in C_c^\infty(\mathbb{R}^2), \quad F \geq 0.$$

Accordingly, the stationary profile $\widetilde{\rho}$ satisfies

$$P'(\widetilde{\rho}) = F + P'(\overline{\rho}).$$

From the above hypotheses, we deduce

$$(\widetilde{\rho} - \overline{\rho}) \in C_c^\infty(\mathbb{R}^2),\ \ \widetilde{\rho}(x) \geq \overline{\rho} \geq 0, \ \text{ for all } x \in \mathbb{R}^2,\ \widetilde{\rho}(x) = \overline{\rho} \ \text{ for all } x \in \mathbb{R}^2 \setminus supp[F].$$

Remark 3.1. *Similarly to [15], we deduce that*

$$p(\rho) - p'(r)(\rho - r) - p(r) \text{ is dominated by } P(\rho) - P'(r)(\rho - r) - P(r),\ \text{specifically,}$$
$$|\rho - r|^2 \leq c(\delta)(P(\rho) - P'(r)(\rho - r) - P(r)) \quad \text{when } 0 < \delta \leq \rho,\ r \leq \frac{1}{\delta},\ \delta > 0,$$
$$1 + |\rho - r| + P(\rho) \leq c(\delta)(P(\rho) - P'(r)(\rho - r) - P(r)) \quad \text{if } 0 < 2\delta < r < \frac{1}{2\delta},$$
$$\rho \in [0,\delta) \cup (\frac{1}{\delta}, \infty),\ \delta > 0.$$

3.2 Lake equation

The limit velocity $\mathbf{v}$ is expected to satisfy the lake equation

$$\operatorname{div}(\widetilde{\rho}\mathbf{v}) = 0, \tag{3.2}$$
$$\partial_t \mathbf{v} + \mathbf{v}\nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{3.3}$$

supplement with the initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}_0. \tag{3.4}$$

As shown by Oliver [22], the lake equation possesses a unique classical solution

$$\mathbf{v} \in C([0,T]; W^{m,2}(\mathbb{R}^2)),\ \ \Pi \in C([0,T]; W^{m,2}(\mathbb{R}^2)), \quad m \geq 3, \tag{3.5}$$

for any initial solution

$$\mathbf{v}_0 \in W^{m,2}(\mathbb{R}^2),\ \ \operatorname{div}(\widetilde{\rho}\mathbf{v}_0) = 0. \tag{3.6}$$

3.3 Ill prepared initial–data

The *ill–prepared* initial data for the scaled system (1.3) take the form

$$\rho_\epsilon(0,\cdot) = \rho_{0,\epsilon} = \widetilde{\rho} + \epsilon s_{0,\epsilon},\ \mathbf{u}_\epsilon(0,\cdot) = \mathbf{u}_{0,\epsilon},$$

where

$$s_{0,\epsilon} \to s_0 \text{ in } W^{k,2}(\mathbb{R}^2) \cap W^{k,1}(\mathbb{R}^2), \quad \mathbf{u}_{0,\epsilon} \to \mathbf{u}_0 \text{ in } W^{k,2}(\mathbb{R}^2) \cap W^{k,1}(\mathbb{R}^2),\ \ (k > 3),$$

$$\mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,\ \operatorname{div}(\tilde{\rho}\mathbf{v}_0) = 0,$$

cf. [11].

3.4 Singular limit – main result

Now, we are ready to state our main result.

Theorem 3.1. *Let $\{Y^\epsilon_{t,x}\}_{(t,x)\in[0,T]\times\Omega}$ be a family of (DMV) solutions to the scaled Euler system (1.3) satisfying the compatibility condition (2.6) with a function ξ independent of ϵ. Let the initial data $\{Y^\epsilon_{0,x}\}_{x\in\Omega}$ be ill-prepared, namely*

$$\int_{\mathbb{R}^2} \langle Y^\epsilon_{0,x}; \frac{1}{2}\rho|\frac{\mathbf{m}}{\rho} - \mathbf{u}_{0,\epsilon}(x)|^2 + \frac{1}{\epsilon^2}(P(\rho) - P'(\rho_{0,\epsilon})(\rho - \rho_{0,\epsilon}) - P(\rho_{0,\epsilon}))\rangle dx \to 0,$$

where $\rho_{0,\epsilon}$, $\mathbf{u}_{0,\epsilon}$ are ill prepared data specified in Section 3.3.

Then

$$\mathcal{D}^\epsilon \to 0 \ \text{ in } L^\infty(0,T),$$
$$Y^\epsilon_{t,x} \to \delta_{[\widetilde{\rho}(x),\widetilde{\rho}(x)\mathbf{v}(t,x)]} \ \text{ in } L^q(0,T; L^1_{\text{loc}}(R^2; \mathcal{M}^+(\mathcal{Q})_{weak-(\star)})) \text{ for any finite } q \geq 1,$$

where $\mathbf{v}$ is the unique solution of the lake equation starting from the initial data $\mathbf{v}_0$.

Remark 3.2. *In particular, our result is valid for the shallow water equation*

$$\left\{ \begin{array}{l} \partial_t h + div(h\mathbf{u}) = 0, \\ \partial_t(h\mathbf{u}) + div(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\epsilon^2} h\nabla_x h = \frac{1}{\epsilon^2} h\nabla_x b, \end{array} \right. \tag{3.7}$$

where $\mathbf{u}$ denotes the velocity, h is the fluid height and b is a given function depending on the space variables.

The rest of the paper is devoted to the proof of Theorem 3.1.

4 Energy bounds

We start by deriving uniform bounds on solutions to (1.3) independent of ϵ. Similarly to [11], we introduce the decomposition

$$h(\rho,\mathbf{m}) = [h]_{ess}(\rho,\mathbf{m}) + [h]_{res}(\rho,\mathbf{m}), \quad [h]_{ess} = \psi(\rho)h(\rho,\mathbf{m}),\ [h]_{res} = (1-\psi(\rho))h(\rho,\mathbf{m}),$$

where

$$\psi \in C_c^\infty(0,\infty),\ \ 0 \le \psi(\rho) \le 1,\ \ \psi(\rho) = 1 \text{ for all } \rho \in [\frac{1}{2}\min_{\mathbb{R}^2}\widetilde{\rho}, 2\max_{\mathbb{R}^2}\widetilde{\rho}].$$

As the initial data are ill–prepared, the expression on the right–hand side of the energy inequality (2.4) remains bounded uniformly for $\epsilon \to 0$. Consequently, we deduce the following bound:

$$ess \sup_{t\in(0,T)} \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \frac{1}{\epsilon^2}(P(\rho) - P'(\widetilde{\rho})(\rho - \widetilde{\rho}) - P(\widetilde{\rho}))\rangle dx \le C. \tag{4.1}$$

Thus, exactly as in [15], we use the structural properties of the function p to deduce

$$\begin{aligned}
&ess \sup_{t\in(0,T)} \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; |[\frac{\rho-\widetilde{\rho}}{\epsilon}]_{ess}|^2\rangle + \langle Y^\epsilon_{t,x}; [\frac{P(\rho)+1}{\epsilon^2}]_{ess}\rangle dx \le C;\\
&(t,x) \mapsto \langle Y^\epsilon_{t,x}; \mathbf{m}\rangle \text{ bounded in } L^\infty(0,T; L^2(\mathbb{R}^2) + L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^2));\\
&(t,x) \mapsto \langle Y^\epsilon_{t,x}; [\frac{\rho-\widetilde{\rho}}{\epsilon}]_{ess}\rangle \text{ bounded in } L^\infty(0,T; L^2(\mathbb{R}^2));\\
&(t,x) \mapsto \epsilon^{-\frac{2}{\gamma}}\langle Y^\epsilon_{t,x}; [\rho]_{res}\rangle \text{ bounded in } L^\infty(0,T; L^\gamma(\mathbb{R}^2)).
\end{aligned} \tag{4.2}$$

5 Acoustic waves

It is well-known that ill-prepared data give rise to rapidly oscillating acoustic waves. Similarly to [11], the relevant acoustic equation reads

$$\begin{cases} \epsilon\partial_t s_\epsilon + \operatorname{div}(\widetilde{\rho}\nabla_x\Phi_\epsilon) = 0,\\ \epsilon\widetilde{\rho}\partial_t\nabla_x\Phi_\epsilon + \widetilde{\rho}\nabla_x(\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}s_\epsilon) = 0, \end{cases} \tag{5.1}$$

supplemented with the initial data

$$s_\epsilon(0,\cdot) = s_0,\ \nabla_x\Phi_\epsilon(0,\cdot) = \nabla_x\Phi_0$$

where s_0, $\nabla_x\Phi_0$ have been introduced in Section 3.3.

As a matter of fact, the initial data must be smoothed and cut-off via suitable regularization operators, namely

$$s_\epsilon(0,\cdot) = s_{0,\delta} = \frac{\widetilde{\rho}}{p'(\widetilde{\rho})}[\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}s_0]_\delta;\ \ \nabla_x\Phi_\epsilon(0,\cdot) = \nabla_x\Phi_{0,\delta} = \nabla_x[\Phi_0]_\delta,$$

where $[\cdot]_\delta$ denotes the regularization introduced in [11].

Denoting the corresponding solutions $s_{\epsilon,\delta}$, $\Phi_{\epsilon,\delta}$ we report the following energy and dispersive estimates proved in [11, Section 5]:

$$\sup_{t\in[0,T]} [\|\Phi_{\epsilon,\delta}(t,\cdot)\|_{W^{m,2}} + \|s_{\epsilon,\delta}(t,\cdot)\|_{W^{m,2}}] \le C(m,\delta)[\|\nabla_x\Phi_{0,\delta}\|_{L^2} + \|s_{0,\delta}\|_{L^2}], \tag{5.2}$$

where C is independent of ϵ; and

$$\int_0^T [\|\Phi_{\epsilon,\delta}(t,\cdot)\|_{W^{m,\infty}} + \|s_{\epsilon,\delta}(t,\cdot)\|_{W^{m,\infty}}] \le \omega(\epsilon,m,\delta)[\|\nabla_x\Phi_{0,\delta}\|_{L^2} + \|r_{0,\delta}\|_{L^2}], \tag{5.3}$$

where $\omega(\epsilon,m,\delta) \to 0$ as $\epsilon \to 0$ for any fixed $m \ge 0$ and $\delta > 0$.

6 Convergence

The proof of convergence is based on the ansatz

$$r_\epsilon = \widetilde{\rho} + \epsilon s_{\epsilon,\delta}, \ \mathbf{U}_\epsilon = \mathbf{v} + \nabla_x \Phi_{\epsilon,\delta}, \tag{6.1}$$

in the relative energy inequality (2.11). In addition, to avoid technicalities, we shall assume that s_0 and Φ_0 are sufficiently regular so that the $\delta-$regularization is not needed in (5.1–5.3). Accordingly, we have $s_{\epsilon,\delta} = s_\epsilon$, $\Phi_{\epsilon,\delta} = \Phi_\epsilon$. The general case may be handled as in [11].

First note that the relative energy for the scaled system reads

$$\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon) = \int_{\mathbb{R}^2} \langle Y_{t,x}; \frac{1}{2}\rho_\epsilon |\frac{\mathbf{m}_\epsilon}{\rho_\epsilon} - \mathbf{U}_\epsilon|^2 + \frac{1}{\epsilon^2}(P(\rho_\epsilon) - P'(r_\epsilon)(\rho_\epsilon - r_\epsilon) - P(r_\epsilon))\rangle dx, \tag{6.2}$$

with the corresponding relative energy inequality:

$$\begin{aligned}
&\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)|_{t=0}^{t=\tau} + \mathcal{D}^\epsilon(\tau) \le \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle (\partial_t \mathbf{U}_\epsilon + \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla_x \mathbf{U}_\epsilon) dxdt \\
&+ \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; r_\epsilon - \rho_\epsilon\rangle \partial_t P'(r_\epsilon) + \langle Y^\epsilon_{t,x}; r_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle \nabla_x P'(r_\epsilon)] dxdt \\
&+ \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \mathbf{m}_\epsilon - \rho_\epsilon \mathbf{U}_\epsilon\rangle \nabla_x F dxdt - \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y_{t,x}; p(\rho_\epsilon) - p(r_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon dxdt \\
&+ \int_0^\tau \int_{\mathbb{R}^2} \nabla_x \mathbf{U}_\epsilon : d\mu_c.
\end{aligned} \tag{6.3}$$

Our goal is to show that, with the ansatz (6.1), the relative energy $\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)$ tends to zero for $\epsilon \to 0$ uniformly in $t \in [0, T]$. In view of the dispersive estimates (5.2) – (5.3), this will yield the conclusion claimed in Theorem 3.1. To this end, we use a Gronwall type argument showing that all integrals in the right-hand side of (6.3) are either small or can be absorbed by the left-hand side as $\epsilon \to 0$. This programme will be carried over by means of several steps.

6.1 Step 1

First, we compute

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; r_\epsilon - \rho_\epsilon\rangle \partial_t P'(r_\epsilon) + \langle Y^\epsilon_{t,x}; r_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle \nabla_x P'(r_\epsilon) - \langle Y^\epsilon_{t,x}; p(\rho_\epsilon) - p(r_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon] dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; p(r_\epsilon) - p'(r_\epsilon)(r_\epsilon - \rho_\epsilon) - p(\rho_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon + \langle Y^\epsilon_{t,x}; r_\epsilon - \rho_\epsilon\rangle \partial_t P'(r_\epsilon) \\
&+ \langle Y^\epsilon_{t,x}; (r_\epsilon - \rho_\epsilon) p'(r_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon + \langle Y^\epsilon_{t,x}; (r_\epsilon - \rho_\epsilon) \nabla_x P'(r_\epsilon)\rangle \mathbf{U}_\epsilon + \langle Y^\epsilon_{t,x}; (\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) \nabla_x P'(r_\epsilon)\rangle] dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; p(r_\epsilon) - p'(r_\epsilon)(r_\epsilon - \rho_\epsilon) - p(\rho_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon + \langle Y^\epsilon_{t,x}; (r_\epsilon - \rho_\epsilon) P''(r_\epsilon)\rangle (\partial_t r_\epsilon + \mathrm{div}_x (r_\epsilon \mathbf{U}_\epsilon)) \\
&+ \langle Y^\epsilon_{t,x}; (\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) \nabla_x P'(r_\epsilon)\rangle] dxdt,
\end{aligned}$$

where we have used (2.7). Note that, in view of (5.1),

$$\partial_t r_\epsilon + \mathrm{div}_x (r_\epsilon \mathbf{U}_\epsilon) = \epsilon \mathrm{div}(s_\epsilon \mathbf{U}_\epsilon).$$

Next, by virtue of (1.4) and (5.1),

$$\int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle \nabla_x P'(r_\epsilon) + \langle Y^\epsilon_{t,x}; \mathbf{m}_\epsilon - \rho_\epsilon \mathbf{U}_\epsilon\rangle \nabla_x F] dxdt$$

$$
\begin{aligned}
&= \int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; \nabla_x \big(P'(r_\epsilon) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho}) - P'(\widetilde{\rho})\big)\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) \\
&+ \langle Y^\epsilon_{t,x}; \nabla_x \big(P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho})\big)\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon)] dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^2} [\langle Y^\epsilon_{t,x}; \nabla_x \big(P'(r_\epsilon) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho}) - P'(\widetilde{\rho})\big)\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) \\
&- \epsilon^2 \langle Y^\epsilon_{t,x}; \partial_t \nabla_x \Phi_\epsilon\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon)] dxdt.
\end{aligned}
$$

Furthermore, by virtue of the compatibility condition (2.6), we can control the concentration measure,

$$
\int_0^\tau \int_{\mathbb{R}^2} \nabla_x \mathbf{U} : d\mu_c \leq \|\nabla_x \mathbf{U}\|_{L^\infty} \int_0^\tau \xi(t) \mathcal{D}^\epsilon(t) dt.
$$

Finally, as the hypotheses about the ill-prepared initial data, we have

$$
\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)(0) \to 0 \text{ as } \epsilon \to 0.
$$

Thus we may conclude that

$$
\begin{aligned}
&\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)(\tau) + \mathcal{D}^\epsilon(\tau) \leq \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; (\partial_t \mathbf{v} + \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla_x \mathbf{U}_\epsilon)(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon)\rangle dxdt \\
&+ \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \nabla_x \big(P'(r_\epsilon) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho}) - P'(\widetilde{\rho})\big)\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) dxdt \\
&- \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; p(\rho_\epsilon) - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)\rangle \mathrm{div} \mathbf{U}_\epsilon dxdt \\
&- \frac{1}{\epsilon} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; (\rho_\epsilon - r_\epsilon) P''(r_\epsilon)\rangle \mathrm{div}(s_\epsilon \mathbf{U}_\epsilon) dxdt + c \int_0^\tau \xi(t) \mathcal{D}^\epsilon(t) dt + \omega(\epsilon),
\end{aligned}
$$

where $\omega(\epsilon)$ denotes a generic quantity satisfying

$$
\omega(\epsilon) \to 0 \text{ in } L^1(0,T) \text{ as } \epsilon \to 0.
$$

6.2 Step 2

We write

$$
\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^2} [\langle \mathbf{Y}^\epsilon_{t,x}; \rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon\rangle(\partial_t \mathbf{v} + \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla_x \mathbf{U}_\epsilon)] dxdt \\
&\quad = \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}) dxdt \\
&\quad + \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle(\mathbf{v} \cdot \nabla_x \nabla_x \Phi_\epsilon + \nabla_x \Phi_\epsilon \nabla_x \mathbf{U}_\epsilon) dxdt \\
&\quad + \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle(\frac{\mathbf{m}_\epsilon}{\rho_\epsilon} - \mathbf{U}_\epsilon) \nabla_x \mathbf{U}_\epsilon dxdt \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
$$

Using the uniform bounds (4.2), we can split the functions in I_2 into their essential and residual parts obtaining

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon\rangle(\mathbf{v} \cdot \nabla_x \nabla_x \Phi_\epsilon + \nabla_x \Phi_\epsilon \nabla_x \mathbf{U}_\epsilon) \; dx \right| \\
&\leq \|\nabla_x \Phi_\epsilon\|^2_{W^{1,\infty}} \left(\|\mathbf{v}\|_{W^{3,2}} + \|\nabla_x \mathbf{U}_\epsilon\|_{W^{3,2}}\right)^2 + c \mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon),
\end{aligned}
$$

where the first term on the right–hand side can be controlled by means of the dispersive estimate (5.2) and (5.3).

Next, we use the fact that $\mathbf{v}$ solves the lake equation (3.2) – (3.3) to obtain

$$\int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon \rangle (\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v}) dxdt = \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \mathbf{m}_\epsilon \rangle \nabla_x \Pi dxdt - \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon \rangle \nabla_x \Pi dxdt.$$

Moreover, it follows form the energy bounds (4.2) that there exists a function $\mathbf{M} \in L^\infty(0,T; L^2 \cap L^r(\mathbb{R}^2))$ for some $r > 1$, that

$$\langle Y^\epsilon_{t,x}; \mathbf{m}_\epsilon \rangle \to \mathbf{M} \;\text{ weakly} - (\star) \text{ in } L^\infty(0,T; L^2 + L^r(\mathbb{R}^2)),$$
$$\langle Y^\epsilon_{t,x}; \rho_\epsilon \rangle \to \widetilde{\rho} \;\text{ weakly} - (\star) \text{ in } L^\infty(0,T; L^2 + L^\gamma(\mathbb{R}^2)).$$

From the continuity equation (2.2), we can deduce that

$$\int_0^\tau \int_{\mathbb{R}^2} \mathbf{M} \cdot \nabla_x \varphi dxdt = 0,$$

for any $\varphi \in C'([0,T] \times \mathbb{R}^2)$ and a.a $\tau \in (0,T)$. Consequently, for any $1 \le q \le \infty$,

$$\{\tau \to \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \mathbf{m}_\epsilon \rangle \nabla_x \Pi dxdt\} \to 0, \;\text{ in } L^q(0,T) \text{ as } \epsilon \to 0.$$

Finally, we may write

$$- \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \rho_\epsilon \mathbf{U}_\epsilon \rangle \nabla_x \Pi dxdt = -\,\epsilon \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \frac{\rho_\epsilon - \widetilde{\rho}}{\epsilon} \rangle \mathbf{U}_\epsilon \cdot \nabla_x \Pi dxdt - \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \widetilde{\rho}(\mathbf{v} + \nabla_x \Phi_\epsilon) \rangle \nabla_x \Pi dxdt,$$

where the dispersive estimates (5.2) and (5.3) can be used to control the integrals on the right–hand side.

Summing up the previous observations we may infer that the relative energy inequality with the ansatz (6.1) reduces to

$$\begin{aligned}
&\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)(\tau) + \mathcal{D}^\epsilon(\tau) \le \\
&\frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; \nabla_x \big(P'(r_\epsilon) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho}) - P'(\widetilde{\rho}) \big) \rangle (\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon) dxdt \\
&- \frac{1}{\epsilon^2} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; p(\rho_\epsilon) - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon) \rangle \mathrm{div} \mathbf{U}_\epsilon dxdt \\
&- \frac{1}{\epsilon} \int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; (\rho_\epsilon - r_\epsilon) P''(r_\epsilon) \rangle \mathrm{div}(s_\epsilon \mathbf{U}_\epsilon) dxdt + c \int_0^\tau \xi(t) \mathcal{D}^\epsilon(t) dt + \omega(\epsilon),
\end{aligned}$$

6.3 Step 3

Using direct calculation and the Taylor formula, we deduce that

$$|\nabla_x \big(P'(r_\epsilon) - P'(\widetilde{\rho}) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho}) \big)|$$

$$= |\epsilon\big(P''(r_\epsilon) - P''(\widetilde{\rho})\big)\nabla_x s_\epsilon + \big(P''(r_\epsilon) - P''(\widetilde{\rho}) - P'''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho})\nabla_x \widetilde{\rho}|$$
$$\leq \epsilon^2 (s_\epsilon |\nabla_x s_\epsilon| + s_\epsilon^2).$$

Therefore,

$$\frac{1}{\epsilon^2}\int_0^\tau \langle Y^\epsilon_{t,x}; \nabla_x\big(P'(r_\epsilon) - P'(\widetilde{\rho}) - P''(\widetilde{\rho})(r_\epsilon - \widetilde{\rho})\big)\rangle(\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon)dt$$
$$\leq \int_0^\tau \langle Y^\epsilon_{t,x}; s_\epsilon|\nabla_x s_\epsilon| + s_\epsilon^2\rangle |\rho_\epsilon \mathbf{U}_\epsilon - \mathbf{m}_\epsilon| dt$$
$$\leq C\int_0^\tau \mathcal{E}(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, U_\epsilon)dt + \omega(\epsilon).$$

6.4 Step 4

Finally, we deal with the remaining pressure terms. First,

$$|\frac{1}{\epsilon^2}\int_0^\tau \langle Y^\epsilon_{t,x}; p(\rho_\epsilon) - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)\rangle \mathrm{div}\mathbf{U}_\epsilon dt|$$
$$= |\frac{1}{\epsilon^2}\int_0^\tau \langle Y^\epsilon_{t,x}; p(\rho_\epsilon) - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)\rangle(\mathrm{div}\mathbf{v} + \Delta\Phi_\epsilon)dt|$$
$$\leq c|\frac{1}{\epsilon^2}\int_0^\tau \langle Y^\epsilon_{t,x}; P(\rho_\epsilon) - P(r_\epsilon) - P'(r_\epsilon)(\rho_\epsilon - r_\epsilon)\rangle(\mathrm{div}\mathbf{v} + \Delta\Phi_\epsilon)dt|$$
$$\leq C\int_0^\tau \mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon)dt.$$

Second, the last term can be controlled as

$$-\frac{1}{\epsilon}\int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; (\rho_\epsilon - r_\epsilon)P''(r_\epsilon)\rangle \mathrm{div}(s_\epsilon \mathbf{U}_\epsilon)dxdt$$
$$\leq C\int_0^\tau \int_{\mathbb{R}^2} \langle Y^\epsilon_{t,x}; (|\frac{\rho_\epsilon - \widetilde{\rho}}{\epsilon}| + |s_\epsilon|)(|s_\epsilon| + |\nabla_x s_\epsilon|)(|\mathbf{v}| + |\mathrm{div}_x \mathbf{v}| + |\nabla_x \Phi_\epsilon| + |\Delta\Phi_\epsilon|)dxdt.$$

where the right-hand side tends to zero in $L^1(0,T)$ due to the dispersive estimate (5.2), (5.3).

Putting together Step 1 to step 4 we conclude

$$\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon) + \mathcal{D}^\epsilon(\tau) \leq \omega(\epsilon) + \int_0^\tau (1 + \xi(t))[\mathcal{E}(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, \mathbf{U}_\epsilon) + \mathcal{D}^\epsilon(t)]dt,$$

where r_ϵ, $\mathbf{U}_\epsilon$ are given by (6.1). Letting $\epsilon \to 0$ and applying the Gronwall's lemma, we complete the proof of Theorem 3.1.

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