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NONSEPARABLE CLOSED VECTOR SUBSPACES OF SEPARABLE TOPOLOGICAL VECTOR SPACES

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Abstract. In 1983 P. Domański investigated the question: For which separable topological vector spaces E, does the separable space $E^{\mathfrak{c}}$ have a nonseparable closed vector subspace, where $\mathfrak{c}$ is the cardinality of the continuum? He provided a partial answer, proving that every separable topological vector space whose completion is not q-minimal (in particular, every separable infinite-dimensional Banach space) E has this property. Using a result of S.A. Saxon, we show that for a separable locally convex space (lcs) E, the product space $E^{\mathfrak{c}}$ has a nonseparable closed vector subspace if and only if E does not have the weak topology. On the other hand, we prove that every metrizable vector subspace of the product of any number of separable Hausdorff lcs is separable. We show however that for the classical Michael line $\mathbb{M}$ the space of all continuous real-valued functions on $\mathbb{M}$ endowed with the pointwise convergence topology, $C_p(\mathbb{M})$ contains a nonseparable closed vector subspace while $C_p(\mathbb{M})$ is separable.

1. Introduction

All topological spaces and topological vector spaces are assumed to be Hausdorff. We say that a topological space X is *separable* if it contains a dense countable subset. It is well-known that a subspace of a separable metric space is separable but that a closed subspace of a separable Hausdorff topological space need not be separable. For topological groups, G. Itzkowitz [14] proved that any closed subgroup of a separable compact Hausdorff group is separable. This was extended by W. Comfort and G. Itzkowitz [3] to show that any closed subgroup of a separable locally compact Hausdorff group is separable. Recently the second and third named authors of this paper and M. Tkachenko [17] proved that any almost connected closed subgroup of a separable pro-Lie group is separable. Another result of [17] states that for a topological group G which is a product of connected locally compact groups, if G is homeomorphic to a subspace of a separable Hausdorff space X, then G is separable.

In this paper we focus on separable locally convex spaces (lcs) over the fixed field $\mathbb{R}$ of reals. Despite the evident fact that any lcs is linearly connected, our case is quite different as an infinite-dimensional lcs is never locally compact and is a pro-Lie group if and only if it is complete and has the weak topology ([12], [13]).

Probably the first example of a closed (but not complete) nonseparable vector subspace of a separable lcs was given by R.H. Lohman and W.J. Stiles [18].

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We observe that that for a compact Hausdorff space X with a closed subspace Y, the free locally convex space $L(Y)$ on Y is a closed vector subspace of the free locally convex space $L(X)$ on X, see [22]. It could be easily shown that if we take a compact X containing a nonseparable closed subspace Y, then $L(Y)$ is a nonseparable lcs which is a closed subspace of the separable lcs $L(X)$.

L. Drewnowski and R.H. Lohman [10] showed that while a subspace of finite codimension of a separable lcs is separable, this is not the case if the subspace is assumed only to be of countable codimension. Since (as observed in [16, Lemma 3.2]) any algebraic complement of an $\aleph_0$-codimensional closed subspace of a barrelled lcs is a topological complement, any nonseparable closed vector subspaces of a separable barrelled space would have to be of uncountable codimension. An example of a separable lcs E (which is not barrelled by the previous remark) containing a closed $\aleph_0$-codimensional nonseparable subspace has been provided in [10]. We pose the following problem:

Problem 1. Characterize those separable locally convex spaces E which contain nonseparable closed vector subspaces.

If E is an lcs which is a continuous linear image of a separable lcs F, where F has all of its closed vector subspaces separable (for example F is a separable Banach space or a separable Frechet space), then of course E has all of its closed vector subspaces separable. Also, if the topological space underlying E is a continuous image of a separable metrizable topological space, then E has all of its subspaces, including vector subspaces, separable.

The classical Hewitt–Marczewski–Pondiczery theorem implies that the product of no more than $\mathfrak{c}$ separable topological spaces is separable. P. Domański [8] gave an example of a nonseparable complete lcs which can be embedded as a closed vector subspace of a product of $\mathfrak{c}$ copies of the Banach space c_0. Later he extended this result to show that every product of $\mathfrak{c}$ copies of any infinite-dimensional Banach space has nonseparable closed vector subspaces [9]. In fact, P. Domański proved in [9] that if E_i, $i \in I$, with $\mathrm{card}(I) = \mathfrak{c}$ are separable topological vector spaces whose completions are not q-minimal, then the product $\prod_{i\in I} E_i$ has a nonseparable closed vector subspace. (An lcs is called q-minimal if it and all its quotient spaces are minimal.) This prompted the following problem:

Problem 2. What separable Hausdorff locally convex spaces E_i, $i \in I$, with $\mathrm{card}(I) = \mathfrak{c}$ are such that the product $\prod_{i\in I} E_i$ has a nonseparable closed vector subspace?

In this paper we provide a complete answer to Problem 2. We show that E has the property referred to in Problem 2 if and only if E does not have the weak topology. Indeed, we prove that if each E_i, $i \in I$, is an lcs with at least $\mathfrak{c}$ of the E_i not having the weak topology, then $\prod_{i\in I} E_i$ has a nonseparable closed vector subspace. The key to our proof is to consider the space φ, promoted in a number of papers by S.A. Saxon, where φ is the $\aleph_0$-dimensional lcs with the strongest locally convex topology.

We note that G. Vidossich [23] (see also [18]) proved that every metrizable vector subspace of a separable topological vector space is separable. Using techniques developed for the study of varieties of locally convex spaces in J. Diestel, S.A. Morris and S.A. Saxon [7], in contradistinction with that stated above, we prove that for every family of separable lcs E_i, $i \in I$, if X is a vector subspace of $\prod_{i\in I} E_i$

with Y a closed vector subspace of X such that either (a) Y is metrizable and X/Y is separable or (b) Y is separable and X/Y is metrizable, then X is separable. Immediately we obtain a corollary stating that every metrizable vector subspace of the product of any number of separable Hausdorff lcs is separable. Thus we extend an aforementioned result [23], [18]. Consequently, this applies to show that every metrizable vector subspace of the space $C_p(X, E)$ of continuous E-valued maps endowed with the pointwise convergence topology is separable, whenever X is a completely regular topological space and E is a separable lcs.

The following question arises naturally: is it true that the separability of $C_p(X)$ implies the separability of every closed vector subspace of $C_p(X)$? In Example 2.13 we show that the classical Michael line $\mathbb{M}$ provides the negative answer: $C_p(\mathbb{M})$ contains a nonseparable closed vector subspace while $C_p(\mathbb{M})$ is separable.

We conclude with the following example. Let $E = (E, \tau)$ be the product of continuum many finite-dimensional lcs (in fact E is isomorphic to $\mathbb{R}^{\mathfrak{c}}$). There exist two separable locally convex topologies ξ_1 and ξ_2 such that $\tau = \inf\{\xi_1, \xi_2\}$ and each (E, ξ_j) contains a nonseparable closed vector subspace.

2. Separability of subspaces of products of locally convex spaces

The key to the results in this section is to use the notion of a variety of locally convex spaces, introduced in [7] and the locally convex space denoted by φ, which is the $\aleph_0$-dimensional vector space endowed with the finest locally convex topology (which can also be identified with the free locally convex space on the countable discrete space), see [20], [7].

A nonempty class Ω of lcs is said to be a *variety* if it is closed under the operations of taking subspaces, quotient spaces, (arbitrary) cartesian products and isomorphic images. Let $\mathcal{C}$ be a class of lcs and let $\mathfrak{V}(\mathcal{C})$ be the intersection of all varieties containing $\mathcal{C}$. Then $\mathfrak{V}(\mathcal{C})$ is said to be the *variety generated by* $\mathcal{C}$. (Clearly this is indeed a variety.) If $\mathcal{C}$ consists of a single lcs E, then $\mathfrak{V}(\mathcal{C})$ is written as $\mathfrak{V}(E)$ and is said to be singly generated, see [7].

Proposition 2.1. (Theorem 1.4 of [20]) *Let E be an lcs. Then the following are equivalent:*

(i) *E is not a subspace of a product of copies of $\mathbb{R}$;*
(ii) *E is not in $\mathfrak{V}(\mathbb{R})$;*
(iii) *E^I contains φ, for every I with $\mathrm{card}(I) \geq \mathfrak{c}$;*
(iv) *E does not have the weak topology.*

Proposition 2.2. *The space φ is a complete lcs which is not minimal and every vector subspace of φ is closed.*

Proof. The space φ is a complete lcs with every vector subspace closed by [21, Exercise 7 (p.69)].

Proposition 2.1 says that an lcs E does not have the weak topology if E^I has φ as a subspace, for some index set I, with $\mathrm{card}(I) \geq \mathfrak{c}$. In particular, φ itself does not have the weak topology. Let the topology of φ be τ. We now know that there is a topology strictly weaker than τ on the underlying vector space of φ, namely the weak topology. So φ is not a minimal space. □

Proposition 2.3. *Let I be an index set and E_i an lcs for each $i \in I$. If at least $\mathfrak{c}$ of the E_i are not in $\mathfrak{V}(\mathbb{R})$, or equivalently do not have the weak topology, then the product $\prod_{i\in I} E_i$ has $\varphi^{\mathfrak{c}}$ as a closed vector subspace.*

Proof. Write I as a union of disjoint sets J_k, $k \in K$, where $\mathrm{card}(K) = \mathfrak{c}$ and such that at least $\mathfrak{c}$ of the E_j are not in $\mathfrak{V}(\mathbb{R})$ for each $j \in J_k$. So $\prod\limits_{i\in I} E_i = \prod\limits_{k\in K} \Big(\prod\limits_{j\in J_k} E_j \Big)$. By Corollary 1.5 of [20], $\prod\limits_{j\in J_k} E_j$ has φ as a closed vector subspace. So $\prod\limits_{i\in I} E_i$ has $\varphi^{\mathfrak{c}}$ as a closed vector subspace, as required. □

The next theorem follows immediately from Theorems 1 and 2 of [9].

Theorem 2.4. [9] *Let E_i, $i \in I$ with $\mathrm{card}(I) = \mathfrak{c}$, be a family of lcs each of which has its completion non-q-minimal. Then $\prod_{i\in I} E_i$ has a nonseparable closed vector subspace.*

As a corollary of Proposition 2.3 and Theorem 2.4 we have our first main result. It provides a complete and very satisfactory answer to Problem 2.

Theorem 2.5. *Let I be an index set and E_i an lcs for each $i \in I$. If at least $\mathfrak{c}$ of the E_i are not in $\mathfrak{V}(\mathbb{R})$, or equivalently do not have the weak topology, then the product $\prod_{i\in I} E_i$ has a nonseparable closed vector subspace.*

Having settled which products of locally convex spaces have nonseparable closed vector subspaces, we turn our attention to conditions which guarantee that a vector subspace of a product of locally convex spaces is separable. Our work is an extension of that of G. Vidossich [23] who proved that every metrizable vector subspace of a separable topological vector space is separable. In particular we shall see that every metrizable subspace of any product of separable lcs is separable. The key will be to use a powerful result of J. Diestel, S.A. Morris and S.A. Saxon on varieties of locally convex spaces in [7].

Proposition 2.6. (Theorem 4.1(ii) of [7]) *Let $\mathcal{C}$ be any class of Hausdorff lcs and let E be an lcs in the variety $\mathfrak{V}(\mathcal{C})$, of locally convex spaces generated by $\mathcal{C}$. If E is metrizable then E is isomorphic to a vector subspace of a countable product of quotient vector spaces of finite products of lcs in $\mathcal{C}$.*

The most important point in Proposition 2.6 is that we need only *countable products*. We now prove the second main result of this section.

Theorem 2.7. *Let I be any index set and each E_i, $i \in I$, a separable lcs. If X is a vector subspace of $\prod_{i\in I} E_i$ with Y a closed vector subspace of X such that either*

(a) *Y is metrizable and X/Y is separable, or*
(b) *Y is separable and X/Y is metrizable,*

then X is separable.

Proof. With Proposition 2.6 in mind, we observe that lcs which is a countable product of a quotient vector spaces of a finite product of a family of (separable) spaces in $\mathcal{C}$ is separable.

Assume firstly that Y is metrizable and X/Y is separable. By Proposition 2.6, Y is a metrizbale subspace of a separable space and so is separable. As X/Y is also separable, and separability is a three-space property, we have that X is separable.

Assume next that X/Y is metrizable and Y is separable. As X/Y is in $\mathfrak{V}(\mathcal{C})$, it is a vector subspace of a countable product of quotient vector spaces of finite products of lcs in $\mathcal{C}$ by Proposition 2.6. Since such a space is separable, the metrizable space X/Y is also separable. As separability is a three-space property and Y is separable, the space X is separable. □

Corollary 2.8. *Every metrizable vector subspace of the product of any number of separable Hausdorff lcs is separable.*

Let $C_p(X,E)$ denote the space of all continuous E-valued functions on X endowed with the pointwise convergence topology, where E is an lcs. The space $C_p(X,E)$ is a vector subspace of E^X endowed with the product topology.

Corollary 2.9. *Let X be a completely regular topological space. If E is a separable lcs, then every metrizable vector subspace of $C_p(X,E)$ is separable.*

Remark 2.10. The claim in the Corollary 2.9 for the particular case when $E=\mathbb{R}$ is the space of real numbers can be also proved as follows. Let G be a metrizable vector subspace of $C_p(X)=C_p(X,\mathbb{R})$, and let L be its closure in $\mathbb{R}^X$. Clearly L is metrizable in $\mathbb{R}^X$ and is isomorphic to a product $\mathbb{R}^A$ for some set A by [2, Corollary 2.6.5]. Consequently, A is countable, so L is separable. Hence G is separable.

For any lcs E denote by $\sigma(E,E')$ and $\sigma(E',E)$ the weak topologies on E and the dual space E' respectively. Denote by $X=(E',\sigma(E',E))$. Since $(E,\sigma(E,E'))\subset C_p(X)$ holds for every lcs E, Corollary 2.9 yields also the following

Corollary 2.11. *Let E be an lcs and let L be a vector subspace of E. The following assertions are equivalent.*
(i) $(L,\sigma(E,E')|L)$ *is metrizable and separable.*
(ii) $(L,\sigma(E,E')|L)$ *is metrizable.*

We note two relevant facts: 1) if X is discrete, then the separability of $C_p(X)$ implies the separability of every closed vector subspace of $C_p(X)$; 2) if X is a compact Hausdorff space, then the separability of $C_p(X)$ implies the separability of every subspace of $C_p(X)$. Both facts are true for the space of all continuous real-valued functions on X endowed with the compact-open topology, $C_c(X)$.

Is it true that for any completely regular topological space X the separability of $C_p(X)$ or $C_c(X)$ implies the separability of every closed vector subspace of $C_p(X)$ or $C_c(X)$, respectively? Below we give the negative answer to these questions.

Proposition 2.12. *Let X be a completely regular topological space such that every closed vector subspace of $C_p(X)$ is separable. Then every closed subset F of X is a G_δ-set, that is $F=\bigcap_{i=1}^{\infty}U_i$, where each U_i is open in X.*

Proof. Define $L=\{f\in C_p(X): f(x)=0 \text{ for any } x\in F\}$. It is easy to see that L is a closed vector subspace of $C_p(X)$. By the assumption there is a countable sequence $\{f_i\}_{i=1}^{\infty}$ which is dense in L. Define $U_i=f_i^{-1}(-1,1)$. Clearly, each U_i is open in X and $F\subset U_i$. We claim that $F=\bigcap_{i=1}^{\infty}U_i$. Fix any point $x\in X\setminus F$. There exists a function $f\in L$ such that $f(x)=2$. By the denseness of the sequence $\{f_i\}_{i=1}^{\infty}$ in L we have a function f_i such that $\|f_i(x)-f(x)\|<0.1$, hence $x\notin U_i$. □

Example 2.13. We recall a definition of the Michael line $\mathbb{M}$. Let $\mathbb{P}$ and $\mathbb{Q}$ denote respectively the irrationals and the rationals with their usual topologies. The Michael line $\mathbb{M}$ is the refinement of the real line $\mathbb{R}$ obtained by isolating all irrational points. Let us mention that $\mathbb{M}$ was the first ZFC example of a hereditarily paracompact space such that the product $\mathbb{M}\times\mathbb{P}$ is not normal. By construction, $\mathbb{M}$ admits a weaker separable metrizable topology, therefore according to Noble's theorem [19] $C_p(\mathbb{M})$ is a separable space.

The set $\mathbb{Q}$ is closed in $\mathbb{M}$, and it is known, by the Baire category theorem argument, that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{M}$, see [11, 5.5.2]. By Proposition 2.12 it means that the separable space $C_p(\mathbb{M})$ contains a nonseparable closed vector subspace. $C_c(\mathbb{M})$ ia also separable and evidently contains a nonseparable closed vector subspace because its topology is stronger than the pointwise convergence topology.

In view of the last results the following particular case of Problem 1 arises naturally and deserves special attention.

Problem 3. Characterize those completely regular topological spaces X such that all closed vector subspaces of $C_p(X)$ ($C_c(X)$) are separable.

Our results provide necessary conditions: X admits a weaker separable metrizable topology; and every closed subset F of X is a G_δ-set. We do not know if these conditions are sufficient.

We complete the paper with the following example motivated by our Theorem 2.5. We will use the concept of Baire-likeness due to S.A. Saxon [20]. An lcs E is called *Baire-like* if given any increasing sequence $(A_n)_n$ of absolutely convex closed subsets of E, there exists $n \in \mathbb{N}$ such that A_n has a non-void interior (equivalently, A_n is a neighbourhood of zero). Every Baire lcs is Baire-like and the topological product of any family of Baire-like spaces is Baire-like, see [20].

Following [5] for a topological vector space (X, γ) and a vector subspace $L \subset X$, denote by $\gamma|L$ the relative topology induced by γ on L and let γ/L denote the quotient topology on the quotient space X/L.

Example 2.14. Let $E = (E, \tau) = \mathbb{R}^{\mathfrak{c}}$. Although every closed vector subspace of E is separable, there exist two separable locally convex topologies ξ_1 and ξ_2 such that $\tau = \inf\{\xi_1, \xi_2\}$ and each (E, ξ_j) contains a nonseparable closed vector subspace.

Proof. Since E is Baire-like, E contains a dense Baire-like (even Baire) subspace F such that $\dim E = \dim F = \dim(E/F) = 2^{\mathfrak{c}}$, see [15, Proposition 2.10]. Let $G := C[0,1]^{\mathfrak{c}}$ be endowed with the product topology. Then G is a separable lcs which contains a closed subspace L not being separable by Theorem 2.5.

Let $q : E \to E/F$ be the quotient map. Since the quotient topology of E/F is trivial and $\dim G = \dim E/F$, the space E/F admits a stronger separable locally convex topology α such that $(E/F, \alpha)$ is isomorphic to G. Hence there exists on E a coarsest vector topology ξ_1 such that $\tau < \xi_1$, $\xi_1/F = \alpha$, $\xi_1|F = \tau|F$, see [5, (2), p.194]. Note that the sets $U \cap q^{-1}(V)$, where U and V run over τ- and α-neighbourhoods of zero, respectively, form a basis of neighbourhoods of zero for ξ_1. Since separability is a three space property, the topology ξ_1 is separable, see [6, 12.10]. Moreover, since both spaces $(F, \xi_1|F)$ and $(E/F, \xi_1/F)$ are Baire-like, the resulting space (E, ξ_1) is Baire-like by [1] (Baire-likeness is a three space property). Note also that ξ_1 is strictly stronger than τ. Indeed, this follows from the fact that F a proper subspace of E which is τ-dense and ξ_1-closed. Finally, since L is a nonseparable closed subspace of $(E/F, \alpha)$, the space $q^{-1}(L)$ is a nonseparable closed subspace of (E, ξ_1), which completes the proof of the first step.

Now we construct the topology ξ_2. Since (E, ξ_1) is Baire-like, we obtain in the space (E, ξ_1) a dense subspace F_1 of codimension $2^{\mathfrak{c}}$ (similarly to what we did previously using the argument from the proof of [15, Proposition 2.10]). Clearly F_1 is also dense in (E, τ). Let ξ_2 be a separable locally convex topology which is strictly stronger that τ and such that $\xi_2|F_1 = \tau|F_1$ and $(E, \xi_2)/F_1$ is isomorphic to G, constructed similarly to what we have done for ξ_1. Also (E, ξ_2) contains a nonseparable closed vector subspace.

Clearly $\tau \leq \inf\{\xi_1, \xi_2\}$. The following equalities hold

$$\tau|L_1 = \inf\{\xi_1, \xi_2\}|L_1 = \xi_2|L_1, \ \tau/L_1 = \inf\{\xi_1, \xi_2\}/L_1 = \xi_1/L_1,$$

since the topologies τ/L_1 and ξ_1/L_1 are trivial, so they are the same. We proved that

$$\tau|L_1 = \inf\{\xi_1, \xi_2\}|L_1, \ \tau/L_1 = \inf\{\xi_1, \xi_2\}/L_1.$$

This together with [4, Lemma 1] yields the expected equality $\tau = \inf\{\xi_1, \xi_2\}$, and the proof is complete. $\square$

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