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# Separable quotients for less-than-barrelled function spaces

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# Separable quotients for less-than-barrelled function spaces

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*In memory of Professor Joe Diestel*

ABSTRACT. We solve the separable quotient problem for spaces $C_p(X)$ and $C_c(X)$ that are dual locally complete (dlc), greatly extending the positive solution for barrelled $C_c(X)$. Indeed, ours is either the best or next-to-best weak barrelledness extension possible. Yet some interesting *non*-dlc spaces $C_p(X)$ and $C_c(X)$ also admit separable quotients.

## 1. What is best?

In the following, all locally convex topological vector spaces (lcs) and quotients thereof are assumed to be Hausdorff and infinite-dimensional. If $X$ is an infinite Tychonoff (completely regular Hausdorff) space, $C(X)$ denotes the linear space of continuous real-valued functions on $X$, and $C_p(X)$ and $C_c(X)$ are the lcs that $C(X)$ becomes when endowed with the pointwise and compact-open topologies, respectively.

The *separable quotient problem* has many forms. There are solutions for dual Banach spaces, non-Banach Fréchet spaces, $(LF)$-spaces, duals of $C_c(X)$ and $C_p(X)$, lcs in general, topological vector spaces, even certain topological groups [**1, 4, 5, 7, 10, 11, 12, 13, 16, 17, 19, 20, 21, 24**]. Complete solutions are unknown for $C_c(X)$, $C_p(X)$, and Banach spaces. Rosenthal [**18**] showed $C_c(X)$ has a separable quotient if it is a Banach space. One extension [**10**, Theorem 3.1] says every *barrelled* $C_c(X)$ admits a separable quotient.

The *optimal problem* (of many forms/solutions [**23**]) seeks the best extension possible *vis-a-vis* modern weak barrelledness: Which of the sixteen barrelled-type hypotheses displayed in the weak barrelledness table [**23**, Fig. 1.3] is the best (weakest, most general, optimal) among those that suffice?

Here is the table's bottom row:

$$\text{non-}S_\sigma \;\Rightarrow\; \text{inductive} \;\Rightarrow\; \varphi\text{-complemental} \;\Rightarrow\; \text{primitive}.$$

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To simplify the search for optimality, note that all $C_c(X)$ spaces and $C_p(X)$ spaces are non-$S_\sigma$ [**14**], hence enjoy all four bottom-row properties [**8**, Theorem 4].

*Primitive* is the weakest of all sixteen properties, and *Baire-like* (BL) the strongest. Of the twelve entries above the bottom row, *dual locally complete* (dlc) [**22**] is the weakest. Always,

$$\text{BL} \Rightarrow \text{barrelled} \Rightarrow \text{dlc} \Rightarrow \text{primitive}.$$

We begin by observing

Theorem 1. *Every barrelled $C_p(X)$ admits a separable quotient.*

Proof. Since $X$ is infinite (by assumption) and $C_p(X)$ is barrelled, $X$ is not pseudocompact [**3**, (1.8)]. Equivalently [**9**, Theorem 3.1], $C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$, as does $C_p(X)$, by minimality. Hence $C_p(X)$ contains a complemented copy of the separable $\mathbb{R}^{\mathbb{N}}$ [**15**, 2.6.5(iii)]. $\square$

For $C_p(X)$ spaces, we know BL $\Leftrightarrow$ barrelled $\Leftrightarrow$ dlc [**8**, Theorem 10].

Corollary 1. *Every dlc $C_p(X)$ admits a separable quotient.*

Since every $C_p(X)$ enjoys the bottom-row properties and *dlc* is the weakest of the remaining twelve, the *dlc* hypothesis on $C_p(X)$ is not optimal only if every arbitrary $C_p(X)$ has a separable quotient; *i.e.*, the only possible way to extend Corollary 1 in the context of [**23**, Fig. 1.3] would be to solve positively the separable quotient problem for arbitrary $C_p(X)$.

What about $C_c(X)$? Within the class of arbitrary $C_c(X)$ spaces, the sixteen tabular notions coalesce into five equivalence classes represented by *Baire-like, barrelled, $\aleph_0$-barrelled, dlc,* and *primitive* [**8**, Theorem 11]. As noted, every barrelled $C_c(X)$ admits a separable quotient [**10**, Theorem 3.1]. Again, *primitive* is clearly the optimal (tacit) hypothesis if and only if every arbitrary $C_c(X)$ admits a separable quotient. We shortly prove *dlc* suffices. Thus *dlc* is optimal, yet again, if *primitive* is not.

Theorem 2. *Every dlc $C_c(X)$ admits a separable quotient.*

Proof. (i) If $X$ admits an infinite compact subset $Y$, then the Banach space $C_c(Y)$ is a quotient of $C_c(X)$ [**7**, Lemma 15] which, itself, has a separable quotient [**18**], and quotient-taking is transitive.

(ii) When case (i) fails, $C_c(X) = C_p(X)$ and Corollary 1 applies. $\square$

## 2. Examples

Consequently, whether *dlc* is the optimal hypothesis for $C_p(X)$ and/or $C_c(X)$ cannot be determined short of a complete solution to the corresponding general separable quotient problem. On the other hand, we can determine that, in both cases, the sufficient *dlc* is not always necessary.

Let $X_1$ and $X_2$ be disjoint Tychonoff spaces such that both $C_p(X_1)$ and $C_c(X_1)$ are not dlc [**8**, Example 1] and both $C_p(X_2)$ and $C_c(X_2)$ admit separable quotients (*e.g.*, $X_2$ could be any P-space or any non-pseudocompact space [**7**, Corollary 19, Theorem 14]). Give $X = X_1 \bigcup X_2$ the topology that induces the original topology on $X_i$, with each $X_i$ open (and thus also closed) in $X$ $(i = 1, 2)$. If $X_i^{\perp} = \{f \in C(X) : f(X_i) = \{0\}\}$, one easily checks to see that for $i = 1, 2$, the quotients $C_p(X)/X_i^{\perp}$ and $C_c(X)/X_i^{\perp}$ are isomorphic to $C_p(X_{3-i})$ and $C_c(X_{3-i})$, respectively.

Example 1. (Notation as above.) *Since their quotients $C_p(X_2)$ and $C_c(X_2)$ admit separable quotients, so do $C_p(X)$ and $C_c(X)$. Yet $C_p(X)$ and $C_c(X)$ are not dlc, since their quotients $C_p(X_1)$ and $C_c(X_1)$ are not dlc* [**23**, Theorem 4.3].

One could also let $X_2$ be the Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers $\mathbb{N}$, by [**18**] and [**12**]. In fact, the latter affords a direct choice of $X$, without the intermediate $X_i$, as follows.

Haydon's third example [**6**] is a topological subspace of $\beta\mathbb{N}$ we denote by $\mathbb{H}$, consisting of $\mathbb{N}$ and one arbitrarily chosen cluster point of $S$ in $\beta\mathbb{N}$ for each infinite subset $S$ of $\mathbb{N}$. We noted that $C_p(\mathbb{H}) = C_c(\mathbb{H})$ is not dlc [**8**, Example 1].

Now $\mathbb{H}$ contains $\mathbb{N}$ and is pseudocompact [**6**]. Hence each $g \in C(\mathbb{H})$ is bounded on $\mathbb{N}$ and has a unique extension $\overline{g} \in C(\beta\mathbb{N})$, so the restriction map $f \mapsto f|_{\mathbb{H}}$ ($f \in C(\beta\mathbb{N})$) is a continuous [**2**, 0.4.1] linear bijection from $C_p(\beta\mathbb{N})$ onto $C_p(\mathbb{H})$. Consequently,

(a) If $F$ is a dense linear subspace of $C_p(\beta\mathbb{N})$, then $F|_{\mathbb{H}} = \{f|_{\mathbb{H}} : f \in F\}$ is dense in $C_p(\mathbb{H})$.

If $\phi$ is in the dual $C_p(\beta\mathbb{N})'$, define $\phi^* \in C_p(\mathbb{H})'$ by writing $\langle g, \phi^* \rangle = \langle \overline{g}, \phi \rangle$ for each $g \in C(\mathbb{H})$.

(b) The map $\phi \mapsto \phi^*$ is linear (routinely), and injective from $C_p(\beta\mathbb{N})'$ into $C_p(\mathbb{H})'$.

Indeed, if $\langle g, \phi^* \rangle = \langle \overline{g}, \phi \rangle = 0$ for all $g \in C(\mathbb{H})$, then $\langle f, \phi \rangle = 0$ for all $f \in C(\beta\mathbb{N})$; *i.e.*, $\phi^*$ is the zero vector in $C_p(\mathbb{H})'$ only if $\phi$ is the zero vector in $C_p(\beta\mathbb{N})'$.

Kąkol and Śliwa [**12**, Theorem 4] proved $C_p(\beta\mathbb{N})$ has a separable quotient. Equivalently, there is a linearly independent sequence $\{\phi_n\}_n \subset C_p(\beta\mathbb{N})'$ such that the eventually zero linear subspace

$$\mathrm{ez}\{\phi_n\}_n = \{f \in C(\beta\mathbb{N}) : \langle f, \phi_n \rangle = 0 \text{ for all but finitely many } n\}$$

is dense in $C_p(\beta\mathbb{N})$. By (a) and (b), $(\mathrm{ez}\{\phi_n\}_n)|_{\mathbb{H}} = \mathrm{ez}\{\phi_n^*\}_n$ is dense in $C_p(\mathbb{H})$ and $\{\phi_n^*\}_n$ is linearly independent in $C_p(\mathbb{H})'$. Therefore

Example 2. *Haydon's space $C_p(\mathbb{H}) = C_c(\mathbb{H})$ admits a separable quotient, yet is not dlc.*

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