TURBULENCE AND STRUCTURE

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Definition. If C is a class of relational structures and E is an analytic equivalence relation on a Polish space X, say that E is C-*structurable* if there is an analytic structure M on X such that for every equivalence class $A \subset X$, $M \upharpoonright A \in C$.

Example. E is *treeable* if there is a analytic graph H on X such that for every equivalence class A of E, $H \upharpoonright A$ is acyclic and connected.

Theorem. If E is a treeable equivalence relation on X and F is an orbit equivalence relation of a turbulent group action on Y, then every Borel homomorphism from E to F stabilizes on a comeager set.

Explanation. If $h : X \to Y$ is a Borel function such that $x_0 \; E \; x_1$ implies $h(x_0) \; F \; h(x_1)$, then there is a single F-equivalence class with a comeager preimage.

Turbulence characterization. Suppose that a Polish group G acts on a Polish space Y with dense and meager orbits. The following are equivalent:

- the action is generically turbulent;
- $P_G \times P_Y \Vdash V[\dot{y}] \cap V[\dot{g} \cdot \dot{y}] = V$.

Improvement. Let G act on Y in a generically turbulent way, inducing the orbit equivalence relation F. In some forcing extension there are points $y_i \in Y$ for $i \in \omega$ such that

1. the points are separately Cohen-generic over V;
2. they are pairwise F-equivalent;
3. for every set $a \subset \omega$, $V[y_i \colon i \in a] \cap V[y_i \colon i \notin a] = V$.

Terminology. Such a set of points is *independent*.

Proof of Theorem. Let E be a treeable equivalence relation on Polish X, as witnessed by an analytic graph H. Let F be the orbit equivalence of a generically turbulent action on Y. Let $h : X \to Y$ be a Borel homomorphism from F to E.

Let y_i for $i \in 4$ be independent generic points in the space Y. Then $h(y_i)$ for $i \in 4$ are E-related points in X, so they are in the same connected component of the graph H. The unique shortest paths between $h(y_0)$ and $h(y_1)$, and between $h(y_2)$ and $h(y_3)$ must intersect. The point x in the intersection belongs to $V[y_0, y_1]$ and $V[y_2, y_3]$, so to V.

The preimage $h^{-1}[x]_E$ is comeager in Y.

Generalizations. Same conclusion for C-structurable equivalence relations where

- C is the class of connected graphs without a perfect clique as a minor;

- C is the class of connected abstract simplicial complexes which are finitewise contractible.