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Marcello Ortaggio
Alena Pravdová

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Marcello Ortaggio, Alena Pravdová

Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1, Czech Republic

E-mail: `ortaggio@math.cas.cz, pravdova@math.cas.cz`

Abstract.
We summarize the fall-off of electromagnetic and gravitational fields in $n > 5$ dimensional Ricci-flat spacetimes along an asympotically expanding non-singular geodesic null congruence.

1. Introduction

Under suitable assumptions, the well-known peeling-off property characterizes the behavior of the gravitational and electromagnetic fields at null infinity (see, e.g., [1, 2] and references therein). It has been observed [3] that the Weyl tensor peels off differently in $n > 4$ dimensions. Here, we summarize our recent results [4, 5] on the leading-order behavior of gravitational and electromagnetic fields in higher dimensions. Ref. [4] partly recovers the results of [3] but uses a different method and different assumptions. We restrict to Ricci-flat spacetimes with suitable properties at null infinity (a cosmological constant can be included [4,5]), formulated in terms of a geodesic null vector field $\boldsymbol{\ell} = \partial_r$ (r is an affine parameter) and of the Weyl tensor, using a "null" frame [6] based on two null vectors $\boldsymbol{m}_{(0)} = \boldsymbol{\ell}$, $\boldsymbol{m}_{(1)} = \boldsymbol{n}$ and $n-2$ orthonormal spacelike vectors $\boldsymbol{m}_{(i)}$ $(i, j, \dots = 2, \dots, n-1)$. First, we assume that the optical matrix $\rho_{ij} = \ell_{a;b} m^a_{(i)} m^b_{(j)}$ is *asymptotically non-singular and expanding* [4, 5] (this includes asymptotically flat spacetimes [3] but also holds more generally – see [7] in four dimensions). Furthermore, we assume that the boost-weight (b.w.) +2 Weyl components $\Omega_{ij} \equiv C_{0i0j} = C_{abcd} \ell^a m^b_{(i)} \ell^c m^d_{(j)}$ fall off as

$$\Omega_{ij} = O(r^{-\nu}) \qquad (\nu > 2). \tag{1}$$

Again, this is satisfied in asymptotically flat spacetimes [3] (e.g., $\Omega_{ij} = O(r^{-5})$ in the 4D spacetimes of [7]). Under the above conditions, one is able to determine how the Maxwell and Weyl tensors fall off as $r \to \infty$, as we summarize in sections 2 and 3. However, as an intermediate step, one also needs the r-dependence of the Ricci rotation coefficients and of the derivative operators [6], which is given in [4] (it follows from the Ricci identities [8], also using the commutators [9] and the Bianchi identities [10]). For example, $\rho_{ij} = \frac{\delta_{ij}}{r} + \dots$. For brevity, in this paper, we discuss only results in $n > 5$ dimensions – the case $n \geq 5$ is studied in [4, 5].

2. Electromagnetic field

We start from the simpler case of *test* Maxwell fields in the background of an n-dimensional Ricci-flat spacetime satisfying the assumptions of section 1 [5]. The gravitational field (Weyl tensor) can be treated similarly, however, resulting in a larger number of possible cases (section 3).

In the frame of section 1, we assume that for $r \to \infty$ the Maxwell components have a *power-like* behavior described by

$$F_{0i} = O(r^{\alpha}), \qquad F_{01} = O(r^{\beta}), \qquad F_{ij} = O(r^{\gamma}), \qquad F_{1i} = O(r^{\delta}). \tag{2}$$

The empty-space Maxwell equations ${F^a}_{b;a} = 0 = F_{[ab;c]}$ (see [5,11] for their GHP and NP form) determine the possible values of α, β, γ and δ. We assume that if a generic component f behaves as $f = O(r^{-\zeta})$ then $\partial_r f = O(r^{-\zeta-1})$ and $\partial_A f = O(r^{-\zeta})$. As it turns out, α can be chosen arbitrarily, giving raise to two main cases, $\alpha \geq -2$ or $\alpha < -2$. In the latter, one needs to choose whether $\gamma \geq -2$ or $\gamma < -2$, and then specify more precisely the value of α, as we detail.

2.1. *Case $\alpha \geq -2$.*

In this case, all components fall off at the same speed, i.e.,

$$F_{0i} = O(r^{\alpha}), \qquad F_{01} = O(r^{\alpha}), \qquad F_{ij} = O(r^{\alpha}), \qquad F_{1i} = O(r^{\alpha}). \tag{3}$$

The electromagnetic field does not peel. This describes, e.g., a uniform magnetic field permeating asymptotically flat black holes [12] (or black rings [13] if $n = 5$ is included, cf. [5]).

2.2. *Case $\alpha < -2$.*

Generically, we have

$$F_{0i} = O(r^{\alpha}), \tag{4}$$
$$F_{01} = o(r^{-2}), \qquad F_{ij} = O(r^{-2}), \tag{5}$$
$$F_{1i} = O(r^{-2}). \tag{6}$$

The above behavior includes the special case when $\boldsymbol{\ell}$ is an aligned null direction of the Maxwell field, i.e., $F_{0i} = 0$ (in the formal limit $\alpha \to -\infty$). The leading term is of type II. Examples can be obtained as a "linearized" Maxwell field limit of certain full Einstein-Maxwell solutions given in [14] for even n. Several subcases are possible when $\gamma < -2$.

2.2.1. *Subcase (a): $\gamma < -2$ with $1 - \frac{n}{2} \leq \alpha < -2$.*

In this case, one has the same results as in section 2.1 above. This subcase does not exist for $n = 6$.

2.2.2. *Subcase (b): $\gamma < -2$ with $-\frac{n}{2} \leq \alpha < 1 - \frac{n}{2}$.*

Here, we have

$$F_{0i} = O(r^{\alpha}), \tag{7}$$
$$F_{01} = O(r^{\alpha}), \qquad F_{ij} = O(r^{\alpha}), \tag{8}$$
$$F_{1i} = O(r^{1-n/2}). \tag{9}$$

The leading term falls off as $1/r^{\frac{n}{2}-1}$ and is of type N. This is characteristic of radiative fields (note that $T_{11} \propto F_{1i}F_{1i} \sim 1/r^{n-2}$ and the energy flux along $\boldsymbol{\ell}$ can be directly related to the energy loss, at least in the case of asymptotically flat spacetimes – cf. [15–17] for $n = 4$). As opposed to the well-known four-dimensional case, here, $\boldsymbol{\ell}$ cannot be aligned with F_{ab} if radiation is present (since $\alpha \geq -\frac{n}{2}$). In the case $\alpha = -\frac{n}{2}$, if one assumes that F_{1i} has a power-like behavior also at the subleading order, from the Maxwell equations, one finds $F_{1i} = F^{(0)}_{1i} r^{1-\frac{n}{2}} + O(r^{-n/2})$, which gives the peeling-off behavior

$$F_{ab} = \frac{N_{ab}}{r^{\frac{n}{2}-1}} + \frac{G_{ab}}{r^{\frac{n}{2}}} + \ldots \qquad \left(\alpha = -\frac{n}{2}\right). \tag{10}$$

The subleading term is algebraically general, which is qualitatively different from the 4D case [1,2,16,17]. This resembles the behavior of the Weyl tensor of higher dimensional asymptotically flat spacetimes [3]. See [5] for a possible different peeling-off in five dimensions.

2.2.3. Subcase (c): $\gamma < -2$ with $2-n \leq \alpha < -\frac{n}{2}$. The same results as in section 2.1 apply.

2.2.4. Subcase (d): $\gamma < -2$ with $\alpha < 2-n$. We have

$$F_{0i} = O(r^{\alpha}), \tag{11}$$
$$F_{01} = O(r^{2-n}), \qquad F_{ij} = o(r^{2-n}), \tag{12}$$
$$F_{1i} = O(r^{2-n}). \tag{13}$$

The leading term is of type II and falls off as $1/r^{n-2}$ (it is purely electric in the subcase $F_{1i} = o(r^{2-n})$). This behavior includes the Coulomb field of a weakly charged asymptotically flat black hole [12, 18] (or black ring [13] if $n = 5$ is included [5]). In the special subcase $F_{01} = o(r^{2-n})$, the same results as in section 2.1 again apply (for example, for $n = 5$ and $\alpha = -4$, this is the case of the weak-field limit of the 5D dipole black rings of [19]).

Let us observe that in all cases, type N fields for which $\boldsymbol{\ell}$ is aligned are not permitted [11,20].

2.3. The case of p-forms

The above results for a 2-form F_{ab} can be extended easily [5] to p-form fields satisfying the generalized Maxwell equations (given in [11] in the GHP notation). In *even* dimensions, the special case $p = n/2$ (including $n = 4$, $p = 2$) has unique properties. It peels off as

$$F_{a_1\ldots a_p} = \frac{N_{a_1\ldots a_p}}{r^{\frac{n}{2}-1}} + \frac{II_{a_1\ldots a_p}}{r^{\frac{n}{2}}} + \ldots \qquad \left(p = \frac{n}{2}\right). \tag{14}$$

The (radiative) leading term is of type N and falls off as $1/r^{\frac{n}{2}-1}$. In contrast to the case $p = 2$ discussed above (or, in fact, any other $p \neq n/2$), Maxwell fields of type N aligned with $\boldsymbol{\ell}$ are now permitted [5] and the peeling (14) applies also in the presence of a cosmological constant [5]. Corresponding solutions of the *full Einstein-Maxwell equations* have recently been obtained [21].

3. Gravitational field

The method to be used for the Weyl tensor [4] is essentially similar, now $-\nu$ playing the role that α played above. Instead of the Maxwell equations, one has to integrate the system "Bianchi-Ricci-commutators". However, there is now extra freedom in the choice of possible boundary conditions. In particular, three possible choices for the behavior of b.w. +1 components Ψ_{ijk} are possible (cases (i), (ii) and (iii) below). Once the fall-off of Ω_{ij} and Ψ_{ijk} has been specified, the next step is to determine the fall-off of the b.w. 0 components Φ_{ijkl}

$$\Phi_{ijkl} = O(r^{\beta_c}). \tag{15}$$

The parameter β_c can then be used to label various possible subcases, which we now present.

3.1. Case (i): $\Omega_{ij} = O(r^{-\nu})$, $\Psi_{ijk} = O(r^{-\nu})$

In all cases given here, we have (this will not be repeated every time below)

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2), \qquad \Psi_{ijk} = O(r^{-\nu}). \tag{16}$$

3.1.1. Subcase (A): $\beta_c = -2$. In this case, necessarily $\beta_c > -\nu$ and we have the following possible behaviors, depending on how ν is chosen (cf. [4] for a few further special subcases):

A1:

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = o(r^{-2}), \qquad \Phi^A_{ij} = o(r^{-2}) \qquad (2 < \nu \leq 3),\\
&\Psi'_{ijk} = O(r^{-2}),\\
&\Omega'_{ij} = O(r^{\sigma}) \qquad (-2 \leq \sigma < -1);
\end{aligned} \tag{17}$$

A2:

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-2}), \quad \Phi^S_{ij} = O(r^{-3}), \quad \Phi = O(r^{-\nu}), \quad \Phi^A_{ij} = O(r^{-3}) \quad (3 < \nu < 4),\\
&\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{-3}),\\
&\Omega'_{ij} = O(r^{-2});
\end{aligned} \tag{18}$$

A3:

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}), \qquad \Phi = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}) \qquad (\nu \geq 4),\\
&\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{-3}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{19}$$

with the further restrictions $\Phi^S_{ij} = O(r^{1-\nu})$ for $4 \leq \nu < 5$ and $\Phi^S_{ij} = O(r^{-4})$ for $\nu \geq 5$;

A4:

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-2}), \quad \Phi^S_{ij} = O(r^{1-\nu}), \quad \Phi = O(r^{-\nu}), \quad \Phi^A_{ij} = O(r^{-\nu}) \quad (\nu \geq 4,\, \nu \neq n),\\
&\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{1-\nu}),\\
&\Omega'_{ij} = O(r^{-2});
\end{aligned} \tag{20}$$

A5:

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}) \qquad (\nu \geq n),\\
&\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{1-n}),\\
&\Omega'_{ij} = O(r^{-2}).
\end{aligned} \tag{21}$$

None of the above five cases can describe asymptotically flat spacetimes, cf. [3]. In cases A2–A5, the leading term falls off as $1/r^2$ at infinity and it is of type II(abd). In cases A3–A5, $\boldsymbol{\ell}$ can be a multiple WAND. Examples in case A5 are Robinson-Trautman spacetime [22].

When $\beta_c < -2$, its precise value depends on the value of ν so that we have to consider the following possible cases.

3.1.2. Subcase (B): $\beta_c < -2$ with $\frac{n}{2} < \nu \leq 1 + \frac{n}{2}$. In this case, $\beta_c = -\frac{n}{2}$ and we have

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-n/2}), \qquad \Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu}) \qquad \left(\frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right),\\
&\Psi'_{ijk} = O(r^{-n/2}),\\
&\Omega'_{ij} = O(r^{1-n/2}).
\end{aligned} \tag{22}$$

Here, $\boldsymbol{\ell}$ cannot be a WAND. The leading term at infinity falls off as $1/r^{n/2-1}$ and it is of type N. This includes *radiative spacetimes* [3] that are asymptotically flat in the Bondi definition [23,24]. If one takes for b.w. +2 components $\nu = 1 + \frac{n}{2}$ and additionally *assumes* that $\Omega_{ij} = \Omega_{ij}^{(0)} r^{-n/2-1} + \Omega_{ij}^{(1)} r^{-n/2-2} + o(r^{-n/2-2})$, then one finds [4] the peeling-off behavior

$$C_{abcd} = \frac{N_{abcd}}{r^{n/2-1}} + \frac{II_{abcd}}{r^{n/2}} + o(r^{-n/2}). \tag{23}$$

This agrees with [3] for asymptotically flat spacetimes. See [3, 4] for special properties of the case $n = 5$. When $\beta_c < -2$ but ν is not in the range $\frac{n}{2} < \nu \leq 1 + \frac{n}{2}$ one has the following subcases (B*) and (C).

3.1.3. Subcase (B): $\beta_c < -2$ with $2 < \nu \leq \frac{n}{2}$ or $1 + \frac{n}{2} < \nu \leq n - 1$.* In this case, $\beta_c = -\nu$ and we have (cf. section IV A 5 of [4])

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu}),\\
&\Psi'_{ijk} = O(r^{-2}) \quad \text{if } 2 < \nu \leq 3, \qquad \Psi'_{ijk} = O(r^{-\nu}) \quad \text{if } \nu > 3,\\
&\Omega'_{ij} = o(r^{1-\nu}) \quad \text{if } \nu \neq \frac{n}{2}, \qquad \Omega'_{ij} = O(r^{1-n/2}) \quad \text{if } \nu = \frac{n}{2}.
\end{aligned} \tag{24}$$

Here, $\boldsymbol{\ell}$ cannot be a WAND.

3.1.4. Subcase (C): $\beta_c < -2$ with $\nu > n - 1$. In this case, $\beta_c = 1 - n$ and we have

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = o(r^{1-n}) \qquad (\nu > n-1),\\
&\Psi'_{ijk} = O(r^{1-n}),\\
&\Omega'_{ij} = o(r^{2-n}),
\end{aligned} \tag{25}$$

with $\Phi^A_{ij} = O(r^{-\nu})$ for $n - 1 < \nu < n$ and $\Phi^A_{ij} = O(r^{-n})$ for $\nu \geq n$. Here, $\boldsymbol{\ell}$ can become a multiple WAND, cf. [25]. This includes asymptotically flat spacetimes in the case of *vanishing radiation* [3], such as those for which $\boldsymbol{\ell}$ is a multiple WAND [25], e.g., the Schwarzschild-Tangherlini metric and Kerr-Schild spacetimes [26] with a non-degenerate Kerr-Schild vector.

3.2. Case (ii): $\Omega_{ij} = o(r^{-n})$, $\Psi_{ijk} = O(r^{-n})$

3.2.1. Subcase $\beta_c = -2$. Generically, one has

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\
&\Omega'_{ij} = O(r^{-2}).
\end{aligned} \tag{26}$$

For $\Psi^{(n)}_{ijk} = 0$, this case reduces to (19) (with $\nu > n$). See [4] for possible subcases.

3.2.2. Subcase $\beta_c = 1 - n$. When $\beta_c < -2$ then necessarily $\beta_c = 1 - n$ and generically, one has

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{1-n}), \qquad \Psi'_i = O(r^{1-n}),\\
&\Omega'_{ij} = o(r^{2-n}).
\end{aligned} \tag{27}$$

This includes asymptotically flat spacetimes in the case of vanishing radiation [3]. For $\Psi^{(n)}_{ijk} = 0$, this case reduces to (25) (with $\nu > n$).

3.3. Case (iii): $\Omega_{ij} = o(r^{-3})$, $\Psi_{ijk} = O(r^{-3})$
This case cannot represent asymptotically flat spacetimes [3]. Generically, $\beta_c = -2$ and

$$\begin{aligned}
&\Omega_{ij} = O(r^{-\nu}) \qquad (\nu > 3), \\
&\Psi_{ijk} = O(r^{-3}), \qquad \Psi_i = o(r^{-3}), \\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}), \qquad \Phi = o(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}), \\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}), \\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{28}$$

where $\Psi_i = O(r^{-\nu})$, $\Phi = O(r^{-\nu})$ for $3 < \nu \leq 4$ while $\Psi_i = O(r^{-4})$, $\Phi = O(r^{-4})$ for $\nu > 4$. Here, $\boldsymbol{\ell}$ can be a single WAND and the asymptotically leading term is of type II(abd). For $\Psi^{(3)}_{ijk} = 0$, this case reduces for $3 < \nu < 4$ to (18) (with $\nu > n$), for $4 \leq \nu \leq n$ to (19) and for $\nu > n$ to (26). If $\beta_c < -2$ then $\Phi_{ijkl} = O(r^{-3})$ and the leading term at infinity becomes of type III(a).

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