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DISERTAČNÍ PRÁCE

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Operadické resolventy diagramů

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Název práce: Operadické resolventy diagramů

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Abstrakt: Zkoumáme resolventy operády $\mathcal{A}_{\mathsf{C}}$ popisující diagramy daného tvaru C v kategorii algeber daného typu $\mathcal{A}$. Za jistých předpokladů dokážeme domněnku M. Markla o konstrukci této resolventy pomocí daných resolvent operád $\mathcal{A}$ a C. V případě asociativních algeber dostaneme explicitní popis kohomologické teorie pro příslušné diagramy, která se shoduje s teorií vymyšlenou Gerstenhaberem a Schackem. Obecně také ukážeme, že operadickou kohomologii lze popsat pomocí Extu v abelovské kategorii operadických modulů.

Klíčová slova: operáda, resolventa, kohomologie

Title: Operadic resolutions of diagrams

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Abstract: We study resolutions of the operad $\mathcal{A}_{\mathsf{C}}$ describing diagrams of a given shape C in the category of algebras of a given type $\mathcal{A}$. We prove the conjecture by Markl on constructing the resolution out of resolutions of $\mathcal{A}$ and C, at least in a certain restricted setting. For associative algebras, we make explicit the cohomology theory for the diagrams and recover Gerstenhaber-Schack diagram cohomology. In general, we show that the operadic cohomology is Ext in the category of operadic modules.

Keywords: operad, resolution, cohomology

Contents

1. Foreword

This foreword is intended to be a nonformal motivation for the problem considered in this thesis. We give a quick overview of its time evolution from our point of view without trying to be historically precise or to give full and original references. More detail on the background can be found in e.g. [18], [15], [17] and [11].

Operads are objects encoding algebraic structures. Originally invented in topology in 60-70's, operads were used to understand what are the right structures preserved by homotopy equivalences and to solve the related problem of loop space recognition (work of Stasheff, May, Boardman, Vogt for example).

In algebra, operads initially attracted little attention probably as a more general formalism of *triples* (aka monads) was available. However, In 90's, Ginzburg and Kapranov [9] showed that there is a Koszul duality theory for operads analogous to that for associative algebras. This allowed for explicit calculations of certain small cofibrant replacements of several operads, so called *minimal* resolutions. This was a major improvement over the triple resolution and even over the algebraic analogue of the W-construction by Boardman and Vogt. It explained some algebraic phenomena in rational homotopy theory and also revealed that various ∞-algebras, which appeared earlier in ad-hoc manners, are in fact algebras over minimal resolution of the operad describing the corresponding non-∞-algebras [13]. For example, A_∞-algebra was found to be an algebra over a minimal resolution of the operad for associative algebras. Further, the principle that algebras over cofibrant operads are homotopy invariant (e.g. transfer along homotopy equivalences), observed by Boardman and Vogt in topology, was made rigorous even in algebra (works of Markl [12], Berger, Moerdijk [1]). It was applied with spectacular success by Kontsevich in his proof of the deformation quantization of Poisson manifolds [10].

Another application was a unified construction of cohomology theories for various types of algebras. Given an algebra A over an operad $\mathcal{A}$, the operadic cohomology of A describes infinitesimal (formal) deformations of A in the category of $\mathcal{A}$-algebras, but also in the category of $\mathcal{A}_\infty$-algebras, where $\mathcal{A}_\infty$ is a cofibrant resolution of $\mathcal{A}$. In particular, this gives an interpretation to higher cohomologies of e.g. Hochschild or Chevalley-Eilenberg complex. Later, it was discovered that the deformation complex computing the operadic cohomology carries a L_∞-structure (works of van der Laan [21], Markl [14], Merkulov, Vallette [19]). The solutions of the corresponding generalized Maurer-Cartan equation associated with this L_∞-structure are full formal deformations of A.

All these applications have a common prerequisite: to know a cofibrant replacement of the given operad $\mathcal{A}$ explicitly. In practice, one restricts to *free* resolutions. There is always one such resolution at hand - it is the analogue of the above W-construction, called bar-cobar resolution. However, the bar-cobar resolution is redundant in the unprecise sense that too many cycles get killed by boundaries. A large class of operads admits minimal resolution which doesn't have this deficiency. Furthermore, the minimal resolution of the given operad $\mathcal{A}$ has the property of being unique up to an isomorphism. It is analogous to the Sullivan minimal model in rational homotopy theory. Thus it is a canonical representative of the weak equivalence class of $\mathcal{A}$. A major breakthrough in the construction of minimal resolutions was the above mentioned paper by Ginzburg and Kapranov on Koszul duality. It was later improved (e.g. works of Getzler, Jones [8], Vallette [20]). The theory covers a large class of common operads, so called Koszul operads. These are characterized by the quadraticity of the defining relations and a certain homological condition. The minimal resolution is provided by the cobar construction on Koszul dual cooperad. In most cases, this resolution is easily made explicit. However, there are also naturally appearing operads which are not Koszul, e.g. the operad for alternative algebras [6]. Less natural examples include e.g. the simple operad for anti-associative algebras [16].

To our best knowledge, there are basically only 2 examples of non-Koszul operads with minimal resolution known. The first one is the operad for Batalin-Vilkovisky algebras. This is a very recent work of Drummond-Cole and Vallette [2]. The second one is known for a long time - it is the coloured operad describing a morphism of 2 associative (or Lie) algebras. To generalize this example, we replace the single morphism category by a general small category C. One easily finds a coloured operad $\mathcal{A}_{\mathsf{C}}$ such that $\mathcal{A}_{\mathsf{C}}$-algebras are C-shaped diagrams of $\mathcal{A}$-algebras. A natural question is: If one knows the minimal resolution of $\mathcal{A}$ and also the minimal resolution of the category C (seen as a coloured operad in arity 1), can one construct the minimal resolution of $\mathcal{A}_{\mathsf{C}}$? This problem is studied in this thesis.

The thesis consists of 3 independent articles. The first article (Chapter 2, [5]) is an overview of basic deformation theory. It may serve as a further motivation for the two remaining chapters. In the second article (Chapter 3, [3]), we find the operadic cohomology for $\mathcal{A}_{\mathsf{C}}$-algebras in case $\mathcal{A} = \mathcal{A}ss$ without constructing the resolution explicitly. This recovers the classical diagram cohomology by Gerstenhaber and Schack [7]. To this end, we show that the operadic cohomology can be computed as certain Ext in the abelian category of operadic modules. In the third article (Chapter 4, [4]), we give a partial description of the resolution. More detail is available in the introduction of each article.

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2. Deformation Theory

Notes, taken by Martin Doubek and Petr Zima, from a course given by Martin Markl at the Charles University, Prague, in the Summer semester 2006.

Abstract

First three sections of this overview paper cover classical topics of deformation theory of associative algebras and necessary background material. We then analyze algebraic structures of the Hochschild cohomology and describe the relation between deformations and solutions of the corresponding Maurer-Cartan equation. In Section 2.6 we generalize the Maurer-Cartan equation to strongly homotopy Lie algebras and prove the homotopy invariance of the moduli space of solutions of this equation. In the last section we indicate the main ideas of Kontsevich's proof of the existence of deformation quantization of Poisson manifolds.

Conventions. All algebraic objects will be considered over a fixed field $\mathbf{k}$ of characteristic zero. The symbol $\otimes$ will denote the tensor product over $\mathbf{k}$. We will sometimes use the same symbol for both an algebra and its underlying space.

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Contents

2.1 Algebras and modules

In this section we investigate modules (where module means rather a bimodule than a one-sided module) over various types of algebras.

2.1.1 Example. – The category Ass of associative algebras.
An associative algebra is a $\mathbf{k}$-vector space A with a bilinear multiplication $A\otimes A \to A$ satisfying

$$a(bc) = (ab)c, \qquad \text{for all } a, b, c \in A.$$

Observe that at this moment we do not assume the existence of a unit $1 \in A$.

What we understand by a module over an associative algebra is in fact a bimodule, i.e. a vector space M equipped with multiplications ("actions") by elements of A from both sides, subject to the axioms

$$\begin{aligned} a(bm) &= (ab)m, \\ a(mb) &= (am)b, \\ m(ab) &= (ma)b, \qquad \text{for all } m \in M,\ a, b \in A. \end{aligned}$$

2.1.2 Example. – The category Com of commutative associative algebras.
In this case left modules, right modules and bimodules coincide. In addition to the axioms in Ass we require the commutativity

$$ab = ba, \qquad \text{for all } a, b \in A,$$

and for a module

$$ma = am, \qquad \text{for all } m \in M,\ a \in A.$$

2.1.3 Example. – The category Lie of Lie algebras.
The bilinear bracket $[-,-] : L \otimes L \to L$ of a Lie algebra L is anticommutative and satisfies the Jacobi identity, that is

$$\begin{aligned} [a, b] &= -[b, a], \text{ and} \\ [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= 0, \qquad \text{for all } a, b, c \in L. \end{aligned}$$

A left module (also called a representation) M of L satisfies the standard axiom

$$a(bm) - b(am) = [a, b]m, \qquad \text{for all } m \in M,\ a, b \in L.$$

Given a left module M as above, one can canonically turn it into a right module by setting $ma := -am$. Denoting these actions of L by the bracket, one can rewrite the axioms as

$$\begin{aligned} [a, m] &= -[m, a], \text{ and} \\ [a, [b, m]] + [b, [m, a]] + [m, [a, b]] &= 0, \qquad \text{for all } m \in M,\ a, b \in L. \end{aligned}$$

Examples 2.1.1–2.1.3 indicate how axioms of algebras induce, by replacing one instance of an algebra variable by a module variable, axioms for the corresponding modules. In the rest of this section we formalize, following [41], this recipe. The standard definitions below can be found for example in [32].

2.1.4 Definition. The *product* in a category C is the limit of a discrete diagram. The *terminal object* of C is the limit of an empty diagram, or equivalently, an object T such that for every $X \in \mathsf{C}$ there exists a unique morphism $X \to T$.

2.1.5 Remark. The product of any object X with the terminal object T is naturally isomorphic to X,

$$X \times T \cong X \cong T \times X.$$

2.1.6 Remark. It follows from the universal property of the product that there exists the *swapping morphism* $X \times X \xrightarrow{s} X \times X$ making the diagram

$$\begin{tikzcd} X \times X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, dashed, "s"] & X \\ X & X \times X, \arrow[l, "p_1"] \arrow[u, "p_2"'] \end{tikzcd}$$

in which p_1 (resp. p_2) is the projection onto the first (resp. second) factor, commutative.

2.1.7 Example. In the category of A-bimodules, the product $M_1 \times M_2$ is the ordinary direct sum $M_1 \oplus M_2$. The terminal object is the trivial module 0.

2.1.8 Definition. A category C has *finite products*, if every *finite* discrete diagram has a limit in C.

By [32, Proposition 5.1], C has finite limits if and only if it has a terminal object and products of pairs of objects.

2.1.9 Definition. Let C be a category, $A \in \mathsf{C}$. The *comma category* (also called the *slice category*) C/A is the category whose

– objects (X, π) are C-morphisms $X \xrightarrow{\pi} A$, $X \in \mathsf{C}$, and

– morphisms $(X', \pi') \xrightarrow{f} (X'', \pi'')$ are commutative diagrams of C-morphisms:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\pi'"'] & X'' \arrow[d, "\pi''"] \\ A \arrow[r, equal, "id_A"'] & A. \end{tikzcd}$$

2.1.10 Definition. The *fibered product* (or *pullback*) of morphisms $X_1 \xrightarrow{f_1} A$ and $X_2 \xrightarrow{f_2} A$ in C is the limit D (together with morphisms $D \xrightarrow{p_1} X_1$, $D \xrightarrow{p_2} X_2$) of the lower right corner of the digram:

$$\begin{tikzcd} D \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] & X_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "f_2"'] & A. \end{tikzcd}$$

In the above situation one sometimes writes $D = X_1 \times_A X_2$.

2.1.11 Proposition. If C has fibered products then C/A has finite products.

Proof. A straightforward verification. The identity morphism (A, id_A) is clearly the terminal object of C/A.

Let (X_1, π_1) and (X_2, π_2) be objects of C/A. By assumption, there exists the fibered product

$$\begin{tikzcd} D \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] \arrow[dr, dashed, "\delta"] & X_1 \arrow[d, "\pi_1"] \\ X_2 \arrow[r, "\pi_2"'] & A \end{tikzcd} \tag{2.1}$$

in C. In the above diagram, of course, $\delta := \pi_1 p_1 = \pi_2 p_2$. The maps $p_1 : D \to X_1$ and $p_2 : D \to X_2$ of the above diagram define morphisms (denoted by the same symbols) $p_1 : (D, \delta) \to (X_1, \pi_1)$ and $p_2 : (D, \delta) \to (X_2, \pi_2)$ in C/A. The universal property of the pullback (2.1) implies that the object (D, δ) with the projections (p_1, p_2) is the product of $(X_1, \pi_1) \times (X_2, \pi_2)$ in C/A. $\square$

One may express the conclusion of the above proof by

$$(X_1, \pi_1) \times (X_2, \pi_2) = X_1 \times_A X_2, \tag{2.2}$$

but one must be aware that the left side lives in C/A while the right one in C, therefore (2.2) has only a symbolical meaning.

2.1.12 Example. In Ass, the fibered product of morphisms $B_1 \stackrel{f_1}{\to} A$, $B_2 \stackrel{f_2}{\to} A$ is the subalgebra

$$B_1 \times_A B_2 = \{(b_1, b_2) \mid f_1(b_1) = f_2(b_2)\} \subseteq B_1 \oplus B_2 \tag{2.3}$$

together with the restricted projections. Hence for any algebra $A \in \mathsf{Ass}$, the comma category Ass/A has finite products.

2.1.13 Definition. Let C be a category with finite products and T its terminal object. An *abelian group object* in C is a quadruple $(G, G \times G \stackrel{\mu}{\to} G, G \stackrel{\eta}{\to} G, T \stackrel{e}{\to} G)$ of objects and morphisms of C such that following diagrams commute:

– the *associativity* μ:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times id_G"] \arrow[d, "id_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G, \end{tikzcd}$$

– the *commutativity* of μ (with s the swapping morphism of Remark 2.1.6):

$$\begin{tikzcd}
G \times G \arrow[rr, "s"] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}$$

– the *neutrality* of e:

$$\begin{tikzcd}
T \times G \arrow[r, "e \times id_G"] \arrow[dr, equal, "\cong"'] & G \times G \arrow[d, "\mu"] & G \times T \arrow[l, "id_G \times e"'] \arrow[dl, equal, "\cong"] \\
& G &
\end{tikzcd}$$

– the diagram saying that η is a two-sided *inverse* for the multiplication μ:

$$\begin{tikzcd}
G \arrow[r, "\eta \times id_G"] \arrow[d, "id_G \times \eta"'] \arrow[dr] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G,
\end{tikzcd}$$

in which the diagonal map is the composition $G \to T \xrightarrow{e} G$.

Maps μ, η and e above are called the *multiplication*, the *inverse* and the *unit* of the abelian group structure, respectively.

Morphisms of abelian group objects $(G', \mu', \eta', e') \xrightarrow{f} (G'', \mu'', \eta'', e'')$ are morphisms $G' \xrightarrow{f} G''$ in C which preserve all structure operations. In terms of diagrams this means that

$$\begin{tikzcd}
G' \times G' \arrow[r, "f \times f"] \arrow[d, "\mu'"'] & G'' \times G'' \arrow[d, "\mu''"] \\
G \arrow[r, "f"'] & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
G' \arrow[r, "f"] \arrow[d, "\eta'"'] & G'' \arrow[d, "\eta''"] \\
G \arrow[r, "f"'] & G'
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, equal, "id_T"] \arrow[d, "e'"'] & T \arrow[d, "e''"] \\
G \arrow[r, "f"'] & G'
\end{tikzcd}$$

commute. The category of abelian group objects of C will be denoted C_{ab}.

Let Alg be any of the examples of categories of algebras considered above and $A \in \mathsf{Alg}$. It turns out that the category $(\mathsf{Alg}/A)_{ab}$ is precisely the corresponding category of A-modules. To verify this for associative algebras, we identify, in Proposition 2.1.15 below, objects of $(\mathsf{Ass}/A)_{ab}$ with trivial extensions in the sense of:

2.1.14 Definition. Let A be an associative algebra and M an A-module. The *trivial extension* of A by M is the associative algebra $A \oplus M$ with the multiplication given by

$$(a, m)(b, n) = (ab, an + mb),\ \ a, b \in A \text{ and } m, n \in M.$$

2.1.15 Proposition. The category $(\mathsf{Ass}\,/A)_{ab}$ is isomorphic to the category of trivial extensions of A.

Proof. Let M be an A-module and $A \oplus M$ the corresponding trivial extension. Then $A \oplus M$ with the projection $A \oplus M \overset{\pi_A}{\to} A$ determines an object G of $\mathsf{Ass}\,/A$ and, by (2.2) and (2.3), $G \times G = (A \oplus M \oplus M \overset{\pi_A}{\to} A)$. It is clear that $\mu : G \times G \to G$ given by $\mu(a, m_1, m_2) := (a, m_1 + m_2)$, e the inclusion $A \hookrightarrow A \oplus M$ and $\eta : G \to G$ defined by $\eta(a, m) := (a, -m)$ make G an abelian group object in $(\mathsf{Ass}\,/A)_{ab}$.

On the other hand, let $((B, \pi), \mu, \eta, e)$ be an abelian group object in $\mathsf{Ass}\,/A$. The diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, equal, "id_A"'] & B \arrow[d, "\pi"] \\ & A \end{tikzcd}$$

for the neutral element says that π is a retraction. Therefore one may identify the algebra A with its image $e(A)$, which is a subalgebra of B. Define $M := \operatorname{Ker} \pi$ so that there is a vector spaces isomorphism $B = A \oplus M$ determined by the inclusion $e : A \hookrightarrow B$ and its retraction π. Since M is an ideal in B, the algebra A acts on M from both sides. Obviously, M with these actions is an A-bimodule, the bimodule axioms following from the associativity of B as in Example 2.1.1. It remains to show that $m'm'' = 0$ for all $m', m'' \in M$ which would imply that B is a trivial extension of A. Let us introduce the following notation.

For a morphism $f : (B', \pi') \to (B'', \pi'')$ of $\mathbf{k}$-splitting objects of $\mathsf{Ass}\,/A$ (i.e. objects with specific $\mathbf{k}$-vector space isomorphisms $B' \cong A \oplus M'$ and $B'' \cong A \oplus M''$ such that π' and π'' are the projections on the first summand) we denote by $\tilde{f} : M' \to M''$ the restriction $f|_{M'}$ followed by the projection $B'' \overset{\pi'}{\to} M''$. We call $\tilde{f}$ the *reduction* of f. Clearly, for every diagram of splitting objects in $\mathsf{Ass}\,/A$ there is the corresponding diagram of reductions in Ass.

The fibered product $(A \oplus M, \pi) \times (A \oplus M, \pi)$ in $\mathsf{Ass}\,/A$ is isomorphic to $A \oplus M \oplus M$ with the multiplication

$$(a', m_1', m_2')(a'', m_1'', m_2'') = (a'a'', a'm_1'' + m_1'a'' + m_1'm_1'', a'm_2'' + m_2'a'' + m_2'm_2'').$$

The neutrality of e implies the following diagram of reductions

$$\begin{tikzcd} 0 \oplus M \arrow[r, "\tilde{e} \times id_M"] \arrow[dr, equal, "\cong"'] & M \oplus M \arrow[d, "\tilde{\mu}"] & M \oplus 0 \arrow[l, "id_M \times \tilde{e}"'] \arrow[dl, equal, "\cong"] \\ & M & \end{tikzcd}$$

which in turn implies

$$\tilde{\mu}(0,m) = \tilde{\mu}(m,0) = m, \qquad \text{for all } m \in M.$$

Since μ is a morphism in Ass, it preserves the multiplication and so does its reduction $\tilde{\mu}$. We finally obtain

$$m' \cdot m'' = \tilde{\mu}(m',0) \cdot \tilde{\mu}(0,m'') = \tilde{\mu}((m',0)\cdot(0,m'')) = \tilde{\mu}(m' \cdot 0, 0 \cdot m'') = 0.$$

This finishes the proof. □

We have shown that objects of $(\mathsf{Ass}\,/A)_{ab}$ are precisely trivial extensions of A. Since there is an obvious equivalence between modules and trivial extensions, we obtain:

2.1.16 Theorem. The category $(\mathsf{Ass}\,/A)_{ab}$ is isomorphic to the category of A-modules.

Exercise 2.1.17. Prove analogous statements also for $(\mathsf{Com}\,/A)_{ab}$ and $(\mathsf{Lie}\,/L)_{ab}$.

Exercise 2.1.18. The only property of abelian group objects used in our proof of Proposition 2.1.15 was the existence of a neutral element for the multiplication. In fact, by analyzing our arguments we conclude that in $\mathsf{Ass}\,/A$, every object with a multiplication and a neutral element (i.e. a *monoid* in $\mathsf{Ass}\,/A$) is an abelian group object. Is this statement true in any comma category? If not, what special property of $\mathsf{Ass}\,/A$ makes it hold in this particular category?

2.2 Cohomology

Let A be an algebra, M an A-module. There are the following approaches to the "cohomology of A with coefficients in M."

(1) *Abelian cohomology* defined as $H^*(Lin(R_*, M))$, where R_* is a resolution of A in the category of A-modules.

(2) *Non-abelian cohomology* defined as $H^*(Der(\mathcal{F}_*, M))$, where $\mathcal{F}_*$ is a resolution of A in the category of algebras and $Der(-, M)$ denotes the space of derivations with coefficients in M.

(3) *Deformation cohomology* which is the subject of this note.

The adjective *(non)-abelian* reminds us that (1) is a derived functor in the abelian category of modules while cohomology (2) is a derived functor in the non-abelian category of algebras. Construction (1) belongs entirely into classical homological algebra [30], but (2) requires Quillen's theory of closed model categories [40]. Recall that in this note we work over a field of characteristics 0, over the integers one should take in (2) a suitable simplicial resolution [1]. Let us indicate the meaning of deformation cohomology in the case of associative algebras.

Let $V = \mathrm{Span}\{e_1, \ldots, e_d\}$ be a d-dimensional $\mathbf{k}$-vector space. Denote $Ass(V)$ the set of all associative algebra structures on V. Such a structure is determined by a bilinear map $\mu : V \otimes V \to V$. Relying on Einstein's convention, we write

$\mu(e_i, e_j) = \Gamma^l_{ij} e_l$ for some scalars $\Gamma^l_{ij} \in \mathbf{k}$. The associativity $\mu(e_i, \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), e_k)$ of μ can then be expressed as

$$\Gamma^r_{il}\Gamma^l_{jk} = \Gamma^l_{ij}\Gamma^r_{lk}, \qquad i, j, k, r = 1, \ldots, d.$$

These d^4 polynomial equations define an affine algebraic variety, which is just another way to view $Ass(V)$, since every point of this variety corresponds to an associative algebra structure on V. We call $Ass(V)$ the *variety of structure constants* of associative algebras.

The next step is to consider the quotient $Ass(V)/\,GL(V)$ of $Ass(V)$ modulo the action of the general linear group $GL(V)$ recalled in formula (2.10) below. However, $Ass(V)/\,GL(V)$ is no longer an affine variety, but only a (possibly singular) algebraic stack (in the sense of Grothendieck). One can remove singularities by replacing $Ass(V)$ by a smooth dg-scheme $\mathcal{M}$. Deformation cohomology is then the cohomology of the tangent space of this smooth dg-scheme [6, 8].

Still more general approach to deformation cohomology is based on considering a given category of algebras as the category of algebras over a certain PROP P and defining the deformation cohomology using a resolution of P in the category of PROPs [27, 34, 36]. When P is a Koszul quadratic operad, we get the *operadic cohomology* whose relation to deformations was studied in [3]. There is also an approach to deformations based on *triples* [11].

For associative algebras all the above approaches give the classical Hochschild cohomology (formula 3.2 of [30, §X.3]):

2.2.1 Definition. The *Hochschild cohomology* of an associative algebra A with coefficients in an A-module M is the cohomology of the complex:

$$0 \longrightarrow M \xrightarrow{\delta_{\rm Hoch}} C^1_{\rm Hoch}(A, M) \xrightarrow{\delta_{\rm Hoch}} \cdots \xrightarrow{\delta_{\rm Hoch}} C^n_{\rm Hoch}(A, M) \xrightarrow{\delta_{\rm Hoch}} \cdots$$

in which $C^n_{\rm Hoch}(A, M) := Lin(A^{\otimes n}, M)$, the space of n-multilinear maps from A to M. The coboundary $\delta = \delta_{\rm Hoch} : C^n_{\rm Hoch}(A, M) \to C^{n+1}_{\rm Hoch}(A, M)$ is defined by

$$\begin{aligned}\delta_{\rm Hoch} f(a_0 \otimes \ldots \otimes a_n) \;:=\; & (-1)^{n+1} a_0 f(a_1 \otimes \ldots \otimes a_n) + f(a_0 \otimes \ldots \otimes a_{n-1}) a_n \\ & + \sum_{i=0}^{n-1} (-1)^{i+n} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n),\end{aligned}$$

for $a_i \in A$. Denote $H^n_{\rm Hoch}(A, M) := H^n(C^*_{\rm Hoch}(A, M), \delta)$.

Exercise 2.2.2. Prove that $\delta^2_{\rm Hoch} = 0$.

2.2.3 Example. A simple computation shows that

– $H^0_{\rm Hoch}(A, M) = \{m \in M \mid am - ma = 0 \text{ for all } a \in A\}$,

– $H^1_{\rm Hoch}(A, M) = Der(A, M)/\,IDer(A, M)$, where $IDer(A, M)$ denotes the subspace of internal derivations, i.e. derivations of the form $\vartheta_m(a) = am - ma$ for $a \in A$ and some fixed $m \in M$. Slightly more difficult is to prove that

– $H^2_{\rm Hoch}(A, M)$ is the space of isomorphism classes of singular extensions of A by M [30, Theorem X.3.1].

2.3 Classical deformation theory

As everywhere in this note, we work over a field $\mathbf{k}$ of characteristics zero and $\otimes$ denotes the tensor product over $\mathbf{k}$. By a *ring* we will mean a commutative associative $\mathbf{k}$-algebra. Let us start with necessary preliminary notions.

2.3.1 Definition. Let R be a ring with unit e and $\omega : \mathbf{k} \to R$ the homomorphism given by $\omega(1) := e$. A homomorphism $\epsilon : R \to \mathbf{k}$ is an *augmentation* of R if $\epsilon\omega = id_{\mathbf{k}}$ or, diagrammatically,

$$\begin{tikzcd} R \arrow[r, "\epsilon"] & \mathbf{k} \\ \mathbf{k}. \arrow[u, "\omega"] \arrow[ur, "id"'] & \end{tikzcd}$$

The subspace $\overline{R} := \mathrm{Ker}\,\epsilon$ is called the *augmentation ideal* of R. The *indecomposables* of the augmented ring R are defined as the quotient $Q(R) := \overline{R}/\overline{R}^2$.

2.3.2 Example. The unital ring $\mathbf{k}[[t]]$ of formal power series with coefficients in $\mathbf{k}$ is augmented, with augmentation $\epsilon : \mathbf{k}[[t]] \to \mathbf{k}$ given by $\epsilon(\sum_{i\in\mathbb{N}_0} a_i t^i) := a_0$. The unital ring $\mathbf{k}[t]$ of polynomials with coefficients in $\mathbf{k}$ is augmented by $\epsilon(f) := f(0)$, for $f \in \mathbf{k}[t]$. The truncated polynomial rings $\mathbf{k}[t]/(t^n)$, $n \geq 1$, are also augmented, with the augmentation induced by the augmentation of $\mathbf{k}[t]$.

2.3.3 Example. Recall that the group ring $\mathbf{k}[G]$ of a finite group G with unit e is the space of all formal linear combinations $\sum_{g\in G} a_g g$, $a_g \in \mathbf{k}$, with the multiplication

$$(\textstyle\sum_{g\in G} a'_g g)(\sum_{g\in G} a''_g g) := \sum_{g\in G} \sum_{uv=g} a'_u a''_v g$$

and unit $1e$. The ring $\mathbf{k}[G]$ is augmented by $\epsilon : \mathbf{k}[G] \to \mathbf{k}$ given as

$$\epsilon(\textstyle\sum_{g\in G} a_g g) := \sum_{g\in G} a_g.$$

2.3.4 Example. A rather trivial example of a ring that does not admit an augmentation is provided by any proper extension $K \supsetneq \mathbf{k}$ of $\mathbf{k}$. If an augmentation $\epsilon : K \to \mathbf{k}$ exists, then $\mathrm{Ker}\,\epsilon$ is, as an ideal in a field, trivial, which implies that ϵ is injective, which would imply that $K = \mathbf{k}$ contradicting the assumption $K \neq \mathbf{k}$.

Exercise 2.3.5. If $\sqrt{-1} \notin \mathbf{k}$, then $\mathbf{k}[x]/(x^2+1)$ admits no augmentation.

In the rest of this section, R will be an augmented unital ring with an augmentation $\epsilon : R \to \mathbf{k}$ and the unit map $\omega : \mathbf{k} \to R$. By a module we will understand a *left* module.

2.3.6 Remark. A unital augmented ring R is a $\mathbf{k}$-bimodule, with the bimodule structure induced by the unit map ω in the obvious manner. Likewise, $\mathbf{k}$ is an R bimodule, with the structure induced by ϵ. If V is a $\mathbf{k}$-module, then $R \otimes V$ is an R-module, with the action $r'(r'' \otimes v) := r'r'' \otimes v$, for $r', r'' \in R$ and $v \in V$.

2.3.7 Definition. Let V be a $\mathbf{k}$-vector space and R a unital $\mathbf{k}$-ring. The *free R-module* generated by V is an R-module $R\langle V\rangle$ together with a $\mathbf{k}$-linear map $\iota : V \to R\langle V\rangle$ with the property that for every R-module W and a $\mathbf{k}$-linear map $V \stackrel{\varphi}{\to} W$, there exists a unique R-linear map $\Phi : R\langle V\rangle \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & R\langle V\rangle \arrow[d, dashed, "\Phi"] \\ & W. \end{tikzcd}$$

This universal property determines the free module $R\langle V\rangle$ uniquely up to isomorphism. A concrete model is provided by the R-module $R \otimes V$ recalled in Remark 2.3.6.

2.3.8 Definition. Let W be an R-module. The *reduction* of W is the $\mathbf{k}$-module $\overline{W} := \mathbf{k} \otimes_R W$, with the $\mathbf{k}$-action given by $k'(k'' \otimes_R w) := k'k'' \otimes_R w$, for $k', k'' \in \mathbf{k}$ and $w \in W$.

One clearly has $\mathbf{k}$-module isomorphisms $\overline{W} \cong W/\overline{R}W$ and $\overline{R\langle V\rangle} \cong V$. The reduction clearly defines a functor from the category of R-modules to the category of $\mathbf{k}$-modules.

2.3.9 Proposition. If B is an associative R-algebra, then the reduction $\overline{B}$ is a $\mathbf{k}$-algebra, with the structure induced by the algebra structure of B.

Proof. Since $\overline{B} \simeq B/\overline{R}B$, it suffices to verify that $\overline{R}B$ is a two-sided ideal in B. But this is simple. For $r \in \overline{R}$, $b', b'' \in B$ one sees that $\mu(rb', b'') = r\mu(b', b'') \in \overline{R}B$, which shows that $\mu(\overline{R}B, B) \subset \overline{R}B$. The right multiplication by elements of $\overline{R}B$ is discussed similarly. $\square$

2.3.10 Definition. Let A be an associative $\mathbf{k}$-algebra and R an augmented unital ring. An *R-deformation* of A is an associative R-algebra B together with a $\mathbf{k}$-algebra isomorphism $\alpha : \overline{B} \to A$.

Two R-deformations $(B', \overline{B}' \stackrel{\alpha'}{\to} A)$ and $(B'', \overline{B}'' \stackrel{\alpha''}{\to} A)$ of A are *equivalent* if there exists an R-algebra isomorphism $\phi : B' \to B''$ such that $\overline{\phi} = \alpha''^{-1} \circ \alpha'$.

There is probably not much to be said about R-deformations without additional assumptions on the R-module B. In this note we assume that B *is a free R-module* or, equivalently, that

$$B \cong R \otimes A \text{ (isomorphism of } R\text{-modules).} \tag{2.4}$$

The above isomorphism identifies A with the $\mathbf{k}$-linear subspace $1 \otimes A$ of B and $A \otimes A$ with the $\mathbf{k}$-linear subspace $(1 \otimes A) \otimes (1 \otimes A)$ of $B \otimes B$.

Another assumption frequently used in algebraic geometry [19, Section III.§9] is that the R-module B is flat which, by definition, means that the functor $B \otimes_R -$ is left exact. One then speaks about *flat deformations.*

In what follows, R will be either a power series ring $\mathbf{k}[[t]]$ or a truncation of the polynomial ring $\mathbf{k}[t]$ by an ideal generated by a power of t. All these rings are local Noetherian rings therefore a finitely generated R-module is flat if and only if it is free (see Exercise 7.15, Corollary 10.16 and Corollary 10.27 of [2]). It is clear that B in Definition 2.3.10 is finitely generated over R if and only if A finitely generated as a $\mathbf{k}$-vector space. Therefore, for A finitely generated over $\mathbf{k}$, free deformations are the same as the flat ones.

The R-linearity of deformations implies the following simple lemma. Recall that all deformations in this sections satisfy (2.4).

2.3.11 Lemma. Let $B = (B, \mu)$ be a deformation as in Definition 2.3.10. Then the multiplication μ in B is determined by its restriction to $A \otimes A \subset B \otimes B$. Likewise, every equivalence of deformations $\phi : B' \to B''$ is determined by its restriction to $A \subset B$.

Proof. By (2.4), each element of B is a finite sum of elements of the form ra, $r \in R$ and $a \in A$, and $\mu(ra, sb) = rs\mu(a, b)$ by the R-bilinearity of μ for each $a, b \in A$ and $r, s \in R$. This proves the first statement. The second part of the lemma is equally obvious. $\square$

The following proposition will also be useful.

2.3.12 Proposition. Let $B' = (B', \overline{B}' \stackrel{\alpha'}{\to} A)$ and $B'' = (B'', \overline{B}'' \stackrel{\alpha''}{\to} A)$ be R-deformations of an associative algebra A. Assume that R is either a local Artinian ring or a complete local ring. Then every homomorphism $\phi : B' \to B''$ of R-algebras such that $\overline{\phi} = \alpha''^{-1} \circ \alpha'$ is an equivalence of deformations.

Sketch of proof. We must show that ϕ is invertible. One may consider a formal inverse of ϕ in the form of an expansion in the successive quotients of the maximal ideal. If R is Artinian, this formal inverse has in fact only finitely many terms and hence it is an actual inverse of ϕ. If R is complete, this formal expansion is convergent. $\square$

We leave as an exercise to prove that each R-deformation of A in the sense of Definition 2.3.10 is equivalent to a deformation of the form $(B, \overline{B} \stackrel{can}{\to} A)$, with $B = R \otimes A$ (equality of $\mathbf{k}$-vector spaces) and *can* the canonical map $\overline{B} = \mathbf{k} \otimes_R (R \otimes A) \to A$ given by

$$can(1 \otimes_R (1 \otimes a)) := a, \text{ for } a \in A.$$

Two deformations (B, μ') and (B, μ'') of this type are equivalent if and only if there exists an R-algebra isomorphism $\phi : (B, \mu') \to (B, \mu'')$ which reduces, under the identification $can : \overline{B} \to A$, to the identity $id_A : A \to A$. Since we will be interested only in equivalence classes of deformations, we will assume that all deformations are of the above special form.

2.3.13 Definition. A *formal deformation* is a deformation, in the sense of Definition 2.3.10, over the complete local augmented ring $\mathbf{k}[[t]]$.

Exercise 2.3.14. Is $\mathbf{k}[x, y, t]/(x^2 + txy)$ a formal deformation of $\mathbf{k}[x, y]/(x^2)$?

2.3.15 Theorem. A formal deformation B of A is given by a family

$$\{\mu_i : A \otimes A \to A \mid i \in \mathbb{N}\}$$

satisfying $\mu_0(a,b) = ab$ (the multiplication in A) and
$(D_k)\quad \sum_{i+j=k,\ i,j\geq 0} \mu_i(\mu_j(a,b),c) = \sum_{i+j=k,\ i,j\geq 0} \mu_i(a,\mu_j(b,c))\quad$ for all $a,b,c \in A$

for each $k \geq 1$.

Proof. By Lemma 2.3.11, the multiplication μ in B is determined by its restriction to $A \otimes A$. Now expand $\mu(a,b)$, for $a,b \in A$, into the power series

$$\mu(a,b) = \mu_0(a,b) + t\mu_1(a,b) + t^2\mu_2(a,b) + \cdots$$

for some $\mathbf{k}$-bilinear functions $\mu_i : A \otimes A \to A$, $i \geq 0$. Obviously, μ_0 must be the multiplication in A. It is easy to see that μ is associative if and only if (D_k) are satisfied for each $k \geq 1$. $\square$

2.3.16 Remark. Observe that (D_1) reads

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

and says precisely that $\mu_1 \in \mathit{Lin}(A^{\otimes 2}, A)$ is a Hochschild cocycle, $\delta_{\mathrm{Hoch}}(\mu_1) = 0$, see Definition 2.2.1.

2.3.17 Example. Let us denote by H the group

$$H := \{u = id_A + \phi_1 t + \phi_2 t^2 + \cdots \mid \phi_i \in \mathit{Lin}(A,A)\},$$

with the multiplication induced by the composition of linear maps. By Proposition 2.3.12, formal deformations $\mu' = \mu_0 + \mu_1' t + \mu_2' t^2 + \cdots$ and $\mu'' = \mu_0 + \mu_1'' t + \mu_2'' t^2 + \cdots$ of μ_0 are equivalent if and only if

$$u \circ (\mu_0 + \mu_1' t + \mu_2' t^2 + \cdots) = (\mu_0 + \mu_1'' t + \mu_2'' t^2 + \cdots) \circ (u \otimes u). \tag{2.5}$$

We close this section by formulating some classical statements [13, 14, 15] which reveal the connection between deformation theory of associative algebras and the Hochschild cohomology. As suggested by Remark 2.3.16, the first natural object to look at is μ_1. This motivates the following

2.3.18 Definition. An *infinitesimal deformation* of an algebra A is a D-deformation of A, where

$$D := \mathbf{k}[t]/(t^2)$$

is the local Artinian ring of *dual numbers*.

2.3.19 Remark. One can easily prove an analog of Theorem 2.3.15 for infinitesimal deformations, namely that there is a one-to-one correspondence between infinitesimal deformations of A and $\mathbf{k}$-linear maps $\mu_1 : A \otimes A \to A$ satisfying (D_1), that is, by Remark 2.3.16, Hochschild 2-cocycles of A with coefficients in itself. But we can formulate a stronger statement:

2.3.20 Theorem. There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of A and the second Hochschild cohomology $H^2_{\mathrm{Hoch}}(A,A)$ of A with coefficients in itself.

Proof. Consider two infinitesimal deformations of A given by multiplications $*'$ and $*''$, respectively. As we observed in Remark 2.3.19, these deformations are determined by Hochschild 2-cocycles $\mu_1', \mu_1'' : A \otimes A \to A$, via equations

$$\begin{aligned} a *' b &= ab + t\mu_1'(a,b) \\ a *'' b &= ab + t\mu_1''(a,b), \quad a, b \in A. \end{aligned} \tag{2.6}$$

Each equivalence ϕ of deformations $*'$ and $*''$ is determined by a $\mathbf{k}$-linear map $\phi_1 : A \to A$,

$$\phi(a) = a + t\phi_1(a), \quad a \in A, \tag{2.7}$$

the invertibility of such a ϕ follows from Proposition 2.3.12 but can easily be checked directly. Substituting (2.6) and (2.7) into

$$\phi(a *' b) = \phi(a) *'' \phi(b), \quad a, b \in A, \tag{2.8}$$

one obtains

$$\phi(ab + t\mu_1'(a,b)) = (a + t\phi_1(a)) *'' (b + t\phi_1(b))$$

which can be further expanded into

$$ab + t\phi(\mu_1'(a,b)) = ab + t(a\phi_1(b)) + t(\phi_1(a)b) + t\mu_1''(a + t\phi_1(a), b + t\phi_1(b))$$

so, finally,

$$ab + t\mu_1'(a,b) = ab + t(a\phi_1(b) + \phi_1(a)b) + t\mu_1''(a,b).$$

Comparing the t-linear terms, we see that (2.8) is equivalent to

$$\mu_1'(a,b) = \delta_{\mathrm{Hoch}}\phi_1(a,b) + \mu_1''(a,b).$$

We conclude that infinitesimal deformations given by $\mu_1', \mu_1'' \in C^2_{\mathrm{Hoch}}(A,A)$ are equivalent if and only if they differ by a coboundary, that is, if and only if $[\mu_1'] = [\mu_1'']$ in $H^2_{\mathrm{Hoch}}(A,A)$. $\square$

Another classical result is:

2.3.21 Theorem. Let A be an associative algebra such that $H^2_{\mathrm{Hoch}}(A,A) = 0$. Then all formal deformations of A are equivalent to A.

Sketch of proof. If $*', *''$ are two formal deformations of A, one can, using the assumption $H^2_{\mathrm{Hoch}}(A,A) = 0$, as in the proof of Theorem 2.3.20 find a $\mathbf{k}$-linear map $\phi_1 : A \to A$ defining an equivalence of $(B, *')$ to $(B, *'')$ modulo t^2. Repeating this process, one ends up with an equivalence $\phi = id + t\phi_1 + t^2\phi_2 + \cdots$ of formal deformations $*'$ and $*''$. $\square$

2.3.22 Definition. An *n-deformation* of an algebra A is an R-deformation of A for R the local Artinian ring $\mathbf{k}[t]/(t^{n+1})$.

We have the following version of Theorem 2.3.15 whose proof is obvious.

2.3.23 Theorem. An n-deformation of A is given by a family

$$\{\mu_i : A \otimes A \to A \mid 1 \le i \le n\}$$

of $\mathbf{k}$-linear maps satisfying (D_k) of Theorem 2.3.15 for $1 \le k \le n$.

2.3.24 Definition. An $(n+1)$-deformation of A given by $\{\mu_1, \ldots, \mu_{n+1}\}$ is called an *extension* of the n-deformation given by $\{\mu_1, \ldots, \mu_n\}$.

Let us rearrange (D_{n+1}) into

$$
\begin{aligned}
-a\mu_{n+1}(b, c) + \mu_{n+1}(ab, c) - \mu_{n+1}(a, bc) + \mu_{n+1}(a, b)c = \\
= \sum_{i+j=n+1,\ i,j>0} (\mu_i(a, \mu_j(b, c)) - \mu_i(\mu_j(a, b), c))
\end{aligned}
$$

Denote the trilinear function in the right-hand side by $\mathfrak{O}_n$ and interpret it as an element of $C^3_{\mathrm{Hoch}}(A, A)$,

$$
\mathfrak{O}_n := \sum_{i+j=n+1,\ i,j>0} (\mu_i(a, \mu_j(b, c)) - \mu_i(\mu_j(a, b), c)) \in C^3_{\mathrm{Hoch}}(A, A). \tag{2.9}
$$

Using the Hochschild differential recalled in Definition 2.2.1, one can rewrite (D_{n+1}) as

$$
\delta_{\mathrm{Hoch}}(\mu_{n+1}) = \mathfrak{O}_n.
$$

We conclude that, if an n-deformation extends to an $(n + 1)$-deformation, then $\mathfrak{O}_n$ is a Hochschild coboundary. In fact, one can prove:

2.3.25 Theorem. For any n-deformation, the Hochschild cochain $\mathfrak{O}_n \in C^3_{\mathrm{Hoch}}(A, A)$ defined in (2.9) is a cocycle, $\delta_{\mathrm{Hoch}}(\mathfrak{O}_n) = 0$. Moreover, $[\mathfrak{O}_n] = 0$ in $H^3_{\mathrm{Hoch}}(A, A)$ if and only if the n-deformation $\{\mu_1, \ldots, \mu_n\}$ extends into some $(n+1)$-deformation.

Proof. Straightforward. □

Geometric deformation theory. Let us turn our attention back to the variety of structure constants $Ass(V)$ recalled in Section 2.2, page 13. Elements of $Ass(V)$ are associative $\mathbf{k}$-linear multiplications $\cdot : V \otimes V \to V$ and there is a natural left action $\cdot \mapsto \cdot_\phi$ of $GL(V)$ on $Ass(V)$ given by

$$
a \cdot_\phi b := \phi(\phi^{-1}(a) \cdot \phi^{-1}(b)), \tag{2.10}
$$

for $a, b \in V$ and $\phi \in GL(V)$. We assume that V is finite dimensional.

2.3.26 Definition. Let A be an algebra with the underlying vector space V interpreted as a point in the variety of structure constants, $A \in Ass(V)$. The algebra A is called (geometrically) *rigid* if the $GL(V)$-orbit of A in $Ass(V)$ is Zarisky-open.

Let us remark that, if $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then, by [39, Proposition 17.1], the $GL(V)$-orbit of A in $Ass(V)$ is Zarisky-open if and only if it is (classically) open. The following statement whose proof can be found in [39, § 5] specifies the relation between the Hochschild cohomology and geometric rigidity, compare also Propositions 1 and 2 of [9].

2.3.27 Theorem. Suppose that the ground field is algebraically closed.
(i) If $H^2_{\mathrm{Hoch}}(A, A) = 0$ then A is rigid, and
(ii) if $H^3_{\mathrm{Hoch}}(A, A) = 0$ then A is rigid if and only if $H^2_{\mathrm{Hoch}}(A, A) = 0$.

Three concepts of rigidity. One says that an associative algebra is *infinitesimally rigid* if A has only trivial (i.e. equivalent to A) infinitesimal deformations. Likewise, A is *analytically rigid*, if all formal deformations of A are trivial.

By Theorem 2.3.20, A is infinitesimally rigid if and only if $H^2_{\mathrm{Hoch}}(A, A) = 0$. Together with Theorem 2.3.21 this establishes the first implication in the following display which in fact holds over fields of arbitrary characteristic

$$\text{infinitesimal rigidity} \Longrightarrow \text{analytic rigidity} \Longrightarrow \text{geometric rigidity.}$$

The second implication in the above display is [16, Theorem 3.2]. Theorem 7.1 of the same paper then says that in characteristic zero, the analytic and geometric rigidity are equivalent concepts:

$$\text{analytic rigidity} \overset{\text{char. } 0}{\Longleftrightarrow} \text{geometric rigidity.}$$

Valued deformations. The authors of [18] studied R-deformations of finite-dimensional algebras in the case when R was a valuation ring [2, Chapter 5]. In particular, they considered deformations over the non-standard extension $\mathbb{C}^*$ of the field of complex numbers, and called these $\mathbb{C}^*$-deformations *perturbations*. They argued, in [18, Theorem 4], that an algebra A admits only trivial perturbations if and only if it is geometrically rigid.

2.3.28 Remark. An analysis parallel to the one presented in this section can be made for any class of "reasonable" algebras, where "reasonable" are algebras over quadratic Koszul operads [38, Section II.3.3] for which the deformation cohomology is given by a "standard construction." Let us emphasize that most of "classical" types of algebras (Lie, associative, associative commutative, Poisson, etc.) are "reasonable." See also [3, 4].

2.4 Structures of (co)associative (co)algebras

Let V be a $\mathbf{k}$-vector space. In this section we recall, in Theorems 2.4.16 and 2.4.21, the following important correspondence between (co)algebras and differentials:

$$\begin{gathered}\{\text{coassociative coalgebra structures on the vector space } V\} \\ \updownarrow \\ \{\text{quadratic differentials on the free associative algebra generated by } V\}.\end{gathered}$$

and its dual version:

$$\begin{gathered}\{\text{associative algebras on the vector space } V\} \\ \updownarrow \\ \{\text{quadratic differentials on the ``cofree'' coassociative coalgebra cogenerated by } V\}.\end{gathered}$$

The reason why we put 'cofree' into parentheses will become clear later in this section. Similar correspondences exist for any "reasonable" (in the sense explained in Remark 2.3.28) class of algebras, see [12, Theorem 8.2]. We will in fact need only the second correspondence but, since it relies on coderivations of "cofree" coalgebras, we decided to start with the first one which is simpler to explain.

2.4.1 Definition. The *free associative algebra* generated by a vector space W is an associative algebra $\mathcal{A}(W) \in \mathsf{Ass}$ together with a linear map $W \to \mathcal{A}(W)$ having the following property:

For every $A \in \mathsf{Ass}$ and a linear map $W \xrightarrow{\varphi} A$, there exists a unique algebra homomorphism $\mathcal{A}(W) \to A$ making the diagram:

$$\begin{tikzcd} W \arrow[r] \arrow[dr, "\varphi"'] & \mathcal{A}(W) \arrow[d, dashed] \\ & A \end{tikzcd}$$

commutative.

The free associative algebra on W is uniquely determined up to isomorphism. An example is provided by the *tensor algebra* $T(W) := \bigoplus_{n=1}^{\infty} W^{\otimes n}$ with the inclusion $W = W^{\otimes 1} \hookrightarrow T(W)$. There is a natural grading on $T(W)$ given by the number of tensor factors,

$$T(W) = \bigoplus_{n=0}^{\infty} T^n(W),$$

where $T^n(W) := W^{\otimes n}$ for $n \geq 1$ and $T^0(W) := 0$. Let us emphasize that the tensor algebra as defined above is *nonunital*, the unital version can be obtained by taking $T^0(W) := \mathbf{k}$.

2.4.2 Convention. We are going to consider graded algebraic objects. Our choice of signs will be dictated by the principle that whenever we commute two "things" of degrees p and q, respectively, we multiply the sign by $(-1)^{pq}$. This rule is sometimes called the *Koszul sign convention*. As usual, non-graded (classical) objects will be, when necessary, considered as graded ones concentrated in degree 0.

Let $f' : V' \to W'$ and $f'' : V'' \to W''$ be homogeneous maps of graded vector spaces. The Koszul sign convention implies that the value of $(f' \otimes f'')$ on the product $v' \otimes v'' \in V' \otimes V''$ of homogeneous elements equals

$$(f' \otimes f'')(v' \otimes v'') := (-1)^{\deg(f'')\deg(v')} f'(v') \otimes f''(v'').$$

In fact, the Koszul sign convention is determined by the above rule for evaluation.

2.4.3 Definition. Assume $V = V^*$ is a graded vector space, $V = \bigoplus_{i \in \mathbb{Z}} V^i$. The *suspension operator* $\uparrow$ assigns to V the graded vector space $\uparrow V$ with $\mathbb{Z}$-grading $(\uparrow V)^i := V^{i-1}$. There is a natural degree $+1$ map $\uparrow\colon V \to \uparrow V$ that sends $v \in V$ into its suspended copy $\uparrow v \in \uparrow V$. Likewise, the *desuspension operator* $\downarrow$ changes the grading of V according to the rule $(\downarrow V)^i := V^{i+1}$. The corresponding degree -1 map $\downarrow\colon V \to \downarrow V$ is defined in the obvious way. The suspension (resp. the desuspension) of V is sometimes also denoted sV or $V[-1]$ (resp. $s^{-1}V$ or $V[1]$).

2.4.4 Example. If V is an un-graded vector space, then $\uparrow V$ is V placed in degree $+1$ and $\downarrow V$ is V placed in degree -1.

2.4.5 Remark. In the "superworld" of $\mathbb{Z}_2$-graded objects, the operators $\uparrow$ and $\downarrow$ agree and coincide with the *parity change operator*.

Exercise 2.4.6. Show that the Koszul sign convention implies $(\downarrow \otimes \downarrow) \circ (\uparrow \otimes \uparrow) = -\, id$ or, more generally,

$$\downarrow^{\otimes n} \circ \uparrow^{\otimes n} = \uparrow^{\otimes n} \circ \downarrow^{\otimes n} = (-1)^{\frac{n(n-1)}{2}}\, id$$

for an arbitrary $n \geq 1$.

2.4.7 Definition. A *derivation* of an associative algebra A is a linear map $\theta : A \to A$ satisfying the *Leibniz rule*

$$\theta(ab) = \theta(a)b + a\theta(b)$$

for every $a, b \in A$. Denote $Der(A)$ the set of all derivations of A.

We will in fact need a graded version of the above definition:

2.4.8 Definition. A *degree d derivation* of a $\mathbb{Z}$-graded algebra A is a degree d linear map $\theta : A \to A$ satisfying the graded Leibniz rule

$$\theta(ab) = \theta(a)b + (-1)^{d|a|} a\theta(b) \tag{2.11}$$

for every homogeneous element $a \in A$ of degree $|a|$ and for every $b \in A$. We denote $Der^d(A)$ the set of all degree d derivations of A.

Exercise 2.4.9. Let $\mu : A \otimes A \to A$ be the multiplication of A. Prove that (2.11) is equivalent to

$$\theta\mu = \mu(\theta \otimes id) + \mu(id \otimes \theta).$$

Observe namely how the signs in the right hand side of (2.11) are dictated by the Koszul convention.

2.4.10 Proposition. Let W be a graded vector space and $T(W)$ the tensor algebra generated by W with the induced grading. For any d, there is a natural isomorphism

$$Der^d(T(W)) \cong Lin^d(W, T(W)), \tag{2.12}$$

where $Lin^d(-,-)$ denotes the space of degree d $\mathbf{k}$-linear maps.

Proof. Let $\theta \in Der^d(T(W))$ and $f := \theta|_W : W \to T(W)$. The Leibniz rule (2.11) implies that, for homogeneous elements $w_i \in W$, $1 \leq i \leq n$,

$$\begin{aligned}\theta(w_1 \otimes \ldots \otimes w_n) &= f(w_1) \otimes w_2 \otimes \ldots \otimes w_n + (-1)^{d|w_1|} w_1 \otimes f(w_2) \otimes \ldots \otimes w_n + \cdots \\ &= \sum_{i=1}^{n} (-1)^{d(|w_1|+\cdots+|w_{i-1}|)} w_1 \otimes \ldots \otimes f(w_i) \otimes \ldots \otimes w_n\end{aligned}$$

which reveals that θ is determined by its restriction f on W. On the other hand, given a degree d linear map $f : W \to T(W)$, the above formula clearly defines a derivation $\theta \in Der^d(T(W))$. The correspondence

$$Der^d(T(W)) \ni \theta \longleftrightarrow f := \theta|_W \in Lin^d(W, T(W))$$

is the required isomorphism (2.12). $\square$

Exercise 2.4.11. Let $\theta \in Der^d(T(W))$, $f := \theta|_V$ and $x \in T^2(W)$. Prove that

$$\theta(x) = (f \otimes id + id \otimes f)(x).$$

2.4.12 Definition. A derivation $\theta \in Der^d(T(W))$ is called *quadratic* if $\theta(W) \subset T^2W$. A degree 1 derivation θ is a *differential* if $\theta^2 = 0$.

Exercise 2.4.13. Prove that the isomorphism of Proposition 2.4.10 restricts to

$$Der_2^d(T(W)) \cong Lin^d(W, T^2(W)),$$

where $Der_2^d(T(W))$ is the space of all quadratic degree d derivations of $T(W)$.

2.4.14 Definition. Let V be a vector space. A *coassociative coalgebra* structure on V is given by a linear map $\Delta : V \to V \otimes V$ satisfying

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

(the *coassociativity*).

We will need, in Section 2.6, also a cocommutative version of coalgebras:

2.4.15 Definition. A coassociative coalgebra $A = (V, \Delta)$ as in Definition 2.4.14 is *cocommutative* if

$$T\Delta = \Delta$$

with the swapping map $T : V \otimes V \to V \otimes V$ given by

$$T(v' \otimes v'') := (-1)^{|v'||v''|} v'' \otimes v'$$

for homogeneous $v', v'' \in V$.

2.4.16 Theorem. Let V be a (possibly graded) vector space. Denote $Coass(V)$ the set of all coassociative coalgebra structures on V and $Diff_2^1(T(\uparrow V))$ the set of all quadratic differentials on the tensor algebra $T(\uparrow V)$. Then there is a natural isomorphism

$$Coass(V) \cong Diff_2^1(T(\uparrow V)).$$

Proof. Let $\chi \in Diff_2^1(T(\uparrow V))$. Put $f := \chi|_{\uparrow V}$ so that f is a degree $+1$ map $\uparrow V \to \uparrow V \otimes \uparrow V$. By Exercise 2.4.11 (with $W := \uparrow V$, $\theta := \chi$ and $x := f(\uparrow v)$),

$$0 = \chi^2(\uparrow v) = \chi(f(\uparrow v)) = (f \otimes id + id \otimes f)(f(\uparrow v))$$

for every $v \in V$, therefore

$$(f \otimes id + id \otimes f)f = 0. \tag{2.13}$$

We have clearly described a one-to-one correspondence between quadratic differentials $\chi \in Diff_2^1(T(\uparrow V))$ and degree $+1$ linear maps $f \in Lin^1(\uparrow V, T^2(\uparrow V))$ satisfying (2.13).

Given $f : \uparrow V \to \uparrow V \otimes \uparrow V$ as above, define the map $\Delta : V \to V \otimes V$ by the commutative diagram

$$\begin{tikzcd} \uparrow V \arrow[r, "f"] & \uparrow V \otimes \uparrow V \\ V \arrow[u, "\uparrow"] \arrow[r, "\Delta"] & V \otimes V \arrow[u, "\uparrow \otimes \uparrow"'] \end{tikzcd}$$

i.e., by Exercise 2.4.6,

$$\Delta := (\uparrow \otimes \uparrow)^{-1} \circ f \circ \uparrow = -(\downarrow \otimes \downarrow) \circ f \circ \uparrow.$$

Let us show that (2.13) is equivalent to the coassociativity of Δ. We have

$$\begin{aligned}(\Delta \otimes id)\Delta &= (-(\downarrow \otimes \downarrow) f \uparrow \otimes\, id)\,(-(\downarrow \otimes \downarrow) f \uparrow) = ((\downarrow \otimes \downarrow) f \uparrow \otimes\, id)\,(\downarrow \otimes \downarrow) f \uparrow \\ &= ((\downarrow \otimes \downarrow) f \otimes \downarrow) f \uparrow = -(\downarrow \otimes \downarrow \otimes \downarrow)(f \otimes id) f \uparrow.\end{aligned}$$

The minus sign in the last term appeared because we interchanged f (a "thing" of degree $+1$) with $\downarrow$ (a "thing" of degree -1). Similarly

$$\begin{aligned}(id \otimes \Delta)\Delta &= (id \otimes (-(\downarrow \otimes \downarrow)) f \uparrow)\,(-(\downarrow \otimes \downarrow) f \uparrow) = (id \otimes (\downarrow \otimes \downarrow) f \uparrow)\,(\downarrow \otimes \downarrow) f \uparrow \\ &= (\downarrow \otimes (\downarrow \otimes \downarrow) f) f \uparrow = (\downarrow \otimes \downarrow \otimes \downarrow)(id \otimes f) f \uparrow,\end{aligned}$$

so (2.13) is indeed equivalent to $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$. This finishes the proof. $\square$

We are going to dualize Theorem 2.4.16 to get a description of associative algebras, not *co*algebras. First, we need a dual version of the tensor algebra:

2.4.17 Definition. The underlying vector space $T(W)$ of the tensor algebra with the comultiplication $\Delta : T(W) \to T(W) \otimes T(W)$ defined by

$$\Delta(w_1 \otimes \ldots \otimes w_n) := \sum_{i=1}^{n-1} (w_1 \otimes \ldots \otimes w_i) \otimes (w_{i+1} \otimes \ldots \otimes w_n)$$

is a coassociative coalgebra denoted ${}^cT(W)$ and called the *tensor coalgebra*.

Warning. Contrary to general belief, the coalgebra ${}^cT(W)$ with the projection ${}^cT(W) \to W$ is not cofree in the category of coassociative coalgebras! Cofree coalgebras (in the sense of the obvious dual of Definition 2.4.1) are surprisingly complicated objects [10, 43, 20]. The coalgebra ${}^cT(W)$ is, however, cofree in the subcategory of coaugmented nilpotent coalgebras [38, Section II.3.7]. This will be enough for our purposes.

In the following dual version of Definition 2.4.8 we use *Sweedler's convention* expressing the comultiplication in a coalgebra C as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $c \in C$.

2.4.18 Definition. A *degree d coderivation* of a $\mathbb{Z}$-graded coalgebra C is a linear degree d map $\theta : C \to C$ satisfying the dual Leibniz rule

$$\Delta\theta(c) = \sum \theta(c_{(1)}) \otimes c_{(2)} + \sum (-1)^{d|c_{(1)}|} c_{(1)} \otimes \theta(c_{(2)}), \tag{2.14}$$

for every $c \in C$. Denote the set of all degree d coderivations of C by $CoDer^d(C)$.

As in Exercise 2.4.9 one easily proves that (2.14) is equivalent to

$$\Delta\theta = (\theta \otimes id)\Delta + (id \otimes \theta)\Delta.$$

Let us prove the dual of Proposition 2.4.10:

2.4.19 Proposition. Let W be a graded vector space. For any d, there is a natural isomorphism

$$CoDer^d({}^cT(W)) \cong Lin^d(T(W), W). \tag{2.15}$$

Proof. For $\theta \in CoDer^d(T(W))$ and $s \geq 1$ denote $f_s \in Lin^d(T^s(W), W)$ the composition

$$f_s : T^s(W) \xrightarrow{\theta|_{T^s(W)}} {}^cT(W) \xrightarrow{\text{proj.}} W. \tag{2.16}$$

The dual Leibniz rule (2.14) implies that, for $w_1, \ldots, w_n \in W$ and $n \geq 0$,

$$\theta(w_1 \otimes \ldots \otimes w_n) :=$$

$$\sum_{s \geq 1} \sum_{i=1}^{n-s+1} (-1)^{d(|w_1|+\cdots+|w_{i-1}|)} w_1 \otimes \ldots \otimes w_{i-1} \otimes f_s(w_i \otimes \ldots \otimes w_{i+s-1}) \otimes w_{i+s} \otimes \ldots \otimes w_n,$$

which shows that θ is uniquely determined by $f := f_0 + f_1 + \cdots \in Lin^d(T(W), W)$. On the other hand, it is easy to verify that for any map $f \in Lin^d(T(W), W)$ decomposed into the sum of its homogeneous components, the above formula defines a coderivation $\theta \in CoDer^d(T(W))$. This finishes the proof. $\square$

2.4.20 Definition. The composition $f_s : T^s(W) \to W$ defined in (2.16) is called the sth *corestriction* of the coderivation θ. A coderivation $\theta \in CoDer^d(T(W))$ is *quadratic* if its sth corestriction is non-zero only for $s = 2$. A degree 1 coderivation θ is a *differential* if $\theta^2 = 0$.

Let us finally formulate a dual version of Theorem 2.4.16.

2.4.21 Theorem. Let V be a graded vector space. Denote $CoDiff^1_2({}^cT(\downarrow V))$ the set of all quadratic differentials on the tensor coalgebra ${}^cT(\downarrow V)$. One then has a natural isomorphism

$$Ass(V) \cong CoDiff^1_2({}^cT(\downarrow V)). \tag{2.17}$$

Proof. Let $\chi \in CoDiff^1_2({}^cT(\downarrow V))$ and $f : \downarrow V \otimes \downarrow V \to \downarrow V$ be the 2nd corestriction of χ. Define $\mu : V \otimes V \to V$ by the diagram

$$\begin{tikzcd} \downarrow V \otimes \downarrow V \arrow[r, "f"] & \downarrow V \\ V \otimes V \arrow[u, "\downarrow \otimes \downarrow"] \arrow[r, "\mu"] & V. \arrow[u, "\downarrow"'] \end{tikzcd}$$

The correspondence $\chi \leftrightarrow \mu$ is then the required isomorphism. This can be verified by dualizing the steps of the proof of Theorem 2.4.16 so we can safely leave the details to the reader. $\square$

2.5 dg-Lie algebras and the Maurer-Cartan equation

2.5.1 Definition. A graded Lie algebra is a $\mathbb{Z}$-graded vector space

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^n$$

equipped with a degree 0 bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the *bracket*) which is graded antisymmetric, i.e.

$$[a,b] = -(-1)^{|a||b|}[b,a] \tag{2.18}$$

for all homogeneous $a, b \in \mathfrak{g}$, and satisfies the graded Jacobi identity:

$$[a,[b,c]] + (-1)^{|a|(|b|+|c|)}[b,[c,a]] + (-1)^{|c|(|a|+|b|)}[c,[a,b]] = 0 \tag{2.19}$$

for all homogeneous $a, b, c \in \mathfrak{g}$.

Exercise 2.5.2. Write the axioms of graded Lie algebras in an element-free form that would use only the bilinear map $l := [-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and its iterated compositions, and the operator of "permuting the inputs" of a multilinear map. Observe how the Koszul sign convention helps remembering the signs in (2.18) and (2.19).

2.5.3 Definition. A *dg-Lie algebra* (an abbreviation for *differential graded Lie algebra*) is a graded Lie algebra $L = \bigoplus_{n\in\mathbb{Z}} L^n$ as in Definition 2.5.1 together with a degree 1 linear map $d : L \to L$ which is

– a degree 1 derivation , i.e. $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$ for homogeneous $a, b \in L$, and

– a differential, i.e. $d^2 = 0$.

Our next aim is to show that the Hochschild complex $(C^*_{\mathrm{Hoch}}(A,A), \delta_{\mathrm{Hoch}})$ of an associative algebra recalled in Definition 2.2.1 has a natural bracket which turns it into a dg-Lie algebra. We start with some preparatory material.

2.5.4 Proposition. Let C be a graded coalgebra. For coderivations $\theta, \phi \in CoDer(C)$ define

$$[\theta,\phi] := \theta \circ \phi - (-1)^{|\theta||\phi|}\phi \circ \theta.$$

The bracket $[-,-]$ makes $CoDer(C) = \bigoplus_{n\in\mathbb{Z}} CoDer^n(\mathcal{C})$ a graded Lie algebra.

Proof. The key observation is that $[\theta,\phi]$ is a coderivation (note that *neither* $\theta \circ \phi$ *nor* $\phi \circ \theta$ are coderivations!). Verifying this and the properties of a graded Lie bracket is straightforward and will be omitted. □

2.5.5 Proposition. Let C be a graded coalgebra and $\chi \in CoDer^1(\mathcal{C})$ such that

$$[\chi,\chi] = 0, \tag{2.20}$$

where $[-,-]$ is the bracket of Proposition 2.5.4. Then

$$d(\theta) := [\chi,\theta] \qquad \text{for } \theta \in CoDer(\mathcal{C})$$

is a differential that makes $CoDer(\mathcal{C})$ a dg-Lie algebra.

Observe that, since $|\chi| = 1$, (2.20) does not tautologically follow from the graded antisymmetry (2.18).

Proof of Proposition 2.5.5. The graded Jacobi identity (2.19) implies that, for each homogeneous θ,

$$[\chi, [\chi, \theta]] = -(-1)^{|\theta|+1}[\chi, [\theta, \chi]] - [\theta, [\chi, \chi]].$$

Now use the graded antisymmetry $[\theta, \chi] = (-1)^{|\theta|+1}[\chi, \theta]$ and the assumption $[\chi, \chi] = 0$ to conclude from the above display that

$$[\chi, [\chi, \theta]] = -[\chi, [\chi, \theta]],$$

therefore, since the characteristic of the ground field is zero,

$$d^2(\theta) = [\chi, [\chi, \theta]] = 0,$$

so d is a differential. The derivation property of d with respect to the bracket can be verified in the same way and we leave it as an exercise to the reader. □

In Proposition 2.5.5 we saw that coderivations of a graded coalgebra form a dg-Lie algebra. Another example of a dg-Lie algebra is provided by the Hochschild cochains of an associative algebra (see Definition 2.2.1). We need the following:

2.5.6 Definition. For $f \in Lin(V^{\otimes(m+1)}, V)$, $g \in Lin(V^{\otimes(n+1)}, V)$ and $1 \leq i \leq m+1$ define $f \circ_i g \in Lin(V^{\otimes(m+n+1)}, V)$ by

$$(f \circ_i g) := f\left(id_V^{\otimes(i-1)} \otimes g \otimes id_V^{\otimes(m-i+1)}\right).$$

Define also

$$f \circ g := \sum_{i=1}^{m+1} (-1)^{n(i+1)} f \circ_i g$$

and, finally,

$$[f, g] := f \circ g - (-1)^{mn} g \circ f.$$

The operation $[-,-]$ is called the *Gerstenhaber bracket* (our definition however differs from the original one of [13] by the overall sign $(-1)^n$).

Let A be an associative algebra with the underlying space V. Since, by Definition 2.2.1, $C^{*+1}_{\mathrm{Hoch}}(A, A) = Lin(V^{\otimes(*+1)}, V)$, the structure of Definition 2.5.6 defines a degree 0 operation $[-,-] : C^{*+1}_{\mathrm{Hoch}}(A, A) \otimes C^{*+1}_{\mathrm{Hoch}}(A, A) \to C^{*+1}_{\mathrm{Hoch}}(A, A)$ called again the Gerstenhaber bracket. We leave as an exercise the proof of

2.5.7 Proposition. The Hochschild cochain complex of an associative algebra together with the Gerstenhaber bracket form a dg-Lie algebra $C^{*+1}_{\mathrm{Hoch}}(A, A) = (C^{*+1}_{\mathrm{Hoch}}(A, A), [-,-], \delta_{\mathrm{Hoch}})$.

The following theorem gives an alternative description of the dg-Lie algebra of Proposition 2.5.7.

2.5.8 Theorem. Let A be an associative algebra with multiplication $\mu : V \otimes V \to V$ and $\chi \in \mathit{CoDiff}^1_2({}^cT(\downarrow V))$ the coderivation that corresponds to μ in the correspondence of Theorem 2.4.21. Let $d := [\chi, -]$ be the differential introduced in Proposition 2.5.5. Then there is a natural isomorphism of dg-Lie algebras

$$\xi : \left(C^{(*+1)}_{\mathrm{Hoch}}(A,A), [-,-], \delta_{\mathrm{Hoch}}\right) \xrightarrow{\cong} (\mathit{CoDer}^*({}^cT(\downarrow V)), [-,-], d)\,.$$

Proof. Given $\phi \in C^{n+1}_{\mathrm{Hoch}}(A,A) = \mathit{Lin}(V^{\otimes(n+1)}, V)$, let $f : (\downarrow V)^{\otimes(n+1)} \to \downarrow V$ be the degree n linear map defined by the diagram

$$\begin{tikzcd}
(\downarrow V)^{\otimes(n+1)} \arrow[r, "f"] & \downarrow V \\
V^{\otimes(n+1)} \arrow[u, "\downarrow^{\otimes(n+1)}"] \arrow[r, "\phi"] & V. \arrow[u, "\downarrow"']
\end{tikzcd}$$

By Proposition 2.4.19, there exists a unique coderivation $\theta \in \mathit{CoDer}^n({}^cT(\downarrow V))$ whose $(n+1)$th corestriction is f and other corestrictions are trivial.

The map $\xi : C^{(*+1)}_{\mathrm{Hoch}}(A,A) \to \mathit{CoDer}^*({}^cT(\downarrow V))$ defined by $\xi(\phi) := \theta$ is clearly an isomorphism. The verification that ξ commutes with the differentials and brackets is a straightforward though involved exercise on the Koszul sign convention which we leave to the reader. □

2.5.9 Corollary. Let μ be the multiplication in A interpreted as an element of $C^2_{\mathrm{Hoch}}(A,A)$, and $f \in C^*_{\mathrm{Hoch}}(A,A)$. Then $\delta_{\mathrm{Hoch}}(f) = [\mu, f]$.

Proof. The corollary immediately follows from Theorem 2.5.8. Indeed, because ξ commutes with all the structures, we have

$$\delta_{\mathrm{Hoch}}(f) = \xi^{-1}\xi\delta_{\mathrm{Hoch}}(f) = \xi^{-1}(d(\xi f)) = \xi^{-1}[\chi, \xi f] = [\mu, f].$$

We however recommend as an exercise to verify the corollary directly, comparing $[\mu, f]$ to the formula for the Hochschild differential. □

2.5.10 Proposition. A bilinear map $\kappa : V \otimes V \to V$ defines an associative algebra structure on V if and only if $[\kappa, \kappa] = 0$.

Proof. By Definition 2.5.6 (with $m = n = 1$),

$$\frac{1}{2}[\kappa,\kappa] = \frac{1}{2}\left(\kappa \circ \kappa - (-1)^{mn}\kappa \circ \kappa\right) = \kappa \circ \kappa = \kappa \circ_1 \kappa - \kappa \circ_2 \kappa = \kappa(\kappa \otimes id_V) - \kappa(id_V \otimes \kappa),$$

therefore $[\kappa, \kappa] = 0$ is indeed equivalent to the associativity of κ. □

2.5.11 Proposition. Let A be an associative algebra with the underlying vector space V and the multiplication $\mu : V \otimes V \to V$. Let $\nu \in C^2_{\mathrm{Hoch}}(A,A)$ be a Hochschild 2-cochain. Then $\mu + \nu \in C^2_{\mathrm{Hoch}}(A,A) = \mathit{Lin}(V^{\otimes 2}, V)$ is associative if and only if

$$\delta_{\mathrm{Hoch}}(\nu) + \frac{1}{2}[\nu, \nu] = 0. \tag{2.21}$$

Proof. By Proposition 2.5.10, $\mu + \nu$ is associative if and only if

$$0 = \frac{1}{2}[\mu+\nu, \mu+\nu] = \frac{1}{2}\Big\{[\mu,\mu] + [\nu,\nu] + [\mu,\nu] + [\nu,\mu]\Big\} = \delta_{\mathrm{Hoch}}(\nu) + \frac{1}{2}[\nu,\nu].$$

To get the rightmost term, we used the fact that, since μ is associative, $[\mu,\mu] = 0$ by Proposition 2.5.10. We also observed that $[\mu,\nu] = [\nu,\mu] = \delta_{\mathrm{Hoch}}(\nu)$ by Corollary 2.5.9. □

A bilinear map $\nu : V \otimes V \to V$ such that $\mu+\nu$ is associative can be viewed as a deformation of μ. This suggests that (2.21) is related to deformations. This is indeed the case, as we will see later in this section. Equation (2.21) is a particular case of the Maurer-Cartan equation in a arbitrary dg-Lie algebra:

2.5.12 Definition. Let $L = (L, [-,-], d)$ be a dg-Lie algebra. A degree 1 element $s \in L^1$ is *Maurer-Cartan* if it satisfies the *Maurer-Cartan equation*

$$ds + \frac{1}{2}[s,s] = 0. \tag{2.22}$$

2.5.13 Remark. The Maurer-Cartan equation (also called the *Berikashvili equation*) along with its clones and generalizations is one of the most important equations in mathematics. For instance, a version of the Maurer-Cartan equation describes the differential of a left-invariant form, see [25, I.§4].

Let $\mathfrak{g}$ be a dg-Lie algebra over the ground field $\mathbf{k}$. Consider the dg-Lie algebra L over the power series ring $\mathbf{k}[[t]]$ defined as

$$L := \mathfrak{g} \otimes (t), \tag{2.23}$$

where $(t) \subset \mathbf{k}[[t]]$ is the ideal generated by t. Degree n elements of L are expressions $f_1 t + f_2 t^2 + \cdots$, $f_i \in \mathfrak{g}^n$ for $i \geq 1$. The dg-Lie structure on L is induced from that of $\mathfrak{g}$ in an obvious manner. Denote by $\mathrm{MC}(\mathfrak{g})$ the set of all Maurer-Cartan elements in L. Clearly, a degree 1 element $s = f_1 t + f_2 t^2 + \cdots$ is Maurer-Cartan if its components $\{f_i \in \mathfrak{g}^1\}_{i\geq 1}$ satisfy the equation:

$$(MC_k) \qquad df_k + \frac{1}{2}\sum_{i+j=k} [f_i, f_j] = 0$$

for each $k \geq 1$.

2.5.14 Example. Let us apply the above construction to the Hochschild complex of an associative algebra A with the multiplication μ_0, that is, take $\mathfrak{g} := C^{*+1}_{\mathrm{Hoch}}(A,A)$ with the Gerstenhaber bracket and the Hochschild differential. In this case, one easily sees that (MC_k) for $s = \mu_1 t + \mu_2 t^2 + \cdots$, $\mu_i \in C^2_{\mathrm{Hoch}}(A,A)$ is precisely equation (D_k) of Theorem 2.3.15, $k \geq 1$, compare also calculations on page 20. We conclude that $\mathrm{MC}(\mathfrak{g})$ is the set of infinitesimal deformations of μ_0.

Let us recall that each Lie algebra $\mathfrak{l}$ can be equipped with a group structure with the multiplication given by the *Hausdorff-Campbell formula*:

$$x \cdot y := x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \cdots \tag{2.24}$$

assuming a suitable condition that guarantees that the above infinite sum makes sense in $\mathfrak{l}$, see [42, I.IV.§7]. We denote $\mathfrak{l}$ with this multiplication by $\exp(\mathfrak{l})$. Formula (2.24) is obtained by expressing the right hand side of

$$x \cdot y = \log(\exp(x)\exp(y)),$$

where

$$\exp(a) := 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots, \quad \log(1+a) := a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \cdots,$$

in terms of iterated commutators of non-commutative variables x and y.

Using this construction, we introduce the *gauge group* of $\mathfrak{g}$ as

$$\mathrm{G}(\mathfrak{g}) := \exp(L^0),$$

where $L^0 = \mathfrak{g}^0 \otimes (t)$ is the Lie subalgebra of degree zero elements in L defined in (2.23). Let us fix an element $\chi \in \mathfrak{g}^1$. The gauge group then acts on $L^1 = \mathfrak{g}^1 \otimes (t)$ by the formula

$$x \cdot l := l + [x, \chi + l] + \frac{1}{2!}[x,[x,\chi+l]] + \frac{1}{3!}[x,[x,[x,\chi+l]]] + \cdots, \ x \in \mathrm{G}(\mathfrak{g}), l \in L^1, \tag{2.25}$$

obtained by expressing the right hand side of

$$x \cdot l = \exp(x)(\chi + l)\exp(-x) - \chi \tag{2.26}$$

in terms of iterated commutators. Denoting $d\chi := [\chi, \chi]$, formula (2.25) reads

$$x \cdot l = l + dx + [x,l] + \frac{1}{2}\Big\{[x,dx] + [x,[x,l]]\Big\} + \frac{1}{3}\Big\{[x,[x,dx]] + [x,[x,[x,l]]]\Big\} + \cdots \tag{2.27}$$

2.5.15 Lemma. Action (2.27) of $\mathrm{G}(\mathfrak{g})$ on L^1 preserves the space $\mathrm{MC}(\mathfrak{g})$ of solutions of the Maurer-Cartan equation.

Proof. We will prove the lemma under the assumption that $\mathfrak{g}$ is a dg-Lie algebra whose differential d has the form $d = [\chi, -]$ for some $\chi \in \mathfrak{g}^1$ satisfying $[\chi,\chi] = 0$ (see Proposition 2.5.5). The proof of the general case is a straightforward, though involved, verification.

It follows from (2.26) that $\chi + x \cdot l = \exp(x)(\chi + l)\exp(-x)$, i.e. x transforms $\chi + l$ into $\exp(x)(\chi+l)\exp(-x)$. Under the assumption $d = [\chi, -]$, the Maurer-Cartan equation for l is equivalent to $[\chi + l, \chi + l] = 0$. The Maurer-Cartan equation for the transformed l then reads

$$[\exp(x)(\chi+l)\exp(-x), \exp(x)(\chi+l)\exp(-x)] = 0,$$

which can be rearranged into

$$\exp(x)[\chi + l, \chi + l]\exp(-x) = 0.$$

This finishes the proof. $\square$

Thanks to Lemma 2.5.15, it makes sense to consider

$$\mathfrak{Def}(\mathfrak{g}) := \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g}),$$

the moduli space of solutions of the Maurer-Cartan equation in $L = \mathfrak{g} \otimes (t)$.

2.5.16 Example. Let us return to the situation in Example 2.5.14. In this case

$$\mathfrak{g}_0 = C^1_{\mathrm{Hoch}}(A, A) = Lin(A, A),$$

with the bracket given by the commutator of the composition of linear maps. The gauge group $\mathrm{G}(\mathfrak{g})$ consists of elements $x = f_1 t + f_2 t^2 + \ldots$, $f_i \in Lin(A, A)$. It follows from the definition of the gauge group action that two formal deformations $\mu' = \mu_0 + \mu'_1 t + \mu'_2 t^2 + \cdots$ and $\mu'' = \mu_0 + \mu''_1 t + \mu''_2 t^2 + \cdots$ of μ_0 define the same element in $\mathfrak{Def}(\mathfrak{g})$ if and only if

$$\exp(x)(\mu_0 + \mu'_1 t + \mu'_2 t^2 + \cdots) = (\mu_0 + \mu''_1 t + \mu''_2 t^2 + \cdots)(\exp(x) \otimes \exp(x)) \quad (2.28)$$

for some $x \in \mathrm{G}(\mathfrak{g})$. The above formula has an actual, not only formal, meaning – all power series make sense because of the completeness of the ground ring.

On the other hand, recall that in Example 2.3.17 we introduced the group

$$H := \{u = id_A + \phi_1 t + \phi_2 t^2 + \cdots \mid \phi_i \in Lin(A, A)\}.$$

The exponential map $\exp : \mathrm{G}(\mathfrak{g}) \to H$ is a well-defined isomorphism with the inverse map $\log : H \to \mathrm{G}(\mathfrak{g})$. We conclude that the equivalence relation defined by (2.28) is the same as the equivalence defined by (2.5) in Example 2.3.17, therefore $\mathfrak{Def}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g})$ is the moduli space of equivalence classes of formal deformations of μ_0.

The above analysis can be generalized by replacing, in (2.23), (t) by an arbitrary ideal $\mathfrak{m}$ in a local Artinian ring or in a complete local ring.

2.6 L_∞-algebras and the Maurer-Cartan equation

We are going to describe a generalization of differential graded Lie algebras. Let us start by recalling some necessary notions.

Let W be a $\mathbb{Z}$-graded vector space. We will denote by $\wedge W$ the *free graded commutative associative algebra* over W. It is characterized by the obvious analog of the universal property in Definition 2.4.1 with respect to graded commutative associative algebras. It can be realized as the tensor algebra $T(W)$ modulo the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$. If one decomposes

$$W = W^{\mathrm{even}} \oplus W^{\mathrm{odd}}$$

into the even and odd parts, then

$$\wedge W \cong \mathbf{k}[W^{\mathrm{even}}] \otimes E[W^{\mathrm{odd}}],$$

where the first factor is the polynomial algebra and the second one is the exterior (Grassmann) algebra. The algebra $\wedge W$ can also be identified with the subspace of $T(W)$ consisting of graded-symmetric elements (remember we work over a characteristic zero field).

Denote the product of (homogeneous) elements $w_1, \ldots, w_n \in W$ in $\wedge W$ by $w_1 \wedge \ldots \wedge w_n$. For a permutation $\sigma \in \mathfrak{S}_k$ we define the *Koszul sign* $\varepsilon(\sigma) \in \{-1, +1\}$ by

$$w_1 \wedge \ldots \wedge w_k = \varepsilon(\sigma) w_{\sigma(1)} \wedge \ldots \wedge w_{\sigma(k)}$$

and the *antisymmetric Koszul sign* $\chi(\sigma) \in \{-1, +1\}$ by

$$\chi(\sigma) := \mathrm{sgn}(\sigma)\varepsilon(\sigma).$$

Exercise 2.6.1. Express $\epsilon(\sigma)$ and $\chi(\sigma)$ explicitly in terms of σ and the degrees $|w_1|, \ldots, |w_n|$.

Finally, a permutation $\sigma \in \mathfrak{S}_n$ is called an $(i, n-i)$-*unshuffle* if $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(n)$. The set of all $(i, n-i)$-unshuffles will be denoted $\mathfrak{S}_{(i,n-i)}$.

2.6.2 Definition. An L_∞-*algebra* (also called a *strongly homotopy Lie* or *sh Lie algebra*) is a graded vector space V together with a system

$$l_k : \otimes^k V \to V, \quad k \in \mathbb{N}$$

of linear maps of degree $2-k$ subject to the following axioms.

– Antisymmetry: For every $k \in \mathbb{N}$, every permutation $\sigma \in \mathfrak{S}_k$ and every homogeneous $v_1, \ldots, v_k \in V$,

$$l_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \chi(\sigma) l_k(v_1, \ldots, v_k). \tag{2.29}$$

– For every $n \geq 1$ and homogeneous $v_1, \ldots, v_n \in V$,

$$(L_n) \qquad \sum_{i+j=n+1} (-1)^i \sum_{\sigma \in \mathfrak{S}_{i,n-i}} \chi(\sigma) l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0.$$

2.6.3 Remark. The sign in (L_n) was taken from [17]. With this sign convention, all terms of the (generalized) Maurer-Cartan equation recalled in (2.31) below have $+1$-signs. Our sign convention is related to the original one in [28, 29] via the transformation $l_n \mapsto (-1)^{\binom{n+1}{2}} l_n$. We also used the opposite grading which is better suited for our purposes – the operation l_k as introduced in [28, 29] has degree $k-2$.

Let us expand axioms (L_n) for $n = 1, 2$ and 3.

Case $n = 1$**.** For $n = 1$ (L_1) reduces to $l_1(l_1(v)) = 0$ for every $v \in V$, i.e. l_1 is a degree $+1$ differential.

Case $n = 2$**.** By (2.29), $l_2 : V \otimes V \to V$ is a linear degree 0 map which is graded antisymmetric,

$$l_2(v, u) = -(-1)^{|u||v|} l_2(u, v)$$

and (L_n) for $n = 2$ gives

$$(L_2) \qquad l_1(l_2(u, v)) = l_2(l_1(u), v) + (-1)^{|u|} l_2(u, l_1(v))$$

meaning that l_1 is a graded derivation with respect to the multiplication l_2. Writing $d := l_1$ and $[u,v] := l_2(u,v)$, (L_2) takes more usual form

$$d[u,v] = [du,v] + (-1)^{|u|}[u,dv].$$

Case $n = 3$. The degree -1 graded antisymmetric map $l_3 : \otimes^3 V \to V$ satisfies (L_3):

$$\begin{gathered}(-1)^{|u||w|}[[u,v],w] + (-1)^{|v||w|}[[w,u],v] + (-1)^{|u||v|}[[v,w],u] = \\ (-1)^{|u||w|}(dl_3(u,v,w) + l_3(du,v,w) + (-1)^{|u|}l_3(u,dv,w) + (-1)^{|u|+|v|}l_3(u,v,dw)).\end{gathered}$$

One immediately recognizes the three terms of the Jacobi identity in the left-hand side and the d-boundary of the trilinear map l_3 in the right-hand side. We conclude that the bracket $[-,-]$ satisfies the Jacobi identity modulo the homotopy l_3.

2.6.4 Example. If all structure operations l_k's of an L_∞-algebra $L = (V, l_1, l_2, \ldots)$ except l_1 vanish, then L is just a dg-vector space with the differential $d = l_1$. If all l_k's except l_1 and l_2 vanish, then L is our familiar dg-Lie algebra from Definition 2.5.3 with $d = l_1$ and the Lie bracket $[-,-] = l_2$. In this sense, dg-Lie algebras are particular cases of L_∞-algebras.

2.6.5 Example. Let $L' = (V', l'_1, l'_2, l'_3, \ldots)$ and $L'' = (V'', l''_1, l''_2, l''_3, \ldots)$ be two L_∞-algebras. Define their *direct sum* $L' \oplus L''$ to be the L_∞-algebra $L' \oplus L''$ with the underlying vector space $V' \oplus V''$ and structure operations $\{l_k\}_{k \geq 1}$ given by

$$l_k(v'_1 \oplus v''_1, \ldots, v'_k \oplus v''_k) := l'_k(v'_1, \ldots, v'_k) + l''_k(v''_1, \ldots, v''_k),$$

for $v'_1, \ldots, v'_k \in V'$, $v''_1, \ldots, v''_k \in V''$.

For a graded vector space V denote $\vee_k(V)$ the quotient of $\bigotimes^k V$ modulo the subspace spanned by elements

$$v_1 \otimes \cdots \otimes v_k - \chi(\sigma)\ v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

The antisymmetry (2.29) implies that the structure operations of an L_∞ algebra can be interpreted as maps

$$l_k : \vee_k(V) \to V,\ k \geq 1.$$

We are going to give a description of the set of L_∞-structures on a given graded vector space in terms of coderivations, in the spirit of Theorem 2.4.21. To this end, we need the following coalgebra which will play the role of ${}^cT(W)$.

2.6.6 Proposition. The space $\bigwedge(W)$ with the comultiplication $\Delta : \bigwedge(W) \to \bigwedge(W) \otimes \bigwedge(W)$ defined by

$$\Delta(w_1 \wedge \ldots \wedge w_n) := \sum_{i=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_{i,n-i}} \epsilon(\sigma)(w_{\sigma(1)} \wedge \ldots \wedge w_{\sigma(i)}) \otimes (w_{\sigma(i+1)} \wedge \ldots \wedge w_{\sigma(n)})$$

is a graded coassociative cocommutative coalgebra. We will denote it ${}^c\bigwedge(W)$.

Proof. A direct verification which we leave to the reader as an exercise. $\square$

For the coalgebra ${}^c\bigwedge(W)$, the following analog of Proposition 2.4.19 holds.

2.6.7 Proposition. Let W be a graded vector space. For any d, there is a natural isomorphism

$$CoDer^d({}^c\bigwedge(W)) \cong Lin^d({}^c\bigwedge(W), W).$$

We leave the proof to the reader. Observe that the coalgebra ${}^c\bigwedge(W)$ is a direct sum

$${}^c\bigwedge(W) = \bigoplus_{n\geq 1} {}^c\bigwedge^n(W)$$

of subspaces ${}^c\bigwedge^n(W)$ spanned by $w_1 \wedge \ldots \wedge w_n$, for $w_1, \ldots, w_n \in W$. One may define the sth *corestriction* of a coderivation $\theta \in CoDer({}^c\bigwedge(W))$ as the composition

$$f_s : {}^c\bigwedge^s(W) \xrightarrow{\theta|\bigwedge^s(W)} {}^c\bigwedge(W) \xrightarrow{\text{proj.}} W.$$

As in Definition 2.4.20, a coderivation $\theta \in CoDer^d({}^c\bigwedge(W))$ is *quadratic* if its sth corestriction is non-zero only for $s = 2$. A *differential* is a degree 1 coderivation θ such that $\theta^2 = 0$.

2.6.8 Theorem. Denote by $L_\infty(V)$ the set of all L_∞-algebra structures on a graded vector space V and $CoDiff^1({}^c\bigwedge(\downarrow V))$ the set of differentials on ${}^c\bigwedge(\downarrow V)$. Then there is a bijection

$$L_\infty(V) \cong CoDiff^1({}^c\bigwedge(\downarrow V)).$$

Proof. Let $\chi \in CoDiff^1({}^c\bigwedge(\downarrow V))$ and $f_n : {}^c\bigwedge^n(\downarrow V) \to \downarrow V$ the nth corestriction of χ, $n \geq 1$. Define $\bar{l}_n : \bigvee_n(V) \to V$ by the diagram

$$\begin{tikzcd}
{}^c\bigwedge_n(\downarrow V) \arrow[r, "f_n"] & \downarrow V \\
\bigvee_n(V) \arrow[u, "\bigotimes^n \downarrow"] \arrow[r, "\bar{l}_n"] & V. \arrow[u, "\downarrow"']
\end{tikzcd}$$

It is then a direct though involved verification that the maps

$$l_n := (-1)^{\binom{n+1}{2}} \bar{l}_n \tag{2.30}$$

define an L_∞-structure on V and that the correspondence $\chi \leftrightarrow (l_1, l_2, l_3, \ldots)$ is one-to-one. The reason for the sign change in (2.30) is explained in Remark 2.6.3. $\square$

2.6.9 Remark. By Theorem 2.6.8, L_∞-algebras can be alternatively defined as square-zero differentials on "cofree" cocommutative coassociative coalgebras (the reason why we put 'cofree' into quotation marks is the same as in Section 2.4, see also the warning on page 25). Dual forms of these object, i.e. square-zero differentials on free commutative associative algebras, are *Sullivan models* that have existed in rational homotopy theory since 1977 [45]. The same objects appeared as generalizations of Lie algebras independently in 1982 in a remarkable paper [7]. As homotopy Lie algebras with a coherent system of higher homotopies, L_∞-algebras were recognized much later [22, 29].

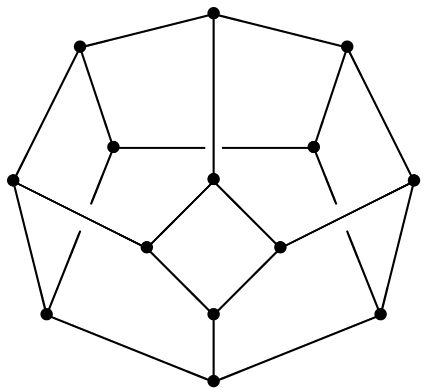

Figure 2.1: Saito's portrait of K_5.

Exercise 2.6.10. Show that the isomorphism of Theorem 2.6.8 restricts to the isomorphism

$$Lie(V) \cong CoDiff^1_2({}^c\wedge(\downarrow V))$$

between the set of Lie algebra structures on V and quadratic differentials on the coalgebra ${}^c\wedge(\downarrow V)$. This isomorphism shall be compared to the isomorphism in Theorem 2.4.21.

Let us make a digression and see what happens when one allows in the right hand side of (2.17) all, not only quadratic, differentials. The above material indicates that one should expect a homotopy version of associative algebras. This is indeed so; one gets the following objects that appeared in 1963 [44] (but we use the sign convention of [33]).

2.6.11 Definition. An A_∞*-algebra* (also called a *strongly homotopy associative algebra*) is a graded vector space V together with a system

$$\mu_k : V^{\otimes k} \to V, \quad k \geq 1,$$

of linear maps of degree $k-2$ such that

$$(A_n)\ \sum_{\lambda=0}^{n-1}\sum_{k=1}^{n-\lambda}(-1)^{\epsilon}\cdot\mu_{n-k+1}(v_1, ..., v_\lambda, \mu_k(v_{\lambda+1}, ..., v_{\lambda+k}), v_{\lambda+k+1}, ..., v_n) = 0$$

for every $n \geq 1$, $v_1, \ldots, v_n \in V$, where $\epsilon = k + \lambda + k\lambda + k(|v_1| + \cdots + |v_\lambda|)$.

One easily sees that (A_1) means that $\partial := \mu_1$ is a degree -1 differential, (A_2) that the bilinear product $\mu_2 : V \otimes V \to V$ commutes with ∂ and (A_3) that μ_2 is associative up to the homotopy μ_3. A_∞-algebras can also be described as algebras over the cellular chain complex of the non-Σ operad $K = \{K_n\}_{n\geq 1}$ whose nth piece is the $(n-2)$-dimensional convex polytope K_n called the *Stasheff associahedron* [38, Section II.1.6]. Let us mention at least that K_2 is the point, K_3 the closed interval and K_4 is the pentagon from Mac Lane's theory of monoidal categories [31]. A portrait of K_5 due to Masahico Saito is in Figure 2.1.

2.6.12 Theorem. For a graded vector space V denote $A_\infty(V)$ the set of all A_∞-algebra structures on V and $\mathit{CoDiff}^1({}^cT(\downarrow V))$ the set of all differentials on ${}^cT(\downarrow V)$. Then there is a natural bijection

$$A_\infty(V) \cong \mathit{CoDiff}^1({}^cT(\downarrow V)).$$

Proof. The isomorphism in the above theorem is of the same nature as the isomorphism of Theorem 2.6.8, but it also involves the 'flip' of degrees since we defined, following [33], A_∞-algebras in such a way that the differential $\partial = \mu_1$ has degree -1. We leave the details to the reader. $\square$

Let us return to the main theme of this section. Our next task will be to introduce morphisms of L_∞-algebras. We start with a simple-minded definition.

Suppose $L' = (V', l'_1, l'_2, l'_3, \ldots)$ and $L'' = (V'', l''_1, l''_2, l''_3, \ldots)$ are two L_∞-algebras. A *strict morphism* is a degree zero linear map $f : V' \to V''$ which commutes with all structure operations, that is

$$f(l'_k(v_1, \ldots, v_k)) = l''_k(f(v_1), \ldots, f(v_k)),$$

for each $v_1, \ldots, v_k \in V'$, $k \geq 1$.

For our purposes we need, however, a subtler notion of morphisms. We give a definition that involves the isomorphism of Theorem 2.6.8.

2.6.13 Definition. Let L' and L'' be L_∞-algebras represented by dg-coalgebras $({}^c\bigwedge(\downarrow V'), \delta')$ and $({}^c\bigwedge(\downarrow V''), \delta'')$. A *(weak) morphism* of L_∞-algebras is then a morphism of dg-coalgebras $F : ({}^c\bigwedge(\downarrow V'), \delta') \to ({}^c\bigwedge(\downarrow V''), \delta'')$.

Definition 2.6.13 can be unwrapped. Let $F_k : {}^c\bigwedge_k(\downarrow V') \to \downarrow V''$ be, for each $k \geq 1$, the composition

$${}^c\bigwedge^k(\downarrow V') \xrightarrow{F} {}^c\bigwedge(\downarrow V'') \xrightarrow{\text{proj.}} \downarrow V''.$$

Define the maps $f_k : \bigvee_k V' \to V''$ by the diagram

$$\begin{tikzcd}
{}^c\bigwedge_k(\downarrow V') \arrow[r, "F_k"] & \downarrow V'' \\
\bigvee_k V' \arrow[u, "\bigotimes^k \downarrow"] \arrow[r, "f_k"] & V.'' \arrow[u, "\downarrow"']
\end{tikzcd}$$

Clearly, f_k is a degree $1 - k$ linear map. The fact that F is a dg-morphism can be expressed via a sequence of axioms (M_n), $n \geq 1$, where (M_n) postulates the vanishing of a combination of n-multilinear maps on V' with values in V'' involving f_i, l'_i and l''_i for $i \leq n$.

We are not going to write (M_n)'s here. Explicit axioms for L_∞-maps can be found in [24], see also [28, Definition 5.2] where the particular case when L'' is a dg-Lie algebra ($l''_k = 0$ for $k \geq 3$) is discussed in detail. The reader is however encouraged to verify that (M_1) says that $f_1 : (V', l'_1) \to (V'', l''_1)$ is a chain map and that (M_2) means that f_1 commutes with the brackets l'_2 and l''_2 modulo the homotopy f_2.

Morphisms of L_∞-algebras L' and L'' with underlying vector spaces V' and V'' can therefore be equivalently defined as systems $f = \{f_k : \bigotimes^k V' \to V''\}_{k\geq 1}$, where f_k is a degree $1-k$ graded antisymmetric linear map, and axioms (M_n), $n \geq 1$, are satisfied. Let us denote by L_∞ the category of L_∞-algebras and their morphisms in the sense of Definition 2.6.13.

Exercise 2.6.14. Show that the category strL_∞ of L_∞-algebras and their strict morphisms can be identified with the (non-full) subcategory of L_∞ with the same objects and morphisms $f = (f_1, f_2, \ldots)$ such that $f_k = 0$ for $k \geq 2$.

Show that the obvious imbedding $\mathsf{dgLie} \hookrightarrow \mathsf{L}_\infty$ is not full. This means that there are more morphisms between dg-Lie algebras considered as elements of the category L_∞ than in the category of dgLie. Observe finally that the forgetful functor $\square : \mathsf{L}_\infty \to \mathsf{dgVect}$ given by forgetting all structure operations is not faithful.

2.7 Homotopy invariance of the Maurer-Cartan equation

Let us start with recalling some necessary definitions.

2.7.1 Definition. A morphism $f = (f_1, f_2, \ldots) : L' = (V', l'_1, l'_2, \ldots) \to L'' = (V'', l''_1, l''_2, \ldots)$ of L_∞-algebras is a *weak equivalence* if the chain map $f_1 : (V', l'_1) \to (V'', l''_1)$ induces an isomorphism of cohomology.

2.7.2 Definition. An L_∞-algebra $L = (V, l_1, l_2, \ldots)$ is *minimal* if $l_1 = 0$. It is *contractible* if $l_k = 0$ for $k \geq 2$ and if $H^*(V, l_1) = 0$.

2.7.3 Proposition. A weak equivalence of minimal L_∞-algebras is an isomorphism.

Proof. Let $f = (f_1, f_2, \ldots) : L' \to L''$ be a weak equivalence of L_∞-algebras. It follows from the minimality of L' and L'' that the linear part f_1 is an isomorphism, thus the corresponding map $F : ({}^c\!\bigwedge(\downarrow V'), \delta') \to ({}^c\!\bigwedge(\downarrow V''), \delta'')$ induces an isomorphism of cogenerators. It can be easily shown that such maps can be inverted. $\square$

The following theorem, which can be found in [26], uses the direct sum of L_∞-algebras recalled in Example 2.6.5.

2.7.4 Theorem. Each L_∞-algebra is the direct sum of a minimal and a contractible L_∞-algebra.

Let $L \cong L_m \oplus L_c$ be a decomposition of an L_∞-algebra L into a minimal L_∞-algebra L_m and a contractible L_∞-algebra L_c. Since the inclusion $\iota : L_m \to L_m \oplus L_c \cong L$ is a weak equivalence, Theorem 2.7.4 implies:

2.7.5 Corollary. Each L_∞-algebra is weakly equivalent to a minimal one.

Corollary 2.7.5 can also be derived from homotopy invariance properties of strongly homotopy algebras proved in [35]. Suppose we are given an L_∞-algebra $L = (V, l_1, l_2, \ldots)$. In characteristic zero, two cochain complexes have the same cochain homotopy type if and only if they have isomorphic cohomology. In particular, the cochain complex (V, l_1) is homotopy equivalent to the cohomology $H^*(V, l_1)$ considered as a complex with trivial differential. Move **(M1)** on page 133 of [35] now implies that there exists an induced minimal L_∞-structure on $H^*(V, l_1)$, weakly equivalent to L. Let us remark that an A_∞-version of Corollary 2.7.5 was known to Kadeishvili already in 1985, see [23].

Remarkably, each L_∞-algebra is, under some mild assumptions, weakly equivalent to a dg-Lie algebra. This can be proved as follows. Suppose L is an L_∞-algebra represented by a dg-coalgebra $({}^c\!\bigwedge(\downarrow V), \delta)$. The bar construction $B({}^c\!\bigwedge(\downarrow V), \delta)$ is a dg-Lie algebra and one may show, under an assumption that guarantees the convergence of a spectral sequence, that $B({}^c\!\bigwedge(\downarrow V), \delta)$ is weakly equivalent to L in the category of L_∞-algebras. This property is an algebraic analog of the *rectification principle* for $W\mathcal{P}$-spaces provided by the M-construction of Boardman and Vogt, see [38, Theorem II.2.9].

Let $\mathfrak{g}$ be an L_∞-algebra over the ground field $\mathbf{k}$, with the underlying $\mathbf{k}$-vector space V. Then $V \otimes (t)$, where $(t) \subset \mathbf{k}[[t]]$ is the ideal generated by t, has a natural induced L_∞-structure. Denote this L_∞-algebra by $L := \mathfrak{g} \otimes (t) = (V \otimes (t), l_1, l_2, l_3, \ldots)$. Let $\mathrm{MC}(\mathfrak{g})$ be the set of all degree $+1$ elements $s \in L^1$ satisfying the *generalized Maurer-Cartan equation*

$$l_1(s) + \frac{1}{2} l_2(s, s) + \frac{1}{3!} l_3(s, s, s) + \cdots + \frac{1}{n!} l_n(s, \ldots, s) + \cdots = 0. \qquad (2.31)$$

When $\mathfrak{g}$ is a dg-Lie algebra, one recognizes the ordinary Maurer-Cartan equation (2.22).

At this moment one needs to introduce a suitable *gauge equivalence* between solutions of (2.31) generalizing the action of the gauge group $\mathrm{G}(\mathfrak{g})$ recalled in (2.25). Since in applications of Section 2.8 all relevant L_∞-algebras are in fact dg-Lie algebras, we are not going to describe this generalized gauge equivalence here, and only refer to [26] instead. We denote $\mathfrak{Def}(\mathfrak{g})$ the set of gauge equivalence classes of solutions of (2.31). Let us, however, mention that there are examples, as bialgebras treated in [36], where deformations are described by a fully-fledged L_∞-algebra.

2.7.6 Example. For $\mathfrak{g}$ contractible, $\mathfrak{Def}(\mathfrak{g})$ is the one-point set consisting of the equivalence class of the trivial solution of (2.31). Indeed,

$$\mathrm{MC}(\mathfrak{g}) = \{s = s_1 t + s_2 t^2 + \ldots \mid ds_1 = ds_2 = \cdots = 0\}$$

so, by acyclicity, $s_i = db_i$ for some $b_i \in \mathfrak{g}^0$, $i \geq 1$. Formula (2.27) (with $x = -b_1 t_1 - b_2 t_2 - \cdots$ and $l = s_1 t + s_2 t^2 + \cdots$) gives

$$(-b_1 t_1 - b_2 t_2 - \cdots) \cdot (s_1 t + s_2 t^2 + \cdots) = 0,$$

therefore $s = s_1 t + s_2 t^2 + \cdots$ is equivalent to the trivial solution.

2.7.7 Example. Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be two L_∞-algebras. Then, for the direct product,

$$\mathfrak{Def}(\mathfrak{g}' \oplus \mathfrak{g}'') \cong \mathfrak{Def}(\mathfrak{g}') \times \mathfrak{Def}(\mathfrak{g}'').$$

Indeed, it follows from definition that $\mathrm{MC}(\mathfrak{g}' \oplus \mathfrak{g}'') \cong \mathrm{MC}(\mathfrak{g}') \times \mathrm{MC}(\mathfrak{g}'')$. This factorization is preserved by the gauge equivalence.

The central statement of this section reads:

2.7.8 Theorem. The assignment $\mathfrak{g} \mapsto \mathfrak{Def}(\mathfrak{g})$ extends to a covariant functor from the category of L_∞-algebras and their weak morphisms to the category of sets. A weak equivalence $f : \mathfrak{g}' \to \mathfrak{g}''$ induces an isomorphism $\mathfrak{Def}(f) : \mathfrak{Def}(\mathfrak{g}') \cong \mathfrak{Def}(\mathfrak{g}'')$.

The above theorem implies that the deformation functor $\mathfrak{Def}$ descends to the localization hoL_∞ obtained by inverting weak equivalences in L_∞. By Quillen's theory [40], hoL_∞ is equivalent to the category of minimal L_∞-algebras and homotopy classes (in an appropriate sense) of their maps. This explains the meaning of homotopy invariance in the title of this section.

Proof of Theorem 2.7.8. For an L_∞-morphism $f = (f_1, f_2, f_3, \ldots) : \mathfrak{g}' \to \mathfrak{g}''$ define $\mathrm{MC}(f) : \mathrm{MC}(\mathfrak{g}') \to \mathrm{MC}(\mathfrak{g}'')$ by

$$\mathrm{MC}(f)(s) := f_1(s) + \frac{1}{2} f_2(s,s) + \cdots + \frac{1}{n!} f_n(s, \ldots, s) + \cdots$$

It can be shown that $\mathrm{MC}(f)$ is a well-defined map that descends to the quotients by the gauge equivalence, giving rise to a map $\mathfrak{Def}(f) : \mathfrak{Def}(\mathfrak{g}') \to \mathfrak{Def}(\mathfrak{g}'')$.

Assume that $f : \mathfrak{g}' \to \mathfrak{g}''$ above is a weak equivalence. By Theorem 2.7.4, $\mathfrak{g}'$ decomposes as $\mathfrak{g}' = \mathfrak{g}'_m \oplus \mathfrak{g}'_c$, with $\mathfrak{g}'_m$ minimal and $\mathfrak{g}'_c$ contractible, and there is a similar decomposition $\mathfrak{g}'' = \mathfrak{g}''_m \oplus \mathfrak{g}''_c$ for $\mathfrak{g}''$. Define the map $\overline{f} : \mathfrak{g}'_m \to \mathfrak{g}''_m$ by the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{g}'_m \oplus \mathfrak{g}'_c \arrow[d, "f"'] & \mathfrak{g}'_m \arrow[l, hook', "i"'] \arrow[d, "\overline{f}"] \\
\mathfrak{g}''_m \oplus \mathfrak{g}''_c \arrow[r, two heads, "p"] & \mathfrak{g}''_m
\end{tikzcd}$$

in which i is the natural inclusion and p the natural projection. Observe that $\overline{f}$ is a weak equivalence so it is, by Proposition 2.7.3, an isomorphism. Therefore, in the following induced diagram, the map $\mathfrak{Def}(\overline{f})$ is an isomorphism, too:

$$\begin{tikzcd}
\mathfrak{Def}(\mathfrak{g}'_m) \times \mathfrak{Def}(\mathfrak{g}'_c) \arrow[d, "\mathfrak{Def}(f)"'] & \mathfrak{Def}(\mathfrak{g}'_m) \arrow[l, hook', "\mathfrak{Def}(i)"'] \arrow[d, "\mathfrak{Def}(\overline{f})"] \\
\mathfrak{Def}(\mathfrak{g}''_m) \times \mathfrak{Def}(\mathfrak{g}''_c) \arrow[r, two heads, "\mathfrak{Def}(p)"] & \mathfrak{Def}(\mathfrak{g}''_m).
\end{tikzcd}$$

Since, by Example 2.7.6, both $\mathfrak{Def}(\mathfrak{g}'_c)$ and $\mathfrak{Def}(\mathfrak{g}''_c)$ are points, the maps $\mathfrak{Def}(i)$ and $\mathfrak{Def}(p)$ are isomorphisms. We finish the proof by concluding that $\mathfrak{Def}(f)$ is also an isomorphism. $\square$

2.8 Deformation quantization of Poisson manifolds

In this section we indicate the main ideas of Kontsevich's proof of the existence of a deformation quantization of Poisson manifolds. Our exposition follows [26]. Let us recall some necessary notions.

2.8.1 Definition. A *Poisson algebra* is a vector space V with operations $\cdot : V \otimes V \to V$ and $\{-,-\} : V \otimes V \to V$ such that:

– $(V, \cdot)$ is an associative commutative algebra,

– $(V, \{-,-\})$ is a Lie algebra, and

– the map $v \mapsto \{u, v\}$ is a $\cdot$-derivation for any $u \in V$, i.e. $\{u, v \cdot w\} = \{u, v\} \cdot w + v \cdot \{u, w\}$.

Exercise 2.8.2. Show that Poisson algebras can be equivalently defined as structures with only one operation $\bullet : V \otimes V \to V$ such that

$$u\bullet(v\bullet w) = (u\bullet v)\bullet w - \frac{1}{3}\Big\{(u\bullet w)\bullet v + (v\bullet w)\bullet u - (v\bullet u)\bullet w - (w\bullet u)\bullet v)\Big\},$$

for each $u, v, w \in V$, see [37, Example 2].

Poisson algebras are 'classical limits' of associative deformations of commutative associative algebras. By this we mean the following. Let $A = (V, \cdot)$ be an associative algebra with multiplication $a, b \mapsto a \cdot b$. Consider a formal deformation $(\mathbf{k}[[t]] \otimes V, \star)$ of A given, as in Theorem 2.3.15, by a family $\{\mu_i : A \otimes A \to A\}_{i \geq 1}$ by the formula

$$a \star b := a \cdot b + t\mu_1(a, b) + t^2\mu_2(a, b) + t^3\mu_3(a, b) + \cdots \tag{2.32}$$

for $a, b \in V$. We have the following:

2.8.3 Proposition. Suppose $A = (V, \cdot)$ is a commutative associative algebra. Then, for an associative deformation (2.32) of A,

$$\{a, b\} := \mu_1(a, b) - \mu_1(b, a),\ a, b \in V,$$

is a Lie bracket such that $P_\star := (V, \cdot, \{-,-\})$ is Poisson algebra.

2.8.4 Definition. In the above situation, $P_\star$ is called the *classical limit* of the $\star$-product and $(\mathbf{k}[[t]] \otimes V, \star)$ a *deformation quantization* of the Poisson algebra $P_\star$.

Proof of Proposition 2.8.3. Let us prove first that $\{-,-\}$ is a Lie bracket. The antisymmetry of $\{-,-\}$ is obvious, one thus only needs to verify the Jacobi identity. It is a standard fact that the antisymmetrization of an associative multiplication is a Lie product [42, Chapter I], therefore $[-,-]$ defined by $[x, y] := x\star y - y\star x$ for $x, y \in \mathbf{k}[[t]] \otimes A$, is a Lie bracket on $\mathbf{k}[[t]] \otimes A$. We conclude by observing that the Jacobi identity for $\{-,-\}$ evaluated at $a, b, c \in A$ is the term at t^2 of the Jacobi identify for $[-,-]$ evaluated at the same elements.

It remains to verify the derivation property. It is clearly equivalent to

$$\mu_1(ab, c) - \mu_1(c, ab) - a\mu_1(b, c) + a\mu_1(c, b) - \mu_1(a, c)b + \mu_1(c, a)b = 0 \tag{2.33}$$

where we, for brevity, omitted the symbol for the $\cdot$-product. In Remark 2.3.16 we observed that μ_1 is a Hochschild cocycle, therefore

$$\rho(a, b, c) := a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c = 0.$$

A straightforward verification involving the commutativity of the $\cdot$-product shows that the left hand side of (2.33) equals $-\rho(a, b, c) + \rho(a, c, b) - \rho(c, a, b)$. This finishes the proof. $\square$

Let us recall geometric versions of the above notions.

2.8.5 Definition. A *Poisson manifold* is a smooth manifold M equipped with a Lie product $\{-,-\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$ on the space of smooth functions such that $(C^\infty(M), \cdot, \{-,-\})$, where $\cdot$ is the standard pointwise multiplication, is a Poisson algebra.

Poisson manifolds generalize symplectic ones in that the bracket $\{-,-\}$ need not be induced by a nondegenerate 2-form. The following notion was introduced and physically justified in [5].

2.8.6 Definition. A *deformation quantization* (also called a *star product*) of a Poisson manifold M is a deformation quantization of the Poisson algebra $(C^\infty(M), \cdot, \{-,-\})$ such that all μ_i's in (2.32) are differential operators.

2.8.7 Theorem (Kontsevich [26])**.** Every Poisson manifold admits a deformation quantization.

Sketch of Proof. Maxim Kontsevich proved this theorem in two steps. He proved first a 'local' version assuming $M = \mathbb{R}^d$, and then he globalized the result to an arbitrary M using ideas of formal geometry and the language of superconnections. We are going to sketch only the first step of Kontsevich's proof.

The idea was to construct two weakly equivalent L_∞-algebras $\mathfrak{g}'$, $\mathfrak{g}''$ such that $\mathfrak{Def}(\mathfrak{g}')$ contained the moduli space of Poisson structures on M and $\mathfrak{Def}(\mathfrak{g}'')$ was the moduli space of star products, and then apply Theorem 2.7.8. In fact, $\mathfrak{g}'$ will turn out to be an ordinary graded Lie algebra and $\mathfrak{g}''$ a dg-Lie algebra.

– *Construction of* $\mathfrak{g}'$. It is the graded Lie algebra of *polyvector fields* with the Shouten-Nijenhuis bracket. In more detail, $\mathfrak{g}' = \bigoplus_{n \geq 0} \mathfrak{g}'^n$ with

$$\mathfrak{g}'^n := \Gamma(M, \wedge^{n+1} TM),\ n \geq 1,$$

where $\Gamma(M, \wedge^{n+1} TM)$ denotes the space of smooth sections of the $(n+1)$th exterior power of the tangent bundle TM. The bracket is determined by

$$[\xi_0 \wedge \ldots \wedge \xi_k, \eta_0 \wedge \ldots \wedge \eta_l] :=$$
$$:= \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j+k} [\xi_i, \eta_j] \wedge \xi_0 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \xi_k \wedge \eta_0 \wedge \ldots \wedge \hat{\eta}_j \wedge \ldots \wedge \eta_l,$$

where $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_l \in \Gamma(M, TM)$ are vector fields, $\hat{}$ indicates the omission and $[\xi_i, \eta_j]$ in the right hand side denotes the classical Lie bracket of vector fields ξ_i and η_j [25, I.§1].

Recall that Poisson structures on M are in one-to-one correspondence with smooth sections $\alpha \in \Gamma(M, \wedge^2 TM)$ satisfying $[\alpha, \alpha] = 0$. The corresponding bracket of smooth functions $f, g \in C^\infty(M)$ is given by $\{f, g\} = \alpha(f \otimes g)$. Since $\mathfrak{g}'$ is just a graded Lie algebra,

$$\mathrm{MC}(\mathfrak{g}') = \{s = s_1 t + s_2 t^2 + \ldots \in \mathfrak{g}'^1 \otimes (t) \mid [s, s] = 0\}$$

therefore clearly $s := \alpha t \in \mathrm{MC}(\mathfrak{g}')$ for each $\alpha \in \Gamma(M, \wedge^2 TM)$ defining a Poisson structure. We see that $\mathfrak{Def}(\mathfrak{g}')$ contains the moduli space of Poisson structures on M.

– *Construction of* $\mathfrak{g}''$. It is the dg Lie algebra of polydiffenential operators,

$$\mathfrak{g}'' = \bigoplus_{n\geq 0} D^n_{\mathrm{poly}}(M),$$

where

$$D^n_{\mathrm{poly}}(M) \subset C^{n+1}_{\mathrm{Hoch}}(C^\infty(M), C^\infty(M))$$

consists of Hochschild cochains (Definition 2.2.1) of the algebra $C^\infty(M)$ given by polydifferential operators. It is clear that $D^*_{\mathrm{poly}}(M)$ is closed under the Hochschild differential and the Gerstenhaber bracket, so the dg-Lie structure of Proposition 2.5.7 restricts to a dg-Lie structure on $\mathfrak{g}''$. The analysis of Example 2.5.16 shows that $\mathfrak{Def}(\mathfrak{g}'')$ represents equivalence classes of star products.

– *The weak equivalence.* Consider the map $f_1 : \mathfrak{g}' \to \mathfrak{g}''$ defined by

$$f_1(\xi_0, \dots, \xi_k)(g_0, \dots, g_k) := \frac{1}{(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \mathrm{sgn}(\sigma) \prod_{i=0}^{k} \xi_{\sigma(i)}(g_i),$$

for $\xi_0, \dots, \xi_k \in \Gamma(M, TM)$ and $g_0, \dots, g_k \in C^\infty(M)$. It is easy to show that $f_1 : (\mathfrak{g}', d = 0) \to (\mathfrak{g}'', \delta_{\mathrm{Hoch}})$ is a chain map. Moreover, a version of the Kostant-Hochschild-Rosenberg theorem for smooth manifolds proved in [26] states that f_1 is a cohomology isomorphism. Unfortunately, f_1 *does not* commute with brackets. The following central statement of Kontsevich's approach to deformation quantization says that f_1 is, however, the linear part of an L_∞-map:

Formality. The map f_1 extends to an L_∞-homomorphism $f = (f_1, f_2, f_3, \dots) : \mathfrak{g}' \to \mathfrak{g}''$.

The formality theorem implies that $\mathfrak{g}'$ and $\mathfrak{g}''$ are weakly equivalent in the category of L_∞-algebras. In other words, the dg-Lie algebra of polydifferential operators is weakly equivalent to its cohomology. The 'formality' in the name of the theorem is justified by rational homotopy theory where formal algebras are algebras having the homotopy type of their cohomology.

Kontsevich's construction of higher f_i's involves coefficients given as integrals over compactifications of certain configuration spaces. An independent approach of Tamarkin [46] based entirely on homological algebra uses a solution of the Deligne conjecture, see also an overview [21] containing references to original sources.

□

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3. Gerstenhaber-Schack diagram cohomology from operadic point of view

Abstract

We show that the operadic cohomology for any type of algebras over a non-symmetric operad $\mathcal{A}$ can be computed as Ext in the category of operadic $\mathcal{A}$-modules. We use this principle to prove that the Gerstenhaber-Schack diagram cohomology is operadic cohomology.

3.1 Introduction

The Operadic Cohomology (OC) gives a systematic way of constructing cohomology theories for algebras A over an operad $\mathcal{A}$. It recovers the classical cases: Hochschild, Chevalley-Eilenberg, Harrison etc. It also applies to algebras over coloured operads (e.g. morphism of algebras) and over PROPs (e.g. bialgebras). The OC first appeared in papers [10], [9] by M. Markl.

Abstractly, the OC is isomorphic to the triple cohomology, at least for algebras over Koszul operads [1]. It is also isomorphic to the André-Quillen Cohomology (AQC). In fact, the definition of OC is analogous to that of AQC: It computes the derived functor of the functor Der of derivations like AQC, but does so in the category of operads. While AQC offers a wider freedom for the choice of a resolution of the given algebra A, OC uses a particular universal resolution for all $\mathcal{A}$-algebras (this resolution is implicit, technically OC resolves the *operad* $\mathcal{A}$). Thus there is, for example, a universal construction of an L_∞ structure on the complex computing OC [13], whose generalized Maurer-Cartan equation describes formal deformations of A.

The success of OC is due to the Koszul duality theory [7], which allows us to construct resolutions of Koszul operads explicitly. Koszul theory has received a lot of attention recently [18] and now goes beyond operads. However, it still has its limitations:

On one hand, it is bound to quadratic relations in a presentation of the operad $\mathcal{A}$. The problem with higher relations can be remedied by using a different presentation, but it comes at the cost of increasing the size of the resolution (e.g. [4]). This is not a major problem in applications, but the minimal resolutions have some nice properties - namely they are unique up to an isomorphism thus providing a cohomology theory unique already at the chain level. So the construction of the minimal resolutions is still of interest.

On the other hand, there are quadratic operads which are not Koszul and for those very little is known [15].

In this paper, we show that OC is isomorphic to Ext in the category of operadic $\mathcal{A}$-modules. Thus instead of resolving the operad $\mathcal{A}$, it suffices to find a projective resolution of a specific $\mathcal{A}$-module $\mathcal{MDA}$ associated to $\mathcal{A}$. The ideas used here were already sketched in the paper [13] by M. Markl.

The resolution of operadic modules are probably much easier to construct explicitly than resolutions of operads, though this has to be explored yet. This simplification allows us to make a small step beyond Koszul theory:

An interesting example of a non-Koszul operad is the coloured operad describing a diagram of a fixed shape consisting of algebras over a fixed operad and morphisms of those algebras. The case of a single morphism between two algebras over a Koszul operad is long well understood. For a morphism between algebras over a general operad as well as for diagrams of a few simple shapes, some partial results were obtained in [12]. These are however not explicit enough to write down the OC.

On the other hand, a satisfactory cohomology for diagrams was invented by Gerstenhaber and Schack [5] in an ad-hoc manner. In [2], the authors proved that the Gerstenhaber-Schack cohomology of a single morphism of associative or Lie algebras is operadic cohomology. We use our theory to extend this result to arbitrary diagrams.

The method used can probably be applied in a more general context to show that a given cohomology theory is isomorphic to OC. The original example is [13] (and similar approach also appears in [17]), where the author proves that Gerstenhaber-Schack *bialgebra* cohomology is the operadic cohomology. Also the method might give an insight into the structure of operadic resolutions themselves, the problem we won't mention in this paper.

On the way, we obtain a modification of the usual OC which includes the quotient by infinitesimal automorphisms (Section 3.3.3).

Also an explicit description of a free resolution of the operad $\mathcal{A}$ *with adjoined derivation* is given if a free resolution of $\mathcal{A}$ is explicitly given. This appeared already in [13] and produces several new examples of minimal resolutions and as such might be of an independent interest.

We assume the reader is familiar with the language of operads (e.g. [16],[8]).

Finally, I would like to thank Martin Markl for many useful discussions.

3.1.1 Convention. As our main object of interest is a diagram of associative algebras, we will get by with *non-symmetric* operads, that is operads with no action of the permutation groups. The results can probably be generalized in a straightforward way to symmetric operads.

Contents

In **Section 3.2**, we briefly recall basic notions of the operad theory with focus on *coloured* operads (see also [11],[12]). We pay special attention to operadic modules. We introduce the notion of tree composition which is just a convenient way to write down complicated operadic compositions. In Section 3.2.3, we discuss free product of operads and obtain a form of the Künneth formula computing homology of the free product.

In **Section 3.3**, we develop the theory sketched by M. Markl in Appendix B of [13]. We give full details for coloured operads. We begin by recalling the operadic cohomology. In Section 3.3.2, we construct an explicit resolution of the operad $\mathcal{DA}$ describing algebras over $\mathcal{A}$ with adjoined derivation assuming we know an explicit resolution of the operad $\mathcal{A}$. In Section 3.3.3, we clarify the significance of operadic derivations on the resolution of $\mathcal{DA}$ with values in $\mathcal{E}nd_A$. This leads to an augmentation of the cotangent complex which has nice interpretation in terms of formal deformation theory. In Sections 3.3.4 and 3.3.5, we realize that all the information needed to construct augmented cohomology is contained in a certain operadic *module*. This module is intrinsically characterized by being a resolution (in the category of operadic modules) of $\mathcal{MDA}$, a certain module constructed from $\mathcal{A}$ in a very simple way.

In **Section 3.4**, we apply the theory to prove that the Gerstenhaber-Schack diagram cohomology is isomorphic to the operadic cohomology. We begin by explaining how a diagram of associative algebras is described by an operad $\mathcal{A}$. We also make the associated module $\mathcal{MDA}$ explicit. Then we recall the Gerstenhaber-Schack cohomology and obtain a candidate for a resolution of $\mathcal{MDA}$. In Section 3.4.3, we verify that the candidate is a valid resolution. This computation is complicated, but still demonstrates the technical advantage of passing to the modules.

3.2 Basics

Fix the following symbols:

- C is a set of colours.
- k is a field of characteristics 0.
- $\mathbb{N}_0$ is the set of natural numbers including 0.

We will also use the following notations and conventions:

- Vector spaces over k are called *k-modules*, chain complexes of vector spaces over k with differential of degree -1 are called *dg-k-modules* and morphisms of chain complexes are called just *maps*. Chain complexes are assumed non-negatively graded unless stated otherwise.
- $|x|$ is the degree of an element x of a dg-k-module.
- $H_*(A)$ is homology of the object A, whatever A is.
- $k\langle S\rangle$ is the k-linear span of the set S.
- $\mathrm{ar}(v)$ is arity of the object v, whatever v is.
- *Quism* is a map f of dg-k-modules such that the induced map $H_*(f)$ on homology is an isomorphism.

3.2.1 Definition. A **dg-C-collection** X is a set

$$\left\{X\binom{c}{c_1,\ldots,c_n} \mid n\in\mathbb{N}_0, c, c_1,\ldots,c_n\in C\right\}$$

of dg-k-modules. We call c the *output* colour of elements of $X\binom{c}{c_1,\ldots,c_n}$, $c_1,\ldots,c_n$ are the *input* colours, n is the *arity*. We also admit $n=0$.

When the above dg-k-modules have zero differentials, we talk just about graded C-collection. If moreover no grading is given, we talk just about C-collection. All notions that follow have similar analogues. If the context is clear, we might omit the prefixes dg-C completely.

A **dg-C-operad** $\mathcal{A}$ is a dg-C-collection $\mathcal{A}$ together with a set of of dg-k-module maps

$$\circ_i : \mathcal{A}\binom{c}{c_1,\ldots,c_k}\otimes\mathcal{A}\binom{c_i}{d_1,\cdots,d_l}\to\mathcal{A}\binom{c}{c_1,\cdots,c_{i-1},d_1,\cdots,d_l,c_{i+1},\cdots c_k},$$

called *operadic compositions*, one for each choice of $k,l\in\mathbb{N}_0$, $1\le i\le k$ and $c,c_1,\ldots,c_k,d_1,\ldots,d_l\in C$, and a set of units

$$e: k\to\mathcal{A}\binom{c}{c},$$

one for each $c\in C$. These maps satisfy the usual associativity and unit axioms, e.g. [16].

The initial dg-C-operad is denoted I.

Equivalently, dg-C-operad is a monoid in the monoidal category of dg-C-collections with the **composition product** $\circ$:

$$(\mathcal{A} \circ \mathcal{B})\binom{c}{c_1, \ldots, c_n} :=$$
$$\bigoplus_{\substack{k \geq 0, \\ i_1, \ldots, i_k \geq 0, \\ d_1, \ldots, d_k \in C}} \mathcal{A}\binom{c}{d_1, \ldots, d_k} \otimes \mathcal{B}\binom{d_1}{c_1, \ldots, c_{i_1}} \otimes \cdots \otimes \mathcal{B}\binom{d_k}{c_{i_1 + \cdots i_{k-1} + 1}, \ldots, c_n}.$$

In contrast to the uncoloured operads, the composition is defined only for the "correct" colours and there is one unit in every colour, i.e. $I\binom{c}{c} = k$ for every $c \in C$. Hence we usually talk about the unit$\underline{\text{s}}$. We denote by $\mathbb{1}_c$ the image of $1 \in k = I\binom{c}{c}$, hence $\operatorname{Im} e = \bigoplus_{c \in C} k\langle \mathbb{1}_c \rangle$. The notation $\mathbb{1}_c$ for units coincides with the notation for identity morphisms. The right meaning will always be clear from the context.

For a dg-C-operad $\mathcal{A}$, we can consider its homology $H_*(\mathcal{A})$. The operadic composition descends to $H_*(\mathcal{A})$. Obviously, the units $\mathbb{1}_c$ are concentrated in degree 0 and by our convention on non-negativity of the grading, $\mathbb{1}_c$ defines a homology class $[\mathbb{1}_c]$. It is a unit in $H^*(\mathcal{A})$. It can happen that $[\mathbb{1}_c] = 0$ in which case it is easily seen that $H_*(\mathcal{A})\binom{c_0}{c_1, \ldots, c_n} = 0$ whenever any of c_i's equals c. If all $[\mathbb{1}_c]$'s are nonzero, then $H_*(\mathcal{A})$ is a graded C-operad.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two dg-C-collections. Then **dg-C-collection morphism** f is a set of dg maps

$$f\binom{c}{c_1, \ldots, c_n} : \mathcal{M}_1\binom{c}{c_1, \ldots, c_n} \to \mathcal{M}_2\binom{c}{c_1, \ldots, c_n},$$

one for each $n \in \mathbb{N}_0, c, c_1, \ldots, c_n \in C$.

The dg-C-collection morphisms are composed "colourwise" in the obvious way.

A **dg-C-operad morphism** is a dg-C-collection morphisms preserving the operadic compositions and units.

Recall that given a dg-k-module A, the *endomorphism operad* $\mathcal{E}nd_A$ comes equipped with the differential

$$\partial_{\mathcal{E}nd_A} f := \partial_A f - (-1)^{|f|} f \partial_{A^{\otimes n}}$$

for $f \in \mathcal{E}nd_A(n)$ homogeneous. Let there be a decomposition

$$(A, \partial_A) = \bigoplus_{c \in C} (A_c, \partial_{A_c}).$$

Then the endomorphism operad is naturally a dg-C-operad via

$$\mathcal{E}nd_A\binom{c}{c_1, \ldots, c_n} := \operatorname{Hom}_k(A_{c_1} \oplus \cdots \oplus A_{c_n}, A_c).$$

An **algebra over a dg-C-operad $\mathcal{A}$** is a dg-C-operad morphism

$$(\mathcal{A}, \partial_{\mathcal{A}}) \to (\mathcal{E}nd_A, \partial_{\mathcal{E}nd_A}).$$

3.2.1 Operadic modules

3.2.2 Definition. Let $\mathcal{A} = (\mathcal{A}, \partial_{\mathcal{A}})$ be a dg-C-operad. An (operadic) **dg-$\mathcal{A}$-module $\mathcal{M}$** is a dg-C-collection

$$\left\{ \mathcal{M}\binom{c}{c_1, \dots, c_n} \mid n \in \mathbb{N}_0, c, c_1, \dots, c_n \in C \right\}$$

with structure maps

$$\circ_i^L : \mathcal{A}\binom{c}{c_1, \cdots, c_k} \otimes \mathcal{M}\binom{c_i}{d_1, \cdots, d_l} \to \mathcal{M}\binom{c}{c_1, \cdots, c_{i-1}, d_1, \cdots, d_l, c_{i+1}, \cdots c_k},$$
$$\circ_i^R : \mathcal{M}\binom{c}{c_1, \cdots, c_k} \otimes \mathcal{A}\binom{c_i}{d_1, \cdots, d_l} \to \mathcal{M}\binom{c}{c_1, \cdots, c_{i-1}, d_1, \cdots, d_l, c_{i+1}, \cdots c_k},$$

one for each choice of $c, c_1, \cdots, d_1, \cdots \in C$ and $1 \leq i \leq k$. These structure maps are required to satisfy the expected axioms:

$$(\alpha_1 \circ_j \alpha_2) \circ_i^L m = \begin{cases} (-1)^{|\alpha_2||m|}(\alpha_1 \circ_i^L m) \circ_{j+\mathrm{ar}(m)-1}^R \alpha_2 & \dots \quad i < j \\ \alpha_1 \circ_j^L (\alpha_2 \circ_{i-j+1}^L m) & \dots \quad j \leq i \leq j + \mathrm{ar}(\alpha_2) - 1 \\ (-1)^{|\alpha_2||m|}(\alpha_1 \circ_{i-\mathrm{ar}(\alpha_2)+1}^L) \circ_j^R \alpha_2 & \dots \quad i \geq j + \mathrm{ar}(\alpha_2), \end{cases}$$

$$m \circ_i^R (\alpha_1 \circ_j \alpha_2) = (m \circ_i^R \alpha_1) \circ_{j+i-1}^R \alpha_2,$$

$$(\alpha_1 \circ_i^L m) \circ_j^R \alpha_2 = \begin{cases} (-1)^{|\alpha_2||m|}(\alpha_1 \circ_j \alpha_2) \circ_{i+\mathrm{ar}(\alpha_2)-1}^L m & \cdots \quad j < i \\ \alpha_1 \circ_i^L (m \circ_{j-i+1}^R \alpha_2) & \cdots \quad i \leq j \leq i + \mathrm{ar}(m) - 1 \\ (-1)^{|\alpha_2||m|}(\alpha_1 \circ_{j-\mathrm{ar}(m)+1} \alpha_2) \circ_i^L m & \cdots \quad j \geq i + \mathrm{ar}(m) \end{cases}$$

and

$$1_c \circ_1 m = m = m \circ_i 1_{c_i} \quad \dots \quad 1 \leq i \leq \mathrm{ar}(m)$$

for $\alpha_1, \alpha_2 \in \mathcal{A}$ and $m \in \mathcal{M}\binom{c}{c_1, \cdots, c_{\mathrm{ar}(m)}}$ in the correct colours. We usually omit the upper indices L, R, writing only $\circ_i$ for all the operations.

A **morphism of dg-$\mathcal{A}$-modules** $\mathcal{M}_1, \mathcal{M}_2$ is a dg-C-collection morphism $\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2$ satisfying

$$f(a \circ_i^L m) = a \circ_i^L f(m),$$
$$f(m \circ_i^R a) = f(m) \circ_i^R a.$$

We expand the definition of dg-$\mathcal{A}$-module. Recall that each $\mathcal{M}\binom{c}{c_1, \dots, c_n}$ is a a dg-k-module. The differentials of these dg-k-modules define a dg-C-collection morphism $\partial_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ of degree -1 satisfying $\partial_{\mathcal{M}}^2 = 0$. The structure maps $\circ_i^L$ and $\circ_i^R$ commute with the differentials on the tensor products. Hence $\partial_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a derivation in the following sense:

$$\partial_{\mathcal{M}}(a \circ_i m) = \partial_{\mathcal{A}} a \circ_i m + (-1)^{|a|} a \circ_i \partial_{\mathcal{M}} m,$$
$$\partial_{\mathcal{M}}(m \circ_i a) = \partial_{\mathcal{M}} m \circ_i a + (-1)^{|m|} m \circ_i \partial_{\mathcal{A}} a.$$

As in the case of modules over a ring, $\mathcal{A}$-modules form an abelian category. We have "colourwise" kernels, cokernels, submodules etc. There is a free $\mathcal{A}$-module generated by a C-collection M, denoted

$$\mathcal{A} \langle M \rangle$$

and satisfying the usual universal property. As an example of an explicit description of $\mathcal{A}\langle M\rangle$, let $\mathcal{A} := \mathbb{F}(M_1)$ be a free C-operad generated by a C-collection M_1. Then $\mathcal{A}\langle M_2\rangle$ is spanned by all planar trees, whose exactly one vertex is decorated by an element of M_2 and all the other vertices are decorated by elements of M_1 such that the colours are respected in the obvious sense.

We warn the reader that the notion of operadic module varies in the literature. For example the monograph [3] uses a different definition.

While dealing with $\mathcal{A}$-modules, it is useful to introduce the following **infinitesimal composition product** $\mathcal{A}\circ'(\mathcal{B},\mathcal{C})$ of C-collections $\mathcal{A},\mathcal{B},\mathcal{C}$:

$$(\mathcal{A}\circ'(\mathcal{B},\mathcal{C}))\binom{c}{c_1,\ldots,c_n} := \tag{3.1}$$

$$\bigoplus_{\substack{k\geq 0, l>0,\\ 0\leq i_1\leq\ldots\leq i_{k-1}\leq n,\\ d_1,\ldots,d_k\in C}} \mathcal{A}\binom{c}{d_1,\ldots,d_k}\otimes\mathcal{B}\binom{d_1}{c_1,\ldots,c_{i_1}}\otimes\cdots\otimes\underbrace{\mathcal{C}\binom{d_l}{c_{i_{l-1}+1},\ldots,c_{i_l}}}_{l^{th}-\text{position}}\otimes\cdots\otimes\mathcal{B}\binom{d_k}{c_{i_{k-1}+1},\ldots,c_n}.$$

See also [8]. We denote by

$$\mathcal{A}\circ'_l(\mathcal{B},\mathcal{C})$$

the projection of $\mathcal{A}\circ'(\mathcal{B},\mathcal{C})$ onto the component with fixed l.

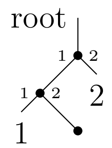
For the free module, we have the following description using the infinitesimal composition product:

$$\mathcal{A}\langle M\rangle \cong \mathcal{A}\circ'(I, M\circ\mathcal{A}).$$

3.2.2 Tree composition

An (unoriented) *graph* (without loops) is a set V of vertices, a set H_v of half edges for every $v\in V$ and a set E of (distinct) unordered pairs (called *edges*) of distinct elements of V. If $e := (v,w)\in E$, we say that the vertices v,w are adjacent to the edge e and the edge e is adjacent to the vertices v,w. Denote E_v the set of all edges adjacent to v. Similarly, for $h\in H_v$, we say that the vertex v is adjacent to the half edge h and vice versa. A path connecting vertices v,w is a sequence $(v,v_1),(v_1,v_2),\ldots,(v_n,w)$ of *distinct* edges. A *tree* is a graph such that for every two vertices v,w there is a path connecting them iff $v\neq w$. A *rooted* tree is a tree with a chosen half edge, called *root*. The *root vertex* is the unique vertex adjacent to the root. The half edges other than the root are called *leaves*. For every vertex v except for the root vertex, there is a unique edge $e_v\in E_v$ contained in the unique path connecting v to the root vertex. The edge e_v is called *output* and the other edges and half edges adjacent to v are called *legs* or *inputs* of v. The root is, by definition, the output of the root vertex. The number of legs of v is called arity of v and is denoted $\mathrm{ar}(v)$. Notice we also admit vertices with no legs, i.e. vertices of arity 0. A *planar* tree is a tree with a given ordering of the set $H_v\sqcup E_v-\{e_v\}$ for each $v\in V$ (the notation $\sqcup$ stands for the disjoint union). The planarity induces an ordering on the set of all leaves, e.g.

by numbering them $1, 2, \ldots$. For example,

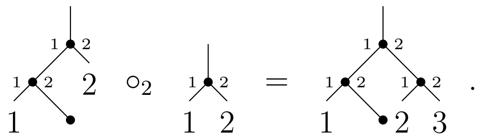


is a planar tree with 3 vertices, 3 half edges and 2 edges. We use the convention that the topmost half edge is always the root. Then there are 2 leaves. The planar ordering of legs of all vertices is denoted by small numbers and the induced ordering of leaves is denoted by big numbers.

Let T_1, T_2 be two planar rooted trees, let T_1 have n leaves and for $j = 1, 2$ let V_j resp. $H_{j,v}$ resp. E_j denote the set of vertices resp. half edges resp. edges of T_j. For $1 \leq i \leq n$ we have the grafting operation $\circ_i$ producing a planar rooted tree $T_1 \circ_i T_2$ defined as follows: first denote l the i^{th} leg of T_1 and denote v_1 the vertex adjacent to l, denote r the root of T_2 and denote v_2 the root vertex of T_2. Then the set of vertices of $T_1 \circ_i T_2$ is $V_1 \sqcup V_2$, the set H_v of half edges is

$$H_v = \begin{cases} H_{1,v} & \ldots \quad v \in V_1 - \{v_1\} \\ H_{1,v_1} - \{l\} & \ldots \quad v = v_1 \\ H_{2,v} & \ldots \quad v \in V_2 - \{v_2\} \\ H_{2,v_2} - \{r\} & \ldots \quad v = v_2 \end{cases},$$

and finally the set of edges is $E_1 \sqcup E_2 \sqcup \{(v_1, v_2)\}$. The planar structure is inherited in the obvious way. For example,



From this point on, *tree* will always mean a planar rooted tree. Such trees can be used to encode compositions of elements of an operad including those of arity 0.

Let T be a tree with n vertices $v_1, \ldots, v_n$. Suppose moreover that the vertices of T are ordered, i.e. there is a bijection $b : \{v_1, \ldots, v_n\} \to \{1, \ldots, n\}$. We denote such a tree with ordered vertices by T_b.

Now we explain how the bijection b induces a structure of tree with levels on T_b such that each vertex is on a different level. Intuitively, b encodes *in what order* are elements of an operad composed. We formalize this as follows:

Let $p_1, \ldots, p_n \in \mathcal{P}$ be elements of a dg-C-operad $\mathcal{P}$ such that if two vertices v_i, v_j are adjacent to a common edge e, which is simultaneously the l^{th} leg of v_i and the output of v_j, then the l^{th} input colour of p_i equals the output colour of p_j. We say that v_i is *decorated* by p_i. Define inductively: Let i be such that v_i is the root vertex. Define

$$T^1 := v_i,$$
$$T_b^1(p_1, \ldots, p_n) := p_i.$$

Here we are identifying v_i with the corresponding corolla. Assume a subtree T^{k-1} of T and $T_b^{k-1}(p_1, \ldots, p_n) \in \mathcal{P}$ are already defined. Consider the set J of all j's such that $v_j \notin T^{k-1}$ and there is an edge e between v_j and some vertex v in T^{k-1}. Let $i \in J$ be such that $b(v_i) = \min\{b(v_j) : j \in J\}$. Let l be the number of the leg e of vertex v in the planar ordering of T and define

$$T^k := T^{k-1} \circ_l v_i,$$
$$T_b^k(p_1, \dots, p_n) := T_b^{k-1}(p_1, \dots, p_n) \circ_l p_i.$$

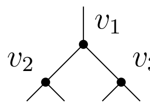
In the upper equation, we are using the operation $\circ_l$ of grafting of trees. Finally

$$T_b(p_1, \dots, p_n) := T_b^n(p_1, \dots, p_n).$$

$T_b(p_1, \dots, p_n)$ is called **tree composition** of $p_1, \dots, p_n$ along T_b.

If T and p_i's are fixed, changing b may change the sign of $T_b(p_1, \dots, p_n)$. Observe that if $\mathcal{P}$ is concentrated in even degrees (in particular 0) then the sign doesn't change. If b is understood and fixed, we usually omit it.

For example, let

$T :=$ 

For $i = 1, 2, 3$, let p_i be an element of degree 1 and arity 2. Let $b(v_1) = 1$, $b(v_2) = 2$, $b(v_3) = 3$ and $b'(v_1) = 1$, $b'(v_2) = 3$, $b(v_3) = 2$. Then

$$T_b(p_1, p_2, p_3) = (p_1 \circ_1 p_2) \circ_3 p_3 \quad \text{and} \quad T_{b'}(p_1, p_2, p_3) = (p_1 \circ_2 p_3) \circ_1 p_2$$

and by the associativity axiom

$$T_b(p_1, p_2, p_3) = -T_{b'}(p_1, p_2, p_3).$$

A useful observation is that we can always reindex p_i's so that

$$T_b(p_1, \dots, p_n) = (\cdots((p_1 \circ_{i_1} p_2) \circ_{i_2} p_3) \cdots \circ_{i_{n-1}} p_n) \tag{3.2}$$

for some $i_1, i_2, \dots, i_{n-1}$.

Tree compositions are a convenient notation for dealing with operadic derivations.

3.2.3 Free product of operads

3.2.3 Definition. Free product $\mathcal{A} * \mathcal{B}$ of dg-C-operads $\mathcal{A}, \mathcal{B}$ is the coproduct $\mathcal{A} \coprod \mathcal{B}$ in the category of dg-C-operads.

Let A, B be dg-k-modules. The usual Künneth formula states that the map

$$\begin{aligned} H_*(A) \otimes H_*(B) &\xrightarrow{\iota} H_*(A \otimes B) \\ [a] \otimes [b] &\mapsto [a \otimes b] \end{aligned} \tag{3.3}$$

is a natural isomorphism of dg-k-modules, where [ ] denotes a homology class. Our aim here is to prove an analogue of the Künneth formula for the free product of operads, that is

$$H_*(\mathcal{A}) * H_*(\mathcal{B}) \cong H_*(\mathcal{A} * \mathcal{B}) \tag{3.4}$$

naturally as C-operads.

First we describe $\mathcal{A}*\mathcal{B}$ more explicitly. Intuitively, $\mathcal{A}*\mathcal{B}$ is spanned by trees whose vertices are decorated by elements of $\mathcal{A}$ or $\mathcal{B}$ such that no two vertices adjacent to a common edge are both decorated by $\mathcal{A}$ or both by $\mathcal{B}$. Unfortunately, this is not quite true - there are problems with units of the operads.

Recall a dg-C-operad $\mathcal{P}$ is called *augmented* iff there is a dg-C-operad morphism $\mathcal{P} \xrightarrow{a} I$ inverting the unit of $\mathcal{P}$ on the left, i.e. the composition $I \xrightarrow{e} \mathcal{P} \xrightarrow{a} I$ is $\mathbb{1}_I$. The kernel of a is denoted by $\overline{\mathcal{P}}$ and usually called augmentation ideal.

If $\mathcal{A}, \mathcal{B}$ are augmented, we let the vertices be decorated by the augmentation ideals $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ instead of $\mathcal{A}, \mathcal{B}$ and the above description of $\mathcal{A}*\mathcal{B}$ works well. In fact, this has been already treated in [11].

However we will work without the augmentation assumption. Choose a sub-C-collection $\overline{\mathcal{A}}$ of $\mathcal{A}$ such that

$$\overline{\mathcal{A}} \oplus \mathbb{1}_{\mathcal{A}} = \mathcal{A}, \tag{3.5}$$

$$\operatorname{Im} \partial_{\mathcal{A}} \subset \overline{\mathcal{A}}, \tag{3.6}$$

where

$$\mathbb{1}_{\mathcal{A}} := \bigoplus_{c \in C} k\langle 1_c \rangle \,.$$

This is possible iff

$$[1_c] \neq 0 \text{ for all } c \in C,$$

that is if $H_*(\mathcal{A})$ is a graded C-operad. This might not be the case generally as we have already seen at the beginning of Section 3.2 so let's assume it. Choose $\overline{\mathcal{B}}$ for $\mathcal{B}$ similarly.

For given $\overline{\mathcal{A}}, \overline{\mathcal{B}}$, a **free product tree** is a tree T together with

$$c(v), c_1(v), c_2(v), \ldots, c_{\mathrm{ar}(v)}(v) \in C \text{ for each vertex } v$$

and a map

$$\overline{\mathcal{P}}: \text{ vertices of } T \to \{\overline{\mathcal{A}}, \overline{\mathcal{B}}\} \tag{3.7}$$

such that if vertices v_1, v_2 are adjacent to a common edge, which is simultaneously the l^{th} leg of v_1 and the output of v_2, then

$$c_l(v_1) = c(v_2) \quad \text{and} \quad \overline{\mathcal{P}}(v_1) \neq \overline{\mathcal{P}}(v_2).$$

Finally, the description of the free product is as follows:

$$\mathcal{A}*\mathcal{B} := \bigoplus_{c \in C} k\langle 1_c \rangle \oplus \bigoplus_T \bigotimes_v \overline{\mathcal{P}}(v) \binom{c(v)}{c_1(v), \ldots, c_{\mathrm{ar}(v)}(v)}, \tag{3.8}$$

where T runs over all isomorphism classes of free product trees and v runs over all vertices of T and 1_c's are of degree 0. If the vertices of T are $v_1, \ldots, v_n$ then every element of $\bigotimes_v \overline{\mathcal{P}}(v)\binom{c(v)}{c_1(v), \ldots, c_{\mathrm{ar}(v)}(v)}$ can be written as a tree composition $T(x_1, \ldots, x_n)$ where $x_i \in \overline{\mathcal{P}}(v_i)$. We say that v_i is decorated by x_i.

The operadic composition

$$T(x_1,\dots,x_n)\circ_i T'(x'_1,\dots,x'_m)$$

in $\mathcal{A}*\mathcal{B}$ is defined in the obvious way by grafting T and T' (the result $T\circ_i T'$ of the grafting may not be a free product tree) and then (repeatedly) applying the following *reducing operations*:

1. Suppose w_1, w_2 are vertices of $T\circ_i T'$ adjacent to a common edge e which is simultaneously the l^{th}leg of w_1 and the output of w_2. Suppose moreover that w_1 is decorated by p_1 and w_2 by p_2. If both p_1, p_2 are elements of $\overline{\mathcal{A}}$ or both of $\overline{\mathcal{B}}$, then contract e and decorate the resulting vertex by the composition of $p_1\circ_l p_2$.

2. If a vertex is decorated by a unit from $\mathbf{1}_{\mathcal{A}}$ or $\mathbf{1}_{\mathcal{B}}$ (this may happen since neither $\overline{\mathcal{A}}$ nor $\overline{\mathcal{B}}$ is generally closed under the composition!), omit it unless it is the only remaining vertex of the tree.

After several applications of the above reducing operations, we obtain a free product tree or a tree with a single vertex decorated by a unit.

Obviously, $\mathbf{1}_c$'s are units for this composition.

The differential ∂ on $\mathcal{A}*\mathcal{B}$ is determined by (3.8) and the requirement that $\partial(\mathbf{1}_c)=0$ for every $c\in C$. It has the derivation property and equals the differential on $\mathcal{A}$ resp. $\mathcal{B}$ upon the restriction on the corresponding sub-C-operad of $\mathcal{A}*\mathcal{B}$. Explicitly, for $T(x_1,\dots,x_n)\in\mathcal{A}*\mathcal{B}$ with $x_i\in\overline{\mathcal{A}}$ or $\overline{\mathcal{B}}$, assuming (3.2), we have

$$\partial(T(x_1,\dots,x_n))=\sum_{i=1}^{n}\epsilon_i T(x_1,\dots,\partial(x_i),\dots,x_n),$$

where $\epsilon_i:=(-1)^{\sum_{j=1}^{i-1}|x_j|}$.

It is easily seen that the dg-C-operad $(\mathcal{A}*\mathcal{B},\partial)$ just described has the required universal property of the coproduct.

Now we are prepared to prove a version of (3.4) in a certain special case:

3.2.4 Lemma. Let $(\mathcal{A},\partial_{\mathcal{A}})\xrightarrow{\alpha}(\mathcal{A}',\partial_{\mathcal{A}'})$ and $(\mathcal{B},\partial_{\mathcal{B}})\xrightarrow{\beta}(\mathcal{B}',\partial_{\mathcal{B}'})$ be quisms of dg-C-operads, that is we assume homology of $\mathcal{A},\mathcal{A}',\mathcal{B},\mathcal{B}'$ are graded C-operads and $H_*(\alpha), H_*(\beta)$ are graded C-operad isomorphisms. Then there are graded C-operad isomorphisms ι,ι' such that the following diagram commutes:

$$\begin{tikzcd}
H_*(\mathcal{A})*H_*(\mathcal{B}) \arrow[r, "\iota"] \arrow[d, "H_*(\alpha)*H_*(\beta)"'] & H_*(\mathcal{A}*\mathcal{B}) \arrow[d, "H_*(\alpha*\beta)"] \\
H_*(\mathcal{A}')*H_*(\mathcal{B}') \arrow[r, "\iota'"] & H_*(\mathcal{A}'*\mathcal{B}')
\end{tikzcd}$$

Proof. Choose $\overline{\mathcal{A}},\overline{\mathcal{B}}$ so that (3.5) and (3.6) hold. Now we want to choose $\overline{\mathcal{A}}'\subset\mathcal{A}'$ so that

$$\overline{\mathcal{A}}'\oplus\mathbf{1}_{\mathcal{A}'}=\mathcal{A}',$$
$$\alpha(\overline{\mathcal{A}})\subset\overline{\mathcal{A}}' \qquad (3.9)$$

and choose $\overline{\mathcal{B}}' \subset \mathcal{B}'$ similarly. To see that this is possible, we observe $\alpha(\overline{\mathcal{A}}) \cap \mathbb{1}_{\mathcal{A}'} = 0$: If $\alpha(\overline{a}) \in \mathbb{1}_{\mathcal{A}'}$ for some $\overline{a} \in \overline{\mathcal{A}}$, there is $u \in \mathbb{1}_{\mathcal{A}}$ such that $\alpha(u) = \alpha(\overline{a})$, hence $\alpha(\overline{a} - u) = 0$ and $\partial_{\mathcal{A}}(\overline{a} - u) = 0$ since both $\overline{a}$ and u are of degree 0. Since α is a quism, $\overline{a} - u = \partial_{\mathcal{A}} a$ for some $a \in \mathcal{A}$ and by the property (3.6) of $\overline{\mathcal{A}}$ we have $\overline{a} - u \in \overline{\mathcal{A}}$. But this implies $u \in \overline{\mathcal{A}}$, a contradiction.

Now use the explicit description (3.8) of the free product $\mathcal{A} * \mathcal{B}$ and the usual Künneth formula (3.3) to obtain an isomorphism

$$H_*(\mathcal{A}) * H_*(\mathcal{B}) = \bigoplus_T \bigotimes_v H_*(\overline{\mathcal{P}}(v)) \xrightarrow{\iota} H_*(\bigoplus_T \bigotimes_v \overline{\mathcal{P}}(v)) = H_*(\mathcal{A} * \mathcal{B})$$

and similarly for $\mathcal{A}', \mathcal{B}'$.

Assume we are given a free product tree T and its vertex v. The tree T comes equipped with $\overline{\mathcal{P}}$ as in (3.7). Let

$$\overline{\mathcal{P}}'(v) := (\overline{\mathcal{P}}(v))' = \begin{cases} \overline{\mathcal{A}}' & \text{for} \quad \overline{\mathcal{P}}(v) = \overline{\mathcal{A}} \\ \overline{\mathcal{B}}' & \text{for} \quad \overline{\mathcal{P}}(v) = \overline{\mathcal{B}} \end{cases}$$

and define a map

$$\pi(v) : \overline{\mathcal{P}}(v) \to \overline{\mathcal{P}}'(v),$$
$$\pi(v) = \begin{cases} \alpha & \text{for} \quad \overline{\mathcal{P}}(v) = \overline{\mathcal{A}} \\ \beta & \text{for} \quad \overline{\mathcal{P}}(v) = \overline{\mathcal{B}} \end{cases}.$$

This is justified by (3.9). Then the following diagram

$$\begin{tikzcd} \bigoplus_T \bigotimes_v H_*(\overline{\mathcal{P}}(v)) \arrow[r, "\iota"] \arrow[d, "\bigoplus_T \bigotimes_v H_*(\pi(v))"'] & H_*(\bigoplus_T \bigotimes_v \overline{\mathcal{P}}(v)) \arrow[d, "H_*(\bigoplus_T \bigotimes_v \pi(v))"] \\ \bigoplus_T \bigotimes_v H_*(\overline{\mathcal{P}}'(v)) \arrow[r, "\iota'"] & H_*(\bigoplus_T \bigotimes_v \overline{\mathcal{P}}'(v)) \end{tikzcd}$$

commutes by the naturality of the usual Künneth formula. The horizontal C-collection isomorphism ι is given in terms of tree compositions by the formula

$$\iota(T([x_1], [x_2], \ldots)) = [T(x_1, x_2, \ldots)],$$

where $x_1, x_2, \ldots \in \overline{\mathcal{A}}$ or $\overline{\mathcal{B}}$. Now we verify that ι preserves the operadic composition:

$$\iota(T_x([x_1], \ldots)) \circ_i \iota(T_y([y_1], \ldots)) = \iota(T_x([x_1], \ldots) \circ_i T_y([y_1], \ldots)).$$

The left-hand side equals $[T_x(x_1, \ldots) \circ_i T_y(y_1, \ldots)]$, so we check

$$[T_x(x_1, \ldots) \circ_i T_y(y_1, \ldots)] = \iota(T_x([x_1], \ldots) \circ_i T_y([y_1], \ldots)).$$

We would like to perform the same reducing operations on $T_x(x_1, \ldots) \circ_i T_y(y_1, \ldots)$ and $T_x([x_1], \ldots) \circ_i T_y([y_1], \ldots)$ parallely. For the first reducing operation, this is OK. For the second one, if, say, $[x_1] \in \mathbb{1}_{H_*(\mathcal{A})}$, then $x_1 = u + \partial_{\mathcal{A}} a$ for some $u \in \mathbb{1}_{\mathcal{A}}$ and $a \in \mathcal{A}$. Hence $T_x(x_1, \ldots) = T_x(u, \ldots) + T_x(\partial_{\mathcal{A}} a, \ldots)$. So we can go on with $T_x(u, \ldots) \circ_i T_y(y_1, \ldots)$ and $T_x([x_1], \ldots) \circ_i T_y([y_1], \ldots)$, ommiting the vertex v_1 decorated by u resp. $[x_1]$, but we also have to apply the reducing operations to $T_x(\partial_{\mathcal{A}} a, \ldots) \circ_i T_y(y_1, \ldots)$. As it turns out, this tree composition is a boundary in $\mathcal{A} * \mathcal{B}$. We leave the details to the reader. $\square$

3.3 Operadic cohomology of algebras

3.3.1 Reminder

Let $(\mathcal{R}, \partial_{\mathcal{R}}) \xrightarrow{\rho} (\mathcal{A}, \partial_{\mathcal{A}})$ be dg-C-operad over $(\mathcal{A}, \partial_{\mathcal{A}})$, i.e. ρ is a dg-C-operad morphism. Let $(\mathcal{M}, \partial_{\mathcal{M}})$ be a dg-$\mathcal{A}$-module. Define a k-module

$$\mathrm{Der}^n_{\mathcal{A}}(\mathcal{R}, \mathcal{M})$$

consisting of all C-collection morphisms $\theta : \mathcal{R} \to \mathcal{M}$ of degree $|\theta| = n$ in all colours satisfying

$$\theta(r_1 \circ_i r_2) = \theta(r_1) \circ^R_i \rho(r_2) + (-1)^{|\theta||r_1|} \rho(r_1) \circ^L_i \theta(r_2)$$

for any $r_1, r_2 \in \mathcal{R}$ and any $1 \le i \le \mathrm{ar}(r_1)$. Denote

$$\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M}) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n_{\mathcal{A}}(\mathcal{R}, \mathcal{M}).$$

For $\theta \in \mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M})$ homogeneous, let

$$\delta\theta := \theta\partial_{\mathcal{R}} - (-1)^{|\theta|}\partial_{\mathcal{M}}\theta. \tag{3.10}$$

Extending by linearity, the above formula defines a map δ from the k-module $\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{M})$.

3.3.1 Lemma. δ maps derivations to derivations and $\delta^2 = 0$.

Proof. The degree of δ obviously equals -1 and

$$\begin{aligned}
\delta^2\theta &= (\theta\partial_{\mathcal{R}} - (-1)^{|\theta|}\partial_{\mathcal{M}}\theta)\partial_{\mathcal{R}} - (-1)^{|\delta\theta|}\partial_{\mathcal{M}}(\theta\partial_{\mathcal{R}} - (-1)^{|\theta|}\partial_{\mathcal{M}}\theta) = \\
&= \theta\partial^2_{\mathcal{R}} - (-1)^{|\theta|}\partial_{\mathcal{M}}\theta\partial_{\mathcal{R}} - (-1)^{|\theta|+1}\partial_{\mathcal{M}}\theta\partial_{\mathcal{R}} - (-1)^{|\theta|+1+|\theta|+1}\partial^2_{\mathcal{M}}\theta = \\
&= 0.
\end{aligned}$$

The following computation shows that δ maps derivations to derivations:

$$\begin{aligned}
(\delta\theta)(r_1 \circ_i r_2) = \theta &\left(\partial r_1 \circ_i r_2 + (-1)^{|r_1|} r_1 \circ_i \partial r_2\right) + \\
&- (-1)^{|\theta|}\partial\left(\theta r_1 \circ_i \rho r_2 + (-1)^{|\theta||r_1|}\rho r_1 \circ_i \theta r_2\right) = \\
&= \theta\partial r_1 \circ_i \rho r_2 + (-1)^{|\theta|(|r_1|+1)}\rho\partial r_1 \circ_i \theta r_2 + \\
&\quad + (-1)^{|r_1|}\theta r_1 \circ_i \rho\partial r_2 + (-1)^{(|\theta|+1)|r_1|}\rho r_1 \circ_i \theta\partial r_2 + \\
&\quad - (-1)^{|\theta|}\partial\theta r_1 \circ_i \rho r_2 - (-1)^{|r_1|}\theta r_1 \circ_i \partial\rho r_2 + \\
&\quad - (-1)^{|\theta|(|r_1|+1)}\partial\rho r_1 \circ_i \theta r_2 - (-1)^{(|\theta|+1)|r_1|+|\theta|}\rho r_1 \circ_i \partial\theta r_2 = \\
&= (\delta\theta) r_1 \circ_i \rho r_2 + (-1)^{|\delta\theta|\cdot|r_1|}\rho r_1 \circ_i (\delta\theta) r_2
\end{aligned}$$

where we have ommited the subscripts of ∂. □

A particular example of this construction is

$$(\mathcal{R}, \partial_{\mathcal{R}}) \underset{\rho}{\xrightarrow{\sim}} (\mathcal{A}, \partial_{\mathcal{A}}),$$

a cofibrant [11] resolution of a dg-C-operad $\mathcal{A}$, and

$$\mathcal{M} := (\mathcal{E}nd_{\mathcal{A}}, \partial_{\mathcal{E}nd_A}),$$

which is a dg-$\mathcal{A}$-module via a dg-C-operad morphism

$$(\mathcal{A}, \partial_{\mathcal{A}}) \xrightarrow{\alpha} (\mathcal{E}nd_A, \partial_{\mathcal{E}nd_A})$$

determining an $\mathcal{A}$-algebra structure on a dg-k-module $(A, \partial_A) = \bigoplus_{c\in C}(A_c, \partial_{A_c})$.

Let $\uparrow C$ denote the *suspension* of a graded object C, that is $(\uparrow C)_n := C_{n-1}$. Analogously $\downarrow$ denote the *desuspension*.

3.3.2 Definition.

$$(C^*(A, A), \delta) := \uparrow \left(\mathrm{Der}^{-*}((\mathcal{R}, \partial_{\mathcal{R}}), \mathcal{E}nd_A), \delta\right) \tag{3.11}$$

is called **operadic cotangent complex of the $\mathcal{A}$-algebra** A and

$$H^*(A, A) := H^*(C^*(A, A), \delta)$$

is called **operadic cohomology of $\mathcal{A}$-algebra** A.

The change of grading $* \mapsto 1 - *$ is purely conventional. For example, if $\mathcal{A}$ is the operad for associative algebras and $\mathcal{R}$ is its minimal resolution, under our convention we recover the grading of the Hochschild complex for which the bilinear cochains are of degree 1.

3.3.2 Algebras with derivation

Let $\mathcal{A}$ be a dg-C-operad. Consider a C-collection $\Phi := k\langle\phi_c | c \in C\rangle$, such that ϕ_c is of arity 1, degree 0 and the input and output colours are both c. Let $\mathfrak{D}$ be the ideal in $\mathcal{A} * \mathbb{F}(\Phi)$ generated by all elements

$$\phi_c \circ_1 \alpha - \sum_{i=1}^{n} \alpha \circ_i \phi_{c_i} \tag{3.12}$$

for $n \in \mathbb{N}_0, c, c_1, \ldots, c_n \in C$ and $\alpha \in \mathcal{A}\binom{c}{c_1,\ldots,c_n}$. Denote

$$\mathcal{D}\mathcal{A} := \left(\frac{\mathcal{A} * \mathbb{F}(\Phi)}{\mathfrak{D}}, \partial_{\mathcal{D}\mathcal{A}}\right),$$

where $\partial_{\mathcal{D}\mathcal{A}}$ is the derivation given by the formulas

$$\partial_{\mathcal{D}\mathcal{A}}(a) := \partial_{\mathcal{A}}(a), \quad \partial_{\mathcal{D}\mathcal{A}}(\phi_c) := 0$$

for $a \in \mathcal{A}$ and $c \in C$.

An algebra over $\mathcal{D}\mathcal{A}$ is a pair (A, ϕ), where $A = \bigoplus_{c\in C} A_c$ is an algebra over $\mathcal{A}$ and ϕ is a derivation of A in the following sense: ϕ is a collection of degree 0 dg-maps $\phi_c : A_c \to A_c$ such that

$$\phi_c(\alpha(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \alpha(a_1, \ldots, \phi_{c_i}(a_i), \ldots, a_n)$$

for $\alpha \in \mathcal{A}\binom{c}{c_1,\dots,c_n}$ and $a_j \in A_{c_j}$, $1 \le j \le n$.

Given a *free* resolution

$$\mathcal{R} := (\mathbb{F}(X), \partial_{\mathcal{R}}) \xrightarrow{\rho_{\mathcal{R}}} (\mathcal{A}, \partial_{\mathcal{A}}), \tag{3.13}$$

where X is a dg-C-collection, it is surprisingly easy to explicitly construct a free resolution of $(\mathcal{DA}, \partial_{\mathcal{DA}})$. Consider the free graded C-operad

$$\mathcal{DR} := \mathbb{F}(X \oplus \Phi \oplus \underline{X}),$$

where $\underline{X} := \uparrow X$. We denote by $\underline{x}$ the element $\uparrow x \in \underline{X}$ corresponding to $x \in X$. To describe the differential, let $s : \mathbb{F}(X) \to \mathcal{DR}$ be a degree $+1$ derivation determined by

$$s(x) := \underline{x} \text{ for } x \in X.$$

Then define a degree -1 derivation $\partial_{\mathcal{DR}} : \mathcal{DR} \to \mathcal{DR}$ by

$$\begin{aligned} &\partial_{\mathcal{DR}}(x) := \partial_{\mathcal{R}}(x), \\ &\partial_{\mathcal{DR}}(\phi_c) := 0, \\ &\partial_{\mathcal{DR}}(\underline{x}) := \phi_c \circ_1 x - \sum_{i=0}^{n} x \circ_i \phi_{c_i} - s(\partial_{\mathcal{R}} x). \end{aligned} \tag{3.14}$$

3.3.3 Convention. From now on we will assume

$$n \in \mathbb{N}_0,\ c, c_1, \dots, c_n \in C,\ x \in X\binom{c}{c_1, \dots, c_n}$$

whenever any of these symbols appears. We will usually omit the lower indices c and c_i's for ϕ.

3.3.4 Lemma. ${\partial_{\mathcal{DR}}}^2 = 0$.

Proof. Using the notion of the tree compositions introduced in Section 3.2.2, let $\partial_{\mathcal{R}}(x) = \sum_i T_i(x_{i1}, \cdots, x_{in_i})$.

$$\begin{aligned} {\partial_{\mathcal{DR}}}^2(x) = \partial_{\mathcal{DR}} &\left(\phi \circ_1 x - \sum_{j=1}^{n} x \circ_j \phi - s(\partial_{\mathcal{R}}(x)) \right) = \\ &= \phi \circ_1 \partial_{\mathcal{DR}}(x) - \sum_{j=1}^{n} \partial_{\mathcal{DR}}(x) \circ_j \phi + \\ &- \partial_{\mathcal{DR}} \left(\sum_i \sum_{j=1}^{n_i} \epsilon_{ij} T_i(x_{i1}, \dots, \underline{x_{ij}}, \dots, x_{in_i}) \right) \end{aligned}$$

If we assume (3.2), then $\epsilon_{ij} = (-1)^{\sum_{l=1}^{j-1} |x_{il}|}$. The last application of $\partial_{\mathcal{DR}}$ on the double sum can be rewritten as

$$\begin{aligned}
&\sum_i \sum_{j=1}^{n_i} \sum_{\substack{1\le k\le n_i,\\ k\neq j}} \tilde{\epsilon}_{ijk} T_i(x_{i1},\ldots,\partial_{\mathcal{R}}(x_{ik}),\ldots,\underline{x_{ij}},\ldots,x_{in_i}) + \\
&+\sum_i \sum_{j=1}^{n_i} T_i(x_{i1},\ldots,\phi\circ_1 x_{ij},\ldots,x_{in_i}) + \\
&-\sum_i \sum_{j=1}^{n_i} \sum_{k=1}^{\mathrm{ar}(x_{ij})} T_i(x_{i1},\ldots,x_{ij}\circ_k \phi,\ldots,x_{in_i}) + \\
&-\sum_i \sum_{j=1}^{n_i} T_i(x_{i1},\ldots,s(\partial_{\mathcal{R}}(x_{ij})),\ldots,x_{in_i}),
\end{aligned}$$

where $\tilde{\epsilon}_{ijk} = \epsilon_{ij}\epsilon_{ik}$ if $k<j$ and $\tilde{\epsilon}_{ijk} = -\epsilon_{ij}\epsilon_{ik}$ if $k>j$. The second and third lines sum to

$$\begin{aligned}
\sum_i \phi\circ_1 T_i(x_{i1},\ldots,x_{in_i}) - \sum_i \sum_{j=1}^{\mathrm{ar}(T_i)} T_i(x_{i1},\ldots,x_{in_i})\circ_j \phi = \\
= \phi\circ_1 \partial_{\mathcal{DR}}(x) - \sum_{j=1}^{n} \partial_{\mathcal{DR}}(x)\circ_j \phi,
\end{aligned}$$

while the first and last rows sum to

$$-s\partial_{\mathcal{DR}}\left(\sum_i T_i(x_{i1},\cdots,x_{in_i})\right) = -s{\partial_{\mathcal{DR}}}^2 = 0$$

and this concludes the computation. □

From now on, we will refer by $\mathcal{DR}$ also to the dg-C-operad $(\mathcal{DR},\partial_{\mathcal{DR}})$. Define a C-operad morphism $\rho_{\mathcal{DR}}:\mathcal{DR}\to\mathcal{DA}$ by

$$\begin{aligned}
\rho_{\mathcal{DR}}(x) &:= \rho_{\mathcal{R}}(x),\\
\rho_{\mathcal{DR}}(\phi_c) &:= \phi_c,\\
\rho_{\mathcal{DR}}(\underline{x}) &:= 0.
\end{aligned}$$

3.3.5 Theorem. $\rho_{\mathcal{DR}}$ is a free resolution of $\mathcal{DA}$.

3.3.6 Example. Let's see what we get for $\mathcal{A} := \mathcal{A}ss = \mathbb{F}(\mu)/(\mu\circ_1\mu - \mu\circ_2\mu)$ and its minimal resolution (see e.g. [12]) $\mathcal{R} := \mathcal{A}ss_\infty = (\mathbb{F}(X),\partial_{\mathcal{R}}) \xrightarrow{\rho_{\mathcal{R}}} (\mathcal{A}ss, 0)$, where

$$X = k\langle x^2, x^3, \ldots\rangle$$

is the collection spanned by x^n in arity n and degree $|x^n| = n-2$ and $\partial_{\mathcal{R}}$ is a derivation differential given by

$$\partial_{\mathcal{R}}(x^n) := \sum_{i+j=n+1} \sum_{k=1}^{i} (-1)^{i+(k+1)(j+1)} x^i \circ_k x^j$$

and the quism $\rho_{\mathcal{R}} : \mathcal{R} = \mathcal{A}ss_\infty \to \mathcal{A}ss = \mathcal{A}$ is given by

$$\rho_{\mathcal{R}}(x^2) := \mu, \quad \rho_{\mathcal{R}}(x^n) := 0 \text{ for } n \geq 3.$$

Then the associated operad with derivation is

$$\mathcal{DA} := \frac{\mathcal{A}ss * \mathbb{F}(\Phi)}{(\phi \circ \mu - \mu \circ_1 \phi - \mu \circ_2 \phi)},$$

where $\Phi := k\langle \phi \rangle$ with ϕ a generator of arity 1. Its free resolution is

$$\mathcal{DR} := (\mathbb{F}(X \oplus \Phi \oplus \underline{X}), \partial_{\mathcal{DR}}) \xrightarrow{\rho_{\mathcal{DR}}} (\mathcal{DA}, 0),$$

where the differential $\partial_{\mathcal{DR}}$ is given by

$$\begin{aligned}
&\partial_{\mathcal{DR}}(x) := \partial_{\mathcal{R}}(x),\\
&\partial_{\mathcal{DR}}(\phi) := 0,\\
&\partial_{\mathcal{DR}}(\underline{x}^n) := \phi \circ_1 x^n - \sum_{i=1}^{n} x^n \circ_i \phi - \sum_{i+j=n+1} \sum_{k=1}^{i} (-1)^{i+(k+1)(j+1)} (\underline{x}^i \circ_k x^j + (-1)^i x^i \circ_k \underline{x}^j)
\end{aligned}$$

and the quism $\rho_{\mathcal{DR}}$ by

$$\rho_{\mathcal{DR}}(x) := \rho_{\mathcal{R}}(x), \quad \rho_{\mathcal{DR}}(\phi) := \phi, \quad \rho_{\mathcal{DR}}(\underline{x}) = 0.$$

of Theorem 3.3.5. Obviously $\rho_{\mathcal{DR}}$ has degree 0 and commutes with differentials because of the relations in $\mathcal{DA}$. Let's abbreviate $\partial_{\mathcal{DR}} =: \partial$. First we want to use a spectral sequence to split ∂ such that ∂^0, the 0^{th} page part of ∂, is nontrivial only on the generators from $\underline{X}$.

Let's put an additional grading gr on the C-collection $X \oplus \Phi \oplus \underline{X}$ of generators:

$$\mathrm{gr}(x) := |x|, \quad \mathrm{gr}(\phi) := 1, \quad \mathrm{gr}(\underline{x}) := |\underline{x}|.$$

This induces a grading on $\mathcal{DR}$ determined by the requirement that the composition is of gr degree 0. Let

$$\mathfrak{F}_p := \bigoplus_{i=0}^{p} \{z \in \mathcal{DR} \mid \mathrm{gr}(z) = i\}.$$

Obviously $\partial_{\mathcal{DR}} \mathfrak{F}_p \subset \mathfrak{F}_p$. Consider the spectral sequence E^* associated to the filtration

$$0 \hookrightarrow \mathfrak{F}_0 \hookrightarrow \mathfrak{F}_1 \hookrightarrow \cdots$$

of $\mathcal{DR}$. On $\mathcal{DA}$ we have the trivial filtration

$$0 \hookrightarrow \mathcal{DA}$$

and the associated spectral sequence E'^*.

We will show that $\rho_{\mathcal{DR}}$ induces quism $(E^1, \partial^1) \xrightarrow{\sim} (E'^1, \partial'^1)$. Then we can use the comparison theorem since both filtrations are obviously bounded below and exhaustive (e.g. [19], page 126, Theorem 5.2.12, and page 135, Theorem 5.5.1).

Then the 0^{th} page satisfies $E^0 \cong \mathbb{F}(X \oplus \Phi \oplus \underline{X})$ and is equipped with the derivation differential ∂^0:

$$\partial^0(x) = 0 = \partial^0(\phi_c), \quad \partial^0(\underline{x}) = \phi_c \circ_1 x - \sum_{i=1}^{\mathrm{ar}(x)} x \circ_i \phi_{c_i}.$$

Denote by $\mathfrak{D}$ the ideal in $\mathbb{F}(X \oplus \Phi)$ generated by

$$\phi_c \circ_1 x - \sum_{i=1}^{\mathrm{ar}(x)} x \circ_i \phi_{c_i} \tag{3.15}$$

for all $x \in X\binom{c}{c_1,\dots,c_{\mathrm{ar}(x)}}$ of arbitrary colours.

3.3.7 Sublemma.

$$H_*(E^0, \partial^0) \cong \frac{\mathbb{F}(X \oplus \Phi)}{\mathfrak{D}}$$

Once this sublemma is proved, ∂^1 on $E^1 \cong H_*(E^0, \partial^0) \cong \frac{\mathbb{F}(X\oplus\Phi)}{\mathfrak{D}}$ will be given by

$$\partial^1(x) = \partial_{\mathcal{R}}(x), \quad \partial^1(\phi_c) = 0. \tag{3.16}$$

We immediately see that $E'^1 \cong \mathcal{DA}$ and it is equipped with the differential $\partial'^1 = \partial_{\mathcal{DA}}$. To see that ${\rho_{\mathcal{DR}}}^1 : E^1 \to E'^1$ induced by $\rho_{\mathcal{DR}}$ is a quism, observe that we can use the relations (3.12) in $\mathcal{DA}$ to "move all the ϕ's to the bottom of the tree compositions", hence, denoting

$$\Phi' := \mathbb{F}(\Phi),$$

we have

$$\mathcal{DA} \cong \mathcal{A} \circ \Phi'.$$

The composition and the differential on $\mathcal{A} \circ \Phi'$ are transferred along this isomorphism from $\mathcal{DA}$. Similarly,

$$\frac{\mathbb{F}(X \oplus \Phi)}{\mathfrak{D}} \cong \mathbb{F}(X) \circ \Phi'. \tag{3.17}$$

Under these quisms

$${\rho_{\mathcal{DR}}}^1 \text{ becomes } \rho_{\mathcal{R}} \circ 1_{\Phi'}. \tag{3.18}$$

It remains to use the usual Künneth formula (3.3) to finish the proof.

of Sublemma 3.3.7. Denote $\phi_c^m := \phi_c \circ_1 \cdots \circ_1 \phi_c$ the m-fold composition of ϕ_c. Let

$$\mathcal{DR}^0 := \mathbb{F}(X \oplus \Phi)$$

and, for $n \geq 0$, let $\mathcal{DR}^{n+1} \subset \mathcal{DR}$ be spanned by elements

$$\phi_c^m \circ_1 x \circ (x_1, \dots, x_{\mathrm{ar}(x)}) \text{ and}$$
$$\phi_c^m \circ_1 \underline{x} \circ (x_1, \dots, x_{\mathrm{ar}(x)})$$

for all $x \in X\binom{c}{c_1,\dots,c_{\mathrm{ar}(x)}}$, $m \geq 0, x_i \in \mathcal{DR}^n\binom{c_i}{\dots}$, $1 \leq i \leq \mathrm{ar}(x)$. In other words,

$$\mathcal{DR}^{n+1} = \Phi' \circ (X \oplus \underline{X}) \circ \mathcal{DR}^n.$$

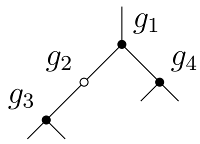
$\mathcal{DR}^n$ is obviously closed under ∂^0 and

$$\mathcal{DR}^0 \hookrightarrow \mathcal{DR}^1 \hookrightarrow \cdots \to \operatorname*{colim}_n \mathcal{DR}^n \cong \mathcal{DR},$$

where the colimit is taken in the category of dg-C-collections.

Before we go further, we must make a short notational digression. Consider a tree composition $T(g_1, \dots, g_m)$ with $g_i \in X \sqcup \Phi \sqcup \underline{X}$ for $1 \leq i \leq m$. Recall the tree T has vertices $v_1, \dots, v_m$ decorated by $g_1, \dots, g_m$ (in that order). We say that g_j **is in depth** d **in** $T(g_1, \dots, g_j, \dots, g_m)$ iff the shortest path from v_j to the root vertex passes through exactly d vertices (including v_j and the root vertex) decorated by elements of $X \sqcup \underline{X}$.

As an example, consider



If $g_1, g_3, g_4 \in X$ and $g_2 \in \Phi$, then g_1, g_2 are in depth 1 and g_3, g_4 are in depth 2.

Using the notion of depth, the definition of $\mathcal{DR}^n$ can be rephrased as follows: $\mathcal{DR}^n$ is spanned by $T(g_1, \dots, g_m)$ with $g_i \in X \sqcup \Phi \sqcup \underline{X}$, $1 \leq i \leq m$, such that if $g_j \in \underline{X}$ for some j, then g_j is in depth $\leq n$ in $T(g_1, \dots, g_m)$.

Consider the quotient $\mathcal{Q}^n$ of $\mathbb{F}(X \oplus \Phi)$ by the ideal generated by elements

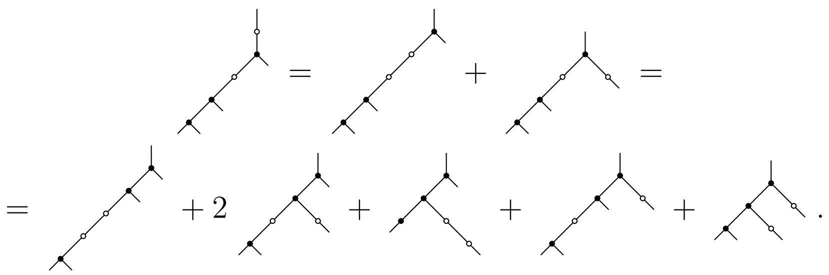
$$T(g_1, \dots, g_{j-1}, \phi_c \circ_1 x_j - \sum_{i=0}^{\mathrm{ar}(x_j)} x_j \circ_i \phi_{c_i}, g_{j+1}, \dots, g_m)$$

for any tree T, any $g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_m \in X \sqcup \Phi$ and any $x_j \in X$ in depth $\leq n$ in $T(g_1, \dots, g_{j-1}, x_j, g_{j+1}, \dots, g_m)$. There are obvious projections

$$\mathbb{F}(X \oplus \Phi) = \mathcal{Q}^0 \twoheadrightarrow \mathcal{Q}^1 \twoheadrightarrow \cdots \to \operatorname*{colim}_n \mathcal{Q}^n \cong \frac{\mathbb{F}(X \oplus \Phi)}{\mathfrak{D}}.$$

To see the last isomorphism, observe that we can use the relations defining $\mathcal{Q}^n$ to "move" the ϕ_c's in tree compositions so that they are all in depth $\geq n$ or in positions such that their inputs are leaves, then use (3.17).

For example, consider the following computation in $\mathcal{Q}^2$, where the black vertices are decorated by X and white vertices by Φ:



Notice that we can't get the white vertices any deeper in $\mathcal{Q}^2$.

In particular,

$$\mathcal{Q}^{n+1} \cong X \circ \mathcal{Q}^n. \tag{3.19}$$

Obviously

$$H_*(\mathcal{DR}^0, \partial^0) \cong \mathcal{Q}^0$$

and we claim that

$$H_*(\mathcal{DR}^n, \partial^0) \cong \mathcal{Q}^n$$

for $n \geq 1$. Suppose the claim holds for n and we prove it for $n+1$. The idea is to use a spectral sequence to get rid of the last sum in the formula

$$\partial^0(\phi_c^m \circ_1 \underline{x} \circ (x_1, \ldots, x_{\mathrm{ar}(x)})) =$$

$$\begin{gathered}\phi_c^{m+1} \circ_1 x \circ (x_1, \ldots, x_{\mathrm{ar}(x)}) + \\ -\sum_{i=1}^{\mathrm{ar}(x)} \phi^m \circ_1 x \circ_i \phi \circ (x_1, \ldots, x_{\mathrm{ar}(x)}) + \\ +(-1)^{|\underline{x}|} \sum_{i=1}^{\mathrm{ar}(x)} (-1)^{\sum_{j=1}^{i-1} |x_j|} \phi^m \circ_1 \underline{x} \circ (x_1, \ldots, \partial^0(x_i), \ldots, x_{\mathrm{ar}(x)}).\end{gathered}$$

Consider the spectral sequence E^{0*} on $\mathcal{DR}^{n+1}$ associated to the filtration

$$0 \hookrightarrow \mathfrak{G}_0 \hookrightarrow \mathfrak{G}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{DR}^n,$$

where $\mathfrak{G}_k$ is spanned by

$$\phi^m \circ_1 g \circ (x_1, \ldots, x_{\mathrm{ar}(x)})$$

for all $m \geq 0$, $g \in X \oplus \underline{X}$, $x_i \in \mathcal{DR}^n$ and $\sum_{i=1}^{\mathrm{ar}(x)} |x_i| \leq k$. Obviously $\partial^0 : \mathfrak{G}_k \to \mathfrak{G}_k$.

We will use the comparison theorem for the obvious projection

$$\mathcal{DR}^{n+1} \xrightarrow{\mathrm{pr}} \mathcal{Q}^{n+1}.$$

We consider the zero differential on $\mathcal{Q}^{n+1}$. It is easily seen that $\mathrm{pr}\, \partial^0 = 0$, hence pr is dg-$C$-collection morphism. We equip $\mathcal{Q}^{n+1}$ with the trivial filtration $0 \hookrightarrow \mathcal{Q}^{n+1}$ and consider the associated spectral sequence E'^{0*}. Again, both filtrations are bounded below and exhaustive.

On the 0^{th} page $E^{00} \cong \mathcal{DR}^{n+1}$, the differential ∂^{00} has the desired form:

$$\begin{gathered}\partial^{00}(\phi^m \circ_1 \underline{x} \circ (x_1, \ldots, x_{\mathrm{ar}(x)})) = \\ \phi_c^{m+1} \circ_1 x \circ (x_1, \ldots, x_{\mathrm{ar}(x)}) - \sum_{i=1}^{\mathrm{ar}(x)} \phi^m \circ_1 x \circ_i \phi \circ (x_1, \ldots, x_{\mathrm{ar}(x)})\end{gathered}$$

and ∂^{00} is zero on other elements. For this differential ∂^{00}, it is (at last!) clear how its kernel looks (compare to ∂^0), namely $\operatorname{Ker} \partial^{00} = \mathbb{F}\,(X \oplus \Phi) \circ \mathcal{DR}^n$. Hence

$$H_*(E^{00}, \partial^{00}) \cong X \circ \mathcal{DR}^n.$$

This is E^{01} and the differential ∂^{01} is equal to the restriction of ∂^0 onto $X \circ \mathcal{DR}^n$.

For E'^{0*} everything is trivial, $E'^{01} \cong \mathcal{Q}^{n+1}$ and $\partial'^{01} = 0$.

Then $\mathrm{pr}^1 : E^{01} \to E'^{01}$ induced by pr is quism, because

$$H_*(E^{01}, \partial^{01}) \cong X \circ H_*(\mathcal{DR}^n, \partial^0) \cong X \circ \mathcal{Q}^n \cong \mathcal{Q}^{n+1},$$

where the first isomorphism follows from the usual Künneth formula (3.3), the second one follows from the induction hypothesis and the last one was already observed in (3.19).

This concludes the proof of the claim $H_*(\mathcal{DR}^n, \partial^0) \cong \mathcal{Q}^n$. Finally

$$H_*(\mathcal{DR}, \partial^0) \cong H_*(\operatorname*{colim}_n \mathcal{DR}^n) \cong \operatorname*{colim}_n H_*(\mathcal{DR}^n) \cong \operatorname*{colim}_n \mathcal{Q}^n \cong \frac{\mathbb{F}\,(X \oplus \Phi)}{\mathfrak{D}}$$

proves Sublemma 3.3.7. $\square$

Now that the sublemma is proved, we easily go through all the isomorphisms to check (3.16) and (3.18). $\square$

3.3.3 Augmented cotangent complex

Let

$$(\mathcal{A}, \partial_{\mathcal{A}}) \xrightarrow{\alpha} (\mathcal{E}nd_A, \partial_{\mathcal{E}nd_A})$$

be an $\mathcal{A}$-algebra structure on A. We begin by extracting the operadic cohomology from $\mathcal{DR}$. Let $\mathbb{F}\,(X \oplus \Phi \oplus \underline{X}) \to \mathcal{A}$ be the dg-C-operad morphism which equals $\rho_{\mathcal{R}}$ on X and vanishes on the other generators. Hence $\mathcal{DR} = \mathbb{F}\,(X \oplus \Phi \oplus \underline{X})$ is a dg-C-operad *over* $\mathcal{A}$.

For M one of the subsets X, $X \oplus \Phi$, $X \oplus \underline{X}$ of $\mathcal{DR}$ define

$$\mathrm{Der}^M_{\mathcal{A}}(\mathcal{DR}, \mathcal{E}nd_A) := \{\theta \in \mathrm{Der}_{\mathcal{A}}(\mathcal{DR}, \mathcal{E}nd_A) \mid \forall m \in M \quad \theta(m) = 0\}\,. \qquad (3.20)$$

We will abbreviate this by Der^M. Let $\overline{\delta}$ be the differential on Der^M defined by

$$\overline{\delta}\theta := \theta\partial_{\mathcal{DR}} - (-1)^{|\theta|}\partial_{\mathcal{E}nd_A}\theta.$$

A check similar to that for (3.10) verifies this is well defined. Obviously

$$\mathrm{Der}^X = \mathrm{Der}^{X \oplus \underline{X}} \oplus \mathrm{Der}^{X \oplus \Phi}\,.$$

Recall we assume the dg-k-module A is graded by the colours, that is $A = \bigoplus_{c \in C} A_c$. Hence we have

$$\mathrm{Der}^{X \oplus \underline{X}} \cong \mathrm{Hom}_{C-\mathrm{coll.}}(\Phi, \mathcal{E}nd_A) \cong \bigoplus_{c \in C} \mathrm{Hom}_k(A_c, A_c).$$

Importantly, $\mathrm{Der}^{X \oplus \Phi}$ is closed under $\overline{\delta}$.

3.3.8 Lemma.

$$(\mathrm{Der}^{X \oplus \Phi}, \overline{\delta}) \cong \downarrow(\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{E}nd_A), \delta)$$

as dg-k-modules.

Proof. Recall $\downarrow \delta = -\delta$. Define a degree $+1$ map

$$\mathrm{Der}^{X\oplus\Phi} \xrightarrow{f_1} \mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{E}nd_A)$$

by the formula

$$(f_1\theta')(x) := \theta'(\underline{x}) \quad \text{for } \theta' \in \mathrm{Der}^{X\oplus\Phi}.$$

Its inverse, f_2 of degree -1, is defined for $\theta \in \mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{E}nd_A)$ by the formulas

$$(f_2\theta)(a) = 0 = (f_2\theta)(\phi), \quad (f_2\theta)(\underline{x}) = \theta(x).$$

Obviously $f_2 f_1 = \mathbf{1}$ and $f_1 f_2 = \mathbf{1}$ and it remains to check $f_1\overline{\delta} = -\delta f_1$.

$$\begin{aligned}(f_1(\overline{\delta}\theta'))(x) &= (\overline{\delta}\theta')(\underline{x}) = \theta'(\partial_{\mathcal{DR}}\underline{x}) - (-1)^{|\theta'|}\partial_{\mathcal{E}nd_A}(\theta'(\underline{x})),\\ (-\delta(f_1\theta'))(x) &= -(f_1\theta')(\partial_{\mathcal{R}}x) + (-1)^{|f_1\theta'|}\partial_{\mathcal{E}nd_A}((f_1\theta')(x)).\end{aligned}$$

Now we check $\theta'(\partial_{\mathcal{DR}}\underline{x}) = -(f_1\theta')(\partial_{\mathcal{R}}x)$. Let $\partial_{\mathcal{R}}x = \sum_i T_i(x_{i1}, \dots, x_{in_i})$.

$$\begin{aligned}\theta'(\partial_{\mathcal{DR}}\underline{x}) &= \theta'(\phi \circ x - \sum_j x \circ_j \phi - s(\partial_{\mathcal{R}}x)) =\\ &= -\theta'(s\sum_i T_i(x_{i1}, \dots, x_{in_i})) =\\ &= -\theta'(\sum_i \sum_{j=1}^{n_i} \epsilon_{ij} T_i(x_{i1}, \dots, \underline{x_{ij}}, \dots, x_{in_i})) =\\ &= -\sum_i \sum_j \epsilon_{ij}^{1+|\theta'|} T_i(\rho_{\mathcal{R}}(x_{i1}), \dots, \theta'(\underline{x_{ij}}), \dots, \rho_{\mathcal{R}}(x_{in_i})),\\ -(f_1\theta')(\partial_{\mathcal{R}}x) &= \ldots = -\sum_i \sum_j \epsilon_{ij}^{|f_1\theta'|} T_i(\rho_{\mathcal{R}}(x_{i1}), \dots, (f_1\theta')(\underline{x_{ij}}), \dots, \rho_{\mathcal{R}}(x_{in_i})),\end{aligned}$$

where we have denoted $\epsilon_{ij} := (-1)^{\sum_{l=1}^{j-1}|x_{il}|}$. $\square$

3.3.9 Definition. We call

$$C^*_{aug}(A, A) := ((\mathrm{Der}^X)^{-*}, \overline{\delta})$$

augmented operadic cotangent complex of A and its cohomology

$$H^*_{aug}(A, A) := H^*(C^*_{aug}(A, A), \overline{\delta})$$

augmented operadic cohomology of A.

An interpretation of the augmentation $(\mathrm{Der}^{X\oplus\underline{X}})^{-*} \xrightarrow{\overline{\delta}} (\mathrm{Der}^{X\oplus\Phi})^{-*} \cong C^*(A, A)$ of the usual cotangent complex $C^*(A, A)$ is via infinitesimal automorphisms of the $\mathcal{A}$-algebra structure on A. This suggests a relation between $H^*_{aug}(A, A)$ and $H^*(A, A)$. It is best seen in an example:

3.3.10 Example. Continuing Example 3.3.6, let A be k-module with a structure of an associative algebra, that is

$$\mathcal{A}ss \xrightarrow{\alpha} \mathcal{E}nd_A.$$

We have

$$\begin{aligned}
C^0_{aug}(A,A) &= \mathrm{Der}^{X\oplus\underline{X}} \cong \mathrm{Hom}_k(A,A),\\
C^n_{aug}(A,A) &= (\mathrm{Der}^{X\oplus\Phi})^{-n} \cong \mathrm{Hom}_{C-\mathrm{coll.}}(\underline{X},\mathcal{E}nd_A)^{-n} \cong\\
&\cong \mathrm{Hom}_{C-\mathrm{coll.}}(\underline{X}_n,\mathcal{E}nd_A) \cong \mathcal{E}nd_A(n+1) = \mathrm{Hom}_k(A^{\otimes n+1},A)
\end{aligned}$$

and, for $f \in C^n_{aug}(A,A)$,

$$\bar{\delta} f = (-1)^{n+1}\mu \circ_2 f + \sum_{k=1}^{n}(-1)^{n+1-k} f \circ_k \mu + \mu \circ_1 f. \tag{3.21}$$

So the augmented cotangent complex is the Hochschild complex without the term $C^{-1}(A,A) = \mathrm{Hom}_k(k,A) \cong A$, while the ordinary cotangent complex would be additionally missing $C^0(A,A)$:

$$\underbrace{C^0(A,A) \xrightarrow{\bar{\delta}} \underbrace{C^1(A,A) \xrightarrow{\bar{\delta}} C^2(A,A) \xrightarrow{\bar{\delta}} \cdots}_{C^*(A,A)}}_{C^*_{aug}(A,A)}$$

To generalize the conclusion of the example, recall from [14] that TJ-grading on a free resolution $\mathcal{R} = (\mathbb{F}(X), \partial) \xrightarrow{\rho} (\mathcal{A}, 0)$ is induced by a grading $X = \bigoplus_{i\geq 0} X^i$ on the C-collection of generators, denoted by upper indices, $\mathcal{R}^i$, and satisfying

1. ∂ maps X^i to $\mathbb{F}\left(\bigoplus_{j<i} X^j\right)$,
2. $H_0(\mathcal{R}^*, \partial) \xrightarrow{H_0(\rho)} \mathcal{A}$ is an isomorphism of graded C-operads.

If we have a TJ-graded resolution $\mathcal{R}$, we can replace the usual grading by the TJ-grading and we let $\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{E}nd_A)^i$ be the k-module of derivations $\mathcal{R} \to \mathcal{E}nd_A$ vanishing on all X^j's except for $j = i$ and let ${}^{TJ}C^*(A,A) := \uparrow(\mathrm{Der}_{\mathcal{A}}(\mathcal{R}, \mathcal{E}nd_A)^*, \delta)$ and ${}^{TJ}H^*(A,A) := H^*({}^{TJ}C^*(A,A))$. In case $\mathcal{A}$ is concentrated in degree 0, the usual grading is TJ and we get the same result as in (3.11), i.e. $C^*(A,A) = {}^{TJ}C^*(A,A)$ and we can forget about the superscripts TJ everywhere.

For a TJ-graded $\mathcal{R}$, we can also equip $C^*_{aug}(A,A)$ with similar TJ-grading ${}^{TJ}C^*_{aug}(A,A)$ as above. On this matter we just remark that ϕ is placed in TJ-degree 0 and leave the details for the interested reader. Finally, the following is obvious:

3.3.11 Theorem.

1. ${}^{TJ}C^n_{aug}(A,A) = 0$ for $n \leq -1$,
2. ${}^{TJ}C^0_{aug}(A,A) = \mathrm{Der}^{\mathcal{A}\oplus\underline{X}} \cong \mathrm{Hom}_k(A,A)$,
3. ${}^{TJ}H^1_{aug}(A,A) \cong$ k-module of formal infinitesimal deformations of the $\mathcal{A}$-algebra structure on A *modulo infinitesimal automorphisms*,
4. ${}^{TJ}H^n_{aug}(A,A) \cong {}^{TJ}H^n(A,A)$ for $n \geq 2$.

Notice that the unaugmented operadic cohomology ${}^{TJ}H^1(A,A)$ is the k-module of formal infinitesimal deformations of the $\mathcal{A}$-algebra structure on A, but the infinitesimal automorphisms are not considered.

Hence the distinction between $H^*(A,A)$ and $H^*_{aug}(A,A)$ is inessential and we will usually not distinguish these two.

3.3.4 Intermediate resolution of $\mathcal{DA}$

Now we construct an intermediate step in the resolution of Theorem 3.3.5:

$$\begin{tikzcd}
\mathcal{DR} \arrow[rr, "\sim", "\rho_{\mathcal{DR}}"'] \arrow[dr, "\iota"', "\sim"] & & \mathcal{DA} \\
& \overline{\mathcal{DR}} \arrow[ur, "\rho_{\overline{\mathcal{DR}}}"', "\sim"] &
\end{tikzcd}$$

Intuitively, ι should "unresolve" the part of $\mathcal{DR}$ corresponding to the $\mathcal{A}$-algebra operations and do nothing in the part corresponding to the derivation ϕ. Let

$$\overline{\mathcal{DR}} := \left(\mathcal{A} * \mathbb{F}\left(\Phi \oplus \underline{X}\right), \partial_{\overline{\mathcal{DR}}}\right).$$

We first define ι to be the composite

$$\mathcal{DR} = \mathbb{F}\left(X \oplus \Phi \oplus \underline{X}\right) \cong \mathbb{F}\left(X\right) * \mathbb{F}\left(\Phi \oplus \underline{X}\right) \xrightarrow{\rho_{\mathcal{R}} * 1} \mathcal{A} * \mathbb{F}\left(\Phi \oplus \underline{X}\right) = \overline{\mathcal{DR}}$$

then $\partial_{\overline{\mathcal{DR}}}$ is the derivation defined by

$$\begin{aligned}
\partial_{\overline{\mathcal{DR}}}(a) &:= \partial_{\mathcal{DA}}(a) = \partial_{\mathcal{A}}(a), \\
\partial_{\overline{\mathcal{DR}}}(\phi_c) &:= 0, \\
\partial_{\overline{\mathcal{DR}}}(\underline{x}) &:= \iota(\partial_{\mathcal{DR}}\underline{x}).
\end{aligned} \tag{3.22}$$

Now we check $\iota\partial_{\mathcal{DR}} = \partial_{\overline{\mathcal{DR}}}\iota$ and this will immediately imply $\partial^2_{\overline{\mathcal{DR}}} = 0$:

$$\begin{aligned}
\iota\partial_{\mathcal{DR}}(x) = \iota\partial_{\mathcal{R}}(x) = \rho_{\mathcal{R}}\partial_{\mathcal{R}}(x) = \partial_{\mathcal{A}}\rho_{\mathcal{R}}(x), \\
\partial_{\overline{\mathcal{DR}}}\iota(x) = \partial_{\overline{\mathcal{DR}}}\rho_{\mathcal{R}}(x) = \partial_{\mathcal{DA}}\rho_{\mathcal{R}}(x) = \partial_{\mathcal{A}}\rho_{\mathcal{R}}(x)
\end{aligned}$$

and similar claim for $\underline{x}$ is an immediate consequence of definitions.

Finally, let $\rho_{\overline{\mathcal{DR}}}$ be the C-operad morphism defined by

$$\begin{aligned}
\rho_{\overline{\mathcal{DR}}}(a) &:= a, \\
\rho_{\overline{\mathcal{DR}}}(\phi_c) &:= \phi_c, \\
\rho_{\overline{\mathcal{DR}}}(\underline{x}) &:= 0.
\end{aligned} \tag{3.23}$$

3.3.12 Lemma. $\rho_{\overline{\mathcal{DR}}}$ is dg-C-operad morphism.

Proof. We only have to check $\rho_{\overline{\mathcal{DR}}}\partial_{\overline{\mathcal{DR}}}(\underline{x}) = 0$:

$$\begin{aligned}
\rho_{\overline{\mathcal{DR}}}\partial_{\overline{\mathcal{DR}}}(\underline{x}) &= \rho_{\overline{\mathcal{DR}}}\iota\left(\phi_c \circ_1 x - \sum_{i=0}^{n} x \circ_i \phi_{c_i} - s(\partial_{\mathcal{R}}x)\right) = \\
&= \phi_c \circ_1 \rho_{\mathcal{R}}(x) - \sum_{i=0}^{n} \rho_{\mathcal{R}}(x) \circ_i \phi_{c_i}.
\end{aligned}$$

The third term in the bracket vanishes since $\iota s\partial_{\mathcal{R}}(x)$ is a sum of compositions each of which contains a generator from $\underline{X}$ and $\rho_{\overline{\mathcal{DR}}}$ vanishes on $\underline{X}$. The above expression vanishes because if $\rho_{\mathcal{R}}(x) \neq 0$, then it is precisely the relator (3.12). $\square$

3.3.13 Lemma. ι is a quism.

Proof. We notice that $\overline{\mathcal{DR}}$ is close to be the 1^{st} term of a spectral sequence computing homology of $\mathcal{DR}$. Now we make this idea precise.

Consider a new grading gr on $\mathcal{DR}$:

$$\mathrm{gr}(x) := 0 =: \mathrm{gr}(\phi), \quad \mathrm{gr}(\underline{x}) := |\underline{x}|$$

and its associated filtration

$$\mathfrak{F}_p := \bigoplus_{i=0}^{p} \{z \in \mathcal{DR} \mid \mathrm{gr}(z) = i\},$$
$$0 \hookrightarrow \mathfrak{F}_0 \hookrightarrow \mathfrak{F}_1 \hookrightarrow \cdots, \quad \partial_{\mathcal{DR}}\mathfrak{F}_p \subset \mathfrak{F}_p$$

and its associated spectral sequence (E^*, ∂^*). There is an analogous spectral sequence (E'^*, ∂'^*) on $\overline{\mathcal{DR}}$ given by the grading

$$\mathrm{gr}'(a) := 0 =: \mathrm{gr}'(\phi), \quad \mathrm{gr}'(\underline{x}) := |\underline{x}|.$$

Since both filtrations are bounded below and exhaustive, we can use the comparison theorem.

We have $E^0 \cong \mathbb{F}(X \oplus \Phi \oplus \underline{X})$. Recalling the formulas (3.14), we immediately see that ∂^0 on E^0 is the derivation differential given by

$$\partial^0 x = \partial_{\mathcal{R}} x, \quad \partial^0 \phi = 0 = \partial^0 \underline{x}.$$

Hence $E^1 \cong H_*(E^0, \partial^0) \cong H_*(\mathbb{F}(X)) * H_*(\mathbb{F}(\Phi \oplus \underline{X})) \cong \mathcal{A} * \mathbb{F}(\Phi \oplus \underline{X})$ by the Künneth formula for a free product of dg-C-operads, see Lemma 3.2.4. Similarly $E'^1 \cong \mathcal{A} * \mathbb{F}(\Phi \oplus \underline{X})$.

Understanding the differentials ∂^1 and ∂'^1 on the 1^{st} pages as well as the induced dg-C-collection morphism ι^1 is easy (though notationally difficult - observe $\partial^1 \underline{x}$ is *not* $\iota(\partial_{\mathcal{DR}} \underline{x})$ in general!) and we immediately see that ι^1 is an isomorphism of dg-C-collections. $\square$

3.3.14 Corollary. $\rho_{\overline{\mathcal{DR}}}$ is a resolution of $\mathcal{DA}$.

3.3.15 Example. Let's continue Example 3.3.6 and make $\overline{\mathcal{DR}}$ explicit:

$$\overline{\mathcal{DR}} := (\mathcal{A}ss * \mathbb{F}(\Phi \oplus \underline{X}), \partial_{\overline{\mathcal{DR}}}),$$

$$\begin{aligned}
&\partial_{\overline{\mathcal{DR}}}(a) := 0 =: \partial_{\overline{\mathcal{DR}}}(\phi),\\
&\partial_{\overline{\mathcal{DR}}}(\underline{x}^2) := \phi \circ \mu - \mu \circ_1 \phi - \mu \circ_2 \phi,\\
&\partial_{\overline{\mathcal{DR}}}(\underline{x}^n) := -(-1)^n \mu \circ_2 \underline{x}^{n-1} - \sum_{k=1}^{n-1} (-1)^{n-k} \underline{x}^{n-1} \circ_k \mu - \mu \circ_1 \underline{x}^{n-1}
\end{aligned}$$

for $n \geq 3$. The last formula is reminiscent to the one for the Hochschild differential. We will make this point precise in Section 3.3.5.

3.3.5 From operads to operadic modules

Associated to the operad $\mathcal{DA} = (\mathcal{A} * \mathbb{F}(\Phi))/\mathfrak{D}$ is the dg-$(\mathcal{A}, \partial_{\mathcal{A}})$-module

$$\mathcal{MDA} := \left(\frac{\mathcal{A}\langle\Phi\rangle}{\mathfrak{D} \cap \mathcal{A}\langle\Phi\rangle}, \partial_{\mathcal{MDA}} \right),$$

where $\mathfrak{D} \cap \mathcal{A}\langle\Phi\rangle$ is the sub-$\mathcal{A}$-module of $\mathcal{A}\langle\Phi\rangle$ generated by the relators (3.12) and $\partial_{\mathcal{MDA}}$ is a dg-$\mathcal{A}$-module morphism given by $\partial_{\mathcal{MDA}}\phi = 0$.

Associated to $\overline{\mathcal{DR}} = (\mathcal{A} * \mathbb{F}(\Phi \oplus \underline{X}), \partial_{\overline{\mathcal{DR}}})$ is the dg-$\mathcal{A}$-module

$$\mathcal{M}\overline{\mathcal{DR}} := (\mathcal{A}\langle\Phi \oplus \underline{X}\rangle, \partial_{\mathcal{M}\overline{\mathcal{DR}}}),$$

where $\partial_{\mathcal{M}\overline{\mathcal{DR}}}$ is a $\mathcal{A}$-module morphism given by the same formulas (3.22) as $\partial_{\overline{\mathcal{DR}}}$. We emphasize that this makes sense because $\mathcal{A}\langle\Phi \oplus \underline{X}\rangle \subset \mathcal{A} * \mathbb{F}(\Phi \oplus \underline{X})$, the dg-$\mathcal{A}$-module structure is induced by the operadic composition and $\partial_{\overline{\mathcal{DR}}}$ maps $\mathcal{A}\langle\Phi \oplus \underline{X}\rangle$ into itself!

Associated to the dg-C-operad morphism $\rho_{\overline{\mathcal{DR}}} : \overline{\mathcal{DR}} \to \mathcal{DA}$ is the dg-$\mathcal{A}$-module morphism

$$\rho_{\mathcal{M}\overline{\mathcal{DR}}} : \mathcal{M}\overline{\mathcal{DR}} \to \mathcal{MDA}$$

again defined by the formulas (3.23) as $\rho_{\mathcal{M}\overline{\mathcal{DR}}}$.

3.3.16 Lemma. $\rho_{\mathcal{M}\overline{\mathcal{DR}}}$ is a quism.

Proof. Let G_p be the sub-C-collection of $\overline{\mathcal{DR}} = \mathcal{A} * \mathbb{F}(\Phi \oplus \underline{X})$ spanned by the compositions containing precisely p generators from $\Phi \oplus \underline{X}$, i.e. $G_0 = \mathcal{A}$, $G_1 = \mathcal{A}\langle\Phi \oplus \underline{X}\rangle$ and

$$\overline{\mathcal{DR}} = \bigoplus_{p \geq 0} G_p.$$

We have analogous grading $\mathcal{A} * \mathbb{F}(\Phi) = \bigoplus_{p\geq 0} G'_p$. Let $\mathrm{pr} : \mathcal{A} * \mathbb{F}(\Phi) \to \mathcal{DA}$ be the natural projection. Since relators (3.12) are homogeneous with respect to this grading, $G''_p := \mathrm{pr}\, G'_p$ defines a grading

$$\mathcal{DA} = \bigoplus_{p \geq 0} G''_p.$$

Observe $G''_0 = \mathcal{A}$ and $G''_1 = \mathcal{MDA}$. By definitions, $\rho_{\overline{\mathcal{DR}}} G_p \subset G''_p$, hence $\rho_{\overline{\mathcal{DR}}}$ decomposes as a sum of ${\rho_{\overline{\mathcal{DR}}}}^p : G_p \to G''_p$. The above direct sums are in fact direct sums of sub-dg-C-collections, $\rho_{\overline{\mathcal{DR}}}$ is a quism by Corollary 3.3.14, hence all the ${\rho_{\overline{\mathcal{DR}}}}^p$'s are quisms, especially ${\rho_{\overline{\mathcal{DR}}}}^1 = \rho_{\mathcal{M}\overline{\mathcal{DR}}}$. □

Now we formalize the statement : $\mathcal{M}\overline{\mathcal{DR}}$ *contains all the information needed to construct the operadic cohomology for $\mathcal{A}$-algebras.*

First observe that $\mathcal{E}nd_A$ is naturally a dg-$\mathcal{A}$-module. Let $\overline{\delta}$ be the differential on $\mathrm{Hom}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{M}\overline{\mathcal{DR}}, \mathcal{E}nd_A)$ defined by the formula

$$\overline{\delta}\theta := \theta\partial_{\mathcal{M}\overline{\mathcal{DR}}} - (-1)^{|\theta|}\partial_{\mathcal{E}nd_A}\theta$$

similar to (3.10).

3.3.17 Lemma.

$$(C^*_{aug}(A,A),\overline{\delta}) \cong \left(\mathrm{Hom}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{M}\overline{\mathcal{DR}}, \mathcal{E}nd_A), \overline{\delta}\right)$$

as dg-k-modules.

Proof. On the level of k-modules, we have

$$C^*_{aug}(A,A) = \mathrm{Der}^X = \{\theta \in \mathrm{Der}_{\mathcal{A}}(\mathbb{F}\left(X \oplus \Phi \oplus \underline{X}\right), \mathcal{E}nd_A) \mid \forall x \in X \quad \theta(x) = 0\} \cong$$
$$\cong \mathrm{Hom}_{\mathrm{dg}-C-\mathrm{coll.}}(\Phi \oplus \underline{X}, \mathcal{E}nd_A)$$

by the defining property of derivations and

$$\mathrm{Hom}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{A}\left\langle \Phi \oplus \underline{X}\right\rangle, \mathcal{E}nd_A) \cong \mathrm{Hom}_{\mathrm{dg}-C-\mathrm{coll.}}(\Phi \oplus \underline{X}, \mathcal{E}nd_A)$$

by the freeness of $\mathcal{A}\left\langle \Phi \oplus \underline{X}\right\rangle$.

The differentials are clearly preserved under the above isomorphism. □

The nice thing is that we have now all the information needed to construct the cohomology for $\mathcal{A}$-algebras encoded in terms of the *abelian* category of dg-$\mathcal{A}$-modules. Hence

3.3.18 Theorem.

$$H^*_{aug}(A,A) \cong H_*(\mathrm{Hom}^{-*}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{M}\overline{\mathcal{DR}}, \mathcal{E}nd_A), \overline{\delta}) \cong$$
$$\cong \mathrm{Ext}^{-*}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{MDA}, \mathcal{E}nd_A)$$

In particular, since the homotopy theory in abelian categories is well known and simple, it is immediate that $\mathrm{Ext}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}$ and hence $H^*_{aug}(A,A)$ doesn't depend on the choice of a projective resolution of $\mathcal{MDA}$ and consequently doesn't depend on the choice of the free resolution $\mathcal{R} \underset{\rho_{\mathcal{R}}}{\xrightarrow{\sim}} \mathcal{A}$ in (3.13).

The main advantage of the above expression is that in order to construct cohomology for $\mathcal{A}$-algebras, we don't need to find a free (or cofibrant) resolution $\mathcal{R} \xrightarrow{\sim} \mathcal{A}$ in the category of dg-C-operads, but it suffices to find a projective resolution of $\mathcal{MDA}$ in the category of dg-$\mathcal{A}$-modules, which is certainly easier.

3.3.19 Example. Let's continue Example 3.3.15:

$$\mathcal{MDA} := \frac{\mathcal{A}ss\left\langle \Phi \right\rangle}{(\phi \circ \mu - \mu \circ_1 \phi - \mu \circ_2 \phi)}$$

and we have the following explicit description of $\mathcal{M}\overline{\mathcal{DR}}$:

$$\mathcal{M}\overline{\mathcal{DR}} := (\mathcal{A}ss\left\langle k\langle \phi^1, \phi^2, \phi^3, \ldots \rangle\right\rangle, \partial_{\mathcal{M}\overline{\mathcal{DR}}}),$$

where $\phi^1 := \phi$ and $\phi^n := \underline{x}^n$, for $n \geq 2$, is of degree $n-1$ and the differential is given by

$$\partial_{\mathcal{M}\overline{\mathcal{DR}}}(\phi^1) := 0,$$
$$\partial_{\mathcal{M}\overline{\mathcal{DR}}}(\phi^n) := -(-1)^n \mu \circ_2 \phi^{n-1} - \sum_{k=1}^{n-1} (-1)^{n-k} \phi^{n-1} \circ_k \mu - \mu \circ_1 \phi^{n-1}.$$

Lemma 3.3.16 states that $\mathcal{M}\overline{\mathcal{DR}} \xrightarrow{\rho_{\mathcal{M}\overline{\mathcal{DR}}}} \mathcal{MDA}$ is a free resolution in the category of $\mathcal{A}ss$-modules.

Notice the similarity to the Hochschild complex. This suggests that if we know a complex computing a cohomology for $\mathcal{A}$-algebras (in this case Hochschild complex) we can read off a candidate for the free resolution of $\mathcal{MDA}$ (in this case we already know a resolution $\mathcal{MDA}$, namely $\mathcal{M}\overline{\mathcal{DR}}$, but this was constructed from the *operadic* resolution $\mathcal{R}$, which is not generally available). If we can prove that this candidate is indeed a resolution, we get that the cohomology in question is isomorphic to the augmented operadic cohomology.

We demonstrate this by constructing a cohomology for diagrams of associative algebras and proving that the (augmented) cotangent complex coincides with that defined by Gerstenhaber and Schack [5].

On the other hand, in the process of constructing $\mathcal{M}\overline{\mathcal{DR}}$ we have discarded much information present in $\mathcal{R}$. Namely $\mathcal{R}$ can be used to define an L_∞ structure on $C^*(A, A)$ governing formal deformations of A (see [13]), which is no longer possible using $\mathcal{M}\overline{\mathcal{DR}}$ (or any other resolution of $\mathcal{MDA}$) only.

3.4 Gerstenhaber-Schack diagram cohomology is operadic cohomology

3.4.1 Operad for diagrams

Let C be a small category. For a morphism f of C, let $I(f)$ be its source (Input) and $O(f)$ its target (Output). Consider the following nerve construction on C:

$$\Sigma^n := \left\{ \left(\xleftarrow{f_n} \cdots \xleftarrow{f_1} \right) \in \mathrm{Hom}_{\mathsf{C}}^{\times n} \mid O(f_i) = I(f_{i+1}) \text{ for } 1 \le i \le n-1 \right\}$$

for $n \ge 1$. For $\sigma = \left(\xleftarrow{f_n} \cdots \xleftarrow{f_1} \right)$, let $|\sigma| := n$, let $I(\sigma) := I(f_1)$ and $O(\sigma) := O(f_n)$. The face maps $\Sigma^n \to \Sigma^{n+1}$ are given by $\sigma \mapsto \sigma_i$, where

$$\begin{aligned}
\sigma_0 &:= \left(\xleftarrow{f_n} \cdots \xleftarrow{f_2} \right), \\
\sigma_i &:= \left(\xleftarrow{f_n} \cdots \xleftarrow{f_{i+1} f_i} \cdots \xleftarrow{f_1} \right) \text{ for } 1 \le i \le n-1, \\
\sigma_n &:= \left(\xleftarrow{f_{n-1}} \cdots \xleftarrow{f_1} \right).
\end{aligned}$$

Denote Σ^0 the set of objects of C and for $\sigma \in \Sigma^0$, let $I(\sigma) = O(\sigma) = \sigma$. Finally let

$$\Sigma := \bigcup_{n=0}^{\infty} \Sigma^n$$

and denote $\Sigma^{\ge 1} := \Sigma - \Sigma^0$.

Let $\mathcal{C}$ be the operadic version of C, that is

$$\mathcal{C} := k\langle \Sigma^1 \rangle .$$

This can be seen as a Σ^0-operad, where each $f \in \Sigma^1$ is an element of $\mathcal{C}\binom{O(f)}{I(f)}$ and the operadic composition is induced by the categorical composition.

A (C-shaped) **diagram** (of associative algebras) is a functor

$$D : \mathsf{C} \to \mathcal{A}ss\text{-algebras}.$$

Now we describe a Σ^0-operad $\mathcal{A}$ such that $\mathcal{A}$-algebras are precisely C-shaped diagrams:

$$\mathcal{A} := \frac{(*_{c\in\Sigma^0}\mathcal{A}ss_c) * \mathcal{C}}{\mathcal{I}},$$

where $\mathcal{A}ss_c$ is a copy of $\mathcal{A}ss$ concentrated in colour c, its generating element is $\mu_c \in \mathcal{A}ss_c\binom{c}{c,c}$ and $\mathcal{I}$ is the ideal generated by

$$f \circ \mu_{I(\sigma)} - \mu_{O(\sigma)} \circ (f, f) \quad \text{for all } f \in \Sigma^1.$$

It should be clear now that the functor D is essentially the same thing as Σ^0-operad morphism $\mathcal{A} \to \mathcal{E}nd_A$, where $A = \bigoplus_{c\in\Sigma^0} D(c)$.

The associated module of Section 3.3.5 is

$$\mathcal{MDA} := \frac{\mathcal{A}\left\langle\bigoplus_{c\in\Sigma^0}\Phi_c\right\rangle}{\mathfrak{D}\cap\mathcal{A}\left\langle\bigoplus_{c\in\Sigma^0}\Phi_c\right\rangle}$$

where $\Phi_c = k\langle\phi_c\rangle$, ϕ_c being an element of colour $\binom{c}{c}$ and of degree 0, and the submodule in the denominator is generated by

$$\phi_c \circ \mu_c - \mu_c \circ_1 \phi_c - \mu_c \circ_2 \phi_c,$$
$$\phi_{O(f)} \circ f - f \circ \phi_{I(f)}$$

for all $c \in \Sigma^0$ and all $f \in \mathcal{C}$ (equivalently $f \in \Sigma^1$). We seek a free resolution $(\mathcal{MR}, \partial) \xrightarrow{\sim} (\mathcal{MDA}, 0)$ to use Theorem 3.3.18. Before constructing $\mathcal{MR}$, let's recall the Gerstenhaber-Schack diagram cohomology. As we have seen in Example 3.3.19, this gives us a candidate for $\mathcal{MR}$.

3.4.2 Gerstenhaber-Schack diagram cohomology

We adapt the notation from the original source [5]. Originally, the diagram D was restricted to be a poset, but this is unnecessary. Also, instead of associative algebras, one may consider any other type of algebras for which a convenient cohomology is known (e.g. Lie algebras, [6]). In this paper we stick to associative algebras, but we believe that other types can be handled in a similar way.

For $\sigma \in \Sigma^0$, denote $\underline{\sigma} := \mathbb{1}_\sigma$. For $\sigma = (\xleftarrow{f_p} \cdots \xleftarrow{f_1}) \in \Sigma^p$, denote

$$\underline{\sigma} := f_p \cdots f_1 : D(I(\sigma)) \to D(O(\sigma))$$

the composition along σ. This algebra morphisms makes $D(O(\sigma))$ a $D(I(\sigma))$-bimodule. For $p, q \geq 0$ let

$$C^{p,q}_{GS}(D, D) := \prod_{\sigma\in\Sigma^p} C^q_{\text{Hoch}}(D(I(\sigma)), D(O(\sigma))),$$

where $C^q_{\text{Hoch}}(D(I(\sigma)), D(O(\sigma))) = \mathrm{Hom}_k(D(I(\sigma))^{\otimes q+1}, D(O(\sigma)))$ are the usual Hochschild cochains. We usually abbreviate $C^{p,q}_{GS} := C^{p,q}_{GS}(D, D)$. There are vertical and horizontal differentials $\delta_V : C^{p,q}_{GS} \to C^{p,q+1}_{GS}$ and $\delta_H : C^{p,q}_{GS} \to C^{p+1,q}_{GS}$.

To write them down, let $\sigma = (\xleftarrow{f_{p+1}} \cdots \xleftarrow{f_1}) \in \Sigma^{p+1}$ and for $\tau \in \Sigma^p$ let $\mathrm{pr}_\tau : \prod_{\lambda\in\Sigma^p} C^q_{\mathrm{Hoch}}(D(I(\lambda)), D(O(\lambda))) \to C^q_{\mathrm{Hoch}}(D(I(\tau)), D(O(\tau)))$ be the projection onto the τ component of $C^{p,q}_{GS}$. Let δ_{Hoch} be the usual Hochschild differential, see (3.21). Finally, for $\theta \in C^{p,q}_{GS}$, let

$$\delta_V := (-1)^p \prod_{\sigma\in\Sigma^p} \delta_{\mathrm{Hoch}}, \tag{3.24}$$

$$\begin{aligned}\mathrm{pr}_\sigma(\delta_H\theta) := (-1)^{p+1}(\mathrm{pr}_{\sigma_0}\theta) \circ (\underbrace{f_1, \dots, f_1}_{q-\text{times}}) + \sum_{i=1}^{p} (-1)^{p+1-i}\,\mathrm{pr}_{\sigma_i}\theta\, + \\ + f_{p+1} \circ (\mathrm{pr}_{\sigma_{p+1}}\theta).\end{aligned} \tag{3.25}$$

It is easy to see that $(C^{*,*}_{GS}, \delta_V, \delta_H)$ is a bicomplex. The Gerstenhaber-Schack cohomology is defined to be the cohomology of the totalization of this bicomplex,

$$H^*_{GS}(D, D) := H^*(\bigoplus_{p+q=*} C^{p,q}_{GS}(D, D), \delta_V + \delta_H).$$

Notice that we have restricted ourselves to the Hochschild complex without C^{-1}_{Hoch} as in Example 3.3.10. Also we consider only the cohomology of D with *coefficients in itself* as this is the case of interest in the formal deformation theory. The general coefficients can be handled using trivial (operadic) extensions.

3.4.3 Resolution of $\mathcal{MDA}$

The preceding section gives us the following candidate for $\mathcal{MR}$:

$$\mathcal{MR} := \left(\mathcal{A}\left\langle \bigoplus_{\sigma\in\Sigma} \Phi_\sigma \oplus X_\sigma \right\rangle, \partial\right),$$

where $\Phi_\sigma := k\langle\phi_\sigma\rangle$ with ϕ_σ of colours $\binom{O(\sigma)}{I(\sigma)}$ and of degree $|\phi_\sigma| := |\sigma|$ and $X_\sigma := \uparrow^{|\sigma|+1} X$ is placed in output colour $O(\sigma)$ and input colours $I(\sigma)$, X being the collection of generators of the minimal resolution of $\mathcal{A}ss$ as in Example 3.3.6. The element of X_σ corresponding to $x \in X$ will be denoted by x_σ, hence $|x^i_\sigma| = i - 1 + |\sigma|$. To define the differential ∂ in an economic way, denote $x^1_\sigma := \phi_\sigma$ for $\sigma \in \Sigma^0$ and let

$$\mathrm{pre}(\phi_\sigma) := 0, \quad \mathrm{pre}(x^i_\sigma) := x^{i-1}_\sigma \text{ for } i \geq 2,$$

and extend linearly to the generators of $\mathcal{MR}$. Further, let's accept the convention that for $\sigma \in \Sigma^0$, the symbol x_{σ_0} stands for zero. Then

$$\begin{aligned}\partial(x_\sigma) := (-1)^{|\sigma|} \Bigg(& (-1)^{|x|}\mu_{O(\sigma)} \circ (\underline{\sigma}, \mathrm{pre}(x_\sigma)) \; + \\ & + \sum_{i=1}^{\mathrm{ar}(x)-1} (-1)^{|x|-i}\,\mathrm{pre}(x_\sigma) \circ_i \mu_{I(\sigma)} + \mu_{O(\sigma)} \circ (\mathrm{pre}(x_\sigma), \underline{\sigma}) \Bigg) + \\ & + (-1)^{|\sigma|} x_{\sigma_0} \circ (\underbrace{f_1, \dots, f_1}_{(\mathrm{ar}(x))\text{-times}}) + \sum_{i=1}^{|\sigma|-1} (-1)^{|\sigma|-i} x_{\sigma_i} + f_{|\sigma|} \circ x_{\sigma_{|\sigma|}}\end{aligned}$$

for any $\sigma \in \Sigma$. Observe that the first part of the above formula corresponds to the vertical differential (3.24) and the second part corresponds to the the horizontal differential (3.25) in $C^*_{GS}(D, D)$. Then it is easily seen that

$$(\mathrm{Hom}_{\mathrm{dg}-\mathcal{A}-\mathrm{mod}}(\mathcal{MR}, \mathcal{E}nd_{\bigoplus_{c\in\Sigma^0} D(c)}), \delta)$$

with $\delta(-) := - \circ \partial$ is, as a dg-k-module, isomorphic to the Gerstenhaber-Schack complex. Once we prove that $\mathcal{MR}$ is a resolution of $\mathcal{MDA}$, we will have, by Theorem 3.3.18,

3.4.1 Theorem. Gerstenhaber-Schack diagram cohomology $H^*_{GS}(D, D)$ is isomorphic to the augmented operadic cohomology $H^*_{aug}(D, D)$.

To prove that $\mathcal{MR}$ is a resolution of $\mathcal{MDA}$ we introduce the dg-$\mathcal{A}$-module morphism $\rho : (\mathcal{MR}, \partial) \to (\mathcal{MDA}, 0)$ given by the formulas

$$\rho(\phi_\sigma) := \begin{cases} \phi_\sigma & \dots \quad |\sigma| = 0 \\ 0 & \dots \quad |\sigma| \geq 1, \end{cases}$$
$$\rho(x_\sigma) := 0.$$

Indeed, it is easy to check that $\rho\partial = 0$. It remains to prove

3.4.2 Lemma. ρ is a quism.

Proof. is basically a reduction to the following two cases:

1. C is a single object with no morphism except for the identity (Lemma 3.4.3),
2. C is arbitrary, but each $D(c)$, $c \in \Sigma^0$, is the trivial algebra k with zero multiplication (Lemma 3.4.4).

We first give a general overview of the proof and postpone technicalities to subsequent lemmas.

Consider a new grading on $\mathcal{MR}$ given by

$$\mathrm{gr}(x_\sigma) := |\sigma| =: \mathrm{gr}(\phi_\sigma)$$

and the usual requirement that the composition is of degree 0. Then we have the associated filtration

$$\mathfrak{F}_n := \bigoplus_{i=0}^{n} \{x \in \mathcal{MR} \mid \mathrm{gr}(x) = i\},$$
$$0 \hookrightarrow \mathfrak{F}_0 \hookrightarrow \mathfrak{F}_1 \hookrightarrow \cdots, \quad \partial\mathfrak{F}_i \subset \mathfrak{F}_i$$

and the spectral sequence (E^*, ∂^*) which is convergent as the filtration is bounded below and exhaustive.

Obviously $E^0 \cong \mathcal{MR}$ and ∂^0 is the derivation differential given by

$$\partial^0(x_\sigma) = (-1)^{|\sigma|} \Bigg((-1)^{|x|}\mu_{O(\sigma)} \circ (\underline{\sigma}, \mathrm{pre}(x_\sigma)) + \tag{3.26}$$
$$+ \sum_{i=1}^{\mathrm{ar}(x)-1} (-1)^{|x|-i}\, \mathrm{pre}(x_\sigma) \circ_i \mu_{I(\sigma)} + \mu_{O(\sigma)} \circ (\mathrm{pre}(x_\sigma), \underline{\sigma}) \Bigg),$$
$$\partial^0(\phi_\sigma) = 0.$$

Now $H_*(E^0, \partial^0) \cong \bigoplus_{\sigma\in\Sigma} H_*(\mathcal{A}\langle \Phi_\sigma \oplus X_\sigma\rangle, \partial^0)$ and we use

3.4.3 Lemma.
$$H_*(\mathcal{A}\langle\Phi_\sigma \oplus X_\sigma\rangle, \partial^0) \cong \frac{\mathcal{A}\langle\Phi_\sigma\rangle}{\mathfrak{D}_\sigma},$$

where $\mathfrak{D}_\sigma$ is the submodule generated by

$$\mu_{O(\sigma)} \circ (\underline{\sigma}, \phi_\sigma) + \mu_{O(\sigma)} \circ (\phi_\sigma, \underline{\sigma}) - \phi_\sigma \circ \mu_{I(\sigma)}. \tag{3.27}$$

This lemma implies

$$E^1 \cong \frac{\mathcal{A}\left\langle\bigoplus_{\sigma\in\Sigma}\Phi_\sigma\right\rangle}{\bigoplus_{\sigma\in\Sigma}\mathfrak{D}_\sigma},$$

$$\partial^1(\phi_\sigma) = (-1)^{|\sigma|}\phi_{\sigma_0} \circ f_1 + \sum_{i=1}^{|\sigma|-1}(-1)^{|\sigma|-i}\phi_{\sigma_i} + f_{|\sigma|} \circ \phi_{\sigma_{|\sigma|}}. \tag{3.28}$$

Denote $\Phi_\Sigma := \bigoplus_{\sigma\in\Sigma}\Phi_\sigma$ and write the nominator in the form

$$\mathcal{A}\langle\Phi_\Sigma\rangle \cong \mathcal{A} \circ' (I, \Phi_\Sigma \circ \mathcal{A}).$$

Here we used the infinitesimal composition product (3.1). Because of the relations $f \circ \mu_{I(f)} - \mu_{O(f)} \circ (f, f)$ in $\mathcal{A}$ for all $f \in \Sigma^1$, we have

$$\mathcal{A} \cong (\bigoplus_{c\in\Sigma^0}\mathcal{A}ss_c) \circ \mathcal{C} \tag{3.29}$$

and hence
$$\mathcal{A}\langle\Phi_\Sigma\rangle \cong \mathcal{A} \circ' (I, \Phi_\Sigma \circ (\bigoplus_{c\in\Sigma^0}\mathcal{A}ss_c) \circ \mathcal{C}).$$

But we are interested in the quotient E^1 of $\mathcal{A}\langle\Phi_\Sigma\rangle$ and the corresponding relations (3.27) give us
$$E^1 \cong \mathcal{A} \circ' (I, \Phi_\Sigma \circ \mathcal{C}).$$

Now we use (3.29) again to obtain

$$E^1 \cong (\bigoplus_{c\in\Sigma^0}\mathcal{A}ss_c) \circ' (\mathcal{C}, \mathcal{C} \circ \Phi_\Sigma \circ \mathcal{C}).$$

Notice that $\mathcal{C} \circ \Phi_\Sigma \circ \mathcal{C} \cong \mathcal{C}\langle\Phi_\Sigma\rangle$. Since ∂^1 is nontrivial only on Φ_Σ and $\mathcal{C}\langle\Phi_\Sigma\rangle$ is closed under ∂^1, to understand $H_*(E^1, \partial^1)$ using the usual Künneth formula, we only have to compute

3.4.4 Lemma.
$$H_*(\mathcal{C}\langle\Phi_\Sigma\rangle, \partial^1) \cong \frac{\mathcal{C}\left\langle\bigoplus_{c\in\Sigma^0}\Phi_c\right\rangle}{\mathfrak{D}'},$$

where $\mathfrak{D}'$ is the submodule generated by

$$f \circ \phi_{I(f)} - \phi_{O(f)} \circ f \tag{3.30}$$

for all $f \in \mathcal{C}$.

Then, tracking back all the above isomorphisms, we get

$$E^2 \cong H_*(E^1, \partial^1) \cong \frac{\mathcal{A}\left\langle \bigoplus_{c\in\Sigma^0} \Phi_c \right\rangle}{\mathfrak{D}''},$$

where $\mathfrak{D}''$ is the submodule generated by relators (3.27) for $\sigma \in \Sigma^0$ and all (3.30)'s. Hence $E^2 \cong \mathcal{MDA}$ and this is concentrated in degree 0, the spectral sequence collapses and this concludes the proof of Lemma 3.4.2. □

of Lemma 3.4.3. We have already seen in Example 3.3.19 that for $c \in \Sigma^0$ the restriction of ρ,

$$\left(\mathcal{A}ss_c \left\langle \Phi_c \oplus X_c \right\rangle, \partial^0\right) \xrightarrow{\rho} \frac{\mathcal{A}ss_c \left\langle \Phi_c \right\rangle}{\mathfrak{J}_c}, \tag{3.31}$$

is a quism, where $\mathfrak{J}_c$ is the submodule generated by (3.27) for $\sigma = c$. We will reduce our problem to this case. Let

$$\mathcal{M}_\sigma := \mathcal{A}ss_{O(\sigma)} \circ' \left(k\langle\underline{\sigma}\rangle, (\Phi_\sigma \oplus X_\sigma) \circ \mathcal{A}ss_{I(\sigma)}\right).$$

This is in fact a sub-Σ^0-collection of $\mathcal{A}\left\langle \Phi_\sigma \oplus X_\sigma \right\rangle$. An easy computation shows that it is closed under ∂^0. It will play a role similar to $\mathcal{A}ss_c \left\langle \Phi_c \oplus X_c \right\rangle$ above:

3.4.5 Sublemma. There is an isomorphism

$$(\mathcal{M}_\sigma, \partial^0) \cong {\uparrow}^{|\sigma|} \left(\mathcal{A}ss_c \left\langle \Phi_c \oplus X_c \right\rangle, \partial^0\right)$$

of dg-collections (we *ignore the colours!*). This induces an isomorphism

$$H_*(\mathcal{M}_\sigma, \partial^0) \cong \frac{\mathcal{A}ss_{O(\sigma)} \circ' \left(k\langle\underline{\sigma}\rangle, \Phi_\sigma \circ \mathcal{A}ss_{I(\sigma)}\right)}{\mathfrak{J}_\sigma}$$

of Σ^0-collections (compare to the right-hand side of (3.31)), where the quotient by $\mathfrak{J}_\sigma$ expresses the fact that ϕ_σ behaves like a derivation with respect to $\mu_{O(\sigma)}$ and $\mu_{I(\sigma)}$ in $\mathcal{M}_\sigma$. The relators in $\mathfrak{J}_\sigma$ are analogous to (3.27), namely $\mathfrak{J}_\sigma$ is sub-Σ^0-collection of $\mathcal{M}_\sigma$ consisting of elements

$$\begin{gathered}
(a_{O(\sigma)} \circ_i (\mu_{O(\sigma)} \circ_1 a^1_{O(\sigma)})) \circ'_{i+\mathrm{ar}(a^1)} (\underline{\sigma}, \phi_\sigma \circ a^2_{I(\sigma)}) + \\
+ (a_{O(\sigma)} \circ_i (\mu_{O(\sigma)} \circ_2 a^2_{O(\sigma)})) \circ'_i (\underline{\sigma}, \phi_\sigma \circ a^1_{I(\sigma)}) + \\
- a_{O(\sigma)} \circ'_i (\underline{\sigma}, \phi_\sigma \circ \mu_{I(\sigma)} \circ (a^1_{I(\sigma)}, a^2_{I(\sigma)}))
\end{gathered}$$

for all $a, a^1, a^2 \in \mathcal{A}ss$ and $1 \le i \le \mathrm{ar}(a)$.

of Sublemma 3.4.5. There is a morphism ψ of collections

$$a_{O(\sigma)} \circ'_i (\underline{\sigma}, x_\sigma \circ (a^1_{I(\sigma)}, \ldots, a^{\mathrm{ar}(x)}_{I(\sigma)})) \mapsto a_c \circ'_i (\mathbf{1}_c, x_c \circ (a^1_c, \ldots, a^{\mathrm{ar}(x)}_c))$$

for $x \in X$ (or $x = \phi$) and $a, a^1, a^2, \ldots \in \mathcal{A}ss$. ψ is obviously an isomorphism of degree $-|\sigma|$. The differential on the suspension is $(-1)^{|\sigma|}\partial^0$, hence we must verify

$$\psi\partial^0 = (-1)^{|\sigma|}\partial^0\psi.$$

This is immediate by the formula (3.26) defining ∂^0. □

Using the relations $f \circ \mu_{I(f)} - \mu_{O(f)} \circ (f, f)$ in $\mathcal{A}$ for $f \in \Sigma^1$, every element $a \in \mathcal{A} \langle \Phi_\sigma \oplus X_\sigma \rangle$ can be written in the form

$$a = a_{top} \circ_m (a_{mod} \circ (c_1, \ldots, c_{\mathrm{ar}(a_{mod})})) \tag{3.32}$$

for some $a_{top} \in \mathcal{A}$, $m \in \mathbb{N}$, $a_{mod} \in \mathcal{M}_\sigma$ and $c_1, c_2, \ldots \in \mathcal{C}$. We want to make this is expression as close to being unique as possible. We will require:

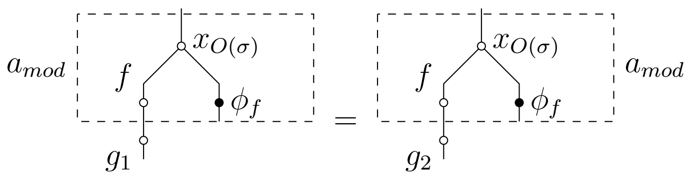
> If a_{top} can be written in the form
>
> $$a_{top} = a' \circ_k (\mu_{O(\sigma)} \circ (a'', 1_{O(\sigma)})) \quad \text{or} \quad a_{top} = a' \circ_k (\mu_{O(\sigma)} \circ (1_{O(\sigma)}, a''))$$
>
> for some $a', a'' \in \mathcal{A}$, $1 \leq k \leq \mathrm{ar}(a')$ satisfying $m = k + \mathrm{ar}(a'')$ resp. $m = k$, then $\mu_{O(\sigma)} \circ (a'', a_{mod})$ resp. $\mu_{O(\sigma)} \circ (a_{mod}, a'')$ can't be written as an element of $\mathcal{M}_\sigma \circ \mathcal{C}$.

It is easily seen that the above requirement can be met for any a so assume it holds in the above expression (3.32). It is easy to see that this determines a_{top} and a_{mod} uniquely up to scalar multiples. The elements $c_1, c_2, \ldots \in \mathcal{C}$ are however not unique as the following example shows:

Let $x \in X$ and let $f, g_1, g_2 \in \mathcal{C}$ be such that $fg_1 = fg_2$, hence

$$a := x_{O(f)} \circ (fg_1, \phi_f) = x_{O(f)} \circ (fg_2, \phi_f).$$

In "the canonical" form (3.32), $a_{mod} = x_{O(f)} \circ (f, \phi_f)$, however c_1 is either g_1 or g_2, in pictures:



So far we have shown that there are, for $m \geq 1$, Σ^0-collections $\mathcal{A}^{top}_{\sigma,m}$ (whose description is implicit in the above discussion) such that there is a Σ^0-collection isomorphism

$$\mathcal{A} \langle \Phi_\sigma \oplus X_\sigma \rangle \begin{pmatrix} c \\ c_1, \ldots, c_N \end{pmatrix} \cong \tag{3.33}$$

$$\bigoplus_{\substack{m,n \geq 1, \\ d \in \Sigma^0}} \mathcal{A}^{top}_{\sigma,m} \begin{pmatrix} c \\ c_1, \ldots, c_{m-1}, d, c_{m+n}, \ldots, c_N \end{pmatrix} \otimes \left(\frac{\mathcal{M}_\sigma \circ \mathcal{C}}{\mathfrak{L}} \right) \begin{pmatrix} d \\ c_m \ldots, c_{m+n-1} \end{pmatrix},$$

where $\mathfrak{L}$ is the sub-Σ^0-collection of $\mathcal{M}_\sigma \circ \mathcal{C}$ describing the non-uniqueness mentioned above. More precisely, it consists of elements

$$\underbrace{(b \circ'_n (\underline{\sigma}, x \circ (b_1, \ldots, b_{\mathrm{ar}(x)})))}_{a_{mod}} \circ (f_1, \ldots, f_{\mathrm{ar}(a_{mod})})$$

for all $a_{mod} \in \mathcal{M}_\sigma$, $f_1, f_2, \ldots \in \mathcal{C}$ satisfying that for some $1 \leq i \leq n-1$ or $n + \sum_{j=1}^{\mathrm{ar}(x)} \mathrm{ar}(b_j) \leq i \leq \mathrm{ar}(a_{mod})$ we have $\underline{\sigma} f_i = 0$. The condition $\underline{\sigma} f_i = 0$ means that $f_i = f - g$ (up to a scalar multiple) for some $f, g \in \Sigma^1$ and $\underline{\sigma} f = \underline{\sigma} g$.

At this point we suggest the reader to go through the above discussion in the case $|\sigma| = 0$ as many things simplify substantially, e.g. $\mathfrak{L} = 0$ and we are essentially done by applying the usual Künneth formula to (3.33) and then using Sublemma 3.4.5 which shows

$$H_*(\mathcal{M}_\sigma \circ \mathcal{C}, \partial^0) \cong \frac{(\mathcal{A}ss_{O(\sigma)} \circ' (k\langle \underline{\sigma} \rangle, \Phi_\sigma \circ \mathcal{A}ss_{I(\sigma)})) \circ \mathcal{C}}{\mathfrak{J}'_\sigma}, \tag{3.34}$$

where $\mathfrak{J}'_\sigma$ is an analogue of $\mathfrak{J}_\sigma$ above. For general σ, we have to get rid of the quotient:

3.4.6 Sublemma.

$$H_*\left(\frac{\mathcal{M}_\sigma \circ \mathcal{C}}{\mathfrak{L}}, \partial^0\right) \cong \frac{(\mathcal{A}ss_{O(\sigma)} \circ' (I, \Phi_\sigma \circ \mathcal{A}ss_{I(\sigma)})) \circ \mathcal{C}}{\mathfrak{J}''_\sigma},$$

where again $\mathfrak{J}''_\sigma$ is the corresponding analogue of $\mathfrak{J}_\sigma$.

of Sublemma 3.4.6. Denote

$$\mathcal{M}_\sigma \circ \mathcal{C} \xrightarrow{\mathrm{pr}} \frac{\mathcal{M}_\sigma \circ \mathcal{C}}{\mathfrak{L}} =: Q$$

the natural projection. The differential ∂^0_Q on Q inherited from $\mathcal{A}\langle \Phi_\sigma \oplus X_\sigma \rangle$ is given, for $\alpha \in Q$, by

$$\partial^0_Q \alpha = \mathrm{pr}\, \partial^0 \tilde{\alpha},$$

where $\tilde{\alpha} \in \mathcal{M}_\sigma \circ \mathcal{C}$ denotes any element such that $\mathrm{pr}\, \tilde{\alpha} = \alpha$.

By (3.34), to prove the sublemma it suffices to show

1. $\mathrm{pr}\, \mathrm{Ker}\, \partial^0 = \mathrm{Ker}\, \partial^0_Q$,
2. $\mathrm{pr}\, \mathrm{Im}\, \partial^0 = \mathrm{Im}\, \partial^0_Q$.

For 1., let $\alpha \in Q$, $\partial^0_Q \alpha = 0$ and we will show there is $\beta \in \mathcal{M}_\sigma \circ \mathcal{C}$ satisfying $\partial^0 \beta = 0$ and $\mathrm{pr}\, \beta = \alpha$. Let $\mathrm{pr}\, \tilde{\alpha} = \alpha$ and let

$$\tilde{\alpha} = \sum_{i \in I} a^i_{mod} \circ (a^i_1, \ldots, a^i_{\mathrm{ar}(a^i_{mod})})$$

for some index set I, $a^i_{mod} \in \mathcal{M}_\sigma$ and $a^i_j \in \mathcal{C}$, $i \in I$ and $1 \leq j \leq \mathrm{ar}(a^i_{mod})$, such that any two ordered $\mathrm{ar}(a^i_{mod})$-tuples $(a^i_1, \ldots, a^i_{\mathrm{ar}(a^i_{mod})})$ are distinct for any two distinct i's. Let $I_1 \subset I$ be the set of i's such that $\partial^0 a^i_{mod} = 0$ and let $I_2 := I - I_1$. By our assumption, $\mathrm{pr}\, \partial^0 \tilde{\alpha} = \partial^0_Q \alpha = 0$, hence

$$\partial^0 \tilde{\alpha} = \sum_{i \in I_2} (\partial^0 a^i_{mod}) \circ (a^i_1, \ldots, a^i_{\mathrm{ar}(a^i_{mod})}) \in \mathfrak{L}.$$

Thus for every i there is j such that $\underline{\sigma} \circ a^i_j = 0$ (by the definition of $\mathfrak{L}$). Because ∂^0 doesn't change a^i_j's, we get $\sum_{i \in I_2} a^i_{mod} \circ (a^i_1, \ldots, a^i_{\mathrm{ar}(a^i_{mod})}) \in \mathfrak{L}$ and we set

$$\beta := \sum_{i \in I_1} a^i_{mod} \circ (a^i_1, \ldots, a^i_{\mathrm{ar}(a^i_{mod})}).$$

Then obviously $\partial^0 \beta = 0$ and $\alpha = \mathrm{pr}\, \beta$, so we have obtained $\mathrm{pr}\, \mathrm{Ker}\, \partial^0 \supset \mathrm{Ker}\, \partial^0_Q$. The opposite inclusion is obvious. Also 2. is easy. $\square$

Now apply the sublemma 3.4.6 to (3.33). This concludes the proof of Lemma 3.4.3. □

of Lemma 3.4.4. This is just a straightforward application of Lemma 3.3.16 to operad $\mathcal{A} := \mathcal{C}$ and its bar-cobar resolution $\mathcal{R} := \Omega \mathrm{B}\mathcal{C}$ (e.g. [18]). To see this, recall that $\Omega \mathrm{B}\mathcal{C}$ is a quasi-free Σ^0-operad generated by Σ^0-collection $k\langle \Sigma^{\geq 1}\rangle$, where the degree of $\sigma \in \Sigma^{\geq 1}$ is $|\sigma| - 1$. The derivation differential is given by

$$\partial \underbrace{(\xleftarrow{f_n} \cdots \xleftarrow{f_1})}_{\sigma} := \sum_{i=1}^{n-1} (-1)^{i+n+1} (\xleftarrow{f_n} \cdots \xleftarrow{f_{i+1}}) \circ (\xleftarrow{f_i} \cdots \xleftarrow{f_1}) + \\ + \sum_{i=1}^{n-1} (-1)^{n-i} \underbrace{(\xleftarrow{f_n} \cdots \xleftarrow{f_{i+1} f_i} \cdots \xleftarrow{f_1})}_{\sigma_i}.$$

The projection $\Omega \mathrm{B}\mathcal{C} \xrightarrow{\rho_{\mathcal{R}}} \Omega \mathrm{B}^1\mathcal{C} \cong \mathcal{C}$ onto the sub-Σ^0-collection consisting of single generators is a quism.

Then $\mathcal{M}\overline{\mathcal{D}\mathcal{R}}$ of Lemma 3.3.16 is

$$\mathcal{M}\overline{\mathcal{D}\Omega\mathrm{B}\mathcal{C}} = \left(\mathcal{A} \left\langle \bigoplus_{c \in \Sigma^0} \Phi_c \oplus \bigoplus_{\sigma \in \Sigma^{\geq 1}} \Phi_\sigma \right\rangle, \partial_{\mathcal{M}\overline{\mathcal{D}\Omega\mathrm{B}\mathcal{C}}} \right),$$

where all the symbols have the same meaning as in the previous parts of this paper and it is easily checked that the differential is given by the formula (3.28). This resolves $\mathcal{M}\mathcal{D}\mathcal{C}$, which is readily seen to be the right-hand side in the statement of Lemma 3.4.4. □

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4. On resolutions of diagrams of algebras

Abstract

We prove a restricted version of a conjecture by M. Markl made in [7] on resolutions of an operad describing diagrams of algebras. We discuss a particular case related to the Gerstenhaber-Schack diagram cohomology.

4.1 Introduction

As explained in [9], the operadic cohomology gives a systematic way of constructing cohomology theories for algebras over an operad $\mathcal{P}$. The corresponding deformation complex carries an L_∞-structure describing deformations of $\mathcal{P}$-algebras. To make this explicit, one has to find a free resolution of $\mathcal{P}$.

In particular, we can apply this to the coloured operad $\mathcal{A}_{\mathsf{C}}$ describing a C-shaped diagram of $\mathcal{A}$-algebras. An important particular case is C consisting of a single morphism. This is discussed in [7],[3] and also, indirectly, in the definition of (weak) A_∞ and L_∞ morphisms. More complicated categories C received very little attention. In [7], M. Markl discussed examples leading to the notions of homotopy of $\mathcal{A}$-algebra morphisms and homotopy isomorphism of $\mathcal{A}$-algebras. In the end of the paper, a conjecture partially describing resolutions of $\mathcal{A}_{\mathsf{C}}$ for any $\mathcal{A}$ and C appears. In particular, it settles the question of the existence of the minimal resolution of $\mathcal{A}_{\mathsf{C}}$. We discuss this conjecture and prove it in the restricted case of $\mathcal{A}$ being a Koszul operad with generating operations concentrated in a single arity and degree, see the main Theorem 4.3.15.

The idea is to glue together a *minimal* resolution of $\mathcal{A}$ and any cofibrant free resolution of C. The generators of the resulting resolution $\mathcal{D}_\infty$ are described explicitly as well as the principal part of the differential ∂. To state the theorem precisely requires some preliminary work.

First, we discuss operadic resolutions $\mathcal{C}_\infty$ of categories. The operads in question are concentrated in arity 1, hence this is just a "coloured" version of classical homological algebra. We deal with maps $[\![-]\!]_n : \mathcal{C}_\infty \to \mathcal{C}_\infty^{\otimes n}$ with certain prescribed properties. These are needed to construct the principal part of ∂. We show that these maps are induced by certain coproducts on $\mathcal{C}_\infty$, thus relating them to (coloured) dg bialgebra structures on $\mathcal{C}_\infty$.

The proof of the main theorem follows the ideas of M. Markl from [7]. It is necessarily more complicated technically and we discuss it in detail in a separate section. We find it convenient to recall some technical results of coloured operad theory, namely a version of the Künneth formula for the composition product $\circ$, which is very useful for homological computations. Hence we spend some time in the initial part of the paper explaining basics, though we expect the reader is already familiar with coloured operads.

The case $\mathcal{C}_\infty$ being the bar-cobar resolution is particularly interesting. Here, $\mathcal{C}_\infty$ has a topological flavour, it is completely explicit and we even make $[\![-]\!]$

explicit. The resulting resolution $\mathcal{D}_\infty$ conjecturally gives rise to the Gerstenhaber-Schack complex for diagram cohomology [4].

Finally, let me thank Martin Markl for many useful discussions.

Contents

In **Section 4.2**, we briefly recall basic notions of coloured operad theory. We focus on the interplay between the colours and Σ action. We prove a version of the Künneth formula in Section 4.2.2. It computes the homology of the composition product.

In **Section 4.3**, we prepare necessary notions to formulate the main theorem. In Section 4.3.1, we discuss operadic resolutions of categories and give several examples. In Section 4.3.2, we introduce $[\![-]\!]_n$ maps, certain combinatorial structures on the resolution of the category. We prove that these maps always exist and recall some examples from the literature. We show that $[\![-]\!]_n$'s are induced by $[\![-]\!]_2$, which is a certain coproduct on the resolution. In Section 4.3.3, we explain how diagrams of algebras are described by coloured operads and show that this construction is functorial and quism-preserving. Section 4.3.4 contains the statement of the main theorem and compares it to the conjecture by M. Markl.

In **Section 4.4**, the main theorem is proved. In Section 4.4.2, we try to explain the structure of the proof and to point out the places where an improvement might be possible.

In **Section 4.5**, we recall the bar-cobar resolution of the category, then we make $[\![-]\!]_n$'s explicit by endowing the resolution with a (coloured) bialgebra structure. Finally, we discuss the conjectural relation to Gerstenhaber-Schack diagram cohomology.

4.2 Basics

4.2.1 Conventions and reminder

We will use the following notations and conventions:

- $\mathbb{N}_0$ is the set of natural numbers including 0.
- k is a fixed field of characteristics 0.
- $k\langle S\rangle$ is the k-linear span of the set S.
- $\otimes$ always means tensor product over k.
- Σ_n is the permutation group on n elements.
- V denotes a set (of colours[1]).
- $\mathrm{ar}(x)$ is arity of the object x, whatever x is.
- Vector spaces over k are called *k-modules*, chain complexes of vector spaces over k with differential of degree -1 are called *dg-k-modules* and morphisms of chain complexes are called just *maps*. Chain complexes are assumed **non-negatively graded** unless stated otherwise. The degree n summand of dg k-mod C is denoted C_n. We let $C_{\leq n} := \bigoplus_{0\leq i\leq n} C_i$ and similarly for other inequality symbols. Similar notation is used e.g. for V-Σ-modules of Definition 4.2.2.
- $\uparrow C$ denotes the *suspension* of the graded object C, that is $(\uparrow C)_n = C_{n-1}$. Similarly, the *desuspension* is defined by $(\downarrow C)_n = C_{n+1}$.
- $|x|$ is the degree of an element x of a dg-k-module.
- $H_*(C)$ is homology of the object C, whatever C is.
- *Quism* is a map f of dg-k-modules such that the induced map $H_*(f)$ on homology is an isomorphism.

We extend the notation introduced in section Basics of [1] for nonsymmetric coloured V-operads to symmetric coloured V-operads.

4.2.1 Definition. A permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ will also be denoted by $[\sigma(1)\sigma(2)\cdots\sigma(n)]$.

Let S be any set. Let $\vec{s} = (s_1,\dots,s_n) \in S^n$. If a context is clear, we may use this vector notation without explanation. S^n carries a right Σ_n action

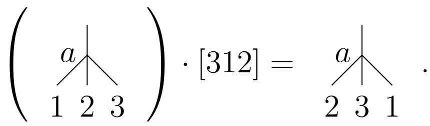
$$\vec{s}\cdot\sigma := (s_{\sigma(1)},\dots,s_{\sigma(n)}).$$

If $f : A^{\otimes n} \to A$ is a linear map, the right Σ_n action on f is defined by

$$(f\cdot\sigma)(\vec{a}) := f(\sigma\cdot\vec{a}) := f(\vec{a}\cdot\sigma^{-1}) = f(a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(n)})$$

[1] V actually stands for Vertices, which will become apparent later.

for $\vec{a} \in A^n$. This is useful for intuitive understanding of the right Σ_n action on elements of an operad. While drawing pictures, we use the convention that into a leaf labelled i, the i^{th} input element is inserted. Hence element $a \cdot \sigma$ is drawn with labels $\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n)$ from left to right, e.g.



For $\vec{v} \in V^n$, let

$$\Sigma_{\vec{v}} := \{\sigma \in \Sigma_n \mid \vec{v} = \vec{v} \cdot \sigma\} = \big\{\sigma \in \Sigma_n \mid v_i = v_{\sigma(i)} \text{ for each } 1 \le i \le n\big\}$$

be the stabilizer of $\vec{v}$ under the action of Σ_n.

4.2.2 Definition. A **dg V-Σ-module** X is a set

$$\{X(n) \mid n \in \mathbb{N}_0\}$$

of dg right $k\langle\Sigma_n\rangle$-modules such that each of them decomposes

$$X(n) = \bigoplus_{\substack{v \in V,\\ v_1,\ldots,v_n \in V}} X\binom{v}{v_1, \ldots, v_n}$$

as a dg k-module and $\sigma \in \Sigma_n$ acts by a dg k-module morphism

$$\cdot\sigma : X\binom{v}{v_1, \ldots, v_n} \to X\binom{v}{v_{\sigma(1)}, \ldots, v_{\sigma(n)}}.$$

It follows that $X\binom{v}{\vec{v}}$ is a dg $k\langle\Sigma_{\vec{v}}\rangle$-module. In particular, the differential commutes with the $k\langle\Sigma_{\vec{v}}\rangle$ action.

A **symmetric dg V-operad** is a dg V-Σ-module with the usual operadic compositions $\circ_i$. The axioms these compositions satisfy are the same as those for nonsymmetric dg V-operad (see [1], Definition 2.1) and we moreover require $\circ_i$'s to be equivariant in the usual sense (see [10], Definition 1.16 for noncoloured case).

We usually omit the prefix "symmetric". If a, b are elements of an operad $\mathcal{A}$ and $\mathrm{ar}(a) = 1$, we usually abbreviate $ab := a \circ b := a \circ_1 b$. If $\mathrm{ar}(a) = n$, we also abbreviate $a(b_1 \otimes \cdots \otimes b_n) := (\cdots((a \circ_1 b_1) \circ_2 b_2) \cdots) \circ_n b_n$. If V is a single element set, we omit the prefix "V-", otherwise we strictly keep the prefix.

Now we discuss the composition product $\circ$ on the category of V-Σ-modules. We need some preliminary notions first.

4.2.3 Definition. Let $l_1, \ldots, l_m$ be nonnegative integers. For $n := l_1 + \cdots + l_m$, there is the inclusion

$$\Sigma_{l_1} \times \cdots \times \Sigma_{l_m} \hookrightarrow \Sigma_n$$

given by

$$(\lambda_1 \times \cdots \times \lambda_m)(l_1 + \cdots + l_{i-1} + j) := l_1 + \cdots + l_{i-1} + \lambda_i(j),$$

where $1 \le i \le m$ and $1 \le j \le l_i$. If $l_i = 0$, we set $\Sigma_{l_i} = \Sigma_0 := \{1\}$.

Let $\tau \in \Sigma_m$. Denote

$$\begin{aligned}\overline{\tau} := \big[\ & l_1 + \cdots + l_{\tau(1)-1} + 1, \dots, l_1 + \cdots + l_{\tau(1)},\\ & l_1 + \cdots + l_{\tau(2)-1} + 1, \dots, l_1 + \cdots + l_{\tau(2)},\\ & \dots,\\ & l_1 + \cdots + l_{\tau(m)-1} + 1, \dots, l_1 + \cdots + l_{\tau(m)}\big].\end{aligned}$$

If $l_i = 0$, the block $l_1 + \cdots + l_{\tau(i)-1} + 1, \dots, l_1 + \cdots + l_{\tau(i)}$ is empty and therefore is omitted in the expression above. Equivalently, the above formula states

$$\overline{\tau}(l_{\tau(1)} + l_{\tau(2)} + \cdots + l_{\tau(i-1)} + j) := l_1 + l_2 + \cdots + l_{\tau(i)-1} + j$$

for any $1 \le i \le m$ and $1 \le j \le l_i$. $\overline{\tau}$ is called $(l_1, \dots, l_m)$-**block permutation** corresponding to τ.

4.2.4 Example.
- $[21] \times 1 \times [312] = [213645]$
- $(2, 1, 3)$-block permutation corresponding to $\tau = [231]$ is $\overline{[231]} = [345612]$:

$$\begin{array}{cccccc} \boxed{1\ \ 2} & & \boxed{3} & \boxed{4\ \ 5\ \ 6} \\ \downarrow\ \downarrow & & \downarrow & \downarrow\ \downarrow\ \downarrow \\ \boxed{3} & & \boxed{4\ \ 5\ \ 6} & \boxed{1\ \ 2} \end{array}$$

4.2.5 Definition. Fix $v \in V$ and $\vec{v} = (v_1, \dots, v_n) \in V^n$. Let $\mathcal{A} = (\mathcal{A}, \partial_{\mathcal{A}})$, $\mathcal{B} = (\mathcal{B}, \partial_{\mathcal{B}})$ be dg V-Σ-modules, let $l_1, \dots, l_m$ be nonnegative integers such that $l_1 + \cdots + l_m = n$. For each $1 \le i \le m$, let $\vec{w}_i = (w_{i,1}, \dots, w_{i,l_i}) \in V^{l_i}$. Denote $\vec{W} = (\vec{w}_1, \dots, \vec{w}_m) = (w_{1,1}, \dots, w_{m,l_m}) \in V^n$. Let

$$\begin{aligned}\Sigma(\vec{W}, \vec{v}) : &= \left\{\sigma \in \Sigma_n \mid \vec{W} \cdot \sigma = \vec{v}\right\} = \qquad (4.1)\\ &= \left\{\sigma \in \Sigma_n \mid w_{i,j} = v_{\sigma^{-1}(l_1 + \cdots + l_{i-1} + j)} \text{ for every } 1 \le i \le m,\ 1 \le j \le l_i\right\}.\end{aligned}$$

For fixed $l_1, \dots, l_m$ and $\vec{w} = (w_1, \dots, w_m)$

$$\bigoplus_{\vec{W}} \mathcal{B}\binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B}\binom{w_m}{\vec{w}_m} \otimes k\left\langle \Sigma(\vec{W}, \vec{v})\right\rangle$$

is a dg[2] right $k\langle \Sigma_{l_1} \times \cdots \times \Sigma_{l_m}\rangle$-module via

$$(b_1 \otimes \cdots \otimes b_m \otimes \sigma) \cdot (\lambda_1 \times \cdots \times \lambda_m) := (b_1 \cdot \lambda_1) \otimes \cdots \otimes (b_m \cdot \lambda_m) \otimes (\lambda_1 \times \cdots \times \lambda_m)^{-1} \sigma.$$

Denote the space of coinvariants of this $k\langle \Sigma_{l_1} \times \cdots \times \Sigma_{l_m}\rangle$-module by the lower index $\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}$.

Now assume only m is fixed and consider

$$\bigoplus_{\substack{l_1, \dots, l_m\\ \vec{w}}} \mathcal{A}\binom{v}{\vec{w}} \otimes \left(\bigoplus_{\vec{W}} \mathcal{B}\binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B}\binom{w_m}{\vec{w}_m} \otimes k\left\langle \Sigma(\vec{W}, \vec{v})\right\rangle\right)_{\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}}.$$

[2] $k\left\langle \Sigma(\vec{W}, \vec{v})\right\rangle$ is concentrated in degree 0.

This is dg right $k\langle\Sigma_m\rangle$-module via

$$(a \otimes b_1 \otimes \cdots \otimes b_m \otimes \sigma) \cdot \tau = (a \cdot \tau) \otimes b_{\tau(1)} \otimes \cdots \otimes b_{\tau(m)} \otimes \overline{\tau^{-1}}\sigma,$$

where the bar denotes the corresponding $(l_1, \dots, l_m)$-block permutation of Definition 4.2.3. It is easy to verify that this action is well defined.

Finally, by taking the Σ_m coinvariants and summing over m in the above formula, we get the desired **composition product** of V-Σ-modules:

$$(\mathcal{A} \circ \mathcal{B})\binom{v}{\vec{v}} := \tag{4.2}$$

$$\bigoplus_m \left(\bigoplus_{\substack{l_1,\dots,l_m \\ \vec{w}}} \mathcal{A}\binom{v}{\vec{w}} \otimes \left(\bigoplus_{\vec{W}} \mathcal{B}\binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B}\binom{w_m}{\vec{w}_m} \otimes k\Big\langle \Sigma(\vec{W}, \vec{v}) \Big\rangle \right)_{\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}} \right)_{\Sigma_m},$$

where

- m runs through nonnegative integers,
- $l_1, \dots, l_m$ run through nonnegative integers so that $l_1 + \cdots + l_m = n$,
- $\vec{w} = (w_1, \dots, w_m)$ runs through V^m,
- $\vec{W} = (\vec{w}_1, \dots, \vec{w}_m)$ runs through m-tuples of $\vec{w}_i$'s, where $\vec{w}_i \in V^{l_i}$,
- $\Sigma(\vec{W}, \vec{v})$ is given by (4.1).

To finish the definition of $\mathcal{A} \circ \mathcal{B}$, we let $\pi \in \Sigma_n$ act by

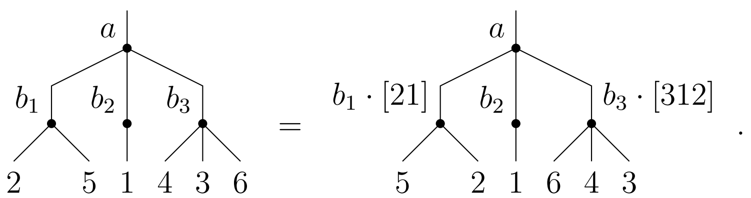
$$(a \otimes b_1 \otimes \cdots \otimes b_m \otimes \sigma) \cdot \pi := a \otimes b_1 \otimes \cdots \otimes b_m \otimes \sigma\pi.$$

We usually omit the coinvariants from the notation while dealing with elements of $\mathcal{A} \circ \mathcal{B}$.

The purpose of $\Sigma(\vec{W}, \vec{v})$ is to label the leaves so that for each i, the leaf labelled by i is of colour v_i. The purpose of the coinvariants is the usual one:

4.2.6 Example. By looking at the pictures, we find that we certainly want the equality

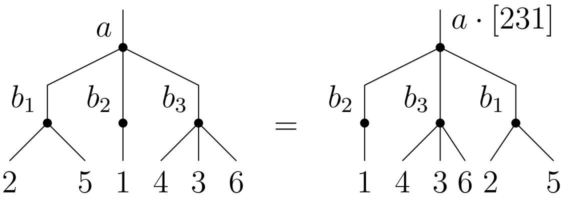
$$a \otimes b_1 \otimes b_2 \otimes b_3 \otimes [251436]^{-1} = a \otimes b_1[21] \otimes b_2 \otimes b_3[312] \otimes [521643]^{-1}$$



But since $[521643] = [251436]([21] \times 1 \times [312])$, the above equality is forced by taking the $\Sigma_{l_1} \times \cdots \times \Sigma_{l_m}$ coinvariants.

We also want

$$a \otimes b_1 \otimes b_2 \otimes b_3 \otimes [251436]^{-1} = a[231] \otimes b_2 \otimes b_3 \otimes b_1 \otimes [143625]^{-1}$$



But $[143625] = [251436]\overline{[231]}$, hence this equality is forced by the Σ_m coinvariants.

4.2.7 Definition. Let X be a V-Σ-module. The free V-operad generated by X carries the *weight grading*

$$\mathbb{F}(X) = \bigoplus_{i \geq 0} \mathbb{F}^i(X),$$

where $\mathbb{F}^i(X)$ is spanned by free compositions of exactly i generators. If X is moreover dg V-Σ-module, the dg structure is inherited to $\mathbb{F}(X)$ in the obvious way and we obtain a free dg V-operad. However, $\mathbb{F}(X)$ can be equipped with a differential which doesn't come from X and in this case, $(\mathbb{F}(X), \partial)$ is called *quasi-free.*

Recall a quasi-free dg V-operad $(\mathbb{F}(X), \partial)$ is called **minimal** iff $\operatorname{Im} \partial \subset \mathbb{F}^{\geq 2}(X)$. As usual, *free resolution* means a quism $(\mathbb{F}(X), \partial) \xrightarrow{\sim} (\mathcal{A}, \partial)$ with a quasi-free source. A *minimal resolution* is a resolution with a minimal source.

4.2.2 A Künneth formula

Our next task is to prove a version of the Künneth formula:

4.2.8 Lemma. Let $(\mathcal{A}, \partial_{\mathcal{A}})$, $(\mathcal{B}, \partial_{\mathcal{B}})$ be dg V-Σ-modules. Then there is a graded V-Σ-module isomorphism

$$H_*((\mathcal{A} \circ \mathcal{B}), \partial) \cong H_*(\mathcal{A}, \partial_{\mathcal{A}}) \circ H_*(\mathcal{B}, \partial_{\mathcal{B}}).$$

Proof. Let G be a finite group, let (M, ∂) be a dg $k\langle G\rangle$-module. Obviously, ∂ descends to coinvariants, hence (M_G, ∂) is a dg $k\langle G\rangle$-module too. We claim

$$H_*(M_G, \partial) \cong (H_*(M, \partial))_G. \tag{4.3}$$

By Maschke's theorem,

$$M = \bigoplus_{i \in I} M^i,$$

where M^i's are irreducible $k\langle G\rangle$-modules. ∂ is G-equivariant, hence for each i either $\partial M^i = 0$ or $\partial : M^i \xrightarrow{\cong} M^j$ is an isomorphism for some $j \neq i$. Denote

$$I_P := \left\{ i \in I \mid \partial M^i = 0 \text{ and there is no } j \text{ such that } \partial M^j = M^i \right\}.$$

Also, for each i, either $\partial M^i_G = 0$ (iff $\partial M^i = 0$) or $\partial : M^i_G \xrightarrow{\cong} M^j_G$ is isomorphism for some $j \neq i$ (iff $\partial : M^i \xrightarrow{\cong} M^j$). Then

$$\begin{aligned} H_*(M_G, \partial) = H_*((\bigoplus_{i \in I} M^i)_G, \partial) = H_*(\bigoplus_{i \in I} M^i_G, \partial) \cong \bigoplus_{i \in I_P} M^i_G \cong \\ \cong (\bigoplus_{i \in I_P} M^i)_G = (H_*(\bigoplus_{i \in I} M^i, \partial))_G \cong (H_*(M, \partial))_G. \end{aligned}$$

(4.3) is proved.

Let's set some shorthand notation. In (4.2), denote $B(\mathcal{B}) := \bigoplus_{\vec{W}} \mathcal{B}\binom{w_1}{\vec{w}_1} \otimes \cdots \otimes \mathcal{B}\binom{w_m}{\vec{w}_m} \otimes k\left\langle \Sigma(\vec{W}, \vec{v}) \right\rangle$. Denote $A(\mathcal{A}) := \mathcal{A}\binom{v}{\vec{w}}$ and $\Sigma := \Sigma_{l_1} \times \cdots \times \Sigma_{l_m}$. Notice that we are suppressing the dependency on $l_1, \ldots, l_m$ and $\vec{w}$. Omit v and $\vec{v}$ too. Hence (4.2) becomes

$$\mathcal{A} \circ \mathcal{B} = \bigoplus_m \big[\bigoplus_{\substack{l_1,\ldots,l_m \\ \vec{w}}} A(\mathcal{A}) \otimes B(\mathcal{B})_\Sigma \big]_{\Sigma_m}.$$

Let's compute:

$$\begin{aligned} H_*(\mathcal{A} \circ \mathcal{B}, \partial) &= \bigoplus_m H_*\big(\big[\bigoplus_{\substack{l_1,\ldots,l_m \\ \vec{w}}} A(\mathcal{A}) \otimes B(\mathcal{B})_\Sigma \big]_{\Sigma_m}\big) \cong \\ &\cong \bigoplus_m \big[\bigoplus_{\substack{l_1,\ldots,l_m \\ \vec{w}}} H_*(A(\mathcal{A}) \otimes B(\mathcal{B})_\Sigma) \big]_{\Sigma_m} \cong \\ &\cong \bigoplus_m \big[\bigoplus_{\substack{l_1,\ldots,l_m \\ \vec{w}}} H_*(A(\mathcal{A})) \otimes \big(H_*(B(\mathcal{B}))\big)_\Sigma \big]_{\Sigma_m} \cong \cdots \end{aligned}$$

The last isomorphism is provided by the usual Künneth formula and (4.3). Now trivially $H_*(A(\mathcal{A})) = A(H_*(\mathcal{A}, \partial_\mathcal{A}))$ and another application of the Künneth formula gives $H_*(B(\mathcal{B})) \cong B(H_*(\mathcal{B}, \partial_\mathcal{B}))$ and we finish:

$$\cdots \cong \bigoplus_m \big[\bigoplus_{\substack{l_1,\ldots,l_m \\ \vec{w}}} A(H_*(\mathcal{A}, \partial_\mathcal{A})) \otimes B(H_*(\mathcal{B}, \partial_\mathcal{B}))_\Sigma \big]_{\Sigma_m} = H_*(\mathcal{A}, \partial_\mathcal{A}) \circ H_*(\mathcal{B}, \partial_\mathcal{B}).$$

□

4.3 Statement of main theorem

4.3.1 Operadic resolution of category

Let C be a small category and denote

$$V := \mathrm{Ob}\, \mathsf{C}$$

the set of its objects. For a morphism $f \in \mathrm{Mor}\, \mathsf{C}$, let $I(f)$ be its source (Input) and $O(f)$ its target (Output). Let $\mathcal{C}$ be the operadic version of $\mathcal{C}$, that is

$$\mathcal{C} := k\langle \mathrm{Mor}\, \mathsf{C} \rangle \tag{4.4}$$

is seen as a coloured V-operad concentrated in arity 1, where each $f \in \mathrm{Mor}\, \mathsf{C}$ is an element of $\mathcal{C}\binom{O(f)}{I(f)}$ and the operadic composition is induced by the categorical composition. Obviously, $\mathcal{C}$ can be presented as

$$\mathcal{C} = \frac{\mathbb{F}(k\langle \mathrm{Mor}\, \mathsf{C} - \{\text{identities}\}\rangle)}{(\text{relations})},$$

where each relator is generated by those of the form $r_1 - r_2$ with r_1, r_2 being operadic compositions of elements of $\operatorname{Mor} \mathsf{C}$. Recall that the elements corresponding to the identities become a part of the free operad construction.

Every such V-operad $\mathcal{C}$ has a free resolution of the form

$$\mathcal{C}_\infty := (\mathbb{F}(F), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

where the graded V-Σ-module[3] $F = \bigoplus_{i \geq 0} F_i$ satisfies

Assumptions 4.3.1.

1. $F_0 = k\langle M \rangle$ for some $M \subset \operatorname{Mor} \mathsf{C} - \{\text{identities}\}$,
2. $F_1 = k\langle R \rangle$, where for each $r \in R$, $\partial r = r_1 - r_2$ for some free operadic compositions r_1, r_2 of elements of $M \cup \{\text{identities}\}$.

The existence of such a resolution is quite obvious and we will give several examples below. A general example is given by the bar-cobar resolution, which will be discussed later in Section 4.5 in detail. Before giving the examples, we note that

$$\mathcal{C} \cong \frac{\mathbb{F}(F_0)}{(\partial F_1)}. \tag{4.5}$$

4.3.2 Example. Let C be the category generated by 2 distinct morphisms between 3 distinct objects as in the picture:

$$V_1 \xrightarrow{\quad f \quad} V_2 \xrightarrow{\quad g \quad} V_3$$

Then $\operatorname{Ob} \mathsf{C} = V = \{V_1, V_2, V_3\}$, $\operatorname{Mor} \mathsf{C} = \{1_{V_1}, 1_{V_2}, 1_{V_3}, f, g,\ h := gf\}$. The composition is obvious. The V-operad $\mathcal{C}$ has colour decomposition $\mathcal{C}\binom{V_2}{V_1} = k\langle f \rangle$, $\mathcal{C}\binom{V_3}{V_2} = k\langle g \rangle$, $\mathcal{C}\binom{V_3}{V_1} = k\langle h \rangle$. $\mathcal{C}$ has the following 2 obvious resolutions:

1. Directly from the obvious presentation of $\mathcal{C}$, we get

   $$(\mathbb{F}(k\langle f, g, h, H \rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

   where f, g, h are copies of the corresponding generators of $\mathcal{C}$ and $I(H) = V_1$, $O(H) = V_3$. The degrees are as follows : $|f| = |g| = |h| = 0$ and $|H| = 1$. The differential ∂ vanishes on f, g, h and $\partial H = gf - h$.

2. A "smaller" resolution of $\mathcal{C}$ is

   $$(\mathbb{F}(k\langle f, g \rangle), 0) \xrightarrow{\sim} (\mathcal{C}, 0).$$

   It has less generators because the existence of h is already forced by the existence of f, g. This is an example of a minimal resolution of Definition 4.2.7.

[3]Of course, the action of Σ_1 carries no information and can be omitted.

4.3.3 Example. The category

$$V_1 \xrightarrow{f} V_2$$

has, apart from the obvious one, a free resolution

$$\mathcal{C}_\infty := (\mathbb{F}(k\langle f, g, H\rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

where $I(g) = I(H) = V_1$, $O(g) = O(H) = V_2$, $|g| = 0$, $|H| = 1$ and $\partial H = f - g$. It was observed in [7] that every algebra over $\mathcal{C}_\infty$ corresponds to a pair of dg k-modules, a pair of morphisms f, g between these and a homotopy H between f and g. Hence even resolutions of boring categories, such as $\mathcal{C}$ in this example, may lead to interesting concepts.

4.3.4 Example. Probably the simplest example of C which can't be resolved in degrees 0 and 1 only is given by the commutative cube:

$$\begin{tikzcd}
& \mathbf{7} \arrow[rr] & & \mathbf{8} \\
\mathbf{5} \arrow[rr] \arrow[ur] & & \mathbf{6} \arrow[ur] & \\
& \mathbf{3} \arrow[rr] \arrow[uu] & & \mathbf{4} \arrow[uu] \\
\mathbf{1} \arrow[rr] \arrow[ur] \arrow[uu] & & \mathbf{2} \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Objects (i.e. vertices) are denoted $\mathbf{1}, \cdots, \mathbf{8}$, edges (and the corresponding generators of the resolution below) are denoted (ab) with $\mathbf{8} \geq a > b \geq \mathbf{1}$. The faces are denoted $(abcd)$ with $\mathbf{8} \geq a > b > c > d \geq \mathbf{1}$. Then

$$(\mathbb{F}(k\langle(\mathbf{21}), \cdots, (\mathbf{4321}), \ldots, H\rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0)$$

is generated by all edges and faces and H so that the edges are of degree 0 and $I((ab)) = b$, $O((ab)) = a$; faces are of degree 1 and $I((abcd)) = d$, $O((abcd)) = a$; finally $|H| = 2$ and $I(H) = \mathbf{1}$, $O(H) = \mathbf{8}$. The differential is given by

$$\begin{aligned}
\partial(ab) &= 0, \\
\partial(abcd) &= (ac)(cd) - (ab)(bd), \\
\partial H &= (\mathbf{84})(\mathbf{4321}) + (\mathbf{8743})(\mathbf{31}) - (\mathbf{8642})(\mathbf{21}) + \\
&\quad + (\mathbf{87})(\mathbf{7531}) - (\mathbf{8765})(\mathbf{51}) - (\mathbf{86})(\mathbf{6521}).
\end{aligned}$$

The resolving morphism maps edges to edges and all other generators to 0. It is easy to verify that this is a minimal resolution.

We let the reader convince himself that $\mathcal{C}$ can't indeed be resolved just in degrees 0 and 1. Rigorously, this would follow from the uniqueness of the minimal resolution together with a theorem asserting that any free resolution decomposes into a free product of a minimal resolution and an acyclic dg V-operad[4]. These theorems however go beyond the scope of this paper.

4.3.5 Example. An explicit resolution of the category generated by

$$\begin{tikzcd} V_1 \arrow[r, shift left, "f"] & V_2 \arrow[l, shift left, "g"] \end{tikzcd}$$

with relations

$$fg - 1_{V_2}, \quad gf - 1_{V_1}$$

was found in [8]. It contains a generator of *each* nonnegative degree.

[4]An analogue exists in rational homotopy theory - see [2], Theorem 14.9.

4.3.2 $[\![-]\!]_n$ maps

Since $\mathcal{C}_\infty$ is concentrated in arity 1, we won't distinguish between $\mathcal{C}_\infty$ and $\mathcal{C}_\infty(1)$. Also observe, that V-operad concentrated in arity 1 is just a coloured[5] dg associative algebra.

Consider the usual dg structure on $\mathcal{C}_\infty^{\otimes n}$. There is also a right action of Σ_n generated by transpositions as follows. Let $\tau \in \Sigma_n$ exchange i and j. Then

$$(r_1 \otimes \cdots \otimes r_i \otimes \cdots \otimes r_j \otimes \cdots \otimes r_n) \cdot \tau := \\ = (-1)^{|r_i||r_j| + (|r_i|+|r_j|)\sum_{i<k<j}|r_k|} r_1 \otimes \cdots \otimes r_j \otimes \cdots \otimes r_i \otimes \cdots \otimes r_n$$

for any $r_1, \ldots, r_n \in \mathcal{C}_\infty$ such that $I(r_i) = O(s_i)$ for all $1 \leq i \leq n$. Further, there is the factorwise composition on $\mathcal{C}_\infty^{\otimes n}$:

$$(r_1 \otimes \cdots \otimes r_n) \circ (s_1 \otimes \cdots \otimes s_n) := (-1)^{\sum_{n \geq i > j \geq 1} |r_i||s_j|} (r_1 s_1) \otimes \cdots \otimes (r_n s_n). \tag{4.6}$$

It is easily seen that ∂ is a degree -1 derivation with respect to $\circ$:

$$\partial(R \circ S) = (\partial R) \circ S + (-1)^{|R|} R \circ \partial S$$

for any $R, S \in \mathcal{C}_\infty^{\otimes n}$. Also, $\circ$ is Σ_n equivariant:

$$(R \circ S) \cdot \tau = (R \cdot \tau) \circ (S \cdot \tau).$$

The following lemma is a straightforward generalization of Definition 23 of [7]:

4.3.6 Lemma. For every integer $n \geq 1$, there is a linear map

$$[\![-]\!]_n : \mathcal{C}_\infty \to \mathcal{C}_\infty^{\otimes n}$$

satisfying for every $r, r' \in \mathcal{C}_\infty$

(C1) $[\![r]\!]_n$ is Σ_n-stable,

(C2) $[\![r]\!]_n \in \mathcal{C}_\infty\binom{O(r)}{I(r)}^{\otimes n}$,

(C3) $\deg [\![r]\!]_n = \deg r$,

(C4) $[\![f]\!]_n = f^{\otimes n}$ for every morphism $f \in M \subset F_0$ (recall 4.3.1),

(C5) $[\![r \circ r']\!]_n = [\![r]\!]_n \circ [\![r']\!]_n$,

(C6) $\partial [\![r]\!]_n = [\![\partial r]\!]_n$.

Proof. Fix n. We proceed by induction on degree d. (C4) defines $[\![-]\!]_n$ for M, we extend linearly to F_0 and then extend by (C5) to all of $\mathbb{F}(F_0)$. Obviously, (C1)–(C6) hold for $r, r' \in \mathbb{F}(F_0)$. Assume we have already defined $[\![-]\!]_n$ on $\mathbb{F}(F_{<d})$ so that (C2)–(C6) hold.

[5]The operations are defined only partially, respecting the colours.

1. Let $d = 1$. By the assumptions 4.3.1, $F_1 = k\langle R\rangle$ and $f \in R$. We have $\partial f = r_1 - r_2$ as in 4.3.1, hence $[\![\partial f]\!]_n = r_1^{\otimes n} - r_2^{\otimes n}$. Define

$$[\![f]\!]_n^{\mathrm{NS}} := \sum_{i=0}^{n-1} r_1^{\otimes i} \otimes f \otimes r_2^{\otimes n-i-1}. \tag{4.7}$$

An easy computation shows $[\![f]\!]_n^{\mathrm{NS}}$ is a degree 1 element of $\mathcal{C}_\infty^{\otimes n}\binom{O(f)}{I(f)}$ satisfying $\partial [\![f]\!]_n^{\mathrm{NS}} = [\![\partial f]\!]_n$.

2. Let $d \geq 2$. Since $|\partial f| < d$, $[\![\partial f]\!]_n$ is already constructed and we are solving the equation
$$\partial [\![f]\!]_n = [\![\partial f]\!]_n$$
for an unknown $[\![f]\!]_n$ in the standard way. By the induction assumption, $\partial [\![\partial f]\!]_n = [\![\partial^2 f]\!]_n = 0$. By the usual Künneth formula, $\mathcal{C}_\infty^{\otimes n}$ is acyclic in positive degrees. Since $|[\![\partial f]\!]_n| = d - 1 > 0$, we obtain a degree d element $[\![f]\!]_n^{\mathrm{NS}} \in \mathcal{C}_\infty\binom{O(f)}{I(f)}^{\otimes n}$ such that $\partial [\![f]\!]_n^{\mathrm{NS}} = [\![\partial f]\!]_n$.

Making $[\![f]\!]_n^{\mathrm{NS}}$ to satisfy (C1) in characteristics 0 is easy:

$$[\![f]\!]_n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} [\![f]\!]_n^{\mathrm{NS}} \cdot \sigma. \tag{4.8}$$

We now have $[\![f]\!]_n$ satisfying (C1)–(C4) and (C6) for every $f \in F_d$. Extend this to $\mathbb{F}(F_{\leq d})$ by (C5). By Σ_n equivariance of $\circ$, (C1) holds on $\mathbb{F}(F_{\leq d})$. Verifying (C2) and (C3) is trivial, hence it remains to check (C6). Let $f_1, \dots, f_m \in F_{\leq d}$:

$$\begin{aligned}
\partial [\![f_1 \cdots f_m]\!]_n &= \partial \left([\![f_1]\!]_n \circ \cdots \circ [\![f_m]\!]_n\right) = \\
&= \sum_{i=1}^{m} (-1)^{\epsilon_i} [\![f_1]\!]_n \circ \cdots \circ \partial [\![f_i]\!]_n \circ \cdots \circ [\![f_m]\!]_n = \\
&= \sum_{i=1}^{m} (-1)^{\epsilon_i} [\![f_1]\!]_n \circ \cdots \circ [\![\partial f_i]\!]_n \circ \cdots \circ [\![f_m]\!]_n = \\
&= [\![\sum_{i=1}^{m} (-1)^{\epsilon_i} f_1 \cdots \partial f_i \cdots f_m]\!]_n = [\![\partial (f_1 \cdots f_m)]\!]_n,
\end{aligned}$$

where $\epsilon_i := |f_1| + \cdots + |f_{i-1}|$. Hence (C6) is valid for all elements of $\mathbb{F}(F_{\leq d})$ and the induction is finished. □

4.3.7 Example. If $\mathcal{C}_\infty$ is concentrated in degrees ≤ 1, then we have explicit formulas (4.7) and (4.8) for $[\![-]\!]$ given in the proof.

4.3.8 Example. For the resolution of Example 4.3.5, the construction of $[\![-]\!]_n$'s using Lemma 4.3.6 is not explicit. In this case, $[\![-]\!]$ was found explicitly in [7], Remark 25.

The following lemma shows that $[\![-]\!]_2$ induces $[\![-]\!]_n$ for all $n \geq 3$. $[\![-]\!]_2$ can be thought of as a coproduct on $\mathcal{C}_\infty$. If $[\![-]\!]_2$ is moreover coassociative, then (C2),(C5) and (C6) means that $(\mathcal{C}_\infty, \circ, [\![-]\!]_2)$ is a coloured dg *bialgebra*.

4.3.9 Lemma. Let $[\![-]\!]_2^{\mathrm{NS}} : \mathcal{C}_\infty \to \mathcal{C}_\infty \otimes \mathcal{C}_\infty$ be a linear map satisfying the conditions (C2)–(C6) of Lemma 4.3.6. Set

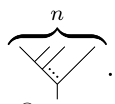
$$[\![-]\!]_n^{\mathrm{NS}} := ([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-3}) \cdots ([\![-]\!]_2^{\mathrm{NS}} \otimes 1)[\![-]\!]_2^{\mathrm{NS}}. \tag{4.9}$$

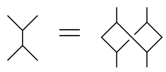
Then for each $n \geq 3$, $[\![-]\!]_n^{\mathrm{NS}}$ satisfies (C2)–(C6) and $[\![-]\!]_n$ defined by (4.8) satisfies (C1)–(C6). If $[\![-]\!]_2^{\mathrm{NS}}$ is coassociative, i.e. $([\![-]\!]_2^{\mathrm{NS}} \otimes 1)[\![-]\!]_2^{\mathrm{NS}} = (1 \otimes [\![-]\!]_2^{\mathrm{NS}})[\![-]\!]_2^{\mathrm{NS}}$, then

$$(1^{\otimes i} \otimes [\![-]\!]_a^{\mathrm{NS}} \otimes 1^{\otimes b-i-1})[\![-]\!]_b^{\mathrm{NS}} = [\![-]\!]_{a+b-1}^{\mathrm{NS}}$$

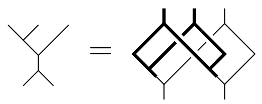
for every $a, b \geq 2$, $0 \leq i \leq b-1$.

Proof. Conditions (C2)–(C4) for $[\![-]\!]_n^{\mathrm{NS}}$ are easily seen to be satisfied.

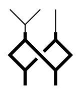
We sketch a proof of (C5) by the standard flow diagrams. Let $[\![-]\!]_2^{\mathrm{NS}}$ be represented by , then $[\![-]\!]_n^{\mathrm{NS}}$ is represented by . Let $\circ : \mathcal{C}_\infty \otimes \mathcal{C}_\infty \to \mathcal{C}_\infty$ of (4.6) be represented by , then $\circ : \mathcal{C}_\infty^{\otimes n} \otimes \mathcal{C}_\infty^{\otimes n} \to \mathcal{C}_\infty^{\otimes n}$ is represented, e.g. for $n = 3$, by . Observe that the signs are handled by the Koszul sign convention. The property (C5) for $[\![-]\!]_2^{\mathrm{NS}}$ states


(4.10)

For $n = 3$, we have to prove $[\![a \circ b]\!]_3^{\mathrm{NS}} = [\![a]\!]_3^{\mathrm{NS}} \circ [\![b]\!]_3^{\mathrm{NS}}$, i.e.



Applying (4.10) to the bold subgraph, we obtain



and another application of (4.10) on the bold subgraph gives the left hand side of the desired equality. The general case is analogous.

We prove (C6) by induction on n. $n = 2$ is the hypothesis. Let (C6) be true for $n-1$ and let's compute:

$$\begin{aligned}
\partial[\![-]\!]_n^{\mathrm{NS}} &= \partial([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2}) \cdots ([\![-]\!]_2^{\mathrm{NS}} \otimes 1)[\![-]\!]_2^{\mathrm{NS}} = \\
&= (\partial[\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})[\![-]\!]_{n-1}^{\mathrm{NS}} + \sum_{i=0}^{n-3}([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes i} \otimes \partial \otimes 1^{\otimes n-3-i})[\![-]\!]_{n-1}^{\mathrm{NS}} = \\
&= ([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})(\partial \otimes 1^{\otimes n-2})[\![-]\!]_{n-1}^{\mathrm{NS}} + \\
&\quad + \sum_{i=0}^{n-3}([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})(1^{\otimes i+1} \otimes \partial \otimes 1^{\otimes n-3-i})[\![-]\!]_{n-1}^{\mathrm{NS}} = \\
&= ([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})\partial[\![-]\!]_{n-1}^{\mathrm{NS}} = ([\![-]\!]_2^{\mathrm{NS}} \otimes 1^{\otimes n-2})[\![-]\!]_{n-1}^{\mathrm{NS}}\partial = [\![-]\!]_n^{\mathrm{NS}}\partial.
\end{aligned}$$

The proof of the coassociativity statement is easy and we leave it to the reader. $\square$

Later, in Theorem 4.5.1, we will construct $[\![-]\!]$ on the bar-cobar resolution $\Omega\mathrm{B}\mathcal{C}$ of $\mathcal{C}$ using this lemma out of a coassociative coproduct on $\Omega\mathrm{B}\mathcal{C}$.

4.3.10 Example. Our assumptions 4.3.1 are important. Consider the category generated by a single morphism between two distinct objects as in Example 4.3.3. Then $\mathcal{C}$ has yet another resolution: Take the same generators as in 4.3.3,

$$\mathcal{C}_\infty := (\mathbb{F}(k\langle f, g, H\rangle), \partial) \xrightarrow{\sim} (\mathcal{C}, 0),$$

but let

$$\partial H = f + g.$$

Then an elementary linear algebra shows that $f^{\otimes 2} + g^{\otimes 2}$ is a cycle but not a boundary in $\mathcal{C}_\infty^{\otimes 2}$. Hence our proof of Lemma 4.3.6 would fail.

4.3.3 Operad describing diagrams

Let a small category C (together with its operadic version (4.4)) and a dg operad $\mathcal{A}$ be given. A (C-shaped) **diagram** of ($\mathcal{A}$-algebras) is a functor

$$D : \mathsf{C} \to \mathcal{A}-\mathsf{algebras}.$$

Now we describe a dg V-operad $\mathcal{D}$ such that $\mathcal{D}$-algebras are precisely C-shaped diagrams. We denote by $*$ the free product of dg V-operads, i.e. the coproduct in the category of dg V-operads.

4.3.11 Definition. For any (noncoloured) dg operad $(\mathcal{A}, \partial_\mathcal{A})$, define

$$(\mathcal{A}, \partial_\mathcal{A})_\mathsf{C} := \left(\frac{(*_{v\in V}\mathcal{A}_v) * \mathcal{C}}{(f a_{I(f)} - a_{O(f)} f^{\otimes \operatorname{ar}(a)} \mid a \in \mathcal{A},\ f \in \operatorname{Mor}\mathsf{C})}, \partial \right), \tag{4.11}$$

where $\mathcal{A}_v$ is a copy of $\mathcal{A}$ concentrated in colour v and symbols for its elements are decorated with lower index v. Let the differential ∂ be defined by formulas

$$\partial a_v = (\partial_\mathcal{A} a)_v,$$
$$\partial f = 0$$

for any $a \in \mathcal{A}$, $v \in V$ and $f \in C$. For a dg operad morphism $(\mathcal{A}, \partial_\mathcal{A}) \xrightarrow{\xi} (\mathcal{B}, \partial_\mathcal{B})$, a dg V-operad morphism

$$(\mathcal{A}, \partial_\mathcal{A})_\mathsf{C} \xrightarrow{\xi_\mathsf{C}} (\mathcal{B}, \partial_\mathcal{B})_\mathsf{C}$$

is defined by

$$\begin{aligned} \xi_\mathsf{C}(a_v) &:= (\xi(a))_v, \\ \xi_\mathsf{C}(f) &:= f. \end{aligned} \tag{4.12}$$

It is easy to verify that the defining ideal of $\mathcal{A}_\mathsf{C}$ is sent to the defining ideal of $\mathcal{B}_\mathsf{C}$ and also that $\xi_\mathsf{C}\partial = \partial\xi_\mathsf{C}$, thus ξ_C is well defined. It is also easily seen that

$$\xi_\mathsf{C}\zeta_\mathsf{C} = (\xi\zeta)_\mathsf{C}$$

for any two dg operad morphisms ξ, ζ, hence

$$-_\mathsf{C} : \mathsf{dg\ operads} \to \mathsf{dg}\ V\mathsf{-operads}$$

is a functor.

Set

$$\mathcal{D} := (\mathcal{A}, \partial_{\mathcal{A}})_{\mathsf{C}}.$$

It is immediately seen that the functor D above is essentially the same thing as $\mathcal{D}$-algebra, i.e. dg V-operad morphism $\mathcal{D} \to \mathcal{E}nd_W$, where $W = \bigoplus_{v \in V} D(v)$ and each $D(v)$ is a dg k-module of colour v.

The following lemma generalizes Proposition 5 of [7].

4.3.12 Lemma. $-_{\mathsf{C}}$ preserves quisms.

Proof. Let $\mathcal{A} = (\mathcal{A}, \partial)$ be a dg V-operad and let $v, v_1, \dots, v_n \in V$. We claim that there is an isomorphism

$$\mathcal{A}_{\mathsf{C}}\binom{v}{v_1, \dots, v_n} \cong \mathcal{A}_v(n) \otimes_{\Sigma_n} \left(\bigoplus_{\vec{W}} \mathcal{C}\binom{v}{w_{1,1}} \otimes \cdots \otimes \mathcal{C}\binom{v}{w_{n,1}} \otimes k\left\langle \Sigma(\vec{W}, \vec{v}) \right\rangle \right) =$$
$$= (\mathcal{A}_v \circ \mathcal{C})\binom{v}{v_1, \dots, v_n}$$

of dg $k\langle \Sigma_{\vec{v}} \rangle$-modules, where $\vec{W} = (w_{1,1}, \dots, w_{n,1}) \in V^n$.

The isomorphism assigns a *canonical form* to an element $x \in \mathcal{A}_{\mathsf{C}}\binom{v}{v_1,\dots,v_n}$: Assume x is an equivalence class of a composition of the generators from $(*_{v \in V} \mathcal{A}_v) * \mathcal{C}$. Now use the defining relations to "move" the generators from $*_{v \in V} A_v$ to the left, so that $x = a \otimes f_1 \otimes \cdots \otimes f_n \otimes \sigma$ for some $a \in \mathcal{A}$, $f_1, \dots, f_n \in \mathcal{C}$ and $\sigma \in \Sigma_n$. Then $a \otimes f_1 \otimes \cdots \otimes f_n \otimes \sigma$ is called the canonical form of x. By freeness, it is uniquely determined by x. It is immediate that we get an isomorphism of dg $k\langle \Sigma_{\vec{v}} \rangle$-modules above and also $\mathcal{A}_{\mathsf{C}}(n) \cong (\mathcal{A}_v \circ \mathcal{C})(n)$ as dg $k\langle \Sigma_n \rangle$-modules.

Let $(\mathcal{A}, \partial_{\mathcal{A}}) \xrightarrow{\xi} (\mathcal{B}, \partial_{\mathcal{B}})$ be a quism. It is easy to see that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}_{\mathsf{C}}\binom{v}{v_1, \dots, v_n} \arrow[r, "\xi_{\mathsf{C}}"] \arrow[d, "\cong"'] & \mathcal{B}_{\mathsf{C}}\binom{v}{v_1, \dots, v_n} \arrow[d, "\cong"] \\
(\mathcal{A}_v \circ \mathcal{C})\binom{v}{v_1, \dots, v_n} \arrow[r, "\xi \otimes 1 \otimes \cdots \otimes 1 \otimes 1"] & (\mathcal{B}_v \circ \mathcal{C})\binom{v}{v_1, \dots, v_n}
\end{tikzcd}$$

The diagram descends to homology, the lower horizontal arrow becomes an isomorphism by Lemma 4.2.8, thus the upper horizontal arrow becomes an isomorphism as well. $\square$

4.3.4 Main theorem

Suppose we are given a resolution

$$\mathcal{C}_\infty = (\mathbb{F}(F), \partial) \xrightarrow[\phi_{\mathcal{C}}]{\sim} (\mathcal{C}, 0)$$

of $\mathcal{C}$ satisfying the *assumptions* 4.3.1 and a *minimal* resolution

$$\mathcal{A}_\infty = (\mathbb{F}(X), \partial) \xrightarrow[\phi_{\mathcal{A}}]{\sim} (\mathcal{A}, \partial)$$

of $\mathcal{A}$. We will use the same symbol ∂ for all the involved differentials. The correct meaning will always be clear from the context. Denote

$$X_V := X \otimes k\langle V\rangle$$
$$X_F := \uparrow X \otimes F.$$

These are V-Σ-modules by Σ action on the X factor. An element $x\otimes v \in X\otimes k\langle V\rangle$ is denoted by x_v. Analogously, $\uparrow x \otimes f \in \uparrow X \otimes F$ is denoted x_f. Hence

$$|x_v| = |x|, \quad |x_f| = |x| + |f| + 1.$$

Obviously $X_V = \bigoplus_{v\in V} X \otimes k\langle v\rangle$ and for any $v \in V$ we denote

$$X_v := X \otimes k\langle v\rangle\,.$$

Finally, let

$$\mathcal{D}_\infty := \mathbb{F}(X_V \oplus F \oplus X_F).$$

We also extend the notation x_v for $x \in X$ and $v \in k\langle V\rangle$ to an operad morphism

$$\begin{aligned} -_v : \mathbb{F}(X) \to \mathbb{F}(X_v) \hookrightarrow \mathcal{D}_\infty \\ x \mapsto x_v. \end{aligned}$$

We will be interested in differentials of a special form on $\mathcal{D}_\infty$. To state it precisely, we introduce the following maps:

4.3.13 Definition. For any $x \in X(n)$, the linear map

$$\mathcal{P}(x, -) : \mathcal{C}_\infty \to \mathcal{D}_\infty(n)$$

is uniquely given by requiring

$$\begin{aligned} &\mathcal{P}(x, f) = x_f, \\ &\mathcal{P}(x, r_1r_2) = \mathcal{P}(x, r_1)[\![r_2]\!]_n + (-1)^{|r_1|(|x|+1)} r_1\mathcal{P}(x, r_2) \end{aligned}$$

for every $f \in F$ and $r_1, r_2 \in \mathcal{C}_\infty$.

Thus $\mathcal{P}(x, -)$ behaves much like a derivation of degree $|x| + 1$. Checking it is well defined boils down to verify $\mathcal{P}(x, r_1(r_2r_3)) = \mathcal{P}(x,(r_1r_2)r_3)$, which is easy. Note that $\mathcal{P}(x, 1) = 0$ for any unit in the V-operad $\mathcal{C}_\infty$.

4.3.14 Definition. Let $\mathcal{A}$ be a graded operad. Recall that a presentation

$$\frac{\mathbb{F}(E)}{(R)} \cong \mathcal{A} \tag{4.13}$$

is called *quadratic* iff $R \subset \mathbb{F}^2(E)$, i.e. elements of R are sums of operadic compositions of exactly 2 generators from E. The elements of the Σ-module E are called *generating operations.*

Recall $\mathcal{A}$ is called *Koszul* iff there is a quadratic presentation (4.13) such that the cobar construction on the Koszul dual $\mathcal{A}^¡$ of $\mathcal{A}$ is a resolution of $\mathcal{A}$, i.e.

$$\Omega(\mathcal{A}^¡) \xrightarrow[\phi_{\mathcal{A}}]{\sim} (\mathcal{A}, 0).$$

See [5] for the notation and more details.

We are now finally able to state our main result:

4.3.15 Theorem. Let $\mathcal{A}$ be a *Koszul* operad with generating operations concentrated in a *single arity* ≥ 2 and a *single degree* ≥ 0. Let C be a small category and let $(\mathcal{C}_\infty, \partial_\mathcal{C}) \xrightarrow{\phi_\mathcal{C}} (\mathcal{C}, 0)$ be its resolution (in the sense explained in Section 4.3.1) satisfying the assumptions 4.3.1. Then the graded V-operad $\mathcal{D} = (\mathcal{A}, 0)_\mathsf{C}$ of (4.11), describing C-shaped diagrams of $\mathcal{A}$-algebras, has a free resolution

$$(\mathcal{D}_\infty, \partial) \underset{\Phi}{\xrightarrow{\sim}} (\mathcal{D}, 0)$$

of the form

$$\mathcal{D}_\infty := \mathbb{F}(X_V \oplus F \oplus X_F)$$

with the differential ∂ given by

$$\begin{aligned}
&\partial x_v = (\partial x)_v,\\
&\partial f = \partial_\mathcal{C} f,\\
&\partial x_f = (-1)^{1+|x|}\mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)}[\![f]\!]_n + \omega(x, f),
\end{aligned} \tag{4.14}$$

where $x \in X(n)$, $v \in V$, $f \in F$ and $\omega(x, f)$ lies in the arity n part of the

$$\text{ideal } \mathcal{I}^{<n} \text{ generated by } F_{\geq 1} \oplus X_F(< n) \tag{4.15}$$

in

$$\mathcal{D}_\infty^{<n} := \mathbb{F}(X_V(< n) \oplus F \oplus X_F(< n)). \tag{4.16}$$

The differential ∂ on $\mathcal{D}_\infty$ is minimal iff ∂ on $\mathcal{C}_\infty$ is. The dg V-operad morphism Φ is given by

$$\begin{aligned}
\Phi(x_v) &= (\phi_\mathcal{A}(x))_v,\\
\Phi(f) &= \phi_\mathcal{C}(f),\\
\Phi(x_f) &= 0.
\end{aligned}$$

4.3.16 Remark. This is a weaker form of Conjecture 31 of [7]. First, we are restricted to Koszul operad $\mathcal{A}$ with generating operations in a single arity and a single degree, while the conjecture lets $\mathcal{A}$ be any dg operad. Second, the ideal $\mathcal{I}^{<n}$ is larger, generated by $F_{\geq 1} \oplus X_F(< n)$, while the conjectured

$$\text{ideal } \mathcal{I}^{<n}_{\text{orig}} \text{ is generated just by } X_F(< n). \tag{4.17}$$

In particular, we recover, at least for $\mathcal{A}$ as above, Theorem 7 of [7] dealing with the case of C being a single morphism between two distinct objects and $\mathcal{C}_\infty$ its trivial resolution. Observe that in this case, $\mathcal{I}^{<n}$ is in fact generated just by $X_F(< n)$ since $F_{\geq 1} = 0$ (of course, similar statement holds for any $\mathcal{C}_\infty$ concentrated in degree 0, which corresponds to a *free* category C). We also recover Theorems 18 and 24 of [7], again with the above mentioned restrictions.

However, there seems to be a completely unclear statement at the very end of the proof of Theorem 7, page 11 of [7]. As the proofs of Theorems 18 and 24 of [7] are only sketched, there is probably the same problem. To remedy it, we had to introduce our assumptions. We will discuss these assumptions in detail after proving our main theorem. However, we don't know any counterexample to the original theorems of [7].

4.4 Proof of main theorem

4.4.1 Lemmas

4.4.1 Lemma. Let C be a small category, let $(\mathcal{C}_\infty, \partial_\mathcal{C}) \xrightarrow{\phi_\mathcal{C}} (\mathcal{C}, 0)$ be its resolution satisfying the assumptions 4.3.1. For *any minimal* dg V-operad of the form $(\mathbb{F}(X), \partial)$ with $X(0) = X(1) = 0$, let

$$\mathcal{D}_\infty := \mathbb{F}(X_V \oplus F \oplus X_F)$$

and *assume* there is a differential ∂ on $\mathcal{D}_\infty$ satisfying

$$\begin{aligned}
&\partial x_v = (\partial x)_v,\\
&\partial f = \partial_\mathcal{C} f,\\
&\partial x_f = (-1)^{1+|x|}\mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} [\![f]\!]_n + \omega(x, f),
\end{aligned}$$

where $x \in X(n)$, $v \in V$, $f \in F$ and

$$\omega(x, f) \in \mathcal{D}_\infty^{<n} = \mathbb{F}(X_V(< n) \oplus F \oplus X_F(< n))(n).$$

Assume ϕ is a dg V-operad morphism

$$(\mathcal{D}_\infty, \partial) \xrightarrow{\phi} (\mathbb{F}(X), \partial)_\mathsf{C}$$

satisfying

$$\begin{aligned}
\phi(x_v) &= x_v, \\
\phi(f) &= f
\end{aligned} \tag{4.18}$$

for $f \in F_0$ and vanishing on all the other generators. Then ϕ is a quism.

We use the symbol x_v either for $x_v \in X_V \subset \mathcal{D}_\infty$ or $x_v \in \mathbb{F}(X)_v \subset (\mathbb{F}(X), \partial)_\mathsf{C}$. Similarly for $f \in F_0$. The correct meaning will always be clear from the context.

Proof. Let $\mathfrak{F}_i$ be the sub V-Σ-module of $\mathcal{D}_\infty$ spanned by free compositions containing at least $-i$ generators from $X_V \oplus X_F$. $\mathfrak{F}_i$'s form a filtration

$$\cdots \subset \mathfrak{F}_{-2} \subset \mathfrak{F}_{-1} \subset \mathfrak{F}_0 = \mathcal{D}_\infty.$$

$\mathcal{P}(x, \partial f) \in \mathfrak{F}_{-1}$ is obvious and $\mathrm{ar}(\omega(x, f)) = \mathrm{ar}(x) \geq 2$ implies $\omega(x, f) \in \mathfrak{F}_{-1}$. Hence $\partial \mathfrak{F}_i \subset \mathfrak{F}_i$. Since X contains no elements of arity 0 and 1, for *a fixed arity* n the arity n part $\mathfrak{F}_i(n)$ of this filtration is bounded below. Consider the corresponding spectral sequence $(E^*(n), \partial^*(n))$. For each n, $(E^*(n), \partial^*(n))$ converges by the classical convergence theorem. We collect these spectral sequences into (E^*, ∂^*). Recall that each (E^i, ∂^i) is a dg V-operad. In the sequel, such arity-wise constructions will be understood without mentioning the arity explicitly. For the 0^{th} term, we have

$$E^0 \cong \mathcal{D}_\infty$$

as graded V-operad. Now we make ∂^0 explicit. Let $x \in X(n)$, $n \geq 2$. By the minimality, each summand of ∂x_v contains at least 2 generators from X_V, hence $\partial x_v \in \mathfrak{F}_{-2}$ and $\partial^0 x_v = 0$. Next, observe that for $n = 2$, $\omega(x, f) = 0$ by arity reasons. Let $n \geq 3$. Each summand of $\omega(x, f) \neq 0$ contains only generators of arity $< n$, hence at least 2 of these are of arities ≥ 2. But generators of arity ≥ 2 come from $X_V \oplus X_F$, i.e. $\omega(x, f) \in \mathfrak{F}_{-2}$. Hence the differential ∂^0 is the derivation determined by formulas

$$\begin{aligned}
\partial^0 x_v &= 0\\
\partial^0 f &= \partial f\\
\partial^0 x_f &= (-1)^{1+|x|}\mathcal{P}(x, \partial f) + (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)}[\![f]\!]_n
\end{aligned}$$

for $x \in X(n)$ and $f \in F$.

There is a similar construction on $(\mathbb{F}(X), \partial)_{\mathsf{C}}$. Denote ∂' its differential. Let $\mathfrak{F}'_i$ be the sub V-Σ-module of $(\mathbb{F}(X), \partial)_{\mathsf{C}}$ spanned by free compositions containing at least $-i$ generators from X_V. Then these form a filtration

$$\cdots \subset \mathfrak{F}'_{-2} \subset \mathfrak{F}'_{-1} \subset \mathfrak{F}'_0 = (\mathbb{F}(X), \partial)_{\mathsf{C}}.$$

Obviously $\partial'\mathfrak{F}'_i \subset \mathfrak{F}'_i$. By the same argument as above, this filtration is bounded below and hence the corresponding spectral sequence (E'^*, ∂'^*) converges. For the 0^{th} page, we have

$$(E'^0, \partial'^0) \cong (\mathbb{F}(X), 0)_{\mathsf{C}}$$

as dg V-operad, i.e.

$$\partial'^0 = 0.$$

The dg V-operad morphism ϕ satisfies $\partial\mathfrak{F}_i \subset \mathfrak{F}'_i$, hence it induces a morphism $\phi^* : (E^*, \partial^*) \to (E'^*, \partial'^*)$ of spectral sequences. By [12], Theorems 5.2.12 and 5.5.1, to prove that ϕ is quism, it suffices to show that ϕ^0 is a quism. We will prove

$$H_*(E^0, \partial^0) = \frac{\mathbb{F}(X_V \oplus F_0)}{\left(\left\{-f x_{I(f)} + x_{O(f)}[\![f]\!]_{\mathrm{ar}(x)} \mid x \in X,\ f \in F_0\right\} \cup \partial F_1\right)}, \tag{4.19}$$

compare with (4.5) and Definition 4.3.11. This implies $H_*(\phi^0)$ is the identity and we are done.

The dg V-operad (E^0, ∂^0) carries a filtration

$$0 = \mathfrak{F}''_{-1} \subset \mathfrak{F}''_0 \subset \mathfrak{F}''_1 \subset \cdots,$$

where $\mathfrak{F}''_i$ is sub V-Σ-module of E^0 spanned by compositions with

$$(\text{degree} + \text{number of generators from } X_V) \leq i.$$

Obviously $\partial^0\mathfrak{F}''_i \subset \mathfrak{F}''_i$. This filtration is bounded below and exhaustive, hence the corresponding spectral sequence (E^{0*}, ∂^{0*}) converges by [12], Theorem 5.2.12. We have

$$E^{00} \cong \mathcal{D}_\infty$$

as graded V-operad and

$$\begin{aligned}
\partial^{00} x_v &= 0,\\
\partial^{00} f &= 0,\\
\partial^{00} x_f &= (-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)}[\![f]\!]_n
\end{aligned}$$

for $x \in X(n)$ and $f \in F$. We will show

$$H_*(E^{00}, \partial^{00}) = \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)}[\![f]\!]_{\mathrm{ar}(x)} \mid x \in X,\ f \in F)}. \tag{4.20}$$

Assume this is already done and let's prove (4.19). We proceed to the 1st page E^{01} of E^{0*}:

$$E^{01} \cong H_*(E^{00}, \partial^{00})$$

and under this isomorphism, ∂^{01} is given by

$$\partial^{01} x_v = 0,$$
$$\partial^{01} f = \partial f.$$

By the same argument as in the proof of Lemma 4.3.12,

$$(E^{01}, \partial^{01}) \cong (\mathbb{F}(X_V), 0) \circ (\mathbb{F}(F), \partial).$$

By Lemma 4.2.8 and (4.5),

$$H_*(E^{01}, \partial^{01}) \cong \mathbb{F}(X_V) \circ \mathcal{C} \cong \mathbb{F}(X_V) \circ \frac{\mathbb{F}(F_0)}{(\partial F_1)}$$

and by the argument of Lemma 4.3.12 again,

$$H_*(E^{01}, \partial^{01}) \cong \frac{\mathbb{F}(X_V \oplus F_0)}{\left(\left\{-f x_{I(f)} + x_{O(f)}[\![f]\!]_{\mathrm{ar}(x)} \mid x \in X,\ f \in F_0\right\} \cup \partial F_1\right)}. \tag{4.21}$$

This is the 2nd page E^{02} and we claim that all the higher differentials vanish: $\partial^{0k} = 0$ for $k \geq 2$. To see this, let's assign *inner degree*, denoted by $||-||$, to generators of E^0:

$$||x_v|| = 0, \quad ||f|| = |f|, \quad ||x_f|| = |f| + 1.$$

This extends to E^0 by requiring the operadic composition to be of inner degree 0. Now notice that ∂^0 is of inner degree -1 and so are all the differentials ∂^{0k}. But (4.21) is concentrated in inner degree 0, hence the spectral sequence (E^{0*}, ∂^{0*}) collapses as claimed. We conclude that $E^{02} = E^{0\infty} \cong H_*(E^0, \partial^0)$, thus proving (4.19).

It remains to prove (4.20). Let $\mathfrak{F}'''_i$ be the sub V-Σ-module of $E^{00} \cong \mathcal{D}_\infty$ spanned by compositions with at least $-i$ generators from $F \oplus X_F$. Then

$$\cdots \subset \mathfrak{F}'''_{-2} \subset \mathfrak{F}'''_{-1} \subset \mathfrak{F}'''_0 = E^{00}$$

is a filtration with $\partial^{00}\mathfrak{F}'''_i \subset \mathfrak{F}'''_i$. Denote $(E^{00*}, \partial^{00*})$ the corresponding spectral sequence. The convergence of this spectral sequence will be discussed later. We have

$$E^{000} \cong \mathcal{D}_\infty,$$
$$\partial^{000} x_v = 0,$$
$$\partial^{000} f = 0,$$
$$\partial^{000} x_f = (-1)^{1+|x||f|} f x_{I(f)}.$$

Now we prove

$$H_*(E^{000}, \partial^{000}) = \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} \mid x \in X,\ f \in F)}. \tag{4.22}$$

First observe that $\mathbb{F}(F) \circ (X_V \oplus X_F)$ is closed under ∂^{000} and

$$H_*(\mathbb{F}(F) \circ (X_V \oplus X_F), \partial^{000}) = X_V. \tag{4.23}$$

In the sequel, we may drop the differentials from the notation "$H_*(-, \partial)$" if no confusion can arise. Let

$$\begin{aligned} P_0 &:= k\langle 1 \rangle\,, \\ P_{n+1} &:= \mathbb{F}(F) \ \oplus\ \mathbb{F}(F) \circ (X_V \oplus X_F) \circ P_n \end{aligned}$$

for $n \geq 0$. We immediately see that P_n's are closed under ∂^{000} and

$$P_n = \bigoplus_{i=0}^{n-1} (\mathbb{F}(F) \circ (X_V \oplus X_F))^{\circ i} \circ \mathbb{F}(F) \ \oplus\ (\mathbb{F}(F) \circ (X_V \oplus X_F))^{\circ n}, \tag{4.24}$$

where we used the (iterated) composition product (4.2). By Lemma 4.2.8 and (4.23),

$$H_*(P_n) \cong \bigoplus_{i=0}^{n-1} (X_V)^{\circ i} \circ \mathbb{F}(F) \ \oplus\ (X_V)^{\circ n}.$$

(4.24) provides a chain of inclusions

$$P_0 \hookrightarrow P_1 \hookrightarrow \cdots \to \operatorname*{colim}_{\overrightarrow{n}} P_n \cong E^{000}$$

with direct limit E^{000}, as easily seen. Since direct limits commute with homology,

$$\begin{aligned} H_*(E^{000}, \partial) &\cong \operatorname*{colim}_{\overrightarrow{i}} H_*(P_n) = \operatorname*{colim}_{\overrightarrow{i}} \left(\bigoplus_{i=0}^{n-1} (X_V)^{\circ i} \circ \mathbb{F}(F) \ \oplus\ (X_V)^{\circ n} \right) \cong \\ &\cong \mathbb{F}(X_V) \circ \mathbb{F}(F) \cong \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} \mid x \in X,\ f \in F)}. \end{aligned}$$

The 1st page E^{001} is therefore described by (4.22). An argument with inner degree analogous to the one above shows that $\partial^{00k} = 0$ for $k \geq 1$: In (E^{00}, ∂^{00}), set

$$||x_v|| = ||f|| = 0, \quad ||x_f|| = 1.$$

Hence $E^{001} = E^{00\infty}$ is the stable term.

Although we don't know how to prove the convergence of the spectral sequence $(E^{00*}, \partial^{00*})$ directly (the filtration is bounded above but not below, we only have the Hausdorff property $\cap_i \mathfrak{F}_i''' = 0$), there is a weaker statement which follows from Lemma 5.5.7 of [12]: The i^{th} graded part $\mathfrak{F}_i''' H_*(E^{00}, \partial^{00}) / \mathfrak{F}_{i-1}''' H_*(E^{00}, \partial^{00})$ of the filtration on homology[6] is isomorphic to a subspace e_i of $E_i^{00\infty}$.[7] We have

[6] Recall the usual notation $\mathfrak{F}_i''' H_*(E^{00}, \partial^{00}) := \operatorname{Im}(H_*(\mathfrak{F}_i''', \partial^{00}) \to H_*(E^{00}, \partial^{00}))$.

[7] The lower index denotes the grading associated to the filtration $\mathfrak{F}_i'''$.

(simplifying the notation)

$$\frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)} + x_{O(f)} [\![f]\!]_{\mathrm{ar}(x)})} \subset H_*(E^{00}) \cong \bigoplus_{i \leq 0} \frac{\mathfrak{F}_i''' H_*(E^{00})}{\mathfrak{F}_{i-1}''' H_*(E^{00})} \cong$$

$$\cong \bigoplus_{i \leq 0} e_i \subset \bigoplus_{i \leq 0} E_p^\infty \cong \frac{\mathbb{F}(X_V \oplus F)}{((-1)^{1+|x||f|} f x_{I(f)})},$$

where the first inclusion is the obvious part of (4.20) and the second inclusion has just been discussed. It is not difficult to map the left-hand side through all the isomorphisms and to see that it is mapped *onto* the right-hand side. Hence the first inclusion is in fact equality and we are done proving (4.20) and consequently the whole Lemma 4.4.1. □

4.4.2 Lemma. Let $\omega(x, f)$ of Lemma 4.4.1 moreover satisfies $\omega(x, f) \in \mathcal{I}^{<n}$ (recall (4.15)). Then ϕ, uniquely determined by (4.18) as a graded V-operad morphism, is automatically a dg V-operad morphism (i.e. ϕ commutes with the differentials).

Proof. We have to verify $\phi\partial = \partial\phi$ for generators from $X_V \oplus F \oplus X_F$. The only nontrivial case concerns X_F: we have to verify $\phi\partial x_f = 0$. We have

$$\begin{aligned}\phi\partial x_f = (-1)^{1+|x|} \phi(\mathcal{P}(x, \partial f)) + \\ + (-1)^{1+|x||f|} \phi(f)\phi(x_{I(f)}) + \phi(x_{O(f)})\phi([\![f]\!]) + \phi(\omega(x, f)).\end{aligned}$$

If $\partial f = 0$, then the first term vanishes trivially. If $\partial f \neq 0$, then each summand of $\mathcal{P}(x, \partial f)$ contains a generator from X_F and hence the first term vanishes too. By the definition of $\mathcal{I}^{<n}$, we have $\phi(\omega(x, f)) = 0$.

Hence it remains to prove

$$(-1)^{1+|x||f|} \phi(f)\phi(x_{I(f)}) + \phi(x_{O(f)})\phi([\![f]\!]) = 0.$$

If $|f| > 0$, we have $\phi f = 0$ by definition. Also $|[\![f]\!]| > 0$ and hence each summand of $[\![f]\!]$ contains a generator from $F_{\geq 1}$ and consequently $\phi([\![f]\!]) = 0$. For $|f| = 0$, we may assume $f \in M$ and we want to prove $-f x_{I(f)} + x_{O(f)} f^{\otimes n} = 0$ in $(\mathbb{F}(X), \partial)_{\mathsf{C}}$. But this is exactly one of the defining relations of $(\mathbb{F}(X), \partial)_{\mathsf{C}}$. □

4.4.3 Lemma. Let an operad $\mathcal{A}$ be Koszul with generating operations concentrated in a single arity $N \geq 2$ and a single degree $D \geq 0$. Then for every generator x of the minimal resolution of $\mathcal{A}$ there is $k \geq 1$ such that

$$\mathrm{ar}(x) = a_k := 1 + (N-1)k, \quad |x| = d_k := -1 + (D+1)k.$$

Moreover, there is K (possibly $K = +\infty$) such that a generator of arity a_k and degree d_k exists iff $k < K$.

Proof. By Koszulity, we have the minimal resolution

$$\Omega(\mathcal{A}^{\text{!`}}) \xrightarrow{\sim} (\mathcal{A}, 0)$$

given by the cobar construction $\Omega(\mathcal{A}^{\text{!`}}) = (\mathbb{F}(\downarrow\overline{\mathcal{A}^{\text{!`}}}), \partial)$. Assume $\mathcal{A}$ has the quadratic presentation (4.13). Recall that the Koszul dual $\mathcal{A}^{\text{!`}}$ is the quadratic cooperad cogenerated by $\uparrow E$ with corelations $\uparrow^2 E$, see [5], 7.1.4. Thus $\mathcal{A}^{\text{!`}}$ is a sub Σ-module of

$\mathbb{F}(\uparrow E)$, hence it is concentrated in arities $1+(N-1)k$ and degrees $(D+1)k$. Hence $\downarrow\overline{\mathcal{A}^{¡}}$ is concentrated in arities $a_k = 1+(N-1)k$ and degrees $d_k = -1+(D+1)k$.

We give only a brief proof of the last claim of this lemma, since we won't need it in the sequel. Suppose that for every $k < K$ a generator of arity a_k and degree d_k exists. Further let there be no generator in arity a_K. By the inductive construction of the minimal resolution, as described in the proof of Theorem 3.125 of [10], the generators in the next possible arity a_{K+1} have degree $\leq d_{K-1} + 2D + 1 = d_{K+1} - 1$. But the existence of any such generator would contradict the previous part of this lemma. In the next arity, a_{K+2}, the generators would have to have degree $\leq d_{K-1} + 3D + 1 < d_{K+2}$. And so on, hence there are no generators in arity a_k for $k \geq K$. We encourage the reader to go through the cases $D = 0$ and $D = 1$. $\square$

4.4.4 Lemma. Let an operad $\mathcal{A}$ be Koszul with generating operations concentrated in a single arity ≥ 2 and a single degree ≥ 0. Then for every $x \in X$ and $f \in F$, there is $\omega(x, f) \in \mathcal{I}^{<\mathrm{ar}(x)}$ as stated in Theorem 4.3.15, i.e. the derivation ∂ defined by (4.14) is indeed a differential on $\mathcal{D}_\infty$.

To prove this lemma, it is convenient to extend $\omega(x, f)$'s to a linear map as follows. Fix $x \in X(n)$. The linear map

$$\omega(x, -) : \mathcal{C}_\infty \to \mathcal{D}_\infty(n)$$

is uniquely determined by

$$\begin{aligned}&\text{(arbitrary) values } \omega(x, f) \text{ on } f \in F \text{ and}\\ &\omega(x, r_1 r_2) = \omega(x, r_1)[\![r_2]\!]_n + (-1)^{|r_1||x|} r_1 \omega(x, r_2)\end{aligned}$$

for any $r_1, r_2 \in \mathcal{C}_\infty$.

Thus $\omega(x, -)$ behaves much like a derivation of degree $|x|$. Checking it is well defined is similar to 4.3.13.

4.4.5 Lemma. For any $r \in \mathcal{C}_\infty$, the formula (4.14) with r in place of f still holds:

$$\partial\mathcal{P}(x, r) = (-1)^{1+|x|}\mathcal{P}(x, \partial r) + (-1)^{1+|r||x|} r x_{I(r)} + x_{O(r)}[\![r]\!]_n + \omega(x, r).$$

The proof explains the $\pm$ signs in the definition (4.14) of ∂ on $\mathcal{D}_\infty$.

Proof. It suffices to prove the lemma for r of the form $r = f_1 f_2 \cdots f_k$, where $f_i \in F$. We proceed by induction on k. The case $k = 1$ is exactly formula (4.14). Let $k \geq 2$ and suppose the lemma holds for every sum of compositions of at most $k-1$ elements and let $r = r_1 r_2$, where r_1, r_2 are compositions of at most $k-1$ generators from F. Now we want to prove

$$\partial\mathcal{P}(x, r_1 r_2) =$$
$$= (-1)^{1+|x|}\mathcal{P}(x, \partial(r_1 r_2)) + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)}[\![r_1 r_2]\!]_n + \omega(x, r_1 r_2).$$

It is a straightforward computation, we will compare Left-Hand Side and Right-Hand Side:

$$
\begin{aligned}
\mathrm{LHS} = {} & \partial\left(\mathcal{P}(x,r_1)[\![r_2]\!]_n + (-1)^{|r_1|(|x|+1)} r_1\mathcal{P}(x,r_2)\right) = \\
= {} & \left((-1)^{1+|x|}\mathcal{P}(x,\partial r_1) + (-1)^{1+|r_1||x|} r_1 x_{I(r_1)} + x_{O(r_1)}[\![r_1]\!]_n + \omega(x,r_1)\right)[\![r_2]\!]_n + \\
& + (-1)^{|x|+|r_1|+1}\mathcal{P}(x,r_1)[\![\partial r_2]\!]_n + (-1)^{|r_1|(|x|+1)}(\partial r_1)\mathcal{P}(x,r_2) + \\
& + (-1)^{|r_1|(|x|+1)+|r_1|} r_1\left((-1)^{1+|x|}\mathcal{P}(x,\partial r_2) + (-1)^{1+|r_2||x|} r_2 x_{I(r_2)} + \right.\\
& \left. + x_{O(r_2)}[\![r_2]\!]_n + \omega(x,r_2)\right) \\
\mathrm{RHS} = {} & (-1)^{1+|x|}\mathcal{P}(x,(\partial r_1)r_2)) + (-1)^{1+|x|+|r_1|}\mathcal{P}(x,r_1\partial r_2) + \\
& + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)}[\![r_1]\!]_n[\![r_2]\!]_n + \\
& + \omega(x,r_1)[\![r_2]\!]_n + (-1)^{|x||r_1|} r_1\omega(x,r_2) = \\
= {} & (-1)^{1+|x|}\mathcal{P}(x,\partial r_1)[\![r_2]\!]_n + (-1)^{1+|x|+(|x|+1)(|r_1|-1)}(\partial r_1)\mathcal{P}(x,r_2) + \\
& + (-1)^{1+|x|+|r_1|}\mathcal{P}(x,r_1)[\![\partial r_2]\!]_n + (-1)^{1+|x|+|r_1|+(|x|+1)|r_1|} r_1\mathcal{P}(x,\partial r_2) + \\
& + (-1)^{1+(|r_1|+|r_2|)|x|} r_1 r_2 x_{I(r_2)} + x_{O(r_1)}[\![r_1]\!]_n[\![r_2]\!]_n + \\
& + \omega(x,r_1)[\![r_2]\!]_n + (-1)^{|x||r_1|} r_1\omega(x,r_2)
\end{aligned}
$$

The proof is finished by a careful sign inspection. □

of Lemma 4.4.4. Let $x \in X(n)$ and $f \in F_d$. First, we make a preliminary computation using the formula of Lemma 4.4.5:

$$
\begin{aligned}
\partial^2 x_f = {} & (-1)^{1+|x|}\partial\mathcal{P}(x,\partial f) + (-1)^{1+|x||f|}(\partial f)x_{I(f)} + (-1)^{1+|x||f|+|f|} f\partial x_{I(f)} + \\
& + (\partial x_{O(f)})[\![f]\!]_n + (-1)^{|x|} x_{O(f)}[\![\partial f]\!]_n + \partial\omega(x,f) = \cdots \\
= {} & (-1)^{1+|x|}\omega(x,\partial f) + (-1)^{1+|x||f|+|f|} f\partial x_{I(f)} + (\partial x_{O(f)})[\![f]\!]_n + \partial\omega(x,f)
\end{aligned}
$$

The condition $\partial^2 x_f = 0$ is equivalent to

$$\partial\omega(x,f) = (-1)^{|x|}\omega(x,\partial f) + (-1)^{|f|(|x|+1)} f\partial x_{I(f)} - (\partial x_{O(f)})[\![f]\!]_n =: \varphi(x,f).$$

To construct $\omega(x,f)$ so that $\partial^2 x_f = 0$, we will inductively solve the equation

$$\partial\omega(x,f) = \varphi(x,f) \tag{4.25}$$

for unknown $\omega(x,f)$. We proceed by induction on arity n of x and simultaneously by induction on degree d of f.

For $n = N$ (the arity of the generating operations of $\mathcal{A}$) and $d = 0$, we have $\partial f = 0 = \partial x$, hence (4.25) becomes $\partial\omega(x,f) = 0$, which has the trivial solution.

Fix n and d. Assume we have already constructed $\omega(x,f) \in \mathcal{I}^{<\mathrm{ar}(x)}$ for every $x \in X(<n)$ and f of any degree and also for $x \in X(n)$ and $f \in F_{<d}$. Let $x \in X(n)$ and $f \in F_d$. Observe that $\varphi(x,f) \in \mathcal{D}_\infty^{<n}$ (recall (4.16)) by the induction assumption and minimality. When we restrict $\phi : \mathcal{D}_\infty \to (\mathbb{F}(X),\partial)_{\mathsf{C}}$ to $\mathcal{D}_\infty^{<n}$, we get the graded V-operad morphism

$$\mathcal{D}_\infty^{<n} \xrightarrow{\phi} (\mathbb{F}(X(<n)),\partial)_{\mathsf{C}}$$

denoted by the same symbol. By the induction assumption, $\partial^2 = 0$ on $\mathcal{D}_\infty^{<n}$. By Lemma 4.4.2, ϕ is dg V-operad morphism. By Lemma 4.4.1, ϕ is a quism. In a moment, we will show

$$\partial\varphi(x,f) = 0, \tag{4.26}$$
$$\phi\varphi(x,f) = 0. \tag{4.27}$$

This will imply the existence of $\omega(x,f) \in \mathcal{D}_\infty^{<n}$ such that $\partial\omega(x,f) = \varphi(x,f)$. In fact, $\omega(x,f) \in \mathcal{I}^{<n}$. To see this, assume a summand S of $\omega(x,f)$ is a composition of generators none of which comes from X_F. Hence S is an operadic composition of $x_1, \cdots, x_a \in X_V(<n)$ and $f_1, \cdots, f_b \in F$. By a degree count, we now show that at least one of f_j's lies in $F_{\geq 1}$. By Lemma 4.4.3, let x_i have arity $1+(N+1)k_i$ and degree $-1+(D+1)k_i$. We have

$$|x| + |f| = |\omega(x,f)| = |S| = \sum_{i=1}^{a} |x_i| + \sum_{j=1}^{b} |f_j| = \sum_i (-1 + (D+1)k_i) + \sum_j |f_j|$$

hence

$$\sum_j |f_j| = |x| + |f| + a - (D+1)\sum_i k_i. \tag{4.28}$$

Now

$$\mathrm{ar}(x) = \mathrm{ar}(S) = 1 + \sum_i (\mathrm{ar}(x_i) - 1) = 1 + (N-1)\sum_i k_i$$

hence

$$|x| = -1 + (D+1)\sum_i k_i.$$

Substituting this into (4.28), we get

$$\sum_j |f_j| = |f| + a - 1.$$

We have the trivial estimate $|f| \geq 0$. Since $\mathrm{ar}(f_j) = 1$ for any j and $\mathrm{ar}(x_i) < n = \mathrm{ar}(S)$ for any i, we have $a \geq 2$. Hence

$$\sum_j |f_j| \geq 1$$

and therefore one of f_j's lies in $F_{\geq 1}$.

It remains to verify the conditions (4.26), (4.27). For (4.26), we have

$$\partial\varphi(x,f) = (-1)^{|x|}\partial\omega(x,\partial f) + (-1)^{|f|(|x|+1)}(\partial f)\partial x_{I(f)} + (-1)^{|x|}(\partial x_{O(f)})[\![\partial f]\!]_n.$$

Lemma 4.4.5 and the induction hypothesis imply

$$\partial\omega(x,\partial f) = (-1)^{|x|(|f|-1)+|f|-1}(\partial f)\partial x_{I(f)} - (\partial x_{O(f)})[\![\partial f]\!]_n$$

and after substituting this into the previous equation, we get $\partial\varphi(x,f) = 0$.

For (4.27), let $d = 0$ first. Then $\varphi(x,f) = f\partial x_{I(f)} - (\partial x_{O(f)})[\![f]\!]_n$, hence we have to verify

$$f\partial x_{I(f)} - (\partial x_{O(f)})f^{\otimes n} = 0 \quad \text{in } (\mathbb{F}(X(<n)),\partial)_{\mathsf{C}}.$$

This follows by the same argument as Lemma 4.3.12. Now let $d > 0$. By induction assumption, $\omega(x,\partial f) \in \mathcal{I}^{<n}$ and therefore $\phi\omega(x,\partial f) = 0$. Finally $\phi f = 0 = \phi[\![f]\!]_n$ by definition of ϕ since $|f| = |[\![f]\!]_n| = d > 0$. $\square$

Now we can finally prove the main theorem:

of Theorem 4.3.15. Decompose Φ into

$$(\mathcal{D}_\infty, \partial) \xrightarrow{\phi} (\mathbb{F}(X), \partial)_{\mathsf{C}} \xrightarrow{(\phi_{\mathcal{A}})_{\mathsf{C}}} (\mathcal{A}, 0)_{\mathsf{C}} = (\mathcal{D}, 0).$$

The dg V-operad morphism $(\phi_{\mathcal{A}})_{\mathsf{C}}$ (recall Definition 4.3.11) is a quism by Lemma 4.3.12. ϕ is the graded V-operad morphism of Lemma 4.4.1. By Lemma 4.4.4, there are $\omega(x, f)$'s in $\mathcal{I}^{<\mathrm{ar}(x)}$ such that ∂ on $\mathcal{D}_\infty$ is indeed a differential. By Lemma 4.4.2, ϕ is a dg V-operad morphism and finally, by Lemma 4.4.1, ϕ is a quism. □

4.4.2 Discussion

It is a remarkable observation that in many cases, only a "principal" part of the differential determines what the homology is. This was exploited in [7] and also e.g. in [6] to partially resolve the PROP for bialgebras. Lemma 4.4.1 is an application of this principle. Here the minimality of $\mathcal{A}_\infty$ and the mild assumption $\omega(x, f) \in \mathcal{D}_\infty^{<n}$ (which in fact only formalizes what we mean by the principal part) are crucial for the spectral sequence argument to separate the principal part of ∂. Apart from the minimality, arbitrary $\mathcal{A}_\infty$ with $X(0) = X(1) = 0$ is allowed (unfortunately, this excludes e.g. unital algebras). Notice, however, that we *assume* that ϕ commutes with differentials.

To guarantee this, we need a stronger constraint on $\omega(x, f)$. An easy sufficient way to ensure this is described in Lemma 4.4.2. It leads to the definition (4.15) of $\mathcal{I}^{<n}$.

Next, we have to construct a differential ∂ on $\mathcal{D}_\infty$ such that the assumptions of Lemma 4.4.2 are satisfied. This is achieved in Lemma 4.4.4. To begin with, one obtains $\omega(x, f) \in \mathcal{D}_\infty^{<n}$ by an inductive argument on the arity of the generators from X using Lemma 4.4.1. Then we have to improve this result. This is where the proof of Theorem 7 of [7] is unclear. We were not able to get the originally desired result $\omega(x, f) \in \mathcal{I}_{\mathrm{orig}}^{<n}$ (recall (4.17)). But if one is able to control the interplay between arity and degree of the generators from X in a suitable way, one obtains at least $\omega(x, f) \in \mathcal{I}^{<n}$ by a simple degree count. A sufficient control is achieved for the Koszul resolution of a Koszul operad with generating operations bound in a single arity and degree. This is explained in Lemma 4.4.3. We note that Lemma 4.4.4 can be proved under a weaker control over X, but the resulting conditions dont't seem to be of any practical interest.

Still, it might be possible to improve the proof of Lemma 4.4.4 to get $\omega(x, f) \in \mathcal{I}_{\mathrm{orig}}^{<n}$ even without the restrictions imposed on $\mathcal{A}$, thus proving the original Conjecture 31 of [7]. However, to our best knowledge, explicit examples of resolutions of diagrams $\mathcal{D} = \mathcal{A}_{\mathsf{C}}$ are known only for free categories C and for operads satisfying the assumptions of Theorem 4.3.15. Moreover, in these cases $\mathcal{I}^{<n} = \mathcal{I}_{\mathrm{orig}}^{<n}$. Hence these do not decide whether the conjecture is still plausible.

Notice a slightly stronger statement about what generators are needed to compose $\omega(x, f)$ can be made. For example, if $|f| = 0$, then $\omega(x, f)$ lies in the ideal generated by $X_f(< n)$ in $\mathbb{F}(X_{O(f)}(< n) \oplus X_{I(f)}(< n) \oplus k\langle f \rangle)$. This can be deduced from the proof of Lemma 4.4.4. However this doesn't seem to be important.

Finally notice that Lemma 4.4.1 is already quite a big achievement - it reduces the problem of resolving $\mathcal{D}$ to finding $\omega(x,f)$'s from $\mathcal{D}_\infty^{<\mathrm{ar}(x)}$ so that $\partial^2=0$ and the differential commutes with ϕ. Alternatively, by Lemma 4.4.2, the problem is reduced to finding $\omega(x,f)$'s from $\mathcal{I}^{<\mathrm{ar}(x)}$ so that $\partial^2=0$.

4.5 Bar-cobar resolution of $\mathcal{C}$

Now we make the content of Theorem 4.3.15 more explicit in the case $\mathcal{C}_\infty=\Omega\mathrm{B}\mathcal{C}$. We apply Lemma 4.3.9 on the bar-cobar resolution $\mathcal{C}_\infty=\Omega\mathrm{B}\mathcal{C}$. Denote

$$\Sigma^n := \left\{ (\xleftarrow{f_n}\cdots\xleftarrow{f_1}) \in (\mathrm{Mor}\,\mathcal{C})^{\times n} \mid O(f_i)=I(f_{i+1}) \text{ for } 1\le i\le n-1 \right\}$$

the set of chains of composable morphisms in C of length n, e.g. $\Sigma^1=\mathrm{Mor}\,\mathsf{C}$. Denote $\Sigma:=\bigcup_{i\geq 1}\Sigma^i$.

Recall that the bar-cobar resolution $\Omega\mathrm{B}\mathcal{C}$ (e.g. [11], where the noncoloured case is treated - but the coloured case is completely analogous) is a quasi-free V-operad generated by V-Σ-module $k\langle\Sigma\rangle$, where the degree of $\sigma\in\Sigma^n$ is $n-1$. The derivation differential is given by

$$\begin{aligned}\partial(\xleftarrow{f_n}\cdots\xleftarrow{f_1}) &:= \sum_{i=1}^{n-1}(-1)^{i+n+1}(\xleftarrow{f_n}\cdots\xleftarrow{f_{i+1}})\circ(\xleftarrow{f_i}\cdots\xleftarrow{f_1}) + \\ &+\sum_{i=1}^{n-1}(-1)^{n-i}(\xleftarrow{f_n}\cdots\xleftarrow{f_{i+1}f_i}\cdots\xleftarrow{f_1}).\end{aligned}$$

The projection $\phi_{\mathcal{C}}:\Omega\mathrm{B}\mathcal{C}\to\Omega\mathrm{B}^1\mathcal{C}\cong\mathcal{C}$ onto the sub V-Σ-module of weight 1 elements is a quism.

4.5.1 Theorem. Let $[\![-]\!]_2^{\mathrm{NS}}:\Omega\mathrm{B}\mathcal{C}\to\Omega\mathrm{B}\mathcal{C}\otimes\Omega\mathrm{B}\mathcal{C}$ be a linear map satisfying $[\![a\circ b]\!]_2^{\mathrm{NS}}=[\![a]\!]_2^{\mathrm{NS}}\circ[\![b]\!]_2^{\mathrm{NS}}$ for all $a,b\in\Omega\mathrm{B}\mathcal{C}$ and determined by its values on generators:

$$\begin{gathered}[\![(\xleftarrow{f_n}\cdots\xleftarrow{f_1})]\!]_2^{\mathrm{NS}} := \\ (\xleftarrow{f_n}\cdots\xleftarrow{f_1})\otimes(\xleftarrow{f_n\cdots f_1}) + \\ +\sum_{\substack{1\le m\le n-1\\ 1\le j_1<\cdots<j_m\le n-1}}(-1)^\epsilon(\xleftarrow{f_n}\cdots\xleftarrow{f_{j_m+1}})\cdots(\xleftarrow{f_{j_1}}\cdots\xleftarrow{f_1})\otimes(\xleftarrow{f_n\cdots f_{j_m+1}}\cdots\xleftarrow{f_{j_1}\cdots f_1}),\end{gathered}$$

where $\epsilon:=mn+\frac{1}{2}m(m-1)+\sum_{i=1}^k j_i$. Then $[\![-]\!]_2^{\mathrm{NS}}$ induces, via (4.9) and (4.8), the maps $[\![-]\!]_n^{\mathrm{NS}}$ and $[\![-]\!]_n$ of Lemma 4.3.6. Moreover,

$$(1^{\otimes i}\otimes[\![-]\!]_a^{\mathrm{NS}}\otimes 1^{\otimes b-i-1})[\![-]\!]_b^{\mathrm{NS}}=[\![-]\!]_{a+b-1}^{\mathrm{NS}}.$$

Proof. We apply Lemma 4.3.9. The only nontrivial properties to verify are $\partial[\![-]\!]_2^{\mathrm{NS}}=[\![\partial-]\!]_2^{\mathrm{NS}}$ and $([\![-]\!]_2^{\mathrm{NS}}\otimes 1)[\![-]\!]_2^{\mathrm{NS}}=(1\otimes[\![-]\!]_2^{\mathrm{NS}})[\![-]\!]_2^{\mathrm{NS}}$. This can be done directly, but it is annoying and doesn't explain the origin of $[\![-]\!]_2^{\mathrm{NS}}$. Thus we go another way. There is the following description of $\Omega\mathrm{B}\mathcal{C}$. Let

$$C_*(I):=k\langle(\mathbf{0}),(\mathbf{1}),(\mathbf{01})\rangle$$

be the simplicial chain complex of the interval, i.e. $|(\mathbf{0})| = |(\mathbf{1})| = 0$, $|(\mathbf{01})| = 1$ and $\partial(\mathbf{0}) = \partial(\mathbf{1}) = 0$, $\partial(\mathbf{01}) = (\mathbf{1}) - (\mathbf{0})$. Then

$$\Omega\mathrm{B}\mathcal{C} = \frac{\bigoplus_{n\geq 0} \mathcal{C}^{\circ(n+1)} \otimes C_*(I)^{\otimes n}}{M}, \tag{4.29}$$

where the subspace M is spanned by

$$\begin{gathered} f_n \otimes \cdots \otimes f_{i+1} \otimes f_i \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_{i+1} \otimes (\mathbf{0}) \otimes c_{i-1} \otimes \cdots \otimes c_1 + \\ -f_n \otimes \cdots \otimes f_{i+1} f_i \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_{i+1} \otimes c_{i-1} \otimes \cdots \otimes c_1 \end{gathered}$$

for any $f_n, \ldots, f_1 \in \mathcal{C}$ of right colours and any $c_{n-1}, \ldots, c_1 \in C_*(I)$. Let the grading and the differential ∂ on $\Omega\mathrm{B}\mathcal{C}$ be induced by $C_*(I)$ ($\mathcal{C}$ is concentrated in degree 0) in the standard way. The operadic composition is defined by

$$\begin{gathered} (f_n \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_1) \circ (g_m \otimes \cdots \otimes g_1 \otimes d_{m-1} \otimes \cdots \otimes d_1) := \\ (f_n \otimes \cdots \otimes f_1 \otimes g_m \otimes \cdots \otimes g_1 \otimes c_{n-1} \otimes \cdots \otimes c_1 \otimes d_{m-1} \otimes \cdots \otimes d_1). \end{gathered}$$

A dg V-operad isomorphism with the previous description is easily seen to be

$$f_n \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_1 \mapsto (\xleftarrow{f_n} \cdots \xleftarrow{f_{j_m+1}}) \cdots (\xleftarrow{f_{j_1}} \cdots \xleftarrow{f_1}), \tag{4.30}$$

where $c_{j_m} = c_{j_{m-1}} = \ldots = c_{j_1} = (\mathbf{1})$ and all other c_i's equal $(\mathbf{01})$ (remember we can get rid of $(\mathbf{0})$ using the defining relations). The point is that $C_*(I)$ carries the obvious coassociative coproduct

$$\Delta(\mathbf{0}) = (\mathbf{0}) \otimes (\mathbf{0}), \quad \Delta(\mathbf{1}) = (\mathbf{1}) \otimes (\mathbf{1}), \quad \Delta(\mathbf{01}) = (\mathbf{0}) \otimes (\mathbf{01}) + (\mathbf{01}) \otimes (\mathbf{1})$$

and there is also the trivial coproduct on $\mathcal{C}$ given by $\Delta(c) = c \otimes c$. These induce coproduct on $\Omega\mathrm{B}\mathcal{C}$ by

$$\begin{gathered} \Delta(f_n \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_1) := \\ = \left((\underbrace{\Delta \otimes \cdots \otimes \Delta}_{2n-1 \text{ times}})(f_n \otimes \cdots \otimes f_1 \otimes c_{n-1} \otimes \cdots \otimes c_1) \right) \cdot \tau, \end{gathered}$$

where $\tau \in \Sigma_{4n-2}$ rearranges the factors in the expected way, which will be obvious from the following computation. Denote $c^0 := (\mathbf{0}) \otimes (\mathbf{01})$, $c^1 := (\mathbf{01}) \otimes (\mathbf{1})$ and for the rest of the proof, let's order the factor of the tensor products from right to left, i.e. $(\mathbf{01})$ is in position 2 in c^1 and $(\mathbf{1})$ is in position 1. Then

$$\Delta(f_n \otimes \cdots \otimes f_1 \otimes \underbrace{(\mathbf{01}) \otimes \cdots \otimes (\mathbf{01})}_{n-1 \text{ times}}) =$$

$$= \sum_{\substack{0 \leq m \leq n-1 \\ 1 \leq j_1 < \cdots < j_m \leq n-1}} \left(f_n \otimes f_n \otimes \cdots \otimes f_1 \otimes f_1 \otimes c^0 \otimes \cdots \otimes \underbrace{c^1}_{\text{position } j_m} \otimes \cdots \otimes \underbrace{c^1}_{\text{position } j_1} \otimes \cdots \otimes c^0 \right) \cdot \tau,$$

where c^1 appears only at positions $j_1, \ldots, j_m$. Applying τ and multiplying yields

$$\begin{aligned} \sum_{\substack{0 \leq m \leq n-1 \\ 1 \leq j_1 < \cdots < j_m \leq n-1}} (-1)^\epsilon & \left(f_n \otimes \cdots \otimes f_1 \otimes (\mathbf{0}) \otimes \cdots \otimes \underbrace{(\mathbf{01})}_{j_m} \otimes \cdots \otimes \underbrace{(\mathbf{01})}_{j_1} \otimes \cdots \otimes (\mathbf{0}) \right) \otimes \\ & \otimes \left(f_n \otimes \cdots \otimes f_1 \otimes (\mathbf{01}) \otimes \cdots \otimes \underbrace{(\mathbf{1})}_{j_m} \otimes \cdots \otimes \underbrace{(\mathbf{1})}_{j_1} \otimes \cdots \otimes (\mathbf{01}) \right), \end{aligned}$$

where $\epsilon = mn + \frac{1}{2}m(m-1) + \sum_{i=1}^{m} j_i$ comes from the Koszul convention. This is exactly the claimed formula under the isomorphisms (4.30).

It is easily seen that $\Delta(a \circ b) = \Delta(a) \circ \Delta(b)$. Δ is the coproduct induced on the quotient (4.29) by the tensor product of coassociative dg coalgebras $C_*(I)$ and $\mathcal{C}$. It is a standard fact that the tensor product is also a coassociative dg coalgebra, hence $\partial\Delta = \Delta\partial$, $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. Then $[\![-]\!]_2^{\mathrm{NS}} := \Delta$ has the properties (C2)–(C6).

Originally, we found the coproduct of this lemma by hand. We are indebted to Benoit Fresse for suggesting its origin in $C_*(I)$. $\square$

A completely explicit cofibrant resolution $\mathcal{D}_\infty$ of $\mathcal{D} = \mathcal{A}_{\mathsf{C}}$ gives rise to a cohomology theory for $\mathcal{A}_{\mathsf{C}}$-algebras (i.e. C-shaped diagrams of $\mathcal{A}$-algebras) describing their deformations. This is explained in [9]. Unfortunately, the description of ∂ on $\mathcal{D}_\infty$ given in Theorem 4.3.15 is not even explicit enough to write down the codifferential δ on the corresponding deformation complex $\mathrm{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W)$, not to mention the rest of the L_∞-structure. For the basic example $\mathcal{A} = \mathcal{A}ss$, we already proved in [1] that $(\mathrm{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W), \delta)$ is isomorphic to the Gerstenhaber-Schack complex (see [4]) $(C^*_{\mathrm{GS}}(D, D), \delta_{\mathrm{GS}})$ (of a diagram D) for *some* resolution $\mathcal{D}_\infty$. The method, however, doesn't allow to find $\mathcal{D}_\infty$ explicitly. We conjecture that this $\mathcal{D}_\infty$ has the form given by Theorem 4.3.15:

4.5.2 Conjecture. In Theorem 4.3.15, let $\mathcal{A} := \mathcal{A}ss$, let $\mathcal{A}_\infty := \mathcal{A}ss_\infty$ be the minimal resolution of $\mathcal{A}ss$ and let $\mathcal{C}_\infty = \Omega\mathrm{B}\mathcal{C}$. Then there are $\omega(x, f)$'s such that

$$(\mathrm{Der}^*(\mathcal{D}_\infty, \mathcal{E}nd_W), \delta) \cong (C^*_{\mathrm{GS}}(D, D), \delta_{\mathrm{GS}}).$$

Another very interesting problem is to find an operadic interpretation of Cohomology Comparison Theorem: Recall that CCT, proved in [4], is a theorem relating deformations of the diagram of associative algebras to deformations of a single associative algebra. The point is that the deformations of the single algebra are described by Hochschild complex equipped with a dg Lie algebra structure given by Hochschild differential and Gerstenhaber bracket. On the other hand, in known examples (see [3]), the L_∞-structure on operadic deformation complex of the diagram has nontrivial higher brackets (see [3]). This suggest that this L_∞-algebra can be rectified to the one given by CCT.

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