# On bounded solutions to the compressible isentropic Euler system

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# Setting of the problem

We study the compressible isentropic Euler system in the whole 2D space

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) \; = \; 0 \\ \partial_t(\rho v) + \mathrm{div}_x \left(\rho v \otimes v\right) + \nabla_x[p(\rho)] \; = \; 0 \\ \rho(\cdot, 0) \; = \; \rho^0 \\ v(\cdot, 0) \; = \; v^0 \, . \end{cases} \tag{1}$$

Unknowns:

- $\rho(x, t)$ … density
- $v(x, t)$ … velocity

The pressure $p(\rho)$ is given.

# Introduction

- It is a hyperbolic system of conservation laws
- The theory of hyperbolic conservation laws is far frome being completely understood
- Solutions develop singularities in finite time even for smooth initial data
- Admissibility comes into play due to the entropy inequality ("selector" of physical solutions in case of existence of many solutions)
- There are satisfactory results in the case of scalar conservation laws (in 1D as well as in multi-D), there is a lot of entropies: $\Rightarrow$ Kruzkov, 1970: Well–posedness theory in $BV$.
- There are also satisfactory results in the case of systems of conservation laws in 1D: Lax, Glimm, Bianchini, Bressan

# Introduction II

Back to our case, the Euler system:

- In more than 1D there is only one (entropy, entropy flux) pair, which is

$$\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}, \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right)$$

  with the internal energy $\varepsilon(\rho)$ given through

$$p(\rho) = \rho^2\varepsilon'(\rho).$$

- Local existence of strong (and therefore admissible) solutions is proved
- On the other hand global existence of weak solutions in general (it is a system in multi D!) is still an open problem, there are only partial results
- The weak–strong uniqueness property holds for this system

# Weak solution

## Definition 1

By a *weak solution* of Euler system on $\mathbb{R}^2 \times [0,\infty)$ we mean a pair $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0,\infty))$ such that the following identities hold for every test functions $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi] \, dxdt + \int_{\mathbb{R}^2} \rho^0(x)\psi(x,0)dx \; = \; 0$$

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho v \cdot \partial_t \phi + \rho v \otimes v : \nabla_x \phi + p(\rho) \operatorname{div}_x \phi] \, dxdt$$
$$+ \int_{\mathbb{R}^2} \rho^0(x) v^0(x) \cdot \phi(x,0)dx \; = \; 0.$$

# Admissible weak solution

## Definition 2

A bounded weak solution $(\rho, v)$ of Euler system is *admissible* if it satisfies the following inequality for every nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} \left[ \left( \rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi \right.$$

$$\left. + \left( \rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dxdt$$

$$+ \int_{\mathbb{R}^2} \left( \rho^0(x)\varepsilon(\rho^0(x)) + \rho^0(x) \frac{|v^0(x)|^2}{2} \right) \varphi(x,0)\, dx \ \geq \ 0\,.$$

# Previous results

- De Lellis, Székelyhidi: There exist initial data $(\rho^0, v^0) \in L^\infty$ such that there are infinitely many bounded admissible weak solutions
- Chiodaroli: ill-posedness of bounded admissible weak solutions for initial data with regular density ($\rho^0 \in C^1$) and irregular velocity ($v^0 \in L^\infty$)

# Main Theorem of Part I

**Theorem 3 (Chiodaroli, De Lellis, K.)**

*Let $p(\rho) = \rho^2$. There exist Lipschitz initial data $(\rho^0, v^0)$ for which there are infinitely many bounded admissible weak solutions $(\rho, v)$ of Euler system on $\mathbb{R}^2 \times [0, \infty)$ with $\inf \rho > 0$. These solutions are all locally Lipschitz on a finite interval of time where they therefore all coincide with the unique classical solution.*

The proof is based on analysis of the Riemann problem and a suitable application of the theory of De Lellis and Székelyhidi for incompressible Euler equations.

# Riemann problem

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{2}$$

where $\rho_\pm, v_\pm$ are constants.

In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of $BV$ functions).

In the case of system (1), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

# First observation

If $v_{-1} = v_{+1}$, then any self-similar solution to (1), (2) has to satisfy $v_1(t,x) = v_{-1} = v_{+1}$ and in particular there is no contact discontinuity in the self-similar solution.
The initial singularity then resolves into at most 2 structures (rarefaction waves or admissible shocks) connected by constant states.

# Classification of self-similar solutions I

1) If

$$v_{+2} - v_{-2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state is vacuum, i.e. $\rho_m = 0$.

2) If

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state has $\rho_m > 0$.

3) If $\rho_- > \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}} < v_{+2} - v_{-2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and an admissible 3−shock.

4) If $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} < v_{+2} - v_{-2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of an admissible 1−shock and a 3−rarefaction wave.

# Classification of self-similar solutions III

5) If

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} \qquad (3)$$

then the self-similar solution consists of an admissible $1-$shock and an admissible $3-$shock.

# Proof of Theorem 3

Step 1 - self-similar solution to Riemann:

- Observe that if $(\rho, v)$ is a solution then also

$$(\tilde{\rho}(x_2, t), \tilde{v}(x_2, t)) := (\rho(-x_2, -t), v(-x_2, -t))$$

  is.
- If $(\rho, v)$ is locally Lipschitz, it satisfies the admissibility condition with *equality*, so does $(\tilde{\rho}, \tilde{v})$.
- Consider Riemann data resolving into a single 1-rarefaction wave and switch left and right.

Thus it holds the following.

**Lemma 4**

*Let* $0 < \rho_- < \rho_+$, $v_- = (-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}))$ *and* $v_+ = (-\frac{1}{\rho_+}, 0)$. *Then there is a pair* $(\rho, v) \in W^{1,\infty}_{loc} \cap L^\infty(\mathbb{R}^2 \times (-\infty, 0), \mathbb{R}^+ \times \mathbb{R}^2)$ *such that*

(i) $\rho_+ \geq \rho \geq \rho_- > 0$;

(ii) *The pair solves the Euler system with* $p(\rho) = \rho^2$ *in the classical sense (pointwise a.e. and distributionally);*

(iii) *for* $t \uparrow 0$ *the pair* $(\rho(\cdot, t), v(\cdot, t))$ *converges pointwise a.e. to* $(\rho^0, v^0)$ *as in (2);*

(iv) $(\rho(\cdot, t), v(\cdot, t)) \in W^{1,\infty}$ *for every* $t < 0$.

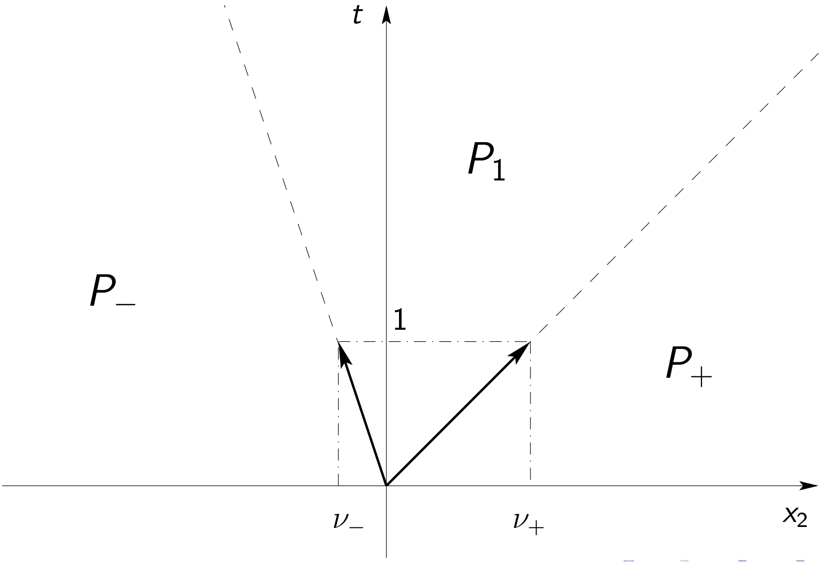
**Definition 5 (Fan partition)**

A *fan partition* of $\mathbb{R}^2 \times (0, \infty)$ consists of three open sets $P_-, P_1, P_+$ of the following form

$$\begin{aligned} P_- &= \{(x,t) : t > 0 \quad \text{and} \quad x_2 < \nu_- t\} \\ P_+ &= \{(x,t) : t > 0 \quad \text{and} \quad x_2 > \nu_+ t\} \\ P_1 &= \{(x,t) : t > 0 \quad \text{and} \quad \nu_- t < x_2 < \nu_+ t\} \end{aligned}$$

where $\nu_- < \nu_+$ is an arbitrary couple of real numbers.

# Picture of fan partition



# Fan subsolution

## Definition 6 (Fan subsolution I)

A *fan subsolution* to the compressible Euler equations with initial data (2) is a triple $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times (0,\infty) \to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ of bounded measurable functions satisfying the following requirements.

(i) There is a fan partition $P_-, P_1, P_+$ of $\mathbb{R}^2 \times (0,\infty)$ such that

$$(\overline{\rho}, \overline{v}, \overline{u}) = (\rho_-, v_-, u_-)\mathbf{1}_{P_-} + (\rho_1, v_1, u_1)\mathbf{1}_{P_1} + (\rho_+, v_+, u_+)\mathbf{1}_{P_+}$$

where $\rho_1 > 0, v_1, u_1$ are constants and

$$u_\pm = v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2 \mathrm{Id}$$

# Fan subsolution II

(ii) There exists a positive constant $C$ such that

$$v_1 \otimes v_1 - u_1 < \frac{C}{2}\mathrm{Id} \text{ a.e.}$$

(iii) The triple $(\overline{\rho}, \overline{v}, \overline{u})$ solves the following system in the sense of distributions:

$$\begin{aligned}
\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\,\overline{v}) &= 0 \\
\partial_t(\overline{\rho}\,\overline{v}) + \mathrm{div}_x\,(\overline{\rho}\,\overline{u}) & \\
+ \nabla_x \left( p(\overline{\rho}) + \frac{1}{2}\left( C\rho_1 \mathbf{1}_{P_1} + \overline{\rho}|\overline{v}|^2 \mathbf{1}_{P_+ \cup P_-} \right) \right) &= 0
\end{aligned}$$

# Admissible fan subsolution

**Definition 7 (Admissible fan subsolution)**

A fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ is said to be *admissible* if it satisfies the following inequality in the sense of distributions

$$\begin{aligned}
&\partial_t \left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \mathrm{div}_x \left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho})\right)\overline{v}\right] \\
&+ \partial_t \left(\overline{\rho}\frac{|\overline{v}|^2}{2}\mathbf{1}_{P_+\cup P_-}\right) + \mathrm{div}_x \left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v}\mathbf{1}_{P_+\cup P_-}\right) \\
&+ \partial_t \left(\rho_1 \frac{C}{2}\mathbf{1}_{P_1}\right) + \mathrm{div}_x \left(\rho_1 \overline{v}\frac{C}{2}\mathbf{1}_{P_1}\right) \leq 0\,.
\end{aligned}$$

# Key Lemma

The true core of our construction is the following Lemma.

**Lemma 8**

*Let* $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ *and* $C > 0$ *be such that* $\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C}{2}\mathrm{Id}$. *For any open set* $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ *there are infinitely many maps* $(\underline{v}, \underline{u}) \in L^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ *with the following property*

(i) $\underline{v}$ *and* $\underline{u}$ *vanish identically outside* $\Omega$

(ii)
$$\begin{cases} \mathrm{div}_x\, \underline{v} &= 0 \\ \partial_t \underline{v} + \mathrm{div}_x\, \underline{u} &= 0 \end{cases}$$

(iii) $(\tilde{v} + \underline{v}) \otimes (\tilde{v} + \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C}{2}\mathrm{Id}$ *a.e. on* $\Omega$.

Proof of this statement is (up to minor modifications) in the first two papers of De Lellis and Székelyhidi on incompressible Euler equations. There is no time to present it here.

# From subsolution to solutions

Using this Lemma we easily prove the following

**Proposition 9**

*Let $p$ be any $C^1$ function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ of the Euler equations with initial data (2). Then there are infinitely many bounded admissible solutions $(\rho, v)$ to (1),(2) such that $\rho = \overline{\rho}$.*

To prove Theorem 12 we therefore need to find an admissible fan subsolution.

Introduce the real numbers $\alpha, \beta, \gamma_1, \gamma_2, v_{-1}, v_{-2}, v_{+1}, v_{+2}$ such that

$$\begin{aligned} v_1 &= (\alpha, \beta) \\ v_- &= (v_{-1}, v_{-2}) \\ v_+ &= (v_{+1}, v_{+2}) \\ u_1 &= \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}. \end{aligned}$$

**Proposition 10**

*Let $P_-, P_1, P_+$ be a fan partition. The constants $v_1, v_-, v_+, u_1, \rho_-, \rho_+, \rho_1$ define an admissible fan subsolution if and only if the following identities and inequalities hold:*

# Algebra II

(i) Rankine-Hugoniot conditions on the left interface

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1 \beta$$
$$\nu_-(\rho_- v_{-1} - \rho_1 \alpha) = \rho_- v_{-1} v_{-2} - \rho_1 \gamma_2$$
$$\nu_-(\rho_- v_{-2} - \rho_1 \beta) = \rho_- v_{-2}^2 + \rho_1 \gamma_1 + p(\rho_-) - p(\rho_1) - \rho_1 \frac{C}{2}$$

(ii) Rankine-Hugoniot conditions on the right interface

$$\nu_+(\rho_1 - \rho_+) = \rho_1 \beta - \rho_+ v_{+2}$$
$$\nu_+(\rho_1 \alpha - \rho_+ v_{+1}) = \rho_1 \gamma_2 - \rho_+ v_{+1} v_{+2}$$
$$\nu_+(\rho_1 \beta - \rho_+ v_{+2}) = -\rho_1 \gamma_1 - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+) + \rho_1 \frac{C}{2}$$

(iii) Subsolution condition

$$\alpha^2 + \beta^2 < C$$
$$\left(\frac{C}{2} - \alpha^2 + \gamma_1\right)\left(\frac{C}{2} - \beta^2 - \gamma_1\right) - (\gamma_2 - \alpha\beta)^2 > 0$$

(iv) Admissibility condition on the left interface

$$\nu_-(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1)) + \nu_-\left(\rho_-\frac{|v_-|^2}{2} - \rho_1\frac{C}{2}\right)$$
$$\leq [(\rho_-\varepsilon(\rho_-) + p(\rho_-))v_{-2} - (\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta]$$
$$+ \left(\rho_- v_{-2}\frac{|v_-|^2}{2} - \rho_1\beta\frac{C}{2}\right)$$

(v) Admissibility condition on the right interface

$$
\begin{aligned}
&\nu_+(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+)) + \nu_+\left(\rho_1\frac{C}{2} - \rho_+\frac{|v_+|^2}{2}\right) \\
\leq &\left[(\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta - (\rho_+\varepsilon(\rho_+) + p(\rho_+))v_{+2}\right] \\
&+ \left(\rho_1\beta\frac{C}{2} - \rho_+ v_{+2}\frac{|v_+|^2}{2}\right).
\end{aligned}
$$

# Algebra V

The proof of the main Theorem is finished as soon as we prove the following:

**Lemma 11**

*Let $p(\rho) = \rho^2$. There exist $(\rho_-, v_-)$, $(\rho_+, v_+)$ producing a compression wave on time interval $(-\infty, 0)$ and $\rho_1, C_1, v_1, u_1, \nu_\pm$ satisfying the algebraic identities and inequalities in the previous Proposition.*

**Proof**:

- We already know how should the initial data look:
  - $0 < \rho_- < \rho_+$
  - $v_- = \left(-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-})\right)$
  - $v_+ = \left(-\frac{1}{\rho_+}, 0\right)$

# Algebra VI

- Plug these values to the set of identities and inequalities
- Try the choice $\nu_+ = 0$ and observe that then $\beta = \gamma_2 = 0$
- Observe that the second admissibility condition is now satisfied as an equality!
- Try the choice $\rho_- = 1$ and $\rho_+ = 4$ and observe that
  - $\alpha = -\frac{1}{4}$
  - $\nu_- = -\frac{7}{2\sqrt{2}}$
  - $\rho_1 = \frac{15}{7}$
  - $\frac{C_1}{2} - \gamma_1 = \frac{559}{105}$
- Finally observe that the remaining three inequalities yield a nonempty interval for $C_1$ and therefore we have miraculously found a solution!

# First Theorem of Part II

**Theorem 12 (Chiodaroli, K.)**

*Let* $p(\rho) = \rho^\gamma$ *with* $\gamma \geq 1$. *For every Riemann data (2) such that the self-similar solution to the Riemann problem (1), (2) consists of an admissible* $1-$*shock and an admissible* $3-$*shock, i.e.* $v_{-1} = v_{+1}$ *and*

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}}, \tag{4}$$

*there exist infinitely many admissible solutions to (1), (2).*

- Compared to Theorem 3, the new Theorem 12 widely extends the set of initial data for which there exist infinitely many admissible solutions to the Riemann problem.
- Moreover Theorem 12 gives this result for any pressure law $p(\rho) = \rho^\gamma$, instead of the specific case $\gamma = 2$ in Theorem 3

# Entropy rate admissibility I

Define the *total energy* of the solutions $(\rho, v)$ to (1) as

$$E[\rho, v](t) = \int_{\mathbb{R}^2} \left( \rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \mathrm{d}x \tag{5}$$

and the *energy dissipation rate* of $(\rho, v)$ at time $t$:

$$D[\rho, v](t) = \frac{\mathrm{d}_+ E[\rho, v](t)}{\mathrm{d}t}. \tag{6}$$

In our case the energy is always infinite, so we restrict the integrals to a finite box:

$$E_L[\rho, v](t) = \int_{(-L,L)^2} \left( \rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \mathrm{d}x \tag{7}$$

$$D_L[\rho, v](t) = \frac{\mathrm{d}_+ E_L[\rho, v](t)}{\mathrm{d}t}. \tag{8}$$

# Entropy rate admissibility II

**Definition 13 (Entropy rate admissible solution)**

A weak solution $(\rho, v)$ of (1) is called *entropy rate admissible* if there exists $L^* > 0$ such that there is no other weak solution $(\overline{\rho}, \overline{v})$ with the property that for some $\tau \geq 0$, $(\overline{\rho}, \overline{v})(x, t) = (\rho, v)(x, t)$ on $\mathbb{R}^2 \times [0, \tau]$ and $D_L[\overline{\rho}, \overline{v}](\tau) < D_L[\rho, v](\tau)$ for all $L \geq L^*$.

This definition is motivated by Dafermos. He proved that for a single equation the entropy rate criterion is equivalent to the viscosity criterion in the class of piecewise smooth solutions. Following the approach of Dafermos, Hsiao proved, in the class of piecewise smooth solutions, the equivalence of the entropy rate criterion and the viscosity criterion for the one-dimensional system of equations of nonisentropic gas dynamics in lagrangian formulation with pressure laws $p(\rho) = \rho^\gamma$ for $\gamma \geq 5/3$ while the same equivalence is disproved for $\gamma < 5/3$.

**Theorem 14 (Chiodaroli, K.)**

*Let* $p(\rho) = \rho^\gamma$, $1 \leq \gamma < 3$. *There exist Riemann data* (2) *for which the self-similar solution to* (1) *emanating from these data is not entropy rate admissible.*

Theorem 14 ensures that for $1 \leq \gamma < 3$ there exist initial Riemann data (2) for which some of the infinitely many nonstandard solutions constructed as in Theorem 12 dissipate more total energy than the self-similar solution, suggesting in particular that the Dafermos entropy rate admissibility criterion would not pick the self-similar solution as the admissible one.

# Algebra revisited I

We are trying to find a solution to the above mentioned system of algebraic identities and inequalities.

- Since $v_{-1} = v_{+1}$ it is easy to prove that $\alpha = v_{-1} = v_{-2}$ and $\gamma_2 = \alpha\beta$.
- Then observe that the subsolution inequalities are equivalent to existence of $\varepsilon_1$, $\varepsilon_2$ such that

$$0 < \varepsilon_1 = \frac{C}{2} - \gamma_1 - \beta^2$$
$$0 < \varepsilon_2 = C - \alpha^2 - \beta^2 - \varepsilon_1$$

- Reformulate the system of identities and inequalities in terms of $\varepsilon_1$ and $\varepsilon_2$
- Achieve 4 identities and 4 inequalities for 6 unknowns $(\nu_\pm, \rho_1, \beta, \varepsilon_1, \varepsilon_2)$. Moreover $\varepsilon_2$ appears only in inequalities.

# Algebra revisited II

- Take $\rho_1$ as a parameter and express $\nu_\pm, \beta$ and $\varepsilon_1$ as functions of initial data and $\rho_1$
- Crucial observation: $\varepsilon_1 > 0$ if and only if $\rho_1 < \rho_m$ with $\rho_m$ being the density of the intermediate state of the self-similar solution.
- Moreover, for fixed $\rho_\pm$ the value of $\rho_m$ grows as $(v_{-2} - v_{+2})^{\frac{2}{\gamma}}$
- Both admissibility inequalities yield $\varepsilon_2 < A \pm \varepsilon_1 B$ with $A(\rho_\pm, \rho_1)$ strictly positive. Therefore by continuity the inequalities are satisfied at least in some left neighborhood of $\rho_m$.

The proof of Theorem 12 is finished.

# Proof of Theorem 14 I

Set $v_{\pm 1} = \alpha = 0$. Denote the energy of the constant state $(\rho_*, v_*)$ by $E_*$:

$$E_* := \rho_* \varepsilon(\rho_*) + \rho_* \frac{|v_*|^2}{2} \tag{9}$$

If $(\rho_c, v_c)$ is the self-similar solution of the Riemann problem consisting of two admissible shocks, the dissipation rate is given by

$$D_L[\rho_c, v_c](t) = 2L\left(\nu_-(E_- - E_m) + \nu_+(E_m - E_+)\right) \tag{10}$$

at least for $t \leq T^*$ with some $T^*$ depending on $L^*$.

# Proof of Theorem 14 II

Consider any solution $(\rho_n, v_n)$ constructed from a subsolution achieved as above. Even though the velocity $v_n$ is highly oscillating in the region $P_1$, it holds $|v_n|^2 |_{P_1} = C = \beta^2 + \varepsilon_1 + \varepsilon_2$ and thus the energy of all such solutions is given by the subsolution

$$E_1 = \rho_1 \varepsilon(\rho_1) + \rho_1 \frac{\beta^2 + \varepsilon_1 + \varepsilon_2}{2}. \tag{11}$$

Therefore we can again write down the dissipation rate similarly as above:

$$D_L[\rho_n, v_n](t) = 2L\left(\nu_-(E_- - E_1) + \nu_+(E_1 - E_+)\right) \tag{12}$$

again at least for small $t$.
Therefore we study properties of function

$$f(\rho_1) := \nu_-(\rho_1)(E_- - E_1(\rho_1)) + \nu_+(\rho_1)(E_1(\rho_1) - E_+) \tag{13}$$

# Proof of Theorem 14 III

Our aim is to make $f(\rho_1)$ as small as possible. Concerning dependence on $\varepsilon_2$ it is easy to observe that smallest possible value of $f(\rho_1)$ is achieved by taking $\varepsilon_2 = 0$.

Moreover

$$\lim_{\rho_1 \to \rho_m} \lim_{\varepsilon_2 \to 0} 2Lf(\rho_1) = D_L[\rho_c, v_c](t). \tag{14}$$

We conclude therefore that Theorem 14 is a direct consequence of the following Lemma.

**Lemma 15**

*Let $1 \leq \gamma < 3$. There exist initial data $\rho_\pm$, $v_{\pm 2}$ for which the function $f(\rho_1)$ defined in (13) is increasing in the neighborhood of $\rho_m$.*

# Proof of Theorem 14 IV

Some easy manipulation yields that $f(\rho_1)$ has the following structure

$$f(\rho_1) = C_1(data) + C_2(data)g(\rho_1) \tag{15}$$

and we prove

**Proposition 16**

*For any $1 \leq \gamma < 3$ and any couple of densities $\rho_- > \rho_+$ there exists a unique local minimum $\overline{\rho} > \rho_-$ of the function $g(\rho_1)$. For fixed $\gamma, \rho_-, \rho_+$ the value of $\overline{\rho}$ grows asymptotically as $(v_{-2} - v_{+2})^{\frac{2}{\gamma+1}}$.*

# Proof of Theorem 14 V

Calculating the derivative of $g(\rho_1)$ and setting it equal to zero we achieve that any critical point of $g(\rho_1)$ satisfies equation

$$z(\rho_1) = \rho_- \rho_+ (v_{-2} - v_{+2})^2 - (\rho_- - \rho_+)(p(\rho_-) - p(\rho_+)) \quad (16)$$

and we show that $z(\rho_1)$ has the following properties

- it is strictly increasing
- $z(\rho_-) = 0$ (assuming $\rho_- > \rho_+$)
- $z(\rho_1) \sim \rho_1^{\gamma+1}$

Therefore for given data there is a unique point of local minimum $\overline{\rho}$ of $f(\rho_1)$. Finally $\overline{\rho}$ grows with respect to $(v_{-2} - v_{+2})$ slower than $\rho_m$, so it is enough to take $(v_{-2} - v_{+2})$ large enough to get $\overline{\rho} < \rho_m$. The proof is finished.

# Uniqueness of rarefaction wave solutions I

First we define the appropriate class of weak solutions. Consider

$$\Omega = \mathcal{T}^1 \times (-a, a), \tag{17}$$

with $a > 0$ sufficiently large and $\mathcal{T}^1$ is a 1D torus. We will consider weak solutions periodic in $x_1$ and having the same boundary fluxes on $x_2 = \pm a$ as has the self-similar solution.

We consider Riemann data satisfying $v_{\pm 1} = 0$.

# Uniqueness of rarefaction wave solutions II

More specifically we work with weak solutions satisfying:

$$\rho v_2(t, x_1, -a) = \rho_- v_{-2}, \ \ \rho v_2(t, x_1, a) = \rho_+ v_{+2}; \tag{18}$$

$$\left(\rho v_j v_2 + p(\rho)\right)(t, x_1, -a) = \left(\rho_- v_{-j} v_{-2} + p(\rho_-)\right) \tag{19}$$

$$\left(\frac{1}{2}\rho|v|^2 + \rho\varepsilon(\rho) + p(\rho)\right) v_2(t, x_1, -a) = \tag{20}$$

$$\left(\frac{1}{2}\rho_-|v_-|^2 + \rho_-\varepsilon(\rho_-) + p(\rho_-)\right) v_{-2} \tag{21}$$

and similarly for $x_2 = a$.

# Uniqueness of rarefaction wave solutions III

This means in particular that in the weak formulation of the Euler system appear additional boundary integrals on $x_2 = \pm a$, for example the equation of continuity in the weak formulation looks as follows:

$$\int_\Omega [\rho(\tau,x)\varphi(\tau,x) - \rho_0(x)\varphi(0,x)]\,\mathrm{d}x$$
$$+ \int_0^\tau \int_{\mathcal{T}^1} \rho_+ v_{+2}\varphi(t,x_1,a)\ \mathrm{d}x_1\mathrm{d}t$$
$$- \int_0^\tau \int_{\mathcal{T}^1} \rho_- v_{-2}\varphi(t,x_1,-a)\ \mathrm{d}x_2\mathrm{d}t$$
$$= \int_0^\tau \int_\Omega [\rho(t,x)\partial_t\varphi(t,x) + \rho v(t,x)\cdot\nabla\varphi(t,x)]\,\mathrm{d}x\mathrm{d}t$$

**Theorem 17 (Feireisl, K.)**

*Let* $p(\rho) = \rho^\gamma$, $\gamma > 1$. *Let* $\tilde{\rho}(t,x) = R(x_2/t)$, $\tilde{v}(t,x) = (0, V(x_2/t))$ *be the self-similar solution to the Riemann problem consisting of rarefaction waves (locally Lipschitz for* $t > 0$*) and such that*

$$\operatorname{ess\,inf}_{(0,t)\times\Omega} \tilde{\rho} > 0. \tag{22}$$

*Let* $(\rho, v)$ *be a bounded admissible weak solution such that*

$$\rho \geq 0 \text{ a.a. in } (0,T)\times\Omega.$$

*Then*

$$\rho \equiv \tilde{\rho},\ \ v \equiv \tilde{v} \text{ in } (0,T)\times\Omega.$$

Note that according to our earlier study the self-similar solution to the Riemann problem consists only of rarefaction waves and satisfies (22) if and only if the initial Riemann data satisfy

$$\left|\int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau\right| \leq v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau.$$

# Uniqueness of rarefaction wave solutions VI

The proof is based on the relative entropy inequality. Define the relative entropy functional

$$\mathcal{E}\left(\rho, v \middle| r, V\right) = \frac{1}{2}\rho|v - V|^2 + \left(H(\rho) - H'(r)(\rho - r) - H(r)\right),$$

where $H(s) = s\varepsilon(s)$.

Similarly as in papers by Feireisl, Novotný and others in the case of Navier-Stokes equations we first prove that any bounded admissible weak solution satisfies the relative entropy inequality with any couple of functions $(r, V)$ such that

$$r \in C^1([0,T] \times \overline{\Omega}),\ \ V \in C^1([0,T] \times \overline{\Omega}),\ \ r > 0.$$

# Uniqueness of rarefaction wave solutions VII

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| r, V\right)(\tau, x)\mathrm{d}x - \int_\Omega \mathcal{E}\left(\rho_0, v_0 \middle| r(0,x), V(0,x)\right)\mathrm{d}x$$
$$+ \text{boundary terms} \leq$$
$$\int_0^\tau \int_\Omega \left[\rho\left(\partial_t V + v \cdot \nabla V\right) \cdot (V - v) + \Big(p(r) - p(\rho)\Big)\,\mathrm{div}\, V\right]\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[(r-\rho)\partial_t H'(r) + (rV - \rho v) \cdot \nabla H'(r)\Big](t,x)\mathrm{d}x\mathrm{d}t$$

# Uniqueness of rarefaction wave solutions VIII

Observe that the rarefaction wave solution $(\tilde{\rho}, \tilde{v})$ may be taken as the test couple $(r, V)$ in the relative entropy inequality as

- $\rho, \tilde{\rho}, v, \tilde{V}$ bounded,
- $\partial_t \tilde{\rho}, \partial_t \tilde{v}_2, \partial_{x_2} \tilde{\rho}, \partial_{x_2} \tilde{v}_2 \in L^\infty(0, T; L^1(\Omega))$

and such step thus can be justified by a density argument and Lebesgue dominated convergence theorem.
Therefore the initial term and the boundary terms in the relative entropy inequality vanish.

Thus we get

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| \tilde{\rho}, \tilde{v}\right)(\tau, x)\mathrm{d}x \leq$$

$$\int_0^\tau \int_\Omega \left[\rho\left(\partial_t \tilde{v}_2 + v_2 \partial_{x_2} \tilde{v}_2\right)\left(\tilde{v}_2 - v_2\right) + \Big(p(\tilde{\rho}) - p(\rho)\Big)\partial_{x_2}\tilde{v}_2\right](t, x)\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big[(\tilde{\rho} - \rho)\partial_t H'(\tilde{\rho}) + (\tilde{\rho}\tilde{v}_2 - \rho v_2)\partial_{x_2} H'(\tilde{\rho})\Big](t, x)\mathrm{d}x\mathrm{d}t$$

This further simplifies to

$$\int_{\Omega} \mathcal{E}\left(\rho, v \middle| \tilde{\rho}, \tilde{v}\right)(\tau, x)\mathrm{d}x \leq$$
$$-\int_0^{\tau}\int_{\Omega}\left[\rho\left(\tilde{v}_2 - v_2\right)^2 + \Big(p(\rho) - p'(\tilde{\rho})(\rho - \tilde{\rho}) - p(\tilde{\rho})\Big)\right]\partial_{x_2}\tilde{v}_2(t,x)\mathrm{d}x\mathrm{d}t$$

Since $p(\rho)$ is convex the theorem follows from the fact that $\partial_{x_2}\tilde{v}_2(t,x) \geq 0$ which is a consequence of the classical theory of the self-similar solutions in the case of rarefaction waves.

# Thank you

Thank you for your attention.