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THE CZECH ACADEMY OF SCIENCES

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Preprint No. 51-2018

PRAHA 2018

AN EXTENSION OF COMPACT OPERATORS BY COMPACT OPERATORS WITH NO NONTRIVIAL MULTIPLIERS

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Abstract. We construct a nonhomogeneous, separably represented, type I and approximately finite dimensional C^*-algebra such that its multiplier algebra is equal to its unitization. This algebra is an essential extension of the algebra $\mathcal{K}(\ell_2(\mathfrak{c}))$ of compact operators on a nonseparable Hilbert space by the algebra $\mathcal{K}(\ell_2)$ of compact operators on a separable Hilbert space, where $\mathfrak{c}$ denotes the cardinality of continuum. Although both $\mathcal{K}(\ell_2(\mathfrak{c}))$ and $\mathcal{K}(\ell_2)$ are stable, our algebra is not. This sheds light on the permanence properties of the stability in the nonseparable setting. Namely, unlike in the separable case, an extension of a stable nonseparable C^*-algebra by $\mathcal{K}(\ell_2)$ does not have to be stable. Our construction can be considered as a noncommutative version of Mrówka's Ψ-space; a space whose one point compactification is equal to its Čech-Stone compactification and is induced by a special uncountable family of almost disjoint subsets of $\mathbb{N}$.

1. introduction

Perhaps the simplest example of a locally compact space whose one-point compactification is equal to the Čech-Stone compactification is the first uncountable ordinal ω_1 with the order topology. This follows from the well-known fact that every real or complex valued continuous function on ω_1 is eventually constant. Another example of such spaces is $K \setminus \{x\}$, where K is a compact extremally disconnected space and x is a nonisolated point (Exercise 1H of [14]). A noncommutative version of this fact was proved in [25] in the context of II_1 factors. In [20] Mrówka constructed a locally compact space with the same property that the one-point compactification and Čech-Stone compactification coincide which moreover has the simplest nontrivial Cantor-Bendixson decomposition, i.e., after removing a countable dense subset of isolated points we are left with an uncountable discrete space. In other words, it is a separable scattered space of Cantor-Bendixson height 2 (see 6.4. of [15]). Such spaces are induced by uncountable almost disjoint families of infinite subsets of $\mathbb{N}$ (every two distinct members of the family have finite intersection). On the level of Banach spaces of continuous functions or commutative C^*-algebras Mrówka's space X satisfies the following short exact sequence

$$0 \to c_0 \xrightarrow{\iota} C_0(X) \to c_0(\mathfrak{c}) \to 0,$$

where $\iota[c_0]$ is an essential ideal $C_0(X)$, i.e., $\mathbb{N}$ is a dense open subset of X. Here $\mathfrak{c}$ denotes the cardinality of the continuum. In other words, $C_0(X)$ is an essential extension of $c_0(\mathfrak{c})$ by c_0 (see II.8.4 of [6]).

The research of the second named author was partially supported by grant PVE Ciência sem Fronteiras - CNPq (406239/2013-4).

In this paper we produce a noncommutative version of this phenomenon. It is widely accepted that the noncommutative version of the ideal of finite subsets of $\mathbb{N}$, or the commutative C^*-algebra c_0, is the C^*-algebra of all compact operators on a separable Hilbert space. The same analogy exists for finite subsets of $\mathfrak{c}$ and the C^*-algebra of all compact operators on the Hilbert space $\ell_2(\mathfrak{c})$ of density $\mathfrak{c}$. The roles of the one point compactification and the Čech-Stone compactification of a locally compact, noncompact space X are played by the unitization of a nonunital C^*-algebra $\mathcal{A}$ (which will be denoted by $\tilde{A}$) and the multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$, respectively. Thus we are interested in an essential extension of the algebra of compact operators $\mathcal{K}(\ell_2(\mathfrak{c}))$ by $\mathcal{K}(\ell_2)$, i.e., a C^*-algebra $\mathcal{A}$ satisfying the short exact sequence

$$(*) \qquad 0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{A} \to \mathcal{K}(\ell_2(\mathfrak{c})) \to 0,$$

where $\iota[\mathcal{K}(\ell_2)]$ is an essential ideal of $\mathcal{A}$. In the main theorem of this paper, Theorem 6.1, we construct such an algebra $\mathcal{A}$ with the required additional property that the multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is *-isomorphic to the unitization of $\mathcal{A}$. In other words, the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ of our $\mathcal{A}$ is *-isomorphic to $\mathbb{C}$. In fact $\mathcal{A}$ has the property that the space $\mathcal{QM}(\mathcal{A})$ of all quasi-multipliers of $\mathcal{A}$ coincides with $\mathcal{M}(\mathcal{A})$ and therefore $\mathcal{QM}(\mathcal{A})/\mathcal{A}$ is also *-isomorphic to $\mathbb{C}$. The algebra $\mathcal{A}$ of Theorem 6.1 is a nonseparable subalgebra of $\mathcal{B}(\ell_2)$, which is type I and approximately finite dimensional in the sense that any finite subset can be approximated from a finite dimensional subalgebra. Moreover $\mathcal{A}$ is a scattered C^*-algebra (see [13]), which means all of its subalgebras are also approximately finite dimensional ([18]). Note that the various equivalent definitions of approximately finite dimensional C^*-algebras which are equivalent in the separable case are no longer equivalent in the nonseparable context (see [12] where a different terminology is used).

For C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, an extension of $\mathcal{B}$ by $\mathcal{C}$ is a short exact sequence of C^*-algebras

$$0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{B} \to 0.$$

The goal of the extension theory is, given $\mathcal{B}$ and $\mathcal{C}$, to classify all the extensions of $\mathcal{B}$ by $\mathcal{C}$ up to a suitable equivalence relation. The set of all equivalence classes of extensions of $\mathcal{B}$ by $\mathcal{C}$ can be equipped with a proper addition which turns it into an abelian semigroup, usually denoted by $Ext(\mathcal{B},\mathcal{C})$, or simply $Ext(\mathcal{B})$ if $\mathcal{C} = \mathcal{K}(\ell_2)$. The reader may refer to [5] for the details and various definitions regarding extensions of C^*-algebras, however in this paper we are not concerned about the structure of Ext semigroups, although we hope that the extension we construct is a contribution to a more general and future project of understanding the semigroup $Ext(\mathcal{K}(\ell_2(\kappa))$ for $\omega_1 \leq \kappa \leq \mathfrak{c}$. The "extension questions" for C^*-algebras ask whether the C^*-algebra $\mathcal{A}$ in the extension

$$0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{B} \to 0,$$

satisfies property $\mathcal{P}$, given that both $\mathcal{B}$ and $\mathcal{C}$ satisfy $\mathcal{P}$. One of the features of our extension is that $\mathcal{B}$ and $\mathcal{C}$ are as simple as possible (besides $\mathcal{B}$ being nonseparable), while $\mathcal{A}$ is quite pathological, which makes it interesting for the questions of this sort.

In particular if $\mathcal{P}$ is the stability property of a C^*-algebra (recall that a C^*-algebra is stable if it is isomorphic to its tensor product by $\mathcal{K}(\ell_2)$) then the above question is usually called "the extension question for stable C^*-algebras" (see [24]). If $\mathcal{C} = \mathcal{K}(\ell_2)$ and $\mathcal{B}$ is a separable C^*-algebra, then $\mathcal{A}$ is stable if and only if $\mathcal{B}$ is

stable (see Proposition 6.12 of [24]; this is essentially a result of BDF-theory ([7])). In fact a result of Blackadar ([4]) shows that this holds also if $\mathcal{B}$ and $\mathcal{C}$ in the above short exact sequence are any separable AF-algebras. This result is extended to extensions of more general separable C^*-algebras in [24]. Therefore our example shows that these results do not hold even in quite basic nonseparable context, as the C^*-algebra $\mathcal{A}$ from Theorem 6.1 satisfies $(*)$ while it is nonstable. The latter is because the multiplier algebra of any stable algebra with a projection contains a copy of $\mathcal{B}(\ell_2)$ (by 3.8. of [1]). However, $\mathcal{A}$ and therefore $\tilde{A}$ (which is isomorphic to $\mathcal{M}(\mathcal{A})$) are scattered C^*-algebras as mentioned above, and consequently all of their subalgebras are AF ([18]). Hence $\mathcal{M}(\mathcal{A})$ does not contain a copy of $\mathcal{B}(\ell_2)$. We need to add however, that a result of Rørdam shows that there are separable extensions of $\mathcal{K}(\ell_2)$ which are not stable ([23]).

On a different note, it is worth noticing that our C^*-algebra $\mathcal{A}$ is complemented in the Banach space $\mathcal{M}(\mathcal{A})$ as it is co-one-dimensional closed subspace. This fact does not hold for many nonseparable C^*-algebras (see 3.7 of [26]). It is also interesting to note that any separable subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ is included in a separable subalgebra $\mathcal{B} \subseteq \mathcal{A}$ satisfying

$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{B} \to \mathcal{K}(\ell_2) \to 0,$$

where $\iota[\mathcal{K}(\ell_2)]$ is essential. All such algebras $\mathcal{B}$ are isomorphic to $\widetilde{\mathcal{K}(\ell_2)} \otimes \mathcal{K}(\ell_2)$, the noncommutative version of $C_0(\omega^2)$ (Proposition 2.16), where ω^2 is the ordinal $\omega \times \omega$ with the order topology. Also note that $\widetilde{\mathcal{K}(\ell_2)} \otimes \mathcal{K}(\ell_2(\mathfrak{c}))$ is a stable C^*-algebra which satisfies the short exact sequence from $(*)$, and clearly is not isomorphic to our algebra which is not stable. These facts have well-known analogues in the commutative context which is surveyed in [15] devoted to applications of almost disjoint families in topology. One should add that there are many noncommutative constructions based on almost disjoint families (see the begining of Section 2.2 for the definition) like in this paper or in papers [8], [27], [3].

The structure of the paper is as follows. In Section 2 we recall and prove preliminary results concerning liftings of "systems of almost matrix units" $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$, which form systems of matrix units in the Calkin algebra. The results are related to the liftings of families of almost orthogonal projections (families of orthogonal projections in the Calkin algebra), which were analyzed in [28] and [10]. Our C^*-algebra $\mathcal{A}$ from Theorem 6.1 is generated by a specific "maximal" system of almost matrix units $\mathcal{T}$ and all operators in $\mathcal{K}(\ell_2)$. A result of [28] states that maximal almost disjoint families of subsets of $\mathbb{N}$ do not necessarily give rise to maximal families of almost orthogonal projections. This is enough to suggest that Mrówka's original almost disjoint family ([20]) can not be directly used for our purpose in the noncommutative setting.

In Section 3, for any system of almost matrix units $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$ and an operator $R \in \mathcal{B}(\ell_2)$ which is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$, we assign a $\kappa \times \kappa$-matrix $\Lambda^{\mathcal{T}}(R)$. The matrix $\Lambda^{\mathcal{T}}(R)$ carries a great load of information about R, and its analysis is crucial in the remaining parts of the paper.

In Section 4 we prove some results related to a system of almost matrix units labeled by pairs of branches of the Cantor tree. In particular, it is essential later to use the Borel structure of the standard topology on the Cantor tree in the form of the "prefect set property" of Borel subsets of the tree.

Section 5 is devoted to a simple method of modifying a system of almost matrix units called *pairing*. Finally in Section 6 we present the main construction which uses all the previously developed theory.

The general scenario of the construction and the proof of the properties of our algebra follows the main steps of [20]. However there are two-fold complications. The usual problems related to passing from the commutative to the noncommutative context, and the combinatorial difficulties related to the fact that the objects corresponding to almost disjoint families, namely the systems of almost matrix units, are labeled by pairs and not single indices. The natural idea is to construct a system of almost matrix units $\mathcal{S} \subset \mathcal{B}(\ell_2)$ such that the C^*-subalgebra $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{K}(\ell_2)$ and the elements of $\mathcal{S}$ has no nontrivial (quasi-)multipliers, meaning that multipliers of the algebra $\mathcal{A}(\mathcal{S})$ are the elements of $\mathcal{A}(\mathcal{S})$ and the compact perturbations of the multiples of the identity. The method of eliminating (or "killing" as it is usually called in set theory) is the above-mentioned pairing from Section 6.

The notation and terminology should be standard and attempts to follow texts like [21], [2], [6], [10]. For $T, S \in \mathcal{B}(\ell_2)$, we often write $T =^{\mathcal{K}} S$ if $T - S \in \mathcal{K}(\ell_2)$. The map δ is always defined so that $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. $[X]^{<\omega}$ denotes the family of all finite subsets of a set X and $[X]^2$ denotes the family of all two-elements sets of X. For C^*-algebras $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ we identify the unitization $\tilde{\mathcal{A}}$ with the subalgebra of $\mathcal{B}(\ell_2)$ generated by $\mathcal{A}$ and the identity operator $1_{\mathcal{B}(\ell_2)}$.

We would like to thank Hannes Thiel for bringing the paper [26] to our attention and to Ilijas Farah for pointing out some gaps in the previous versions of the paper and for valuable comments.

2. PRELIMINARIES

2.1. **Compact operators.** The following elementary lemma sums up the basic properties of the compact operators which will be used throughout this paper.

Lemma 2.1. *Suppose that $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for the Hilbert space ℓ_2 and S is a bounded linear operator on ℓ_2.*

1. *If $\Sigma_{n\in\mathbb{N}}\|S(e_n)\| < \infty$, then S is compact,*
2. *If S is compact, $w_k \in span(e_n : n \in F_k)$ are norm 1 vectors, for pairwise disjoint finite $F_k \subseteq \mathbb{N}$ and all $k \in \mathbb{N}$, then $(\|S(w_k)\|)_{k\in\mathbb{N}} \to 0$,*
3. *If S is noncompact, then there is $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there is a finite subset $F_k \subseteq \mathbb{N}$ with $k < \min(F_k)$ and $w_k \in span(e_n : n \in F_k)$ of norm 1 such that $\|S(w_k)\| > \varepsilon$.*

Proof. The above clauses easily follow from the fact that an operator S is compact if and only if $\lim_{n\to\infty} \|S(1 - R_n)\| = 0$, where R_n is the projection on the span of $\{e_i : i \leq n\}$. □

Note that there are noncompact linear operators $S : \ell_2 \to \ell_2$ satisfying $S(e_n) \to 0$. For example, consider the operator S defined by $S((x_n)_{n\in\mathbb{N}})(k) = \Sigma_{i\in I_k} x_i/\sqrt{k}$, where $(I_k)_{k\in\mathbb{N}}$ form pairwise disjoint consecutive intervals in $\mathbb{N}$ of size k. Considering $w_k = \frac{1}{\sqrt{k}}\chi_{I_k}$, where χ_{I_k} is the characteristic function on I_k, one can easily verify that (2) fails.

2.2. **Families of almost orthogonal projections.** A family $\{A_\xi : \xi < \kappa\}$ of subsets of $\mathbb{N}$ is called an almost disjoint family if $A_\xi \cap A_\eta$ is finite for distinct $\xi, \eta < \kappa$. Suppose $\wp(\mathbb{N})$ denotes the Boolean algebra of all subsets of $\mathbb{N}$ and Fin

is the ideal of all finite subsets of $\mathbb{N}$. Almost disjoint families correspond to sets of pairwise incomparable elements (antichains) of $\wp(\mathbb{N})/Fin$. An almost disjoint family is called maximal if it is maximal with respect to the inclusion. For a fixed orthonormal basis for the Hilbert space ℓ_2, $\wp(\mathbb{N})/Fin$ naturally embeds in the poset of projection of the Calkin algebra. In other words, any almost disjoint family $\{A_\xi : \xi < \kappa\}$ would naturally give rise to a family of diagonalized projections $\{P_\xi : \xi < \kappa\}$ on ℓ_2 such that $P_\xi P_\eta$ is a compact (finite dimensional) projection, for any distinct $\xi, \eta < \kappa$. The following is a natural generalization of such families.

Definition 2.2 ([28])**.** *For a Hilbert space $\mathcal{H}$, a family $\mathcal{P}$ of noncompact projections of $\mathcal{B}(\mathcal{H})$ is called almost orthogonal if the product of any two distinct elements of it is compact. Such a family $\mathcal{P}$ is called maximal if for every noncompact projection $Q \in \mathcal{B}(\mathcal{H})$ the operator PQ is noncompact, for some $P \in \mathcal{P}$.*

Having fixed an orthonormal basis $(e_n : n \in \mathbb{N})$ for $\ell_2(\mathbb{N})$ and given a family $\mathcal{F} \subseteq \wp(\mathbb{N})$ one can consider the orthogonal projections P_A for $A \in \mathcal{F}$ onto the closed span of $\{e_n : n \in A\}$. As it was observed in [28], almost orthogonal families of projections corresponding in the above sense to maximal almost disjoint families do not have to be maximal.

Recall that a "masa" of $\mathcal{B}(\ell_2)$ is a maximal abelian subalgebra of $\mathcal{B}(\ell_2)$. A masa is called atomic if it is isomorphic to ℓ_∞, the algebra of all operators that are diagonalized by a fixed basis for ℓ_2. The following is Lemma 5.34 of [10].

Lemma 2.3. *Let $\pi : \mathcal{B}(\ell_2) \to \mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$ be the quotient map. Given any sequence $\{P_n : n \in \mathbb{N}\}$ of projections in $\mathcal{B}(\ell_2)$ such that $\pi(P_i)$ and $\pi(P_j)$ commute for all $i, j \in \mathbb{N}$, there is an atomic masa $\mathcal{A}$ in $\mathcal{B}(\ell_2)$ such that $\pi[\mathcal{A}]$ contains each $\pi(P_i)$ for $i \in \mathbb{N}$.*

Lemma 2.4. *Suppose that $\{P_n : n \in \mathbb{N}\}$ is an almost orthogonal family of projections of $\mathcal{B}(\ell_2)$. Then there are pairwise orthogonal projections $\{R_n : n \in \mathbb{N}\}$ in $\mathcal{B}(\ell_2)$ such that $P_n =^{\mathcal{K}} R_n$, for every $n \in \mathbb{N}$.*

Proof. By Lemma 2.3 there is an atomic masa $\mathcal{A}$ in $\mathcal{B}(\ell_2)$ such that $\pi[\mathcal{A}] \supseteq \{\pi(P_n) : n \in \mathbb{N}\}$. Since $\mathcal{A}$ is isomorphic to $\ell_\infty \cong C(\beta\mathbb{N})$, the ideal of compact operators in $\mathcal{A}$ is isomorphic to $c_0 \cong C_0(\beta\mathbb{N}, \mathbb{N}^*) = \{f \in C(\beta\mathbb{N}) : f|\mathbb{N}^* = 0\}$, where $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. Therefore $\mathcal{A}/(\mathcal{K}(\ell_2) \cap \mathcal{A}) \cong C(\beta\mathbb{N})/C_0(\beta\mathbb{N}, \mathbb{N}^*) \cong C(\mathbb{N}^*)$. As $\pi(P_n)$ are orthogonal projections, they correspond to the characteristic functions of pairwise disjoint clopen subsets of $\mathbb{N}^*$. Such sets are given by pairwise disjoint elements of $\wp(\mathbb{N})/Fin$. For any such family in $\wp(\mathbb{N})/Fin$ we can choose pairwise disjoint representatives in $\wp(\mathbb{N})$, which define disjoint clopen subsets of $\beta\mathbb{N}$ and therefore pairwise orthogonal projections R_n in $\mathcal{A}$ such that $\pi(R_n) = \pi(P_n)$. □

In the following $R|X$ denotes the restriction of the operator R to the closed subspace X of ℓ_2.

Lemma 2.5. *Suppose that $R \in \mathcal{B}(\ell_2)$ is noncompact and self-adjoint. Then there is a closed infinite dimensional subspace $X \subseteq \ell_2$ such that $R|X$ is invertible in $\mathcal{B}(X)$ and R commutes with the orthogonal projection P_X onto X.*

Proof. By the spectral theorem there are a measure space $(\mathcal{M}, \mu)$, an isomorphism of Hilbert spaces $U : \ell_2 \to L_2(\mathcal{M}, \mu)$, and a measurable function f such that URU^* is equal to the operator M_f on $L_2(\mathcal{M}, \mu)$ acting by multiplication by f. Since R is noncompact, we have $f \neq 0$. Putting $A_n = \{x \in \mathcal{M} : |f(x)| > 1/n\}$, we have

$M_f = \lim_{n\to\infty} M_{f\chi_{A_n}}$, where the convergence is in the operator norm as $f - f\chi_{A_n}$ is a function bounded by $1/n$. Let $P_n = U^* M_{\chi_n} U$. Each P_n is a projection which commutes with R and R is bounded away from zero on the range of P_n for each $n \in \mathbb{N}$. Moreover $(RP_n)_{n\in\mathbb{N}}$ converges (in the norm) to R. As R is noncompact, for some n the projection P_n is infinite dimensional. Hence we obtain the lemma by letting $X = P_n[\ell_2]$. □

Lemma 2.6. *Suppose that $\mathcal{P}$ is a maximal almost orthogonal family of projections of $\mathcal{B}(\ell_2)$ and $S \in \mathcal{B}(\ell_2)$ is a self-adjoint and noncompact operator. Then there are $P_1, P_2 \in \mathcal{P}$ such that P_2SP_1 is noncompact.*

Proof. Let $X \subseteq \ell_2$ be an infinite dimensional subspace such that $S|X$ is invertible and P_X commutes with S, which exists by Lemma 2.5. By the maximality of $\mathcal{P}$ we find $P_1 \in \mathcal{P}$ such that $P_X P_1$ is noncompact. Therefore $P_X P_1 P_X$ is a self-adjoint and noncompact operator. Using Lemma 2.5 again for $P_X P_1 P_X$, there is an infinite dimensional subspace $Y \subseteq X$ such that $(P_X P_1 P_X)|Y$ is invertible. So $P_X P_1 P_X$ acts on Y as an isomorphism of Banach spaces, transforming Y into its image $(P_X P_1 P_X)[Y]$ which is an infinite dimensional subspace of X. Since S acts as an isomorphism of Banach spaces on X, it follows that $SP_X P_1 P_X$ is noncompact. Also since S commutes with P_X, the operator $P_X S P_1 P_X$ is noncompact, and therefore SP_1 is noncompact. Working with SP_1 instead of S, similarly we find $P_2 \in \mathcal{P}$ such that $P_2 S P_1$ is noncompact. □

2.3. **Systems of almost matrix units.** Let κ be a cardinal and $\mathcal{A}$ be a C^*-algebra. A family $\{a_{\beta,\alpha} : \alpha, \beta < \kappa\}$ of nonzero elements of $\mathcal{A}$ satisfying the following matrix units relations:

- $a^*_{\beta,\alpha} = a_{\alpha,\beta}$ for all $\alpha, \beta < \kappa$,
- $a_{\beta,\alpha} a_{\gamma,\eta} = \delta_{\alpha,\gamma} a_{\beta,\eta}$ for all $\alpha, \beta, \gamma, \eta < \kappa$,

is called a system of matrix units in $\mathcal{A}$.

Proposition 2.7. *Let $\mathcal{A}$ be the C^*-algebra generated by a system of its matrix units $\{a_{\eta,\xi} : \xi, \eta < \kappa\}$. Then $\mathcal{A}$ is *-isomorphic to the algebra $\mathcal{K}(\ell_2(\kappa))$ of all compact operators on $\ell_2(\kappa)$.*

Proof. Let $\{e_\xi : \xi < \kappa\}$ be an orthonormal basis for $\ell_2(\kappa)$ and the operators $\{T_{\eta,\xi} : \xi, \eta < \kappa\}$ are the system of matrix units in $\mathcal{B}(\ell_2(\kappa))$ defined by $T_{\eta,\xi}(e_\xi) = e_\eta$ and $T_{\eta,\xi}(e'_\xi) = 0$ for $\xi' \neq \xi$. For every finite subset F of κ, let $\mathcal{B}_F$ be the C^*-subalgebra generated by $\{T_{\eta,\xi} : \xi, \eta \in F\}$, which is clearly isomorphic to $M_{|F|}$, the algebra of all $|F| \times |F|$ matrices. Let $\mathcal{B}$ be the inductive limit of the algebras $\mathcal{B}_F$, along the set $\mathcal{F}_\kappa$ of finite subsets of κ and the *-homomorphisms $\phi_{G,F} : \mathcal{B}_F \to \mathcal{B}_G$ for $F \subseteq G$ and $F, G \in \mathcal{F}_\kappa$, defined by $\phi_{G,F}(T_{\eta,\xi}) = T_{\eta,\xi}$, for $\xi, \eta \in F$. Clearly $\mathcal{B} = \mathcal{K}(\ell_2(\kappa))$. The map which sends $a_{\eta,\xi}$ to $T_{\eta,\xi}$ extends to a $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. □

Definition 2.8. *Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$ is a family of noncompact operators. We say that $\mathcal{T}$ is a system of almost matrix units if and only if for every $\alpha, \beta, \xi, \eta < \kappa$,*

(1) $T^*_{\eta,\xi} =^{\mathcal{K}} T_{\xi,\eta}$,
(2) $T_{\beta,\alpha}\ T_{\eta,\xi} =^{\mathcal{K}} \delta_{\alpha,\eta} T_{\beta,\xi}$.

Definition 2.9. *Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$ is a system of almost matrix units and $\{P_\xi : \xi < \kappa\}$ is a collection of almost orthogonal projections in*

$\mathcal{B}(\ell_2)$. *We say that* $\mathcal{T}$ *is based on* $\mathcal{P}$ *if* $T_{\xi,\xi} =^{\mathcal{K}} P_\xi$ *for all* $\xi < \kappa$. *We say that* $\mathcal{T}$ *is a maximal system of almost matrix units if it is based on a maximal family of almost orthogonal projections.*

Lemma 2.10. *Every system of almost matrix units is based on a family of almost orthogonal projections.*

Proof. By the almost matrix units relations 2.8 we have that $T^*_{\xi,\xi} =^{\mathcal{K}} T_{\xi,\xi}$ and $T_{\xi,\xi}T_{\xi,\xi} =^{\mathcal{K}} T_{\xi,\xi}$, so $[T_{\xi,\xi}]_{\mathcal{K}(\ell_2)}$ is a projection in the Calkin algebra. Therefore we can find a projection $P_\xi \in \mathcal{B}(\ell_2)$ such that $P_\xi =^{\mathcal{K}} T_{\xi,\xi}$ (see Lemma 5.3. of [10]). □

In the rest of this section we use some elementary facts about partial isometries i.e., elements of $\mathcal{B}(\ell_2)$ which are isometries on a subspace of ℓ_2 and zero on its orthogonal complement. For an element $U \in \mathcal{B}(\ell_2)$ being a partial isometry is equivalent to each of the conditions (i) $U = UU^*U$, (ii) $U^* = U^*UU^*$, (iii) U^*U is a projection, (iv) UU^* is a projection (2.3.3. [21]). Moreover recall that by the polar decomposition, any $T \in \mathcal{B}(\ell_2)$ can be written as $T = U|T|$, where U is a partial isometry whose kernel is equal to the kernel of T (2.3.4. [21]).

Lemma 2.11. *Suppose that* $\mathcal{P} = \{P_\xi : \xi < \kappa\} \subseteq \mathcal{B}(\ell_2)$ *is a family of almost orthogonal projections. Then there is a system of almost matrix units* $\mathcal{T}$ *based on* $\mathcal{P}$.

Proof. Since P_ξ for $\xi < \kappa$ are infinite dimensional projections, there are partial isometries $T_{\xi,0} \in \mathcal{B}(\ell_2)$ such that $T^*_{\xi,0}T_{\xi,0} = P_0$ and $T_{\xi,0}T^*_{\xi,0} = P_\xi$, for each $\xi < \kappa$. Let $T_{0,\xi} = T^*_{\xi,0}$. We have $T_{\xi,0} = P_\xi T_{\xi,0}$ and $T_{0,\xi} = T_{0,\xi}P_\xi$. For $\xi, \eta < \kappa$, define

$$T_{\xi,\eta} = T_{\xi,0}T_{0,\eta}.$$

It is clear that $\{T_{\xi,\eta} : \xi, \eta < \kappa\}$ satisfies the condition (1) of Definition 2.8. For (2) note that if $\alpha, \eta < \kappa$ then

$$T_{0,\alpha}T_{\eta,0} = T_{0,\alpha}P_\alpha P_\eta T_{\eta,0},$$

which is compact if $\xi \neq \eta$, by the almost orthogonality of P_ξs and $T_{\beta,\alpha}T_{\alpha,\xi} = T_{\beta,\xi}$. □

Lemma 2.12. *Every system of almost matrix units can be extended to a maximal one.*

Proof. Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\}$ is a system of almost matrix units. Let $\mathcal{P} = \{P_\xi : \xi \in \kappa\}$ be a family of projections such that $T_{\xi,\xi} =^{\mathcal{K}} P_\xi$ as in Lemma 2.10. Extend $\mathcal{P}$ to a maximal family of almost orthogonal projections $\mathcal{P}' = \{P_\xi : \xi \in \kappa\} \cup \{P_\xi : \xi \in X\}$ for some set X disjoint from κ. Use Lemma 2.11 to construct a system of almost matrix units $\{T_{\eta,\xi} : \xi, \eta \in X \cup \{0\}\}$ based on $\{P_\xi : \xi \in X \cup \{0\}\}$. For $\xi \in \kappa$ and $\eta \in X$ define

$$T_{\eta,\xi} = T_{\eta,0}T_{0,\xi}, \quad T_{\xi,\eta} = T_{\xi,0}T_{0,\eta}.$$

It is straightforward to check that $\{T_{\eta,\xi} : \xi, \eta \in \kappa \cup X\}$ forms a system of almost matrix units based on $\mathcal{P}'$.

□

The next lemma is a version of Lemma III 6.2 from [9].

Lemma 2.13. *Suppose that $\{T_{j,i} : i,j \in \mathbb{N}\}$ is a system of almost matrix units in $\mathcal{B}(\ell_2)$. Then there is a system $\{E_{j,i} : i,j \in \mathbb{N}\}$ of matrix units in $\mathcal{B}(\ell_2)$ such that $E_{j,i} =^{\mathcal{K}} T_{j,i}$ for every $i,j \in \mathbb{N}$.*

Proof. Let $\mathcal{A}$ be a C^*-algebra generated in $\mathcal{B}(\ell_2)$ by $\{T_{j,i} : i,j \in \mathbb{N}\}$ and the compact operators. Then since $\mathcal{B} = \mathcal{A}/\mathcal{K}(\ell_2)$ is generated by $\{[T_{j,i}]_{\mathcal{K}(\ell_2)} \in \mathcal{A}/\mathcal{K}(\ell_2) : i,j \in \mathbb{N}\}$, it is isomorphic to $\mathcal{K}(\ell_2)$ (see Lemma 2.14). Since $K_0(\mathcal{B}) \cong \mathbb{Z}$ is a free abelian group, $\mathcal{A}$ is a trivial extension (see Exercise 16.4.7 of [5]), i.e., the short exact sequence

$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{A} \to \mathcal{B} \to 0$$

splits, which means $\{[T_{j,i}]_{\mathcal{K}(\ell_2)} \in \mathcal{B} : i,j \in \mathbb{N}\}$ lift. □

2.4. **Ψ-type C^*-algebras.** If $\mathcal{T} = \{T_{\xi,\eta} : \xi,\eta < \kappa\}$ is a system of almost matrix units, we use $\mathcal{A}(\mathcal{T})$ to denote the C^*-subalgebra of $\mathcal{B}(\ell_2)$ generated by $\{T_{\xi,\eta} : \xi,\eta < \kappa\}$ and the compact operators in $\mathcal{B}(\ell_2)$.

Lemma 2.14. *Suppose that $\mathcal{T} = \{T_{\xi,\eta} : \xi,\eta < \kappa\}$ is a system of almost matrix units in $\mathcal{B}(\ell_2)$. The C^*-algebra $\mathcal{A}(\mathcal{T})$ satisfies the short exact sequence*

$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{A}(\mathcal{T}) \xrightarrow{\pi} \mathcal{K}(\ell_2(\kappa)) \to 0,$$

where $\iota[\mathcal{K}(\ell_2)]$ is an essential ideal of $\mathcal{A}(\mathcal{T})$. If κ is uncountable, then the extension is not split, i.e., there is no $\sigma : \mathcal{K}(\ell_2(\kappa)) \to \mathcal{A}(\mathcal{T})$ such that $\pi \circ \sigma$ is the identity on $\mathcal{K}(\ell_2(\kappa))$.

Proof. The map ι is the inclusion. Since $\mathcal{K}(\ell_2) \subseteq \mathcal{A}(\mathcal{T}) \subseteq \mathcal{B}(\ell_2)$ and $\mathcal{K}(\ell_2)$ is an essential ideal in $\mathcal{B}(\ell_2)$, we conclude that $\iota[\mathcal{K}(\ell_2)]$ is an essential ideal of $A(\mathcal{T})$.

The operators $\{[T_{\xi,\eta}]_{\mathcal{K}(\ell_2)} \in \mathcal{A}/\mathcal{K}(\ell_2) : \xi,\eta < \kappa\}$ generate $\mathcal{A}(\mathcal{T})/\mathcal{K}(\ell_2)$ (by the definition of $\mathcal{A}(\mathcal{T})$) and satisfy the matrix unit relations in $\mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$, that is

- $[T_{\xi,\eta}]^*_{\mathcal{K}(\ell_2)} = [T_{\eta,\xi}]_{\mathcal{K}(\ell_2)}$,
- $[T_{\beta,\alpha}]_{\mathcal{K}(\ell_2)}[T_{\eta,\xi}]_{\mathcal{K}(\ell_2)} = \delta_{\alpha,\eta}[T_{\beta,\xi}]_{\mathcal{K}(\ell_2)}$.

Thus $\mathcal{A}(\mathcal{T})/\mathcal{K}(\ell_2) \cong \mathcal{K}(\ell_2(\kappa))$ by Proposition 2.7. If κ is uncountable, then we observe that $\mathcal{K}(\ell_2(\kappa))$ can not be embedded into $\mathcal{B}(\ell_2)$ and so it can not be embedded into $\mathcal{A}(\mathcal{T})$. This follows from the fact that $\mathcal{B}(\ell_2)$ does not contain any uncountable family of pairwise orthogonal projections, while $\mathcal{K}(\ell_2(\kappa))$ clearly does, if κ is uncountable. □

We say a C^*-algebra is *Ψ-type* if it is of the form $\mathcal{A}(\mathcal{T})$ for a system of almost matrix units $\mathcal{T}$. These C^*-algebras are the natural noncommutative analogues of the Ψ-spaces in topology, which are induced by almost disjoint families (see Definition 2.6 of [15]). In topology Ψ-spaces are classical examples of separable locally compact Hausdorff scattered (every nonempty subset has a relative isolated point) spaces with the Cantor-Bendixson height two. Granting the role of isolated points to minimal projections in C^*-algebras, one can define scattered C^*-algebras. A projection p in a C^*-algebra $\mathcal{A}$ is called minimal if $p\mathcal{A}p = \mathbb{C}p$ and a C^*-algebra is scattered if every nonzero subalgebra has a minimal projection (see [13] for more on scattered C^*-algebras). Just like the scattered spaces, these algebras can be analyzed using the "Cantor-Bendixson sequences". For a C^*-algebra $\mathcal{A}$ let $\mathcal{I}^{At}(\mathcal{A})$ denote the subalgebra of $\mathcal{A}$ generated by the minimal projections of $\mathcal{A}$. The subalgebra $\mathcal{I}^{At}(\mathcal{A})$ turns out to be an ideal isomorphic to a subalgebra of all compact operators in any faithful representation of $\mathcal{A}$ and in fact is the largest ideal with

this property (Proposition 3.15 and Proposition 3.16 of [13]). A C^*-algebra $\mathcal{A}$ is scattered if and only if there is an ordinal $ht(\mathcal{A})$ and a increasing sequence of closed ideals $(\mathcal{I}_\alpha)_{\alpha\leq ht(\mathcal{A})}$ such that $\mathcal{I}_0=\{0\}$, $\mathcal{I}_{ht(\mathcal{A})}=\mathcal{A}$, if α is a limit ordinal $\mathcal{I}_\alpha=\overline{\bigcup_{\beta<\alpha}\mathcal{I}_\beta}$, and

$$\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha=\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha),$$

for every $\alpha<ht(\mathcal{A})$ (Theorem 1.4 of [13]). The sequence $(\mathcal{I}_\alpha)_{\alpha\leq ht(\mathcal{A})}$ is called the Cantor-Bendixson sequence for $\mathcal{A}$ and the ordinal $ht(\mathcal{A})$ is called the Cantor-Bendixson height or simply the height of $\mathcal{A}$.

Proposition 2.15. *Suppose that $\mathcal{T}=\{T_{\xi,\eta}:\xi,\eta<\kappa\}$ is a system of almost matrix units. The C^*-algebra $\mathcal{A}(\mathcal{T})$ is a scattered C^*-algebra of height 2. Therefore $\mathcal{A}(\mathcal{T})$ is GCR (type I) and AF, in the sense that every finite set of elements can be approximated from a finite dimensional subalgebra.*

Proof. Since $\mathcal{K}(\ell_2)\subseteq\mathcal{A}(\mathcal{T})\subseteq\mathcal{B}(\ell_2)$, by Proposition 3.21 of [13] we have $\mathcal{I}_1=\mathcal{I}^{At}(\mathcal{A}(\mathcal{T}))=\mathcal{K}(\ell_2)$ and also $\mathcal{I}_2/\mathcal{I}_1=\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}^{At}(\mathcal{A}(\mathcal{T}))\cong\mathcal{K}(\ell_2(\kappa))$ by Lemma 2.14, and therefore $\mathcal{I}_2=\mathcal{A}(\mathcal{T})$.

The composition series $(0,\mathcal{K}(\ell_2),\mathcal{A}(\mathcal{T}))$ witnesses the fact that $\mathcal{A}(\mathcal{T})$ is GCR (see IV.1.3 of [6]). Also every scattered C^*-algebra is AF (see [19], cf. [13]). □

Let us conclude this section by observing the contrast between the separable and nonseparable case for the extensions of an algebra of compact operators by compact operators.

Proposition 2.16. *Suppose that $\mathcal{B}$ is a C^*-algebra satisfying the short exact sequence*

$$0\to\mathcal{K}(\ell_2)\xrightarrow{i}\mathcal{B}\xrightarrow{q}\mathcal{K}(\ell_2)\to 0,$$

*where $i[\mathcal{K}(\ell_2)]$ is an essential ideal of $\mathcal{B}$. Then $\mathcal{B}$ is *-isomorphic to $\widetilde{\mathcal{K}(\ell_2)}\otimes\mathcal{K}(\ell_2)$.*

Proof. It is enough to show that $\mathcal{B}$ is unique up to *-isomorphism. Since $K_0(\mathcal{K}(\ell_2))$ is a free abelian group the sequence above splits (Exercise 16.4.7 of [5]). All the nonunital split essential extensions of $\mathcal{K}(\ell_2)$ by $\mathcal{K}(\ell_2)$ are equivalent and therefore isomorphic (see II.8.4.30 of [6]).

□

3. Multipliers of systems of almost matrix units

Let $\mathcal{A}$ be a nondegenerate subalgebras of $\mathcal{B}(\ell_2)$. A multiplier of (or a multiplier for) $\mathcal{A}$ is an operator T in $\mathcal{B}(\ell_2)$ such that $T\mathcal{A}\subseteq\mathcal{A}$ and $\mathcal{A}T\subseteq\mathcal{A}$. An operator T in $\mathcal{B}(\ell_2)$ is called a quasi-multiplier of $\mathcal{A}$ if $\mathcal{A}T\mathcal{A}\subseteq\mathcal{A}$. We denote the set of multipliers of $\mathcal{A}$ by $\mathcal{M}(\mathcal{A})$ and the set of all quasi-multipliers of $\mathcal{A}$ by $\mathcal{QM}(\mathcal{A})$. It is well-known that $\mathcal{QM}(\mathcal{A})$ is a norm closed *-invariant subspace of $\mathcal{A}''$ and $\mathcal{M}(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{A}''$ and of course $\mathcal{A}\subseteq\mathcal{M}(\mathcal{A})\subseteq\mathcal{QM}(\mathcal{A})$ (see 3.12 of [22]).

Lemma 3.1. *Suppose that $\mathcal{T}=\{T_{\eta,\xi}:\xi,\eta<\kappa\}\subseteq\mathcal{B}(\ell_2)$ is a system of almost matrix units. Then the following are equivalent:*

(1) $R\in\mathcal{QM}(\mathcal{A}(\mathcal{T}))$,
(2) for every $\xi,\eta<\kappa$ there is $\lambda^{\mathcal{T}}_{\xi,\eta}(R)\in\mathbb{C}$ such that

$$T_{\eta,\eta}RT_{\xi,\xi}=^{\mathcal{K}}\lambda^{\mathcal{T}}_{\eta,\xi}(R)T_{\eta,\xi}.$$

Proof. Suppose that R is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$ and $\xi, \eta < \kappa$ are given. Then $S = T_{\eta,\eta} R T_{\xi,\xi}$ is an operator in $\mathcal{A}$ satisfying $S =^{\mathcal{K}} T_{\eta,\eta} S T_{\xi,\xi}$. The only operators in $\mathcal{A}$ with this property are compact perturbations of constant multiples of $T_{\eta,\xi}$. The other implication follows immediately from the definition of $\mathcal{A}(\mathcal{T})$. □

If R is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$, let $\Lambda^{\mathcal{T}}(R)$ denote the $\kappa \times \kappa$ matrix $(\lambda^{\mathcal{T}}_{\eta,\xi}(R))_{\xi,\eta<\kappa}$ over $\mathbb{C}$. If $\mathcal{T}$ is clear from the context, we often drop the superscript $\mathcal{T}$, and write $\Lambda(R) = (\lambda_{\eta,\xi}(R))_{\xi,\eta<\kappa}$.

In the following, we use I_κ to denote the $\kappa \times \kappa$ matrix which has constant 1 on the diagonal and zero everywhere else, where κ is a cardinal. We will also use $1_{B(\ell_2)}$ to denote the unit element of $\mathcal{B}(\ell_2)$. When considering $\kappa \times \kappa$ matrices we can treat some of them as operators in $\mathcal{B}(\ell_2(\kappa))$. Namely, for a fixed (the canonical) orthonormal basis $\{e_\xi : \xi < \kappa\}$ for $\ell_2(\kappa)$, we identify operators $T_M \in \mathcal{B}(\ell_2(\kappa))$ defined by $T_M(e_\xi)(\eta) = m_{\eta,\xi}$ with the $\kappa \times \kappa$ matrix $M = (m_{\xi,\eta})$. So, for example, T_{I_κ} is the unit of $\mathcal{B}(\ell_2(\kappa))$ and T_M is compact if M is a matrix which has only finitely many nonzero entries. In particular, we will say that a matrix M is a matrix of a compact operator if T_M is compact. The operations of addition, multiplication by scalar and the transposition of $\kappa \times \kappa$ matrices should be clear.

Lemma 3.2. *Assume $\mathcal{T}$ is a system of almost matrix units of size κ and $R \in \mathcal{B}(\ell_2)$ is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$. Then $\Lambda^{\mathcal{T}}(R)$ is a matrix of a bounded linear operator on $\ell_2(\kappa)$ of norm not bigger than $\|R\|$. In particular, all rows and columns of the matrix $\Lambda^{\mathcal{T}}(R)$ are in $\ell_2(\kappa)$.*

Proof. It is enough to prove that for any finite $F \subseteq \kappa$ and for any $(c_\xi)_{\xi\in F} \subseteq \mathbb{C}$ such that $\Sigma_{\xi\in F}|c_\xi| \leq 1$ we have

$$(*) \qquad \sqrt{\Sigma_{\eta\in F}|\Sigma_{\xi\in F}\lambda_{\eta,\xi}c_\xi|^2} \leq \|R\|.$$

Using Lemma 2.13 we have a system of matrix units $(E_{\eta,\xi})_{\eta,\xi\in F}$ in $\mathcal{B}(\ell_2)$ such that $T_{\eta,\xi} =^{\mathcal{K}} E_{\eta,\xi}$ for every $\xi, \eta \in F$. It follows that

$$E_{\eta,\eta} R E_{\xi,\xi} = \lambda_{\eta,\xi}(R) E_{\eta,\xi} + S_{\eta,\xi},$$

where $S_{\eta,\xi}$ is a compact operator, for each $\xi, \eta \in F$. For a given $\epsilon > 0$ we will find a norm one vector $w \in \ell_2$ such that $\|R(w)\|^2 \geq \Sigma_{\eta\in F}|\Sigma_{\xi\in F}\lambda_{\eta,\xi}c_\xi|^2 - \varepsilon$, which will prove $(*)$.

By considering an infinite orthonormal basis in the ranges of each $E_{\xi,\xi}$ for $\xi \in F$ and using Lemma 2.1 (2) we can find norm 1 vectors w_ξ in the ranges of $E_{\xi,\xi}$, respectively, such that

$$\Sigma_{\eta\in F}\Sigma_{\xi\in F}|c_\xi|^2\|S_{\eta,\xi}(w_\xi)\|^2 < \varepsilon,$$

and $w_\eta = E_{\eta,\xi}(w_\xi)$ for $\xi, \eta \in F$. The last statement follows from the fact that $E_{\eta,\xi}$s are partial isometries, so all the orthonormal bases may be considered to be the images of a fixed orthonormal basis in $E_{\eta,\eta}$.

So by the pairwise orthogonality of $E_{\xi,\xi}$s for $\xi \in F$ and by the Pythagorean theorem we have

$$\|R(\Sigma_{\xi\in F}c_\xi w_\xi)\|^2 \geq \Sigma_{\eta\in F}|\Sigma_{\xi\in F}\lambda_{\eta,\xi}c_\xi|^2 - \varepsilon,$$

which completes the proof. □

In particular by Lemma 3.2 all columns and rows can have at most countably many nonzero entries. Therefore if κ is an uncountable cardinal and R is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$, then for every $\xi < \kappa$ there is $\eta < \kappa$ such that $\lambda^{\mathcal{T}}_{\eta,\xi}(R) = 0$.

Lemma 3.3. *Assume $\mathcal{T}$ is a system of almost matrix units of size κ. The map $\Lambda^{\mathcal{T}}$ from $\mathcal{QM}(\mathcal{A}(\mathcal{T}))$ into $\mathcal{B}(\ell_2(\kappa))$ is a norm one linear operator such that $\Lambda^{\mathcal{T}}(R^*) = \Lambda^{\mathcal{T}}(R)^*$ for every quasi-multiplier R of $\mathcal{A}(\mathcal{T})$.*

Proof. The linearity of $\Lambda^{\mathcal{T}}$ is immediate. The fact that $\|\Lambda^{\mathcal{T}}\| \leq 1$ follows from Lemma 3.2. For the last part, note that $T_{\eta,\eta}R^*T_{\xi,\xi} = (T_{\xi,\xi}RT_{\eta,\eta})^* = (\lambda^{\mathcal{T}}_{\xi,\eta}(R)T_{\xi,\eta})^* = \overline{\lambda^{\mathcal{T}}_{\xi,\eta}(R)}T_{\eta,\xi}$, and therefore $\lambda^{\mathcal{T}}_{\eta,\xi}(R^*) = \overline{\lambda^{\mathcal{T}}_{\xi,\eta}(R)}$. □

Lemma 3.4. *Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2)$ is a system of almost matrix units. For every $\kappa \times \kappa$ matrix $(\lambda_{\eta,\xi})_{\xi,\eta<\kappa}$ of a compact operator on $\ell_2(\kappa)$ there is $R \in \mathcal{A}(\mathcal{T})$ such that $\lambda_{\eta,\xi}(R) = \lambda_{\eta,\xi}$ for every $\xi, \eta < \kappa$ (see Definition 3.1).*

Proof. Since $(\lambda_{\eta,\xi})_{\xi,\eta<\kappa}$ is a matrix of a compact operator on $\mathcal{B}(\ell_2(\kappa))$ (denote this operator by S), there is a countable $A \subseteq \kappa$ such that $\lambda_{\xi,\eta} = 0$ if $(\xi, \eta) \notin A \times A$. This follows from the fact that the image of the unit ball under a compact operator is compact and metrizable, and hence separable which implies that the matrix of the operator must have at most countably many nonzero rows. Now since each row of the matrix $(\lambda_{\eta,\xi})_{\xi,\eta<\kappa}$ belongs to $\ell_2(\kappa)$, there can be at most countably many nonzero entries.

Apply Lemma 2.13 to obtain a system of matrix units $(E_{\eta,\xi})_{\xi,\eta\in A}$ in $\mathcal{B}(\ell_2)$ such that $E_{\eta,\xi} =^{\mathcal{K}} T_{\eta,\xi}$ for $\xi, \eta \in A$. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite sets such that $\bigcup_{n\in\mathbb{N}} F_n = A$. Let $\mathcal{B}_n$ be the subalgebra of $(\Sigma_{\xi\in F_n} E_{\xi,\xi})\mathcal{B}(\ell_2)(\Sigma_{\xi\in F_n} E_{\xi,\xi})$ of all operators of the form

$$(*) \qquad \Sigma_{\xi,\eta\in F_n} \alpha_{\eta,\xi} E_{\eta,\xi},$$

where $\alpha_{\eta,\xi} \in \mathbb{C}$ for every $\xi, \eta \in F_n$. From the matrix unit relations and Proposition 2.7 it follows that $\mathcal{B}_n$ is $*$-isomorphic to the algebra of $|F_n|\times|F_n|$ matrices. Therefore the norm of the operator as in $(*)$ is equal to the matrix norm of the matrix $(\alpha_{\eta,\xi})_{\xi,\eta}$. The norm of this operator in $\mathcal{B}(\ell_2))$ is the same.

Let $S_n \in \mathcal{B}(\ell_2(\kappa))$ be given by $S_n = P_{F_n} S P_{F_n}$, where P_X is the orthogonal projection from $\ell_2(\kappa)$ onto $\ell_2(X)$ for $X \subseteq \kappa$. Then since $(\lambda_{\eta,\xi})_{\xi,\eta}$ is a matrix of a compact operator S, the operators S_n converge in the norm to S. Consider the operators

$$R_n = \Sigma_{\xi,\eta\in F_n} \lambda_{\eta,\xi} E_{\eta,\xi}.$$

By the above comments about the norms of operators in $\mathcal{B}_n$ we conclude that $\|R_n - R_m\| = \|S_n - S_m\|$ for every $n, m \in \mathbb{N}$, and therefore $(R_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathcal{B}(\ell_2)$ with all elements in $\mathcal{A}(\mathcal{T})$ and hence converges to some operator $R \in \mathcal{A}(\mathcal{T})$. Since for every $\xi, \eta \in A$, there is large enough $n \in \mathbb{N}$ such that $E_{\eta,\eta}R_nE_{\xi,\xi} = \lambda_{\eta,\xi}E_{\eta,\xi}$, we conclude that $T_{\eta,\eta}RT_{\xi,\xi} =^{\mathcal{K}} E_{\eta,\eta}RE_{\xi,\xi} = \lambda_{\eta,\xi}E_{\eta,\xi}$ for all $\xi, \eta \in A$. So $\lambda^{\mathcal{T}}_{\eta,\xi}(R) = \lambda_{\eta,\xi}$ for all $\xi, \eta \in A$. On the other hand if $(\xi, \eta) \notin A \times A$, then $T_{\xi,\xi}R_nT_{\eta,\eta} \in \mathcal{K}(\ell_2)$ for all $n \in \mathbb{N}$ as $\{T_{\xi,\xi} : \xi \notin A\} \cup \{E_{\xi,\xi} : \xi \in A\}$ is still almost orthogonal, and thus $\lambda^{\mathcal{T}}_{\eta,\xi}(R) = 0$. It follows that $\Lambda^{\mathcal{T}}(R) = (\lambda_{\eta,\xi})_{\xi,\eta<\kappa}$ as required. □

The key to the proof of our main theorem is to characterize each quasi-multiplier R of $\mathcal{A}(\mathcal{T})$ based on how "complex" the matrix $\Lambda(R)$ is. This is captured in the following definition.

Definition 3.5. *Assume that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta < \kappa\}$ is a system of almost matrix units, and $R \in \mathcal{QM}(\mathcal{A}(\mathcal{T}))$. We say*

- *R is a trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$, if $\Lambda^{\mathcal{T}}(R) = \lambda I_\kappa + M$, for a matrix of a compact operator M on $\mathcal{B}(\ell_2(\kappa))$ and $\lambda \in \mathbb{C}$,*
- *R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$, if $\Lambda^{\mathcal{T}}(R) = \lambda I_\kappa + M$, for a $\kappa \times \kappa$ matrix M with at most countably many nonzero entries and some $\lambda \in \mathbb{C}$,*
- *R is a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$, if $\Lambda^{\mathcal{T}}(R) = \lambda I_\kappa + M$, for a $\kappa \times \kappa$ matrix M with less than continuum many nonzero entries and some $\lambda \in \mathbb{C}$.*

Lemma 3.6. *Assume that $\mathcal{T}$ is a maximal system of almost matrix units of size κ. Given two quasi-multipliers R, R' of $\mathcal{A}(\mathcal{T})$, if $\Lambda(R) = \Lambda(R')$, then $R =^{\mathcal{K}} R'$. In particular,*

(1) if $\Lambda(R)$ is a compact $\kappa \times \kappa$-matrix, then $R \in \mathcal{A}(\mathcal{T})$,
(2) if $\Lambda(R) = \lambda I_\kappa$, for some $\lambda \in \mathbb{C}$, then $R =^{\mathcal{K}} \lambda 1_{\mathcal{B}(\ell_2)}$,
(3) If R is a trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$, then $R \in \widetilde{\mathcal{A}(\mathcal{T})}$.

Proof. Suppose that $R - R'$ is not compact. Then by Lemma 2.6, there are $\xi, \eta < \kappa$ such that $T_{\xi,\xi}(R - R')T_{\eta,\eta}$ is noncompact, and hence by Lemma 3.3 we have that $\lambda_{\xi,\eta}(R - R') = \lambda_{\xi,\eta}(R) - \lambda_{\xi,\eta}(R') \neq 0$.

(1) Suppose that $R \in \mathcal{QM}(\mathcal{A}(\mathcal{T}))$ is such that $(\lambda_{\xi,\eta}(R))_{\xi,\eta<\kappa}$ is a matrix of a compact operator on $\ell_2(\kappa)$. By Lemma 3.4 we obtain $R' \in \mathcal{A}(\mathcal{T})$ such that $\lambda_{\xi,\eta}(R) = \lambda_{\xi,\eta}(R')$ for every $\xi, \eta < \kappa$. By the first part of the lemma we conclude that $R - R'$ is compact, and therefore $R \in A(\mathcal{T})$, since $A(\mathcal{T})$ includes all compact operators.

(2) Note that $\lambda 1_{\mathcal{B}(\ell_2)}$ is clearly a quasi-multiplier of $\mathcal{A}(\mathcal{T})$, and $\Lambda(\lambda 1_{\mathcal{B}(\ell_2)})$ is the matrix λI_κ. Now use the first part of the lemma to conclude the statement.

(3) Suppose $\Lambda^{\mathcal{T}}(R) = \lambda I_\kappa + M$, for a matrix of a compact operator M and $\lambda \in \mathbb{C}$. By Lemma 3.4 there is $R' \in \mathcal{A}(\mathcal{T})$ such that $\Lambda(R') = M$. Then $\Lambda(R - R') = \lambda I_\kappa$ and by (2) we have $R - R' =^{\mathcal{K}} \lambda I_{\mathcal{B}(\ell_2)}$. Therefore $R \in \widetilde{\mathcal{A}(\mathcal{T})}$. □

Lemma 3.7. *Assume that κ is a cardinal and $\mathcal{T} = \{T_{\xi,\eta} : \xi, \eta < \kappa\}$ is a maximal system of almost matrix units and $R \in \mathcal{QM}(\mathcal{A}(\mathcal{T}))$. If $\Lambda(R)$ has at most countably nonzero entries, then it is a $\kappa \times \kappa$ matrix of a compact operator.*

Proof. Assume $\Lambda(R) = (\lambda_{\eta,\xi})_{\xi,\eta<\kappa}$. By re-enumerating the $T_{\xi,\eta}$s we may assume that if $\lambda_{\xi,\eta} \neq 0$, then $\xi, \eta < \omega$. Also by Lemma 2.4 we may assume that $\{T_{\xi,\xi} : \xi < \omega\}$ are pairwise orthogonal.

By Lemma 3.2, $\Lambda(R)$ is a matrix of a bounded linear operator on $\ell_2(\kappa)$. Suppose that $\Lambda(R)$ is a matrix of a noncompact operator on $\ell_2(\kappa)$. Aiming at a contradiction, we will construct a projection P such that $T_{\xi,\xi}RP$ is compact for all $\xi < \omega$ but RP is noncompact. By the argument similar to Lemma 2.6 this will give an ordinal $\xi_0 < \kappa$ such that $T_{\xi_0,\xi_0}RP$ is noncompact, which by the assumption implies that $\omega \leq \xi_0 < \kappa$. Then $T_{\xi_0,\xi_0}R$ is also noncompact, so again there is $\eta_0 < \kappa$ such that $T_{\xi_0,\xi_0}RT_{\eta_0,\eta_0}$ is noncompact, which means that $\lambda_{\xi_0,\eta_0} \neq 0$ and $(\xi_0, \eta_0) \notin \omega \times \omega$, contradicting the hypothesis of the lemma.

To construct P we will construct its range spanned by its orthonormal basis $(v_k : k \in \omega)$. It will be enough to choose the vectors v_k in such a way that for each $\xi < \omega$ we have that $\|T_{\xi,\xi}R(v_k)\| \leq 1/2^k$ for all $k \in [\xi, \omega)$ and $\|R(v_k)\|$ does not converge to 0 when $k \to \infty$. Then as $P(v_k) = v_k$ and $(v_k : k \in \mathbb{N})$ can be extended

to an orthonormal basis of ℓ_2, we will obtain from Lemma 2.1 (1) that $T_{\xi,\xi}RP$ is compact for every $\xi < \omega$ and from Lemma 2.1 (2) that RP is noncompact.

Using the fact that $\Lambda(R)$ is a matrix of a bounded linear operator (Lemma 3.2) which is not compact, by induction on $k \in \mathbb{N}$ (using Lemma 2.1 (3)), we can construct finite pairwise disjoint $F_k \subseteq \mathbb{N} = \omega$ and $(a_n)_{n\in F_k}$ such that for some $\varepsilon > 0$,

(1) $\Sigma_{n\in F_k}|a_n|^2 \leq 1$,
(2) $\|\Lambda(R)(\Sigma_{n\in F_k}a_n\chi_{\{n\}})\| = \sqrt{\Sigma_{\xi\in\kappa}|\Sigma_{n\in F_k}a_n\lambda_{\xi,n}|^2} > \varepsilon$,
(3) $|\Sigma_{n\in F_k}a_n\lambda_{\xi,n}|^2 \leq \Sigma_{n\in F_k}|\lambda_{\xi,n}|^2 \leq 1/2^k$ for all $\xi < k$.

The condition (3) follows from (1) and the fact that the rows of $\Lambda(R)$ are in $\ell_2(\kappa)$ (Lemma 3.2). Using the compactness of the operators $T_{\xi,\xi}RT_{n,n} - \lambda_{\xi,n}T_{\xi,n}$, we find $w_n \in Im(T_{n,n})$ of norm one such that $\|\Sigma_{\xi<\omega}\Sigma_{n\in F_k}(T_{\xi,\xi}RT_{n,n} - \lambda_{\xi,n}T_{\xi,n})(w_n)\| < \varepsilon/2$. Putting $v_k = \Sigma_{n\in F_k}a_nw_n$ we obtain

(a) $\|R(v_k)\| \geq \|\Sigma_{\xi\in\omega}\Sigma_{n\in F_k}T_{\xi,\xi}RT_{n,n}(a_nw_n)\| \geq \sqrt{\Sigma_{\xi\in\kappa}|\Sigma_{n\in F_k}a_n\lambda_{\xi,n}|^2} - \varepsilon/2 > \varepsilon/2$,
(b) $\|T_{\xi,\xi}R(v_k)\| = \|\Sigma_{n\in F_k}T_{\xi,\xi}RT_{n,n}(v_k)\| \leq 1/2^k$, for all $\xi < k$.

As noted before this is sufficient to obtain a contradiction from the conjunction of the hypothesis that $\Lambda(R)$ is a matrix of a noncompact operator and the set of its nonzero entries is countable. □

It was noted by the referee that the proof of above lemma can be simplified using the countable degree-1 saturation of the Calkin algebra (see [11]).

Corollary 3.8. *Assume that κ is a cardinal and $\mathcal{T} = \{T_{\xi,\eta} : \xi,\eta < \kappa\}$ is a maximal system of almost matrix units and $R \in \mathcal{QM}(\mathcal{A}(\mathcal{T}))$. If R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$, then R is a trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$.*

Proof. Assume that $\Lambda^{\mathcal{T}}(R)$ is of the form $\lambda I_\kappa + M$ where $\lambda \in \mathbb{C}$ and M is a matrix with countably many nonzero entries. Therefore $\Lambda^{\mathcal{T}}(R - \lambda 1_{\mathcal{B}(\ell_2)})$ is M, which has at most countably many nonzero entries, which by Lemma 3.7 means that M is a matrix of a compact operator and so R is a trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$. □

4. The Cantor tree system of almost matrix units

Let $2^{<\mathbb{N}}$ be the set of all maps $s : \{0, ..., n\} \to \{0, 1\}$ for $n \in \mathbb{N}$ or $s = \emptyset$ and by $2^{\mathbb{N}}$ denote the Cantor space, the space of all maps $\xi : \mathbb{N} \to \{0, 1\}$, equipped with the product topology. For each $\xi \in 2^{\mathbb{N}}$ we can associate a set

$$A_\xi = \{s \in 2^{<\mathbb{N}} : s \subseteq \xi\},$$

which is usually called the "branch through ξ". It is easy to see that $\{A_\xi : \xi \in 2^{\mathbb{N}}\}$ is an almost disjoint family of subsets of $2^{<\mathbb{N}}$ of size continuum. In this section $\mathcal{H}$ denotes the separable Hilbert space $\ell_2(2^{<\mathbb{N}})$. For each $\xi \in 2^{\mathbb{N}}$ define a projection $T_{\xi,\xi} \in \mathcal{B}(\mathcal{H})$ by

$$T_{\xi,\xi}(x)(s) = \begin{cases} x(s) & \text{if } s \in A_\xi, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in \mathcal{H}$ and $s \in 2^{<\mathbb{N}}$. Then $\mathcal{P}_{2^{\mathbb{N}}} = \{T_{\xi,\xi} : \xi \in 2^{\mathbb{N}}\}$ is a family of almost orthogonal projections in $\mathcal{B}(\mathcal{H})$.

Let $\{e_s : s \in 2^{<\mathbb{N}}\}$ be the canonical orthonormal basis for $\mathcal{H}$, i.e., $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$, otherwise. For every $\xi, \eta \in 2^{\mathbb{N}}$, define a linear bounded operator

$T_{\eta,\xi} : \mathcal{H} \to \mathcal{H}$ by $T_{\eta,\xi}(e_{\xi|k}) = e_{\eta|k}$ for every $k \in \mathbb{N}$ and $T_{\eta,\xi}(e_t) = 0$ if t is not equal to $\xi|k$ for any $k \in \mathbb{N}$. It is easy to see that $\mathcal{T}_{2^{\mathbb{N}}} = \{T_{\eta,\xi} : \xi, \eta \in 2^{\mathbb{N}}\}$ is a system of almost matrix units (Definition 2.8) based on $\mathcal{P}_{2^{\mathbb{N}}}$ (Definition 2.9). We will call this system of almost matrix units "the Cantor tree system of matrix units". In the rest of this section the operators $T_{\eta,\xi}$ will always refer to the members of $\mathcal{T}_{2^{\mathbb{N}}}$.

Recall that a family $\mathcal{F}$ of subsets of a Polish space (a separable completely metrizable space) is said to have the *perfect set property*, if every uncountable element of $\mathcal{F}$ has a perfect subset. In particular every uncountable element of $\mathcal{F}$ must have cardinality continuum. In the following lemma we use the fact that the family of Borel sets has the perfect set property (e.g., 13.6 of [17]).

Lemma 4.1. *Assume $R \in \mathcal{B}(\mathcal{H})$ is a quasi-multiplier of $\mathcal{A}(\mathcal{T}_{2^{\mathbb{N}}})$ and U is a Borel subset of $\mathbb{C}$, then the set*

$$B_U^R = \{(\eta, \xi) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : \lambda_{\eta,\xi}^{\mathcal{T}_{2^{\mathbb{N}}}}(R) \in U\}$$

is Borel in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. In particular, B_U^R is either countable or of size continuum.

Proof. Let $\lambda_{\eta,\xi} = \lambda_{\eta,\xi}^{\mathcal{T}_{2^{\mathbb{N}}}}(R)$ for every $\xi, \eta \in 2^{\mathbb{N}}$.

Claim. $\psi_n : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{C}$ defined by

$$\psi_n(\eta, \xi) = \left\langle R(e_{\eta|_n}), e_{\xi|_n} \right\rangle$$

is a continuous function, for every $n \in \mathbb{N}$.

Proof of the Claim. Fix $n \in \mathbb{N}$. For $s, t \in 2^n$ let $O_{s,t}$ denote the clopen set $\{(\eta, \xi) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : s \subseteq \xi \ \& \ t \subseteq \eta\}$. Note that ψ_n is constant on $O_{s,t}$, for every $(s, t) \in 2^n \times 2^n$. In fact, $\psi_n(\eta, \xi) = \langle R(e_s), e_t \rangle$ for every $(\eta, \xi) \in O_{s,t}$. Since $2^{\mathbb{N}} \times 2^{\mathbb{N}} = \bigcup_{s,t \in 2^n} O_{s,t}$, the range of ψ_n is finite, and it is continuous, which completes the proof of the claim.

For each $\xi, \eta \in 2^{\mathbb{N}}$, since $W_{\eta,\xi} = \lambda_{\eta,\xi} T_{\eta,\xi} - T_{\eta,\eta} R T_{\xi,\xi}$ is a compact operator in $\mathcal{B}(\mathcal{H})$, we have $\lim_{n\to\infty} \|W_{\eta,\xi}(e_{\xi|_n})\| = 0$ (see Lemma 2.1 (2)), and therefore

$$\lim_{n\to\infty} |\langle W_{\eta,\xi}(e_{\xi|_n}), e_{\eta|_n} \rangle| = 0.$$

This means that

$$\left\langle T_{\eta,\eta} R T_{\xi,\xi}(e_{\xi|_n}), e_{\eta|_n} \right\rangle \to \left\langle \lambda_{\eta,\xi} T_{\eta,\xi}(e_{\xi|_n}), e_{\eta|_n} \right\rangle = \lambda_{\eta,\xi} \left\langle e_{\eta|_n}, e_{\eta|_n} \right\rangle = \lambda_{\eta,\xi}.$$

Thus, for each $\xi, \eta \in 2^{\mathbb{N}}$

$$\psi_n(\eta, \xi) = \left\langle R(e_{\eta|_n}), e_{\xi|_n} \right\rangle = \left\langle T_{\eta,\eta} R T_{\xi,\xi}(e_{\xi|_n}), e_{\eta|_n} \right\rangle,$$

converges to $\lambda_{\eta,\xi}$. So the map $\psi : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{C}$ given by $\psi(\eta, \xi) = \lambda_{\eta,\xi}$ is the pointwise limit of continuous functions ψ_n for $n \in \mathbb{N}$, hence it is Borel (Ex. 11.2 (i) [17]), which means $B_U^R = \psi^{-1}[U]$ is Borel.

□

Corollary 4.2. *If R is a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T}_{2^{\mathbb{N}}})$, then R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T}_{2^{\mathbb{N}}})$.*

Proof. Suppose that $\Lambda^{\mathcal{T}_{2^{\mathbb{N}}}}(R) = \lambda I_{2^{\mathbb{N}}} + M$ where $\lambda \in \mathbb{C}$ and M has less then continuum nonzero entries. Note that $\Lambda^{\mathcal{T}_{2^{\mathbb{N}}}}(R - \lambda 1_{\mathcal{B}(\mathcal{H})}) = \lambda I_{2^{\mathbb{N}}} + M - \lambda I_{2^{\mathbb{N}}} = M$ by Lemma 3.3. So $B_{\mathbb{C}\setminus\{0\}}^{R-\lambda 1_{\mathcal{B}(\mathcal{H})}}$ is a Borel subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ (Lemma 4.1), and of cardinality less then $\mathfrak{c}$. Therefore the perfect set property for Borel sets, implies that $B_{\mathbb{C}\setminus\{0\}}^{R-\lambda 1_{\mathcal{B}(\mathcal{H})}}$ is countable, so M has at most countably many nonzero entries, which means that R is σ-trivial. □

5. Pairing systems of almost matrix units

In this section we introduce a method of eliminating nontrivial quasi-multipliers of $\mathcal{A}(\mathcal{T})$ by pairing the elements of $\mathcal{T}$ into a new system of almost matrix units.

Definition 5.1. *Let X be a set which is partitioned into two subsets of the same cardinality, $X = Y \cup (X \setminus Y)$ and suppose that $\rho : Y \to (X \setminus Y)$ is a bijection. Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ is a system of almost matrix units in $\mathcal{B}(\ell_2)$. We say $\mathcal{U} = \{U_{\eta,\xi} : \xi, \eta \in Y\}$ is a pairing of $\mathcal{T}$ along ρ if and only if for every $\xi, \eta \in Y$ the following holds:*

$$U_{\eta,\xi} =^{\mathcal{K}} T_{\eta,\xi} + T_{\rho(\eta),\rho(\xi)}.$$

Proposition 5.2. *Let X, Y, ρ and $\mathcal{T}$ be as above. Then any pairing $\mathcal{U}$ of $\mathcal{T}$ along ρ is a system of almost matrix units. If $\mathcal{T}$ is maximal, then $\mathcal{U}$ is also maximal.*

Proof. Let $\mathcal{U} = \{U_{\eta,\xi} : \xi, \eta \in Y\}$ be a pairing of $\mathcal{T}$ along ρ. Then for every $\xi, \eta \in Y$ we have

$$U_{\eta,\xi} =^{\mathcal{K}} T_{\eta,\xi} + T_{\rho(\eta),\rho(\xi)}.$$

We check that $\mathcal{U} = \{U_{\eta,\xi} : \xi, \eta \in Y\}$ is a system of almost matrix units: $(U_{\eta,\xi})^* =^{\mathcal{K}} (T_{\eta,\xi} + T_{\rho(\eta),\rho(\xi)})^* =^{\mathcal{K}} T_{\xi,\eta} + T_{\rho(\xi),\rho(\eta)} =^{\mathcal{K}} U_{\xi,\eta}$ for all $\xi, \eta \in Y$. For all $\alpha, \beta, \xi, \eta \in Y$, since $Y \cap \rho[Y] = \emptyset$ and ρ is a bijection, a straightforward calculation show that

$$U_{\beta,\alpha} U_{\eta,\xi} =^{\mathcal{K}} \delta_{\alpha,\eta} U_{\beta,\xi}.$$

Now suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ is a maximal system of almost matrix units, that is, there is a maximal family $\{P_\xi : \xi \in X\}$ of almost orthogonal projections (see Definition 2.9) such that $T_{\xi,\xi} =^{\mathcal{K}} P_\xi$ for each $\xi \in X$. We will show that $\mathcal{U}$ is also a maximal system of almost matrix units. We need to produce a maximal family $\mathcal{Q} = \{Q_\xi : \xi \in Y\}$ of almost orthogonal projections, such that $\mathcal{U}$ is based on $\mathcal{Q}$.

Using Lemma 2.4 for each pair $s = \{\xi, \rho(\xi)\}$, separately for every $\xi \in Y$ find orthogonal projections $P^s_\xi, P^s_{\rho(\xi)} \in \mathcal{B}(\ell_2)$ such that $P^s_\xi =^{\mathcal{K}} P_\xi$ and $P^s_{\rho(\xi)} =^{\mathcal{K}} P_{\rho(\xi)}$ and $P^s_\xi P^s_{\rho(\xi)} = 0$. For each $\xi \in Y$ define $Q_\xi = P^s_\xi + P^s_{\rho(\xi)}$ for $s = \{\xi, \rho(\xi)\}$, which is a projection as it is the sum of two orthogonal projections and moreover

$$(*) \qquad Q_\xi =^{\mathcal{K}} T_{\xi,\xi} + T_{\rho(\xi),\rho(\xi)} =^{\mathcal{K}} U_{\xi,\xi}.$$

It remains to prove that $\mathcal{Q}$ is a maximal family of almost orthogonal projections. Suppose that P is a projection in $\mathcal{B}(\ell_2)$. By the maximality of $\{P_\xi : \xi \in X\}$, there is $\alpha \in X$ such that $P_\alpha P$ is not a compact operator. Let $\xi \in Y$ be such that $\alpha \in \{\xi, \rho(\xi)\}$, so we have

$$T_{\alpha,\alpha} U_{\xi,\xi} =^{\mathcal{K}} T_{\alpha,\alpha}(T_{\xi,\xi} + T_{\rho(\xi),\rho(\xi)}) =^{\mathcal{K}} T_{\alpha,\alpha},$$

by Definition 2.8 (2), as the domain and the range of ρ are disjoint. Therefore $T_{\alpha,\alpha} U_{\xi,\xi} P =^{\mathcal{K}} T_{\alpha,\alpha} P =^{\mathcal{K}} P_\alpha P$. Thus $T_{\alpha,\alpha} U_{\xi,\xi} P$ and consequently by $(*)$ $U_{\xi,\xi} P =^{\mathcal{K}} Q_\xi P$ are noncompact, which shows that $\mathcal{U}$ is maximal as well. □

Lemma 5.3. *Suppose X, Y and ρ are as in Definition 5.1. Let $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ be a system of almost matrix units and $\mathcal{U}$ be a pairing of $\mathcal{T}$ along ρ. Suppose that $R \in \mathcal{B}(\ell_2)$ is a quasi-multiplier for $\mathcal{A}(\mathcal{U})$. Then R is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$ and*

$$\lambda^{\mathcal{U}}_{\eta,\xi}(R) = \lambda^{\mathcal{T}}_{\eta,\xi}(R) = \lambda^{\mathcal{T}}_{\rho(\eta),\rho(\xi)}(R),$$

$$\lambda^{\mathcal{T}}_{\eta,\rho(\xi)}(R) = \lambda^{\mathcal{T}}_{\rho(\eta),\xi}(R) = 0$$

for each $\xi, \eta \in Y$.

Proof. We have $T_{\xi,\xi}U_{\xi,\xi} =^{\mathcal{K}} T_{\xi,\xi}(T_{\xi,\xi} + T_{\rho(\xi),\rho(\xi)}) =^{\mathcal{K}} T_{\xi,\xi}$, as $Y \cap \rho[Y] = \emptyset$ and by the almost matrix units relations. Similarly $U_{\xi,\xi}T_{\xi,\xi} =^{\mathcal{K}} T_{\xi,\xi}$. Then

$$T_{\eta,\eta}RT_{\xi,\xi} =^{\mathcal{K}} \lambda^{\mathcal{U}}_{\eta,\xi}(R)T_{\eta,\xi},$$

again since $Y \cap \rho[Y] = \emptyset$.

Similarly $U_{\xi,\xi}T_{\rho(\xi),\rho(\xi)} =^{\mathcal{K}} T_{\rho(\xi),\rho(\xi)}$ and $T_{\rho(\xi),\rho(\xi)}U_{\xi,\xi} =^{\mathcal{K}} T_{\rho(\xi),\rho(\xi)}$ and so $T_{\rho(\eta),\rho(\eta)}RT_{\rho(\xi),\rho(\xi)} =^{\mathcal{K}} \lambda^{\mathcal{U}}_{\eta,\xi}(R)T_{\rho(\eta),\rho(\xi)}$.

To prove the second part of the lemma note that $\lambda^{\mathcal{U}}_{\eta,\xi}(R)(T_{\eta,\xi} + T_{\rho(\eta),\rho(\xi)}) =^{\mathcal{K}} \lambda^{\mathcal{U}}_{\eta,\xi}(R)U_{\eta,\xi} =^{\mathcal{K}} U_{\eta,\eta}RU_{\xi,\xi} =^{\mathcal{K}} (T_{\eta,\eta}+T_{\rho(\eta),\rho(\eta)})R(T_{\xi,\xi}+T_{\rho(\xi),\rho(\xi)}) =^{\mathcal{K}} \lambda^{\mathcal{T}}_{\eta,\xi}(R)T_{\eta,\xi}+ \lambda^{\mathcal{T}}_{\eta,\rho(\xi)}(R)T_{\eta,\rho(\xi)} + \lambda^{\mathcal{T}}_{\rho(\eta),\xi}(R)T_{\rho(\eta),\xi} + \lambda^{\mathcal{T}}_{\rho(\eta),\rho(\xi)}(R)T_{\rho(\eta),\rho(\xi)}$. Multiplying the above equalities by $T_{\eta,\eta}$ from the left and $T_{\rho(\xi),\rho(\xi)}$ from the right, using 2.8 (2) and the fact that $\rho(\eta) \neq \eta \neq \rho(\xi) \neq \xi$ we obtain that $0 =^{\mathcal{K}} \lambda^{\mathcal{T}}_{\eta,\rho(\xi)}(R)T_{\eta,\rho(\xi)}$. Since $T_{\eta,\rho(\xi)}$ is noncompact, it follows that $\lambda^{\mathcal{T}}_{\eta,\rho(\xi)}(R) = 0$. We obtain $\lambda^{\mathcal{T}}_{\rho(\eta),\xi}(R) = 0$ in a similar way multiplying the above equalities by $T_{\xi,\xi}$ from the right and $T_{\rho(\eta),\rho(\eta)}$ from the left. □

Lemma 5.4. *Suppose* X, Y *and* ρ *are as in Definition 5.1. Let* $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ *be a system of almost matrix units and* $\mathcal{U}$ *be a pairing of* $\mathcal{T}$ *along* ρ. *If* $R \in \mathcal{B}(\ell_2)$ *is a* σ*-trivial (*$\mathfrak{c}$*-trivial) quasi-multiplier for* $\mathcal{A}(\mathcal{U})$, *then* R *is a* σ*-trivial (*$\mathfrak{c}$*-trivial) quasi-multiplier of* $\mathcal{A}(\mathcal{T})$.

Proof. The $X \times X$ matrix $\Lambda^{\mathcal{T}}(R)$ consists of four blocks $Y \times Y$, $(X \setminus Y) \times Y$, $Y \times (X \setminus Y)$ and $(X \setminus Y) \times (X \setminus Y)$. Lemma 5.3 implies that the $Y \times Y$-block is the matrix $\Lambda^{\mathcal{U}}(R)$, that $(X \setminus Y) \times (X \setminus Y)$-block is a copy of the $Y \times Y$-block and the remaining blocks have only zero entries. This clearly implies the lemma. □

Lemma 5.5. *Suppose that* $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ *is a system of almost matrix units where* X *is of size continuum. Then there are* $Y \subseteq X$ *and a bijection* $\rho : Y \to (X \setminus Y)$ *such that for every pairing* $\mathcal{U}$ *of* $\mathcal{T}$ *along* ρ, *whenever* R *is a quasi-multiplier of* $\mathcal{A}(\mathcal{U})$, *then* R *is a* $\mathfrak{c}$*-trivial quasi-multiplier of* $\mathcal{A}(\mathcal{T})$.

Proof. We may assume that $X = \mathfrak{c}$. Let $(R_\xi)_{\xi<\mathfrak{c}}$ be an enumeration (with possible repetitions) of all quasi-multipliers of $\mathcal{A}(\mathcal{T})$ which are not $\mathfrak{c}$-trivial. By induction on $\alpha < \mathfrak{c}$ we construct distinct $\beta^i_\alpha, \gamma^i_\alpha \in \mathfrak{c}$ for $i = 1, 2, 3$ such that $\{\beta^i_\alpha, \gamma^i_\alpha : i \in \{1, 2, 3\}\}$ has six distinct elements for each $\alpha < \mathfrak{c}$ and such that either

(1) $\lambda^{\mathcal{T}}_{\beta^1_\alpha,\beta^2_\alpha}(R_\alpha) \neq 0$ and $\lambda^{\mathcal{T}}_{\gamma^1_\alpha,\gamma^2_\alpha}(R_\alpha) = 0$, or
(2) $\lambda^{\mathcal{T}}_{\beta^3_\alpha,\beta^3_\alpha}(R_\alpha) \neq \lambda^{\mathcal{T}}_{\gamma^3_\alpha,\gamma^3_\alpha}(R_\alpha)$,

and moreover $\{\beta^i_\alpha, \gamma^i_\alpha : i \in \{1, 2, 3\}, \alpha < \mathfrak{c}\} = \mathfrak{c}$.

At stage $\alpha < \mathfrak{c}$ consider the set $A_\alpha = \{\beta^i_\delta, \gamma^i_\delta : i \in \{1, 2, 3\}, \delta < \alpha\}$. Before defining $\{\beta^i_\alpha, \gamma^i_\alpha : i \in \{1, 2, 3\}$ we will identify the reason why a quasi-multiplier R_α is not a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$. If it is because $\Lambda^{\mathcal{T}}(R_\alpha)$ has continuum nonzero entries off the diagonal, then we find such an entry $\lambda^{\mathcal{T}}_{\eta,\xi}(R_\alpha)$ with distinct $\xi, \eta \notin A_\alpha$. This can be achieved because by Lemma 3.2 the cardinality of the set of all nonzero entries $\lambda_{\xi,\eta}$ with $\xi, \eta \in A_\alpha$ is less than $\mathfrak{c}$ and we have assumed that $\Lambda^{\mathcal{T}}(R_\alpha)$ has continuum nonzero entries off the diagonal. Now find distinct $\xi', \eta' \notin A_\alpha \cup \{\xi, \eta\}$ so that $\lambda^{\mathcal{T}}_{\eta',\xi'}(R_\alpha) = 0$. This can be achieved again by Lemma

3.2. Put $\beta^1_\alpha = \xi, \gamma^1_\alpha = \xi', \beta^2_\alpha = \eta, \gamma^2_\alpha = \eta'$ so (1) holds. Now take $\beta^3_\alpha, \gamma^3_\alpha$ be the first two elements of the set $\mathfrak{c} \setminus (A_\alpha \cup \{\beta^1_\alpha, \beta^2_\alpha, \gamma^1_\alpha, \gamma^2_\alpha\})$.

Otherwise if $\Lambda^{\mathcal{T}}(R_\alpha)$ has less then continuum nonzero entries off the diagonal but is not a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T})$. Then it must be the case that $\Lambda^{\mathcal{T}}(R_\alpha)$ has two different entries $\lambda^{\mathcal{T}}_{\xi,\xi}(R_\alpha) \neq \lambda^{\mathcal{T}}_{\eta,\eta}(R_\alpha)$ on the diagonal such that $\xi, \eta \notin A_\alpha$, since A_α has cardinality less than continuum. So we put $\beta^3_\alpha = \xi$, $\gamma^3_\alpha = \eta$ so that (2) holds.

In this case put $\beta^1_\alpha, \gamma^1_\alpha, \beta^2_\alpha, \gamma^2_\alpha$ to be the first four elements of the set $\mathfrak{c} \setminus (A_\alpha \cup \{\beta^3_\alpha, , \gamma^3_\alpha\}))$. The choice of $\beta^3_\alpha, \gamma^3_\alpha$ in the first case and $\beta^1_\alpha, \gamma^1_\alpha, \beta^2_\alpha, \gamma^2_\alpha$ in the second case guarantees that $\{\beta^i_\alpha, \gamma^i_\alpha : i \in \{1,2,3\}, \alpha < \mathfrak{c}\} = \mathfrak{c}$, which completes the inductive construction.

We put $Y = \{\beta^i_\alpha : i \in \{1,2,3\}, \alpha < \mathfrak{c}\}$ and we define $\rho : Y \to (X \setminus Y)$ by $\rho(\beta^i_\alpha) = \gamma^i_\alpha$. Let $\mathcal{U} = \{U_{\eta,\xi} : \xi, \eta \in Y\}$ be a pairing of $\mathcal{T}$ along ρ.

Suppose that R is a quasi-multiplier of $\mathcal{A}(\mathcal{T})$ which is not $\mathfrak{c}$-trivial, so $R = R_\alpha$ for some $\alpha < \mathfrak{c}$. We will show that R is not a quasi-multiplier of $\mathcal{A}(\mathcal{U})$, which will prove the required property of $\mathcal{U}$.

If (1) holds, then $\lambda^{\mathcal{T}}_{\beta^1_\alpha,\beta^2_\alpha}(R) \neq \lambda^{\mathcal{T}}_{\rho(\beta^1_\alpha),\rho(\beta^2_\alpha)}(R)$ as $\rho(\beta^i_\alpha) = \gamma^i_\alpha$ for $i = 1, 2$, but this contradicts Lemma 5.3.

If (2) holds, then $\lambda^{\mathcal{T}}_{\beta^3_\alpha,\beta^3_\alpha}(R) \neq \lambda^{\mathcal{T}}_{\rho(\beta^3_\alpha),\rho(\beta^3_\alpha)}(R)$ as $\rho(\beta^3_\alpha) = \gamma^3_\alpha$, but this contradicts Lemma 5.3. This shows that R is not a quasi-multiplier of $\mathcal{A}(\mathcal{U})$ and completes the proof of the lemma. □

Lemma 5.6. *Suppose that $\mathcal{T} = \{T_{\eta,\xi} : \xi, \eta \in X\}$ is a system of almost matrix units and $\rho : Y \to (X \setminus Y)$ is a bijection where $Y \subseteq X$ and that $\mathcal{U}$ is a pairing of $\mathcal{T}$ along ρ. If $R \in \mathcal{B}(\ell_2)$ is a quasi-multiplier of $\mathcal{A}(\mathcal{U})$ which is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T}_1)$, where $\mathcal{T}_1 = \{T_{\eta,\xi} : \xi, \eta \in Y'\}$, for any Y' satisfying $Y \subseteq Y' \subseteq X$, then R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{U})$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $\Lambda^{\mathcal{T}_1}(R - \lambda 1_{\mathcal{B}(\ell_2)})$ is a matrix with countably many nonzero entries. By Lemma 5.3 there are only countably many nonzero entries of $\Lambda^{\mathcal{U}}(R - \lambda 1_{\mathcal{B}(\ell_2)})$, because they are all equal to some entries of $\Lambda^{\mathcal{T}_1}(R - \lambda 1_{\mathcal{B}(\ell_2)})$, so R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{U})$.

□

6. The final construction

The construction of the C^*-algebra indicated in the title of this paper and described in the introduction starts with the Cantor tree system of almost matrix units $\mathcal{T}_{2^{\mathbb{N}}}$ and follows the scheme:

$$\mathcal{T}_{2^{\mathbb{N}}} \xrightarrow{\text{extending}} \{T_{\xi,\eta} : \xi, \eta \in 2^{\mathbb{N}} \cup X\} \xrightarrow{\text{pairing with } Y \subseteq 2^{\mathbb{N}}} \mathcal{U} \xrightarrow{\text{pairing, Lemma 5.5.}} \mathcal{S}$$

Theorem 6.1. *There is a type I C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\ell_2)$ containing the ideal of compact operators $\mathcal{K}(\ell_2)$ such that $\mathcal{A}/\mathcal{K}(\ell_2)$ is $*$-isomorphic to the algebra $\mathcal{K}(\ell_2(\mathfrak{c}))$ of all compact operators on the Hilbert space of density continuum and the algebra $\mathcal{M}(\mathcal{A})$ of multipliers of $\mathcal{A}$ is equal to the unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$.*

Proof. We work with $\ell_2(2^{<\mathbb{N}})$ instead of ℓ_2, as $2^{<\mathbb{N}}$ is countable. Start with the Cantor tree system of almost matrix units $\mathcal{T}_{2^{\mathbb{N}}}$ of Section 4. Extend it to a maximal system of almost matrix units $\{T_{\xi,\eta} : \xi, \eta \in 2^{\mathbb{N}} \cup X\}$ for some set X, by Lemma 2.12. It is clear that X has cardinality not bigger than continuum. Let $Y \subseteq 2^{\mathbb{N}}$ be such that both Y and $2^{\mathbb{N}} \setminus Y$ have cardinality $\mathfrak{c}$. Fix a bijection $\rho : Y \to (2^{\mathbb{N}} \setminus Y) \cup X$. Now let $\mathcal{U}$ be a pairing of $\{T_{\xi,\eta} : \xi, \eta \in 2^{\mathbb{N}} \cup X\}$ along ρ. Finally apply the pairing from Lemma 5.5 (for $\mathcal{U}$ instead of $\mathcal{T}$), to obtain a system $\mathcal{S}$ of almost matrix units with the special properties mentioned in the Lemma 5.5. We claim that $\mathcal{A}(\mathcal{S})$ is the desired C^*-algebra.

So suppose that R is in the multiplier algebra $\mathcal{M}(\mathcal{A}(\mathcal{S}))$ of $\mathcal{A}(\mathcal{S})$. Then R is a quasi-multiplier of $\mathcal{A}(\mathcal{S})$. Lemma 5.5 implies that R is a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\mathcal{U})$ and so Lemma 5.4 implies that R is a $\mathfrak{c}$-trivial quasi-multiplier of $\mathcal{A}(\{T_{\xi,\eta} : \xi, \eta \in 2^{\mathbb{N}} \cup X\})$ and hence for $\mathcal{A}(\mathcal{T}_{2^{\mathbb{N}}})$. This however implies that R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{T}_{2^{\mathbb{N}}})$ by Corollary 4.2. By Lemma 5.6 (for $Y' = 2^{\mathbb{N}}$) the operator R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{U})$ and again by Lemma 5.6 (for $\mathcal{U}$ as $\mathcal{T} = \mathcal{T}_1$ and $Y' = Y$) it is σ-trivial for $\mathcal{A}(\mathcal{S})$. However $\{T_{\xi,\eta} : \xi, \eta \in 2^{\mathbb{N}} \cup X\}$ was a maximal system of almost matrix units, so by the last part of Lemma 5.2, the system $\mathcal{U}$ and hence $\mathcal{S}$ are maximal systems of almost matrix units. The maximality of $\mathcal{S}$ together with the fact that R is a σ-trivial quasi-multiplier of $\mathcal{A}(\mathcal{S})$ implies that R is a trivial quasi-multiplier of $\mathcal{A}(\mathcal{S})$ (Corollary 3.8). Trivial quasi-multipliers of maximal systems of almost matrix units belong to the unitizations of the algebra generated by them and the compact operators, by Lemma 3.6 (3). Therefore R belongs to the unitization of $\mathcal{A}(\mathcal{S})$, as required.

□

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