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A LETTER CONCERNING LEONETTI'S PAPER 'CONTINUOUS PROJECTIONS ONTO IDEAL CONVERGENT SEQUENCES'

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ABSTRACT. Leonetti proved that whenever $\mathcal{I}$ is an ideal on $\mathbb{N}$ such that there exists an uncountable family of sets that are not in $\mathcal{I}$ with the property that the intersection of any two distinct members of that family is in $\mathcal{I}$, then the space $c_{0,\mathcal{I}}$ of sequences in ℓ_∞ that converge to 0 along $\mathcal{I}$ is not complemented. We provide a shorter proof of a more general fact that the quotient space $\ell_\infty/c_{0,\mathcal{I}}$ does not even embed into ℓ_∞.

Very recently, Leonetti ([3]) distilled a property of ideals $\mathcal{I}$ on the set of natural numbers (shared by ideals that are meagre when regarded as subsets of the Cantor set; see [3, Lemma 2.3], hence for example by the ideal of sets that have density 0) that gives a fairly satisfactory sufficient condition for non-complementability in ℓ_∞, the space of all bounded scalar sequences, of the space $c_{0,\mathcal{I}}$ consisting of sequences that converge to 0 along $\mathcal{I}$. Leonetti's proof is an interesting refinement of Whitley's proof of the Phillips–Sobczyk theorem ([4]; see also [1, Theorem 2.5.4]), which asserts that the space c_0 is not complemented in ℓ_∞. (All necessary terminology will be explained in subsequent paragraphs.)

More specifically, Leonetti's result contributes to the problem of characterisation of those ideals $\mathcal{I}$ of $\mathbb{N}$ for which the space $c_{0,\mathcal{I}}$ is complemented in ℓ_∞ proposed by Pérez Hernández during the Winter School in Abstract Analysis 2017 held in Svratka, Czech Republic. Conspicuously, the classical space c_0 coincides with $c_{0,\mathcal{I}}$ where $\mathcal{I}$ is the ideal of finite sets, so it is not complemented by virtue of the above-mentioned Phillips–Sobczyk theorem. On the other hand, when $\mathcal{I}$ is a maximal ideal (that is when the dual filter is an ultrafilter), the subspace $c_{0,\mathcal{I}}$ has codimension 1 in ℓ_∞, so it is complemented. (Similarly, the intersection $\mathcal{I}$ of finitely many maximal ideals will correspond to $c_{0,\mathcal{I}}$ having finite co-dimension in ℓ_∞, so in particular $c_{0,\mathcal{I}}$ is complemented in this case.) As every non-principal ideal contains the ideal of finite sets and, at the same time, is contained in a maximal ideal, the problem is quite tantalising indeed. Let us record Leonetti's result formally.

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Theorem. Suppose that $\mathcal{J}$ is an ideal on $\mathbb{N}$ with the property that there exists an uncountable family $\mathcal{A} \subset \wp(\mathbb{N}) \setminus \mathcal{J}$ such that $A \cap B \in \mathcal{J}$ for distinct $A, B \in \mathcal{A}$. Then the space $c_{0,\mathcal{J}}$ is not complemented in ℓ_∞.

We strengthen the above result by noticing that not only is $c_{0,\mathcal{J}}$ uncomplemented in ℓ_∞ but the quotient space $\ell_\infty / c_{0,\mathcal{J}}$ does not embed into ℓ_∞. We have then the following result.

Theorem A. Suppose that $\mathcal{J}$ is an ideal on $\mathbb{N}$ with the property that there exists an uncountable family $\mathcal{A} \subset \wp(\mathbb{N}) \setminus \mathcal{J}$ such that $A \cap B \in \mathcal{J}$ for distinct $A, B \in \mathcal{A}$. Then $\ell_\infty / c_{0,\mathcal{J}}$ is not isomorphic to a subspace of ℓ_∞. In particular, $c_{0,\mathcal{J}}$ is not complemented in ℓ_∞.

A closed subspace E of a Banach space X is *complemented* whenever there exists a closed subspace F of X such that $X = E \oplus F$; this, in turn, is equivalent to the existence of a bounded linear map $T\colon X/E \to X$ such that the composite map $T\pi$ is the identity map when restricted to E; here $\pi\colon X \to X/E$ denotes the canonical quotient map.

Let Γ be a set. A family $\mathcal{J}$ of subsets of Γ is an *ideal* on Γ, when it is closed under finite unions and $B \in \mathcal{J}$, whenever $B \subseteq A$ and $A \in \mathcal{J}$. Let $\mathcal{J}$ be an ideal on the set of natural numbers. A sequence $(x_n)_{n=1}^\infty$ in a metric space (X, d) converges to $x \in X$ along $\mathcal{J}$, whenever for every $\varepsilon > 0$ there is $A \in \mathcal{J}$ such that for every $n \notin A$ we have $d(x_n, x) < \varepsilon$. For every ideal $\mathcal{J}$, the subspace $c_{0,\mathcal{J}}$ comprising all bounded sequences that converge to 0 along $\mathcal{J}$ is a closed ideal of ℓ_∞, that is a closed subspace which is closed under multiplication by arbitrary elements of ℓ_∞. There is a standard picture of the space $c_{0,\mathcal{J}}$ as a space of continuous functions on a certain locally compact space that vanish at infinity.

Firstly, let us recall that ℓ_∞ is isometrically isomorphic as an algebra to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Čech–Stone compactification of the integers. Secondly, as $\beta\mathbb{N}$ consists of all ultrafilters on $\mathbb{N}$ that is topologised by the base $\{p \in \beta\mathbb{N}\colon A \in p\}$ $(A \subseteq \mathbb{N})$, one can consider the open subspace $U_\mathcal{J}$ comprising all ultrafilters that extend the filter dual to $\mathcal{J}$. Set $K_\mathcal{J} = \beta\mathbb{N} \setminus U_\mathcal{J}$. By the Tietze–Uryoshn extension theorem, $c_{0,\mathcal{J}}$ is isometric to $C_0(U_\mathcal{J})$, the space of scalar-valued continuous functions on $U_\mathcal{J}$ vanishing at infinity, and $\ell_\infty / c_{0,\mathcal{J}}$ is isometric to $C(K_\mathcal{J})$.

Proof of Theorem A. Let $\pi\colon \ell_\infty \to \ell_\infty / c_{0,\mathcal{J}}$ be the quotient map. Since $c_{0,\mathcal{J}}$ is an algebraic ideal of ℓ_∞, π is, in fact, an algebra homomorphism and $\ell_\infty / c_{0,\mathcal{J}}$ algebraically isomorphic to $C(K_\mathcal{J})$.

For any $A \subseteq \mathbb{N}$ let us consider the indicator function $\mathbb{1}_A \in \ell_\infty$. By the hypothesis,

$$0 = \pi(\mathbb{1}_{A\cap B}) = \pi(\mathbb{1}_A \cdot \mathbb{1}_B) = \pi(\mathbb{1}_A) \cdot \pi(\mathbb{1}_B)$$

for any distinct $A, B \in \mathcal{A}$, yet $\pi(\mathbb{1}_A) \neq 0$. Consequently, $\{\pi(\mathbb{1}_A)\colon A \in \mathcal{A}\}$ is a family of pairwise orthogonal non-zero idempotents in $C(K_\mathcal{J})$—that is $\{0, 1\}$-valued functions—so it spans an isometric copy of the non-separable space $c_0(\mathcal{A})$ (as $\mathcal{A}$ is uncountable). Consequently, $\ell_\infty / c_{0,\mathcal{J}}$ cannot embed into ℓ_∞ (and so $c_{0,\mathcal{J}}$ is not complemented in ℓ_∞) as it contains $c_0(\mathcal{A})$. (Indeed, it is a standard fact that $c_0(\Gamma)$ does not embed into ℓ_∞ because $\ell_\infty^* \cong \ell_1^{**}$ is weak*-separable by Goldstine's theorem, but $c_0(\Gamma)^*$ is not unless Γ is countable and separability is transferred by quotient maps.) □

Closing remark. A quick (however unnecessarily high-tech) way of explaining why c_0 is not complemented is ℓ_∞ is by appealing to the Grothendieck property of the latter space (as proved by Grothendieck himself [2]), while observing that c_0 clearly lacks it and this property is preserved by surjective linear operators. (A Banach space X is *Grothendieck* whenever every sequence in X^* that converges weak* also converges weakly.) It is then natural to ask the following question.

Suppose that $\mathcal{I}$ is an ideal as in the statement of Theorem A. Is it true that $c_{0,\mathcal{I}}$ is *not* a Grothendieck space?

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