

Convergence of graphons and the weak* topology

Martin Doležal

(joint work with J. Grebík, J. Hladký, I. Rocha, V. Rozhoň)

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Motivation

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A **graph** is a pair $G = (V, E)$ where V is a finite set and E is a set of 2-element subsets of V .

But huge networks are never completely known, often not even well defined.

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Graphons and cut distance

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For every measurable function $V: [0, 1]^2 \rightarrow [-1, 1]$ we define the **cut norm** of V by

$$\|V\|_{\square} := \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} V(x, y) \right|$$

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For graphons U, W we define the **cut distance** of U and W by

$$\delta_{\square}(U, W) := \inf_{\varphi: [0, 1] \rightarrow [0, 1]} \|U^{\varphi} - W\|_{\square}$$

where the infimum ranges over all invertible measure preserving maps $\varphi: [0, 1] \rightarrow [0, 1]$ and U^{φ} is defined by

$$U^{\varphi}(x, y) = U(\varphi(x), \varphi(y)).$$

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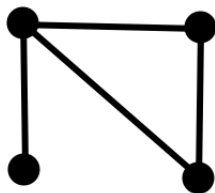
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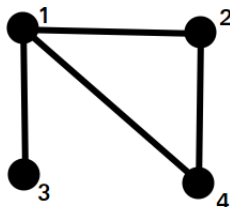
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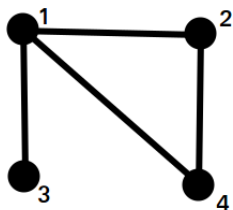
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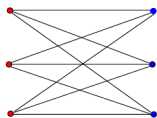
	1	2	3	4
1	0	1	1	1
2	1	0	0	1
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Basic example

$K_{n,n}$ the complete bipartite graph with both partitions of size n

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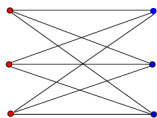
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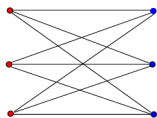


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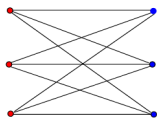


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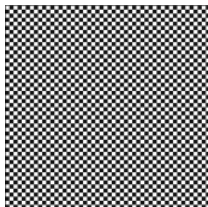
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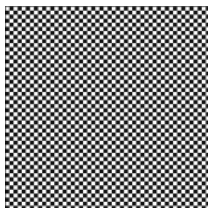
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...a sequence $(W_n)_n$ of graphons weak* converges to a graphon W iff for every measurable set $S \subseteq [0, 1]$ it holds

$$\lim_{n \rightarrow \infty} \int_{S \times S} W_n(x, y) = \int_{S \times S} W(x, y).$$

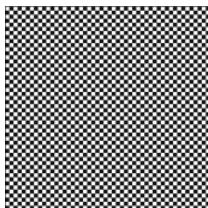
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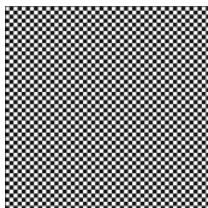
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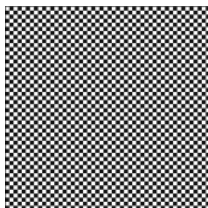
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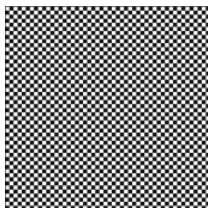


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- ▶ all these graphons belong to the same equivalence class
- ▶ therefore these graphons do not converge to $C_{\frac{1}{2}}$ **in the cut distance**

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$$\text{LIM}_{w^*}((W_n)_n) := \{W: \text{there are invertible measure preserving maps } \varphi_n: [0, 1] \rightarrow [0, 1] \text{ such that } W \text{ is a weak}^* \text{ limit of } (W_n^{\varphi_n})_n\}$$

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We want to take the 'most structured' element of either $\text{LIM}_{w^*}((W_n)_n)$ or $\text{ACC}_{w^*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance δ_{\square} .

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Key Theorem A

For every sequence $(W_n)_n$ of graphons there is a subsequence $(W_{n_k})_k$ of $(W_n)_n$ such that

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If one of these conditions holds then $(W_k)_k$ converges in the cut distance δ_{\square} to the 'most structured' element of $\text{LIM}_{w^}((W_k)_k)$.*

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It turns out that it is a homeomorphism onto a closed subset of the hyperspace. As the hyperspace is compact, the space of all (equivalence classes of) graphons is compact as well.

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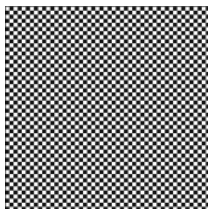
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Basic example once more

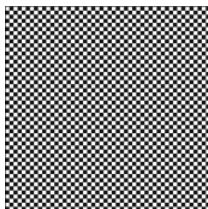
A representation of $K_{n,n}$ (for large n):



W_n

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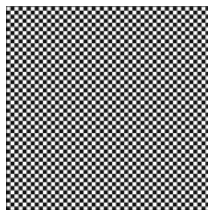


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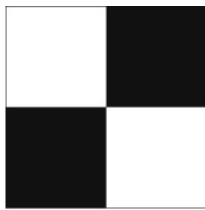
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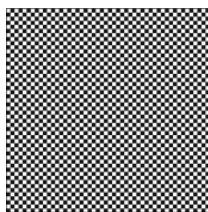
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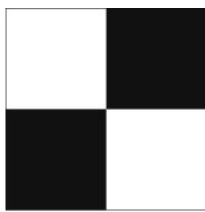
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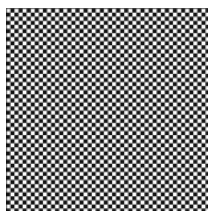
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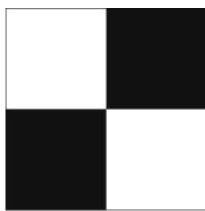
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In fact, W_1 is the most structured element of $\text{LIM}_{w^*}((W_n)_n)$.

Selected references

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