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Martin Doležal

Warren B. Moors

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Martin Doležal[a,b,1], Warren B. Moors[c]

[a] *Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland*
[b] *Institute of Mathematics of the Czech Academy of Sciences. Žitná 25, 115 67 Praha 1, Czech Republic*
[c] *Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland Mail Centre, Auckland 1142, New Zealand*

Abstract

We present a simple generalization of W-spaces introduced by G. Gruenhage. We show that this generalization leads to a strictly larger class of topological spaces which we call $\widetilde{W}$-spaces, and we provide several applications. Namely, we use the notion of $\widetilde{W}$-spaces to provide sufficient conditions for the product of two spaces to be a Baire space, for a semitopological group to be a topological group, or for a separately continuous function to be continuous at the points of a certain large set.

Keywords: Topological games, Baire spaces, semitopological groups, separate continuity
2010 MSC: 91A05, 54E52, 54C08

1. Introduction

All topological spaces considered in this paper are non-empty and Hausdorff.

One of the possible generalizations of first countable spaces are W-spaces which were introduces by G. Gruenhage in [1], and studied in detail by the same author in [2]. W-spaces are defined in terms of the following topological game. Let x be a point of a topological space (X, τ). Player I chooses an open set U_1 containing x, then player II chooses a point $x_1 \in U_1$. Next, player I chooses an open set U_2 containing x, and player II chooses a point $x_2 \in U_2$, and so on. Player I wins if x is an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. We denote this topological game by $G(x)$. We say that a point x of a topological space (X, τ) is a W-point if player I has a winning strategy in the game $G(x)$. Finally, we say that a topological space (X, τ) is a W-space if every point $x \in X$ is a W-point. (Note that the original winning condition for player I in the game $G(x)$ was different in [1]; it was required that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x. Then it was shown in [3] that the winning condition can be relaxed to 'x being in the closure of the set $\{x_n : n \in \mathbb{N}\}$', without altering the property of being a W-point. The winning condition which we use is weaker than the original one from [1] but stronger than the one used in [3], so the property of being a W-point is still unaltered.)

The paper [2] mainly investigates the relationship of W-spaces to some other classes of spaces. As an example, let us recall that every first-countable space is a W-space, and every W-space is countably bi-sequential.

In this paper, we provide a further generalization of W-spaces (and thus of first-countable spaces) which we call $\widetilde{W}$-spaces. To do this, we introduce a new topological game which is very similar to the game $G(x)$ described above. But we do not require player I to choose the open sets such that they contain x. Instead, player I may choose arbitrary non-empty open sets. At first sight, this may look a little weird as one would expect that any reasonable generalization of first-countability should deal with 'neighborhoods'. However, we provide several applications which suggest that this new notion could be useful. First, we prove Theorem 9 which is an improvement of [3, Theorem 4.4]. This theorem is more general than the statement that the product $X \times Y$ of a Baire space (X, τ) and a hereditarily Baire space (Y, τ') is Baire provided (Y, τ') is a $\widetilde{W}$-space. (The proof of Theorem 9 is almost the same as the proof of [3, Theorem 4.4], we only realized that it was not used in the proof of [3, Theorem 4.4] that the moves of player I in the

Email addresses: `dolezal@math.cas.cz` (Martin Doležal), `w.moors@auckland.ac.nz` (Warren B. Moors)

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game $G(x)$ are neighborhoods of x.) In Theorem 11, we provide a new sufficient condition for a separately continuous function to be quasi-continuous at certain points. This is a generalization of [4, Lemma 2] where points with a countable local base are used instead of $\widetilde{W}$-points. Theorem 11 can be used to the study of when a semitopological group is a topological group (see Corollary 12 which is a generalization of [4, Corollary 1]) or when a separately continuous function is continuous at the points of a certain large set (see Corollary 14). To justify our results, we also provide examples of $\widetilde{W}$-spaces which are very far from being W-spaces (see Examples 4 and 5).

The organization of the paper is very simple. In Chapter 2, we introduce all the notation needed to prove our results. In Chapter 3, we define the key notions of $\widetilde{W}$-points and $\widetilde{W}$-spaces, and we prove our results.

2. Notation

First, we recall the definition of W-points and W-spaces. For a point x of a topological space (X, τ), we denote by by $G(x)$ the topological game introduced by G. Gruenhage in [1] (this is the game described in the introduction).

Definition 1 (G. Gruenhage)**.** *We say that a point x of a topological space (X, τ) is a W-*point *in X if player I has a winning strategy in the game $G(x)$.*

*We say that a topological space (X, τ) is a W-*space *if every point $x \in X$ is a W-point in X.*

By a *strategy* for player I in the game $G(x)$ we mean a rule that specifies each move of player I in every possible situation. More precisely, a strategy σ for player I in the game $G(x)$ is a mapping $\sigma\colon X^{<\mathbb{N}} \to \tau$ defined on the set $X^{<\mathbb{N}}$ of all finite (possibly empty) sequences of elements of X whose values are open subsets of X, such that $x \in \sigma(x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in X^{<\mathbb{N}}$. If player I follows a strategy σ then he starts the game by playing $\sigma(\emptyset)$ in his first move. If player II replies by choosing some $x_1 \in \sigma(\emptyset)$ then player I plays $\sigma(x_1)$ in his second move. If player II replies by some $x_2 \in \sigma(x_1)$ then player I continues by $\sigma(x_1, x_2)$, and so on. A strategy σ for player I in the game $G(x)$ is called a *winning strategy* if player I wins each run of the game $G(x)$ when following the strategy. A finite (resp. infinite) sequence $(x_j)_j$ of elements of X is called a *σ-sequence* if player II can play his finitely many first moves (resp. each of his moves) of the game $G(x)$ according to the sequence if player I follows his strategy σ. That means that $(x_j)_j$ is a σ-sequence if and only if $x_j \in \sigma(x_1, \ldots, x_{j-1})$ for every j.

A (winning) strategy for player II in the game $G(x)$ can be defined analogously. One can also similarly define (winning) strategies for either player, as well as σ-sequences (where σ is a strategy for either player), in other topological games (in Chapter 3, we will introduce the game $\widetilde{G}(x)$ and recall the games $BM(R)$ and $\mathcal{G}(X)$).

Suppose that $\{(X_s, \tau_s)\colon\, s \in S\}$ is a nonempty family of topological spaces and $a = (a_s)_{s\in S} \in \Pi_{s\in S} X_s$. Then the Σ-*product* $\Sigma_{s\in S} X_s(a)$ of the family $\{(X_s, \tau_s)\colon\, s \in S\}$ with the *base point a* is the set

$$\{(x_s)_{s\in S} \in \Pi_{s\in S} X_s\colon\, x_s \neq a_s \text{ for at most countably many } s \in S\}$$

endowed with the topology inherited from the product space $\Pi_{s\in S} X_s$.

We also need the notion of a “rich family” which was first defined and used in [5]. Let (X, τ) be a topological space and let $\mathcal{F}$ be a family of nonempty closed separable subspaces of X. Then $\mathcal{F}$ is called a *rich family* if the following two conditions are satisfied:

(i) for every separable subspace Y of X, there is $F \in \mathcal{F}$ such that $Y \subseteq F$,

(ii) for every increasing (with respect to inclusion) sequence $(F_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ it holds $\overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{F}$.

Recall that a subset of a topological space is called *meager* (or a *set of first category*) if it is the union of countably many nowhere dense sets. A *comeager* (also called *residual*) set is the complement of a meager set. In other words, a comeager set is the intersection of countably many sets with dense interiors.

A topological space is called *Baire* if every intersection of countably many open dense subsets of the space is dense.

3. $\widetilde{W}$-points and $\widetilde{W}$-spaces

We define $\widetilde{W}$-spaces in terms of the following topological game. Let x be a point of a topological space (X,τ). Player I chooses a non-empty open set $U_1 \subseteq X$, then player II chooses a point $x_1 \in U_1$. Next, player I chooses a non-empty open set $U_2 \subseteq X$, and player II chooses a point $x_2 \in U_2$, and so on. Player I wins if x is an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. We denote this topological game by $\widetilde{G}(x)$.

Definition 2. *We say that a point x of a topological space (X,τ) is a* $\widetilde{W}$-point *in X if player I has a winning strategy in the game $\widetilde{G}(x)$.*

We say that a topological space (X,τ) is a $\widetilde{W}$-space *if every point $x \in X$ is a $\widetilde{W}$-point in X.*

Lemma 3. *Let (X,τ) be a topological space. Whenever $x \in X$ is in the closure of a countable subset of X consisting of $\widetilde{W}$-points then x is also a $\widetilde{W}$-point.*

Proof. Suppose that $x \in X$ is in the closure of the set $\{y_k : k \in \mathbb{N}\}$ where each y_k is a $\widetilde{W}$-point. We need to show that x is a $\widetilde{W}$-point. For each $k \in \mathbb{N}$, we fix a winning strategy σ_k for player I in the game $\widetilde{G}(y_k)$. We fix a sequence $\{k_n : n \in \mathbb{N}\}$ of natural numbers such that each $k \in \mathbb{N}$ occurs infinitely many times in the sequence. We define a strategy for player I in the game $\widetilde{G}(x)$ as follows. The first move of player I is $\sigma_{k_1}(\emptyset)$. Now suppose that the first n moves of the game $\widetilde{G}(x)$ have been played (and so the points $x_1,\dots,x_n$ chosen by player II are already known). Let $p_1 < \dots < p_m$ be all natural numbers $1 \le p \le n$ for which $k_p = k_{n+1}$. Then in his $(n+1)$th move, player I chooses $\sigma_{k_{n+1}}(x_{k_{p_1}},\dots,x_{k_{p_m}})$. It is easy to see that the described strategy for player I is winning. Indeed, each y_k is an accumulation point of the subsequence $(x_n)_{n\in\mathbb{N},k_n=k}$ since the strategy σ_k is winning for player I in the game $\widetilde{G}(y_k)$. Therefore each y_k is also an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. Now the conclusion immediately follows from the fact that the set of all accumulation points of any given sequence is always a closed set. $\square$

Example 4. *The Stone-Čech compactification $\beta\mathbb{N}$ of natural numbers $\mathbb{N}$ is a $\widetilde{W}$-space but not a W-space.*

Proof. The space $\beta\mathbb{N}$ is a regular separable space which is not first-countable, and so it is not a W-space (see [2, Theorem 3.6]).

On the other hand, every $n \in \mathbb{N}$ is clearly a $\widetilde{W}$-point in $\beta\mathbb{N}$ as it is an isolated point. Therefore by Lemma 3, the space $\beta\mathbb{N}$ is a $\widetilde{W}$-space. $\square$

Note that the space $\beta\mathbb{N}$ contains a dense subspace $\mathbb{N}$ consisting of W-points. Therefore the following example is even stronger than the previous one. Also note that Theorem 9, as a generalization of [3, Theorem 4.4], is not justified just by Example 4. Indeed, $X \times \mathbb{N}$ is a Baire space whenever (X,τ) is a Baire space (this is trivial but it also follows from the statement of [3, Theorem 4.4]). And as $X \times \mathbb{N}$ is a dense subspace of $X \times \beta\mathbb{N}$, we conclude that $X \times \beta\mathbb{N}$ is also a Baire space (and in fact, this follows from the second part of [3, Theorem 4.4]). On the other hand, Example 5 justifies Theorem 9 since the topological space constructed in that example contains no dense subspaces which are W-spaces.

Example 5. *There is a topological space (X,τ) with the following properties:*

- *X is a $\widetilde{W}$-space,*
- *whenever Z is a dense subspace of X then no point $x \in Z$ is a W-point in Z (in particular, no point $x \in X$ is a W-point in X),*
- *X possesses a rich family of Baire subspaces.*

Proof. Let S be an uncountable set. For every $s \in S$, we put $X^s = \beta\mathbb{N}$. For every $a = (a^s)_{s\in S} \in \Pi_{s\in S}\beta\mathbb{N}$, we define a space X_a as the Σ-product $\Sigma_{s\in S} X^s(a)$ with the base point a. First, we prove that each X_a is a $\widetilde{W}$-space. Let us fix $a = (a^s)_{s\in S} \in \Pi_{s\in S}\beta\mathbb{N}$ and a point $x = (x^s)_{s\in S} \in X_a$, we will construct a strategy σ for player I in the game $\widetilde{G}(x)$ played in X_a. For every $y = (y^s)_{s\in S} \in X_a$, we fix an infinite sequence $(s_1(y), s_2(y), \dots)$ of elements of S containing all indices $s \in S$ for which $y^s \neq x^s$ (so for $y = x$, this may be an arbitrary sequence of elements of S). We also fix a sequence $(t_n)_{n\in\mathbb{N}}$ of infinite sequences of natural numbers such that every finite sequence $t \in \mathbb{N}^{<\mathbb{N}}$ of natural numbers is an initial segment of infinitely many sequences t_n. We define the first move of player I as $\sigma(\emptyset) = X_a$. Let $x_1 = (x_1^s)_{s\in S}$

be the first move of player II. We put $r_1 = s_1(x_1)$. Then we define the second move of player I as the set of all points $(y^s)_{s\in S} \in X_a$ for which y^{r_1} is equal to the first element of the sequence t_1, that is

$$\sigma(x_1) = \{(y^s)_{s\in S} \in X_a \colon (y^{r_1}) \text{ is an initial segment of } t_1\}.$$

Let $x_2 = (x_2^s)_{s\in S}$ be the second move of player II. Then we prolong the sequence (r_1) (of length $k_1 = 1$) to a sequence $(r_1, \ldots, r_{k_2})$ of pairwise distinct indices from the set S (for some natural number $k_2 \geq 1$) such that $\{r_1, \ldots, r_{k_2}\} = \{s_i(x_j) \colon i, j \leq 2\}$. Now suppose that for some $n \geq 2$ we have already defined a sequence $(r_1, \ldots, r_{k_n})$ of pairwise distinct indices from the set S (where k_n is some natural number) such that $\{r_1, \ldots, r_{k_n}\} = \{s_i(x_j) \colon i, j \leq n\}$. Suppose also that the first n moves of the game $\widetilde{G}(x)$ have been played. Let $x_i = (x_i^s)_{s\in S}$, $i = 1, \ldots, n$, be the first n moves of player II. Then we define the $(n+1)$th move of player I by

$$\sigma(x_1, \ldots, x_n) = \{(y^s)_{s\in S} \in X_a \colon (y^{r_1}, \ldots, y^{r_{k_n}}) \text{ is an initial segment of } t_n\}.$$

Let $x_{n+1} = (x_{n+1}^s)_{s\in S}$ be the $(n+1)$th move of player II. Then we prolong the sequence $(r_1, \ldots, r_{k_n})$ to a sequence $(r_1, \ldots, r_{k_{n+1}})$ of pairwise distinct indices from the set S (for some natural number $k_{n+1} \geq k_n$) such that $\{r_1, \ldots, r_{k_{n+1}}\} = \{s_i(x_j) \colon i, j \leq n+1\}$. This completes the construction of the strategy σ. Next, we show that σ is a winning strategy for player I. We fix a natural number n_0 and an open neighborhood U of x of the form

$$U = \{(y^s)_{s\in S} \in X_a \colon y^{p_m} \in U_m \text{ for } m = 1, \ldots, q\}$$

for some pairwise distinct indices $p_1, \ldots, p_q$ from S and for some open neighborhoods U_m of x^{p_m} in $\beta\mathbb{N}$, $m = 1, \ldots, q$. We need to find a natural number $n \geq n_0$ such that $x_{n+1} \in U$. We may assume that for every $m = 1, \ldots, q$, there are $i, j \in \mathbb{N}$ such that $p_m = s_i(x_j)$, and so there is also $a_m \in \mathbb{N}$ such that $p_m = r_{a_m}$. Recall that we constructed a nondecreasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers for which there clearly is $n_1 \in \mathbb{N}$ such that $k_n \geq l := \max\{a_1, \ldots, a_q\}$ for every $n \geq n_1$. For every $1 \leq m \leq q$, we fix some $b_m \in U_m \cap \mathbb{N}$. We also fix a finite sequence $f = (f_1, \ldots, f_l)$ of natural numbers of length l such that $f_{a_m} = b_m$ for every $m = 1, \ldots, q$. By the choice of the sequence $(t_n)_{n\in\mathbb{N}}$, there is a natural number $n \geq \max\{n_0, n_1\}$ such that the finite sequence f is an initial segment of t_n. Then

$$\begin{aligned}
\sigma(x_1, \ldots, x_n) &= \{(y^s)_{s\in S} \in X_a \colon (y^{r_1}, \ldots, y^{r_{k_n}}) \text{ is an initial segment of } t_n\} \\
&\subseteq \{(y^s)_{s\in S} \in X_a \colon (y^{r_1}, \ldots, y^{r_l}) \text{ is an initial segment of } t_n\} \\
&\subseteq \{(y^s)_{s\in S} \in X_a \colon y^{r_{a_m}} = f_{a_m} \text{ for } m = 1, \ldots, q\} \\
&= \{(y^s)_{s\in S} \in X_a \colon y^{p_m} = b_m \text{ for } m = 1, \ldots, q\} \subseteq U,
\end{aligned}$$

and so $x_{n+1} \in \sigma(x_1, \ldots, x_n) \subseteq U$. This shows that X_a is a $\widetilde{W}$-space for every $a \in \Pi_{s\in S}\beta\mathbb{N}$.

In the rest of the proof, we use the well known identification of elements of $\beta\mathbb{N}$ with ultrafilters on the set $\mathbb{N}$ of all natural numbers. Let $\mathcal{F}$ be the filter consisting of all subsets A of $\mathbb{N}$ with

$$\liminf_{k\to\infty} \frac{|\{i \in A \colon 1 \leq i \leq k\}|}{k} = 1.$$

We fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the filter $\mathcal{F}$. Now suppose that the point $a = (a^s)_{s\in S}$ from the previous construction is chosen such that each a^s, $s \in S$, is the element of $\beta\mathbb{N}$ corresponding to the ultrafilter $\mathcal{U}$. We will show that whenever Z is a dense subspace of $X := X_a$ then no point $x \in Z$ is a W-point in Z. To this end, we fix a dense subspace Z of X and a point $x = (x^s)_{s\in S} \in Z$. We find a coordinate $s_0 \in S$ such that $x^{s_0} = a^{s_0}$ (i.e., the point $x^{s_0} \in \beta\mathbb{N}$ corresponds to the ultrafilter $\mathcal{U}$). We define a strategy for player II in the game $G(x)$ played in the subspace Z as follows. Suppose that U_n is the nth move of player I (for some $n \in \mathbb{N}$). We find an open subset W_n of X such that $U_n = W_n \cap Z$. Then the projection $\pi^{s_0}(W_n)$ of W_n to the coordinate s_0 is an open neighborhood of x^{s_0} in $\beta\mathbb{N}$, and so it has an infinite intersection with the subset $\mathbb{N}$ of $\beta\mathbb{N}$. Using this fact together with the density of Z in X, player II can choose his nth move $x_n = (x_n^s)_{s\in S} \in U_n$ such that $x_n^{s_0} \in \mathbb{N}$ and $x_n^{s_0} \geq n^2$. We show that this strategy is winning for player II which will complete the proof. It suffices to show that x^{s_0} is not a cluster point of the set $C = \{x_n^{s_0} \colon n \in \mathbb{N}\}$. By the description of the strategy, it holds

$$\limsup_{k\to\infty} \frac{|\{i \in C \colon 1 \leq i \leq k\}|}{k} = 0.$$

This means that $\mathbb{N} \setminus C \in \mathcal{F} \subseteq \mathcal{U}$. Therefore the set

$$U = \{y \in \beta\mathbb{N} \colon y \text{ corresponds to an ultrafilter containing } \mathbb{N} \setminus C\}$$

is an open neighborhood of x^{s_0} in $\beta\mathbb{N}$. But U does not intersect the set C, and so x^{s_0} is not a cluster point of C.

Finally, the family of all subspaces of X of the form $\{(x^s)_{s\in S} \in X \colon x^s = a^s$ for every $s \in S \setminus S'\}$ where S' is an at most countable subset of S is clearly a rich family of Baire subspaces. □

Note that it is not difficult to see that the space (X, τ) constructed in Example 5 is even a Baire space (as it is a Σ-product of compact spaces).

Recall that in the Banach-Mazur game $BM(R)$ played in a topological space (X, τ) with a subset R of X, two players β (who starts the game) and α alternately construct a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X. Player β wins the game if $\bigcap_{n\in\mathbb{N}} B_n \not\subseteq R$.

Theorem 6. *[6] Let R be a subset of a topological space (X, τ). Then R is comeager in X if and only if player α has a winning strategy in the game $BM(R)$.*

Lemma 7. *[3, Lemma 4.2] Let (X, τ), (Y, τ') be topological spaces, and let O be an open dense subset of $X \times Y$. Let U be a nonempty open subset of X, and let $V_1, \dots, V_m$ be nonempty open subsets of Y. Then there exist a nonempty open subset W of U and points $z_i \in V_i$, $1 \le i \le m$, such that $W \times \{z_1, \dots, z_m\} \subseteq O$.*

The following lemma has the same proof as [3, Theorem 4.3]. We only observed that it was not used in the proof of [3, Theorem 4.3] that the moves of player I in the game $G(x)$ are neighborhoods of x.

Lemma 8. *Let (X, τ) be a topological space, and let (Y, τ') be a $\widetilde{W}$-space. Let Z be a separable subspace of Y, and let $\{O_n \colon n \in \mathbb{N}\}$ be a countable system of dense open subsets of $X \times Y$. Then for every rich family $\mathcal{F}$ in Y, the set*

$$R = \{x \in X \colon \text{ there is } F_x \in \mathcal{F} \text{ containing } Z \text{ such that } \{y \in F_x \colon (x, y) \in O_n\} \text{ is dense in } F_x \text{ for all } n \in \mathbb{N}\}$$

is comeager in X.

Proof. Let us fix a rich family $\mathcal{F}$ in Y. We may assume that Y is infinite, otherwise the assertion is trivial. Then we may also assume that all elements of $\mathcal{F}$ are infinite. Moreover, we may assume that the sequence $(O_n)_{n\in\mathbb{N}}$ is decreasing (with respect to inclusion). For every $y \in Y$, we fix a winning strategy σ_y for player I in the game $\widetilde{G}(y)$ played in Y. We will construct a strategy σ for player α in the game $BM(R)$ played in X, and then we will show that this strategy is winning. The rest will follow from Theorem 6.

We start the construction of the strategy σ by fixing a countable subset $F_1 = \{f_{(1,j)} \colon j \in \mathbb{N}\}$ of Y such that $Z \subseteq \overline{F_1} \in \mathcal{F}$. Let $B_1 \subseteq X$ be the first move of player β in the game $BM(R)$. By Lemma 7, there are a nonempty open subset W_1 of B_1 and $z_{(1,1,1)} \in \sigma_{f_{(1,1)}}(\emptyset)$ such that $W_1 \times \{z_{(1,1,1)}\} \subseteq O_1$. We define $Z_1 = \{z_{(1,1,1)}\}$ and $\sigma(B_1) = W_1$. Note that the sequence $(z_{(1,1,1)})$ (of length 1) is a $\sigma_{f_{(1,1)}}$-sequence.

Now suppose that player β has already played his first n moves $B_1, \dots, B_n$ of the game $BM(R)$. Suppose also that for every $1 \le k \le n-1$, we have already defined the kth move $\sigma(B_1, \dots, B_k)$ of player α in the game $BM(R)$, together with a countable subset $F_k = \{f_{(k,j)} \colon j \in \mathbb{N}\}$ of Y and a finite subset $Z_k = \{z_{(i,j,l)} \colon i + j + l \le k + 2\}$ of Y such that

(i) $Z_{k-1} \cup F_{k-1} \subseteq \overline{F_k} \in \mathcal{F}$ (if $k \ge 2$),

(ii) the sequence $(z_{(i,j,1)}, \dots, z_{(i,j,l)})$ is a $\sigma_{f_{(i,j)}}$-sequence for every $i, j, l \in \mathbb{N}$ with $i + j + l = k + 2$,

(iii) $\sigma(B_1, \dots, B_k) \times \{z_{(i,j,l)} \colon i + j + l = k + 2\} \subseteq O_k$.

Then we find a countable subset $F_n = \{f_{(n,j)} \colon j \in \mathbb{N}\}$ of Y such that $Z_{n-1} \cup F_{n-1} \subseteq \overline{F_n} \in \mathcal{F}$. By Lemma 7, there are a nonempty open subset W_n of B_n and points $z_{(i,j,n+2-i-j)} \in \sigma_{f_{(i,j)}}(z_{(i,j,1)}, \dots, z_{(i,j,n+1-i-j)})$ for every $i + j \le n + 1$, such that $W_n \times \{z_{(i,j,n+2-i-j)} \colon i + j \le n + 1\} \subseteq O_n$. We define $Z_n = \{z_{(i,j,l)} \colon i + j + l \le n + 2\}$ and $\sigma(B_1, \dots, B_n) = W_n$. Note that conditions (i)-(iii) are satisfied for $k = n$. This completes the construction of the strategy σ.

It remains to show that σ is a winning strategy for player α. To this end, let $(B_n : n \in \mathbb{N})$ be a σ-sequence. Let us fix $x \in \bigcap_{n\in\mathbb{N}} B_n$, we need to prove that $x \in R$. The subspace $F_x = \overline{\bigcup_{n\in\mathbb{N}} F_n}$ of Y is clearly an element of $\mathcal{F}$ containing Z. Therefore it suffices to show that for every $n \in \mathbb{N}$, the set $A_{x,n} := \{y \in F_x : (x, y) \in O_n\}$ is dense in F_x. So let us fix $n \in \mathbb{N}$. Let U be an open subset of Y intersecting F_x. Then U intersect also $\bigcup_{n\in\mathbb{N}} F_n$, and so there are $i, j \in \mathbb{N}$ such that $f_{(i,j)} \in U$. Since condition (ii) immediately implies that the infinite sequence $(z_{(i,j,l)})_{l=1}^{\infty}$ is a $\sigma_{f_{(i,j)}}$-sequence, there is $l \geq n$ such that $z_{(i,j,l)} \in U$. By condition (iii), it holds $(x, z_{(i,j,l)}) \in O_{i+j+l-2} \subseteq O_l \subseteq O_n$. At the same time, condition (i) implies that $z_{(i,j,l)} \in Z_{i+j+l-2} \subseteq \overline{F_{i+j+l-1}} \subseteq F_x$. Therefore $z_{(i,j,l)} \in A_{x,n} \cap F_x \cap U$, and so $A_{x,n}$ is dense in F_x. □

The next theorem is an improvement of [3, Theorem 4.4]. Its proof is the same as the proof of [3, Theorem 4.4], we only need to use Lemma 8 instead of [3, Theorem 4.3].

Theorem 9. *Let (X, τ) be a Baire space, and let (Y, τ') be a $\widetilde{W}$-space which possesses a rich family $\mathcal{F}$ of Baire subspaces. Then $X \times Y$ is a Baire space.*

Proof. Let $\{O_n : n \in \mathbb{N}\}$ be a countable family of open dense subsets of $X \times Y$, we need to show that $\bigcap_{n\in\mathbb{N}} O_n$ is dense as well. So let U be a nonempty open subset of X and V be a nonempty open subset of Y, we will show that $\bigcap_{n\in\mathbb{N}} O_n \cap (U \times V) \neq \emptyset$. Fix an arbitrary point $z \in V$. By Lemma 8 used on $Z = \{z\}$, the set

$$\begin{aligned}\{x \in X :\ &\text{there is } F_x \in \mathcal{F} \text{ containing } z \text{ such that}\\ &\{y \in F_x : (x, y) \in O_n\} \text{ is dense in } F_x \text{ for all } n \in \mathbb{N}\}\end{aligned}$$

is comeager in X, and so the (bigger) set

$$\begin{aligned}\{x \in X :\ &\text{there is } F_x \in \mathcal{F} \text{ intersecting } V \text{ such that}\\ &\{y \in F_x : (x, y) \in O_n\} \text{ is dense in } F_x \text{ for all } n \in \mathbb{N}\}\end{aligned}$$

is also comeager in X. Therefore there are $x_0 \in U$ and $F_{x_0} \in \mathcal{F}$ intersecting V such that $\{y \in F_{x_0} : (x_0, y) \in O_n\}$ is dense in F_{x_0} for all $n \in \mathbb{N}$. As F_{x_0} is a Baire space, the set $\{y \in F_{x_0} : (x_0, y) \in \bigcap_{n\in\mathbb{N}} O_n\}$ is also dense in F_{x_0}. So there is $y_0 \in F_{x_0} \cap V$ such that $(x_0, y_0) \in \bigcap_{n\in\mathbb{N}} O_n$. In particular, $\bigcap_{n\in\mathbb{N}} O_n \cap (U \times V) \neq \emptyset$. □

For our next application of $\widetilde{W}$-space we need to consider another game.

The *Choquet* game, $\mathcal{G}(X)$, played on a topological space (X, τ) between two players β (who starts the game) and α who alternately construct a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X. Player α wins the game if $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$.

The importance of this definition is revealed next.

Theorem 10 ([7]). *A topological space (X, τ) is a Baire space if, and only if, the player β does not have a winning strategy in the Choquet game played on X.*

The final two notions required for our next theorem are that of separate continuity and quasi-continuity. A function $g : X \times Y \to Z$ that maps from a product of topological spaces (X, τ) and (Y, τ') into a topological space (Z, τ'') is said to be *separately continuous* on $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto g(x_0, y)$ and $x \mapsto g(x, y_0)$ are both continuous on Y and X respectively.

Suppose that $f : X \to Y$ is a function and $x_0 \in X$. Then we say that f is *quasi-continuous at* x_0 if for every open neighborhood U of x_0 and W of $f(x_0)$ there exists a nonempty open subset V of U such that $f(V) \subseteq W$.

Theorem 11. *Suppose that (X, τ), (Y, τ') and (Z, τ'') are topological spaces and $f : X \times Y \to Z$ is a separately continuous function. If (X, τ) is a Baire space, (Z, τ'') is a regular space and $y_0 \in Y$ is a $\widetilde{W}$-point, then f is quasi-continuous at each point of $X \times \{y_0\}$.*

Proof. Suppose, in order to obtain a contradiction, that f is not quasi-continuous at some point $(x_0, y_0) \in X \times \{y_0\}$. Then there exists an open neighborhood U_0 of x_0, an open neighborhood V_0 of y_0 and an open neighborhood W of $f(x_0, y_0)$ such that $f(U' \times V') \nsubseteq \overline{W}$ for any pair of nonempty open subsets U' of U_0 and V' of V_0. Since, $x \mapsto f(x, y_0)$, is continuous and $f(x_0, y_0) \in W$, we may assume, by possibly making U_0 smaller that $f(U_0 \times \{y_0\}) \subseteq W$.

We will now inductively define a strategy t for the player β in the Choquet game $\mathcal{G}(U_0)$ played on U_0. Let σ be a winning strategy for the player I in the $\widetilde{G}(y_0)$ played on Y.

Step 1. If $\sigma(\emptyset) \cap V_0 = \emptyset$ then choose $y_1 \in \sigma(\emptyset)$ - any choice is fine - and define $t(\emptyset) := U_0$. Otherwise, choose $y_1 \in \sigma(\emptyset) \cap V_0$ and $x' \in U_0$ such that $f(x', y_1) \notin \overline{W}$. Since, $x \mapsto f(x, y_1)$, is continuous there exists an open neighborhood U' of x', contained in U_0 such that $f(U' \times \{y_1\}) \subseteq Z \setminus \overline{W}$. Define $t(\emptyset) := U'$.

Now suppose that $y_j \in Y$ and U_j have been defined for each $2 \leq j \leq n$ so that (i) $U_1, U_2, \ldots, U_n$ are the first n moves of the player α in the $\mathcal{G}(U_0)$ played on U_0;
(ii) either $\sigma(y_1, \ldots, y_{j-1}) \cap V_0 = \emptyset$, $y_j \in \sigma(y_1, \ldots, y_{j-1})$ and $t(U_0, \ldots, U_{j-1}) = U_{j-1}$
or
$\sigma(y_1, \ldots, y_{j-1}) \cap V_0 \neq \emptyset$, $y_j \in \sigma(y_1, \ldots, y_{j-1}) \cap V_0$ and $f(t(U_0, \ldots, U_{j-1}) \times \{y_j\})$ is a subset of $Z \setminus \overline{W}$.

Step n+1. If $\sigma(y_1, \ldots, y_n) \cap V_0 = \emptyset$ choose $y_{n+1} \in \sigma(y_1, \ldots, y_n)$ and define $t(U_1, \ldots, U_n) := U_n$. Otherwise, choose $y_{n+1} \in \sigma(y_1, \ldots, y_n) \cap V_0$ and $x' \in U_n$ such that $f(x', y_{n+1}) \notin \overline{W}$. Since, $x \mapsto f(x, y_{n+1})$, is continuous there exists an open neighborhood U' of x', contained in U_n such that $f(U' \times \{y_{n+1}\}) \subseteq Z \setminus \overline{W}$. Define $t(U_1, \ldots, U_n) := U'$.

This completes the definition of t. Since U_0 is itself a Baire space with the relative topology, t is not a winning strategy for the player β in the $\mathcal{G}(U_0)$ game. Hence there exists a play $\{U_n\}_{n=1}^{\infty}$ where player α wins, i.e., $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$. Also, y_0 is an accumulation points of the sequence $(y_n)_{n\in\mathbb{N}}$, since σ is a winning strategy for the player I in the $\widetilde{G}(y_0)$ game. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers such that $\{n \in \mathbb{N} : y_n \in V_0\} = \{n_k : k \in \mathbb{N}\}$. Then $y_0 \in \overline{\{y_{n_k} : k \in \mathbb{N}\}}$. Let $x_\infty \in \bigcap U_n \subseteq U_0$. Thus, $f(x_\infty, y_{n_k}) \notin \overline{W}$ for all $k \in \mathbb{N}$. However, this implies that $f(x_\infty, y_0) \notin W$ since, $y \mapsto f(x_\infty, y)$, is continuous and $y_0 \in \overline{\{y_{n_k} : k \in \mathbb{N}\}}$. This contradicts our assumption that $f(U_0 \times \{y_0\}) \subseteq W$. Hence, f must be quasi-continuous at each point of $X \times \{y_0\}$. □

This result has implications for semitopological groups.

A triple $(G, \cdot, \tau)$ is called a *semitopological group* (*topological group*) if $(G, \cdot)$ is a group, (G, τ) is a topological space and the multiplication operation "·" is separately continuous on $G \times G$ (jointly continuous on $G \times G$ and the inversion mapping, $g \mapsto g^{-1}$, is continuous on G).

Let (X, τ) be a topological space. Following E. Reznichenko, (see, [4]) we shall say that a subset $W \subseteq X \times X$ is *separately open, in the second variable,* if for each $x \in X$, $\{z \in X : (x, z) \in W\} \in \tau$ and we shall say that a topological space (X, τ) is a Δ-*Baire space* if for each separately open, in the second variable set W, containing $\Delta_X := \{(x, y) \in X \times X : x = y\}$, there exists a nonempty open subset U of X such that $U \times U \subseteq \overline{W}$. Many spaces are Δ-Baire spaces. Indeed, all metrisable Baire spaces and all locally Čech-complete spaces are Δ-Baire spaces, see [4].

Corollary 12. *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is: (i) a regular Baire $\widetilde{W}$-space and (ii) a Δ-Baire space, then $(G, \cdot, \tau)$ is a topological group. In particular, if (G, τ) is a metrisable Baire space, then $(G, \cdot, \tau)$ is a topological group.*

Proof. This follows directly from Theorem 11 and Theorem 1 in [4] which states that a semitopological group that is a regular Δ-Baire space and whose multiplication operation is quasi-continuous is a topological group. □

We shall end this paper with another application of Theorem 9 and Theorem 11. To state this corollary we need to recall the following definition. Let (Z, τ) be a topological space and ρ some metric on Z. The space (Z, τ) is said to be *fragmented by the metric* ρ, if for every $\varepsilon > 0$ and every nonempty subset A of Z there exists a nonempty relatively open subset B of A with $\rho -$ diameter$(B) < \varepsilon$. In such a case the space (Z, τ) is called *fragmentable*.

An important theorem concerning fragmentable spaces is given next.

Theorem 13 ([8, Theorem 1])**.** *Let (X, τ) be a Baire space and $f : X \to Z$ be a quasi-continuous map from (X, τ) into a topological space (Z, τ') which is fragmented by some metric ρ. Then there exists a dense G_δ-subset $C \subseteq X$ at the points of which $f : (X, \tau) \to (Z, \rho)$ is continuous. In particular, if the topology generated by the metric ρ contains the topology τ', then $f : (X, \tau) \to (Z, \tau')$ is continuous at every point of the set C.*

Corollary 14. *Suppose that* (X,τ)*,* (Y,τ') *and* (Z,τ'') *are topological spaces and* $f : X \times Y \to Z$ *is a separately continuous function. If: (i)* (X,τ) *is a Baire space; (ii)* (Y,τ') *is a* $\widetilde{W}$*-space which possesses a rich family* $\mathcal{F}$ *of Baire subspaces and (iii)* (Z,τ'') *is a regular space that is fragmented by some metric* ρ *whose topology contains the topology* τ''*. Then* f *is continuous at the points of a dense* G_δ*-subset of* $X \times Y$*.*

Proof. From Theorem 9, $X \times Y$ is a Baire space. From Theorem 11, $f : X \times Y \to Z$ is quasi-continuous. The result then follows from Theorem 13. □

The utility of this result stems from the fact that, in addition to all metrisable spaces, there are many topological spaces (Z,τ) that are fragmented by some metric ρ whose topology contains the original topology τ, see [9].

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