Oscillatory solutions to inviscid fluid flows

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Complete Euler system

Phase variables

$$\text{mass density } \varrho = \varrho(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{(absolute) temperature } \vartheta = \vartheta(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{(bulk) velocity field } \mathbf{u} = \mathbf{u}(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

Standard formulation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \mathbf{u} \right] = 0$$

Impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Complete Euler system in conservative variables

Conservative variables

$$\text{mass density } \varrho = \varrho(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{(total energy) } E = E(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{momentum } \mathbf{m} = \mathbf{m}(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$p = (\gamma - 1)\varrho e,\ p = (\gamma - 1)\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right)$$

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Entropy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \boxed{\geq} 0$$

Entropy in the polytropic case

$$s = S\left(\frac{p}{\varrho^\gamma}\right) = S\left((\gamma - 1)\frac{E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma}\right)$$

Several concepts of solutions

Classical solutions

The phase variables are smooth (differentiable), the equations are satisfied in the standard sense. Classical solutions are often uniquely determined by the data.

Weak (distributional) solutions

Limits of classical solutions, limits of regularized problems. Equations are satisfies in the distributional sense. Weak solutions may not be uniquely determined by the data.

Limits of approximate (numerical) schemes

Zero step limits of numerical schemes.

Admissible (entropy) weak solutions

Field equations

$$\left[\int_\Omega \varrho\varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right] \,\mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T]\times\overline{\Omega})$

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi} \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p\mathrm{div}_x\boldsymbol{\varphi}\right] \,\mathrm{d}x\mathrm{d}t$$

for all $\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega}; R^3),\ \boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$

$$\left[\int_\Omega E\varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[E\partial_t\varphi + \left[(E+p)\frac{\mathbf{m}}{\varrho}\right]\cdot\nabla_x\varphi\right] \,\mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T]\times\overline{\Omega})$

Entropy inequality

$$\left[\int_\Omega \varrho s\varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} \boxed{\geq} \int_0^\tau \int_\Omega \left[\varrho s\partial_t\varphi + s\mathbf{m}\cdot\nabla_x\varphi\right] \,\mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T]\times\overline{\Omega}),\ \varphi \geq 0$

Infinitely many weak solutions

Initial data

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \vartheta(0,\cdot) = \vartheta_0.$$

Existence via convex integration

Let $N = 2, 3$. Let ϱ_0, ϑ_0 be piecewise constant (arbitrary) positive. Then the exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many admissible weak solution in $(0, T) \times \Omega$.

Dissipative measure–valued (DMV) solutions

Parameterized measure

$$\underbrace{\mathcal{F}}_{\text{phase space}} = \left\{ \varrho \geq 0,\ \mathbf{m} \in R^3,\ E \in [0, \infty) \right\},\quad \underbrace{Q_T}_{\text{physicalspace}} = (0, T) \times \Omega$$

$$\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T},\ Y_{t,x} \in \mathcal{P}(\mathcal{F})$$

Field equations

$$\partial_t \left\langle \mathcal{V}_{t,x}; \varrho \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle = 0$$

$$\partial_t \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle + \nabla_x \left\langle \mathcal{V}_{t,x}; p \right\rangle = D_x \mu_C$$

$$\partial_t \int_\Omega \left\langle \mathcal{V}_{t,x}; E \right\rangle \ \mathrm{d}x + \mathcal{D} = 0,\ \partial_t \left\langle \mathcal{V}_{t,x}; \varrho s \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; s\mathbf{m} \right\rangle \geq 0$$

Compatibility

$$\int_0^\tau \int_\Omega |\mu_C| \ \mathrm{d}x\mathrm{d}t \leq C \int_0^\tau \mathcal{D} \mathrm{d}t$$

Why to go measure–valued?

Motto: The larger (class) the better

- Universal limits of *numerical* schemes
- Limits of more complex physical systems - vanishing viscosity/heat conductivity limit
- Singular limits (low Mach etc.)

Weak–strong uniqueness

A (DMV) solution coincides with a smooth solution with the same initial data as long as the latter solution exists

Thermodynamic stability

Thermodynamic stability in the standard variables

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Thermodynamic stability in the conservative variables

$$(\varrho, \mathbf{m}, E) \mapsto \varrho s(\varrho, \mathbf{m}, E)$$

is a (strictly) concave function

Thermodynamic stability in the polytropic case

$$\varrho s = \varrho S \left(\frac{p}{\varrho^\gamma} \right), \ \ p = (\gamma - 1) \varrho e$$

$$S'(Z) > 0, \ (1 - \gamma) S'(Z) - \gamma S''(Z) Z > 0$$

Relative energy

Relative energy in the standard variables

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|\tilde\varrho,\tilde\vartheta,\tilde{\mathbf{u}}\right)$$
$$=\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\partial_\varrho H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)(\varrho-\tilde\varrho)-H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)$$

$$H_{\tilde\vartheta}(\varrho,\vartheta)=\varrho\Big(e(\varrho,\vartheta)-\tilde\vartheta s(\varrho,\vartheta)\Big)$$

Relative energy in the conservative variables

$$\mathcal{E}\left(\varrho,\mathbf{m},E\Big|\tilde\varrho,\tilde{\mathbf{m}},\tilde E\right)$$
$$=-\tilde\vartheta\Big[\varrho s-\partial_\varrho(\varrho s)(\varrho-\tilde\varrho)-\nabla_{\mathbf{m}}(\varrho s)\cdot(\mathbf{m}-\tilde{\mathbf{m}})-\partial_E(\varrho s)(E-\tilde E)$$
$$-\tilde\varrho\tilde s\Big]$$

Relative energy inequality

Relative energy revisited

$$\mathcal{E}\left(\varrho, \mathbf{m}, E \middle| \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}\right) \equiv E - \tilde\vartheta \mathcal{S}(\varrho, \mathbf{m}, E) - \mathbf{m}\cdot\tilde{\mathbf{u}} + \frac{1}{2}\varrho|\tilde{\mathbf{u}}|^2 + p(\tilde\varrho, \tilde\vartheta)$$

$$- \left(e(\tilde\varrho, \tilde\vartheta) - \tilde\vartheta s(\tilde\varrho, \tilde\vartheta) + \frac{p(\tilde\varrho, \tilde\vartheta)}{\tilde\varrho} \right) \varrho$$

Relative energy inequality

$$\left[\int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{E}\left(\varrho, \mathbf{m}, E \middle| \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}\right) \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) \le \int_0^\tau \mathcal{R}(t)\ \mathrm{d}t$$

Stability of strong solutions

Measure–valued strong uniqueness

Suppose the thermodynamic functions p, e, and s comply with the hypothesis of thermodynamic stability. Let $(\varrho, \mathbf{m}, E)$ be a smooth (C^1) solution of the Euler system and let $(Y_{t,x}; \mathcal{D})$ be a dissipative measure–valued solution of the same system with the same initial data, meaning

$$Y_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x), E_0(x)} \text{ for a.a. } x \in \Omega.$$

Then

$$\mathcal{D} \equiv 0, \ \ Y_{t,x} = \delta_{\varrho(t,x), \mathbf{m}(t,x), E(t,x)}$$

for a.a. $(t,x) \in (0,T) \times \Omega$.

Maximal dissipation principle

Entropy production rate

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho \mathbf{m}) = \boxed{\sigma} \geq 0$$

Dissipative ordering

$$\mathcal{V}^1_{t,x} \succeq \mathcal{V}^2_{t,x} \text{ iff } \sigma_1 \geq \sigma_2 \text{ in } [0,T) \times \Omega$$

$$\int_0^T \int_\Omega \left[\left\langle \mathcal{V}^1_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \right\rangle \partial_t \varphi + \left\langle \mathcal{V}^1_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right\rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \int_\Omega \left[\left\langle \mathcal{V}^2_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \right\rangle \partial_t \varphi + \left\langle \mathcal{V}^2_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right\rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t$$

Maximal dissipation principle

A (DMV) solution is admissible if it is *maximal* with respect to the ordering $\succeq$. A maximal (DMV) solution exists.

Generating MV solutions, limits weak → MV

Navier–Stokes–Fourier system

$$
\begin{aligned}
&\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\\
&\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x(p + a p_R) = \nu \mathrm{div}_x \mathbb{S},\\
&\partial_t\left(\varrho(e + a e_R)\right) + \mathrm{div}_x\left(\varrho(e + a e_R)\mathbf{u}\right) + \omega \nabla_x \mathbf{q}\\
&= \nu \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} - \lambda(\vartheta - \overline{\vartheta})^3.
\end{aligned}
$$

Constitutive assumptions, radiative components

$$
\mathbb{S}(\varrho, \nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right),
$$

$$
\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta
$$

$$
p_R = \frac{1}{3}\vartheta^4,\ \ e_R = \frac{\vartheta^4}{\varrho},\ \ s_R = \frac{4}{3}\frac{\vartheta^3}{\varrho}
$$

Limit (weak) → (MV)

Vanishing dissipation limit

Suppose that p and e are interrelated through the polytropic EOS with $\gamma = \frac{5}{3}$, and "other mostly technical conditions". Let

$$\nu = \omega = \varepsilon,\ a\varepsilon^{\alpha},\ \alpha > 1,\ \lambda = \varepsilon^{\beta},\ \beta < 1.$$

Let $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions to the Navier–Stokes–Fourier system periodic in the space variable.
Then $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ generates a Young measure Y and the energy defect measure a function $\mathcal{D}$ - a (DMV) solution of the Euler system.

Limits of Euler flows with strong stratification

Scaled Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \frac{1}{\varepsilon^2} \varrho \nabla_x \Phi,$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right]$$

$$+ \mathrm{div}_x \left(\frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathbf{u} \right) = \frac{1}{\varepsilon^2} \varrho \nabla_x \Phi \cdot \mathbf{u}.$$

Geometry

$$\Omega = \mathcal{T}^2 \times (0,1), \ \mathcal{T}^2 = [0,1]|_{\{0,1\}} \text{ — the two dimensional torus}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} - 0$$

Initial data

Stationary problem

$$p = \varrho\vartheta,\ \ \Phi = \Phi(z) = -z$$

$$\nabla_x(\varrho_s\overline{\Theta}) = -\varrho_s\nabla_x\Phi,\ \ \varrho_s = \exp\left(-\frac{z}{\overline{\Theta}}\right),\ \ \overline{\Theta} > 0$$

Well–prepared initial data

$$\varrho_{0,\varepsilon} = \varrho_s + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \ \vartheta_{0,\varepsilon} = \overline{\Theta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon},\ \ \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^N)} \leq c,$$

$$\varrho^{(1)}_{\varepsilon} \to 0,\ \ \vartheta^{(1)}_{0,\varepsilon} \to 0,\ \ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0,$$

$$\mathbf{U}_0 \in W^{k,2}(\Omega; R^3),\ \ k > 3,\ \ \mathbf{U}_0 = [U^1_0, U^2_0, 0],\ \ \mathrm{div}_h\mathbf{U}_0 = 0.$$

Target problem

Euler system

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_h \mathbf{U} + \nabla_x \Pi = 0, \ \mathrm{div}_h \mathbf{U} = 0, \ x_h \in \mathcal{T}^2,$$

Stratified initial data

$$\mathbf{U}(0, x) = \mathbf{U}_0(x_h, z) = [U_0^1(x_h, z), U_0^2(x_h, z), 0]$$

Singular limit (MV) $\to$ strong

Convergence to the target system

Let $\{Y^{\varepsilon}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, $\mathcal{D}^{\varepsilon}$ be a family of dissipative measure–valued solutions to the scaled system scaled Euler system, with the well prepared initial data

$$Y^{\varepsilon}_{0,x} = \delta_{\varrho_{0,\varepsilon}, \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}, c_v \varrho_{0,\varepsilon}\vartheta_{0,\varepsilon}}.$$

Then

$$\mathcal{D}^{\varepsilon} \to 0 \text{ in } L^{\infty}(0,T),$$

and

$$Y^{\varepsilon} \to \delta_{\varrho_s, \varrho_s \mathbf{U}, c_v \varrho_s \overline{\Theta}} \text{ in } L^{\infty}(0,T; \mathcal{M}^{+}(\mathcal{F})_{\mathrm{weak}-(*)}),$$

where $[\varrho_s, \overline{\Theta}]$ is the static state and $\mathbf{U}$ is the unique solution to the incompressible 2D Euler system