Regulated evolution quasivariational inequalities *

Pavel Krejčí

Matematický ústav Akademie věd ČR

Žitná 25, CZ – 11567 Praha 1

Czech Republic

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Contents

Introduction

This text is intended to serve as some sort of lecture notes for a series of lectures given by the author at the Department of Mathematics of the University of Pavia in May, 2003, originally planned for 2001. The idea at that time was to make an introduction into mathematical methods of modeling and analysis of both continuous and discontinuous rate-independent hysteresis phenomena. It seemed natural to describe the evolution of hysteresis systems in the space $G(a,b\,;X)$ of the so-called *regulated functions* of one real variable $t \in [a,b]$ with values in a Banach or Hilbert space X, that is, functions which at each point of their domain of definition admit both (possibly different) one-sided limits. It turned out in the meantime that the preparatory material on the three main ingredients of the theory, namely convergence concepts in $G(a,b\,;X)$, the Kurzweil integral, and regulated evolution variational inequalities, became an organized system which brings some new aspects into the analysis of discontinuous processes in general and deserves perhaps independent attention. In particular, the rich topological structure of the space $G(a,b\,;X)$ is of interest and might find applications also outside the classical theory of hysteresis.

This is why the reader will actually find little about hysteresis here, also because a certain number of monographs devoted exclusively to the mathematics of hysteresis is already available, for instance [12, 20, 22, 38]. In all these monographs, solution operators of variational inequalities play a central role, although their variational structure remains sometimes hidden under equivalent explicit representations in the scalar-valued situation, but main emphasis is put on applications in hysteresis modeling. Instead, this survey, based substantially on recent results obtained jointly with M. Brokate, J. Kurzweil, and Ph. Laurençot, focuses on the necessary background for the Kurzweil integral formulation of evolution variational inequalities which has been somewhat neglected so far.

The notes are divided into seven sections. Section 1 is devoted to an introduction into the Kurzweil integration including a new generalized variant of the Kurzweil integral (the KN-integral) which will be used throughout the text. In Section 2 we derive some basic properties of regulated functions and of the space $G(a,b\,;X)$, and list classical results on integration of regulated functions. Before we continue with the investigation of oscillatory properties of regulated functions and pointwise convergence in Section 4, we insert Section 3 containing the formulation and first results on regulated variational inequalities which constitute an essential step in the proof of equivalence of oscillation criteria. A detailed discussion about the relationship between various convergence concepts in $G(a,b\,;X)$ is the topic of Section 5. In Section 6 we apply the above methods to proving the existence of a solution to a rather general quasivariational inequality and show that the solution is not unique in general. Uniqueness and continuous data dependence is obtained under additional smoothness assumptions. The last Section is an Appendix collecting some basic concepts from convex analysis in Hilbert spaces, in particular special properties of the Minkowski functional.

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1 The Kurzweil integral

The first section is devoted to a brief overview of the Kurzweil integration theory including also some more recent specific results motivated by evolution (quasi-)variational inequalities.

1.1 Gauges and partitions

We consider a nondegenerate compact interval $[a,b] \subset \mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all divisions of the form

$$d \;=\; \{t_0, \dots, t_m\}\,, \quad a = t_0 < t_1 < \dots < t_m = b\,. \tag{1.1}$$

With a division $a = t_0 < t_1 < \dots < t_m = b$ of the interval $[a,b]$ we associate *partitions* D defined as

$$D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}\,; \qquad \tau_j \in [t_{j-1}, t_j] \quad \forall j = 1, \dots, m\,. \tag{1.2}$$

The basic concept in the Kurzweil integration theory, namely in its original version introduced in [27] which we call below the K-*integral*, as well as in its generalizations (the K^*-*integral* defined in [31] and the KN-integral proposed in [23]), is that of a δ-*fine partition*. We define the set

$$\Gamma(a,b) \;:=\; \{\delta : [a,b] \to \mathbb{R}\,;\, \delta(t) > 0 \quad \text{for every} \quad t \in [a,b]\}\,. \tag{1.3}$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*. For $t \in [a,b]$ and $\delta \in \Gamma(a,b)$ we denote

$$I_\delta(t) \;:=\;]t - \delta(t), t + \delta(t)[\,. \tag{1.4}$$

Definition 1.1 *Let $\delta \in \Gamma(a,b)$ be a given gauge. A partition D of the form (1.2) is said to be δ-*fine *if for every $j = 1, \dots, m$ we have*

$$\tau_j \;\in\; [t_{j-1}, t_j] \;\subset\; I_\delta(\tau_j)\,. \tag{1.5}$$

If moreover a δ-fine partition D satisfies the implications

$$\tau_j \;=\; t_{j-1} \;\Rightarrow\; j = 1\,, \quad \tau_j = t_j \;\Rightarrow\; j = m\,, \tag{1.6}$$

*then it is called a δ-*fine* *partition.*

The set of all δ-fine (δ-fine) partitions is denoted by $\mathcal{F}_\delta(a,b)$ ($\mathcal{F}^*_\delta(a,b)$, respectively).*

We have indeed $\mathcal{F}^*_\delta(a,b) \subset \mathcal{F}_\delta(a,b)$. The next lemma (often referred to as Cousin's Lemma) implies in particular that these sets are nonempty for every $\delta \in \Gamma(a,b)$.

Lemma 1.2 *Let $\delta \in \Gamma(a,b)$ and a dense subset $\Omega \subset]a,b[$ be given. Then there exists $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}^*_\delta(a,b)$ such that $t_j \in \Omega$ for every $j = 1, \dots, m-1$.*

Proof. We have $[a,b] \subset \cup_{t\in[a,b]} I_\delta(t)$, hence there exists a finite covering

$$[a,b] \subset \bigcup_{j=1}^{m} I_\delta(\tau_j)\ , \quad a \le \tau_1 \le \cdots \le \tau_m \le b\,. \tag{1.7}$$

The inclusion remains valid if we eliminate all intervals $I_\delta(\tau_j)$ for which there exists $k \ne j$, $I_\delta(\tau_j) \subset I_\delta(\tau_k)$. We claim that then we have

$$\min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\} \ > \ \max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\} \tag{1.8}$$

for every $j = 1,\ldots,m-1$. Indeed, we obviously have $\tau_{j+1} > \tau_j$, since otherwise $I_\delta(\tau_{j+1}) \subset I_\delta(\tau_j)$ or $I_\delta(\tau_j) \subset I_\delta(\tau_{j+1})$ according to whether $\delta(\tau_{j+1}) \le \delta(\tau_j)$ or $\delta(\tau_{j+1}) \ge \delta(\tau_j)$. Assume now that for some j we have

$$\min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\} \ \le \ \max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\}\,.$$

Then $\tau_{j+1} > \tau_{j+1} - \delta(\tau_{j+1}) \ge \tau_j + \delta(\tau_j) > \tau_j$, hence the points $\tau_j + \delta(\tau_j)$, $\tau_{j+1} - \delta(\tau_{j+1})$ do not belong to $I_\delta(\tau_j) \cup I_\delta(\tau_{j+1})$. Then there exists necessarily either $k < j$ such that $\tau_j + \delta(\tau_j) \in I_\delta(\tau_k)$, hence $I_\delta(\tau_j) \subset I_\delta(\tau_k)$, or $k > j+1$ such that $\tau_{j+1} - \delta(\tau_{j+1}) \in I_\delta(\tau_k)$, hence $I_\delta(\tau_{j+1}) \subset I_\delta(\tau_k)$, which is a contradiction. Inequality (1.8) is thus verified and we may choose arbitrarily

$$t_j \ \in \ \Big]\max\{\tau_j, \tau_{j+1} - \delta(\tau_{j+1})\}\,, \min\{\tau_{j+1}, \tau_j + \delta(\tau_j)\}\Big[\ \cap \ \Omega\,, \quad j = 1,\ldots,m-1\,,$$

$t_0 := a$, $t_m := b$, and the assertion immediately follows. ■

The advantage of the Kurzweil integration is based on the following property of δ-fine partitions which enables us to control the position of the 'tags' τ_j.

Lemma 1.3 *Let $U \subset [a,b]$ be a finite set and let $\delta \in \Gamma(a,b)$ be a gauge such that*

$$\delta(t) \ \le \ \mathrm{dist}\,(t, U \setminus \{t\}) \quad \textit{for every} \ \ t \in [a,b]\,. \tag{1.9}$$

Then for every partition $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\ldots,m\} \in \mathcal{F}_\delta(a,b)$ we have $U \subset \cup_{j=1}^m \{\tau_j\}$ and for every $\tau_j \in \cup_{j=1}^m \{\tau_j\} \setminus U$ we have $[t_{j-1}, t_j] \cap U \ = \ \emptyset$.

Proof. For every $\tau_j \in \cup_{j=1}^m \{\tau_j\} \setminus U$ we have $\delta(\tau_j) \le \mathrm{dist}\,(\tau_j, U)$, hence $I_\delta(\tau_j) \cap U \ = \ \emptyset$. On the other hand, for every $u \in U$ there exists i such that $u \in [t_{i-1}, t_i]$, hence $u = \tau_i$. ■

1.2 Definition of Kurzweil integrals

Having in mind applications to variational inequalities, we introduce the scalar Kurzweil integral for functions with values in a separable Hilbert space X endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|x| = \sqrt{\langle x, x\rangle}$ for $x \in X$. For more general couplings we refer the reader to [32].

For given functions $f, g : [a,b] \to X$ and a partition D of the form (1.2) we define the Kurzweil integral sum $K_D(f,g)$ by the formula

$$K_D(f,g) \ = \ \sum_{j=1}^{m} \langle f(\tau_j), g(t_j) - g(t_{j-1}) \rangle \ . \tag{1.10}$$

Definition 1.4 *Let* $f, g : [a,b] \to X$ *be given. We say that* $J \in \mathbb{R}$ *(*$J^* \in \mathbb{R}$*) is the* K-integral *(*K^*-integral, *respectively) over* $[a,b]$ *of* f *with respect to* g *and denote*

$$J = (K)\int_a^b \langle f(t), dg(t)\rangle\,, \qquad \left(J^* = (K^*)\int_a^b \langle f(t), dg(t)\rangle\,, \quad \textit{respectively}\right), \tag{1.11}$$

if for every $\varepsilon > 0$ *there exists* $\delta \in \Gamma(a,b)$ *such that for every* $D \in \mathcal{F}_\delta(a,b)$ *(*$D^* \in \mathcal{F}^*_\delta(a,b)$*, respectively) we have*

$$|J - K_D(f,g)| \le \varepsilon\,, \qquad \left(\, |J^* - K_{D^*}(f,g)| \le \varepsilon\,, \quad \textit{respectively}\right). \tag{1.12}$$

Using the fact that the implication

$$\delta \le \min\{\delta_1\,,\, \delta_2\} \quad \Rightarrow \quad \begin{cases} \mathcal{F}^*_\delta(a,b) \subset \mathcal{F}^*_{\delta_1}(a,b) \cap \mathcal{F}^*_{\delta_2}(a,b)\,, \\ \mathcal{F}_\delta(a,b) \subset \mathcal{F}_{\delta_1}(a,b) \cap \mathcal{F}_{\delta_2}(a,b) \end{cases} \tag{1.13}$$

holds for every $\delta\,, \delta_1\,, \delta_2 \in \Gamma(a,b)$, we easily check that the values J, J^* in Definition 1.4 are uniquely determined. Since $\mathcal{F}^*_\delta(a,b) \subset \mathcal{F}_\delta(a,b)$ for every gauge δ, we also see that if $(K)\int_a^b \langle f(t), dg(t)\rangle$ exists, then $(K^*)\int_a^b \langle f(t), dg(t)\rangle$ exists and both are equal. To illustrate the difference between the integrals (K) and (K^*), we prove the following easy Lemma. For a subset $B \subset [a,b]$ we denote by χ_B the characteristic function of B, that is, $\chi_B(t) = 1$ for $t \in B$, $\chi_B(t) = 0$ for $t \in [a,b] \setminus B$.

Lemma 1.5 *Let* $g : [a,b] \to X$ *and a vector* $v \in X$ *be given. Then* $(K)\int_a^b \left\langle v\,\chi_{\{a\}}(t), dg(t)\right\rangle$ *exists if and only if* $\lim_{t\to a+} \langle v, g(t)\rangle =: \langle v, g\rangle(a+)$ *exists and we have*

$$(K)\int_a^b \left\langle v\,\chi_{\{a\}}(t), dg(t)\right\rangle = \langle v, g\rangle(a+) - \langle v, g(a)\rangle\,. \tag{1.14}$$

Proof. Assume first that the integral on the left-hand side of (1.14) exists and equals J, and consider an arbitrary $\varepsilon > 0$, We find $\delta \in \Gamma(a,b)$ such that for every $D \in \mathcal{F}_\delta(a,b)$ we have

$$\left|K_D(v\,\chi_{\{a\}}, g) - J\right| < \varepsilon\,, \tag{1.15}$$

and put $\eta = \min\{\delta(a), b-a\}$. Let $\hat{t} \in]a, a+\eta[$ be arbitrary. By Lemma 1.2 we construct an arbitrary partition $\hat{D} \in \mathcal{F}^*_\delta(\hat{t}, b)$, $\hat{D} = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\ldots,m\}$. The partition $D = \{(a, [a,\hat{t}])\} \cup \hat{D}$ belongs to $\mathcal{F}_\delta(a,b)$, and (1.15) yields that $\left|K_D(v\,\chi_{\{a\}}, g) - J\right| = |\left\langle v, g(\hat{t}) - g(a)\right\rangle - J| < \varepsilon$, hence $\langle v, g\rangle(a+) = \langle v, g(a)\rangle + J$.

Conversely, let $\langle v, g\rangle(a+)$ exist and let $\varepsilon > 0$ be arbitrary. We find $\delta_0 > 0$ such that $t - a < \delta_0 \Rightarrow |\langle v, g(t)\rangle - \langle v, g\rangle(a+)| < \varepsilon$ for $t \in]a,b]$, and put $\delta(a) = \delta_0$, $\delta(t) = t - a$ for $t \in]a,b]$. For an arbitrary $D \in \mathcal{F}_\delta(a,b)$, $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\ldots,m\}$ we have by Lemma 1.3 that $\tau_1 = t_0 = a$, $t_1 < a + \delta_0$, $K_D(v\,\chi_{\{a\}}, g) = \langle v, g(t_1) - g(a)\rangle$, and the assertion follows. ■

The following example shows that Lemma 1.5 does not hold for the K^*-integral.

Example 1.6 Consider a sequence $s_k \searrow a$ as $k \to \infty$ and the function

$$g(t) = \sum_{k=1}^{\infty} v\ \chi_{\{s_k\}}(t) \qquad \text{for } \ t \in [a,b]\,, \tag{1.16}$$

with some $v \in X$, $v \neq 0$, and put

$$\delta(t) \ = \ \begin{cases} 1 & \text{for } \ t = a\,, \\ \min\{|t - s_\ell|\,;\, \ell \in \mathbb{N} \setminus \{k\}\} & \text{for } \ t = s_k\,, \quad k \in \mathbb{N}\,, \\ \min\{|t - s_\ell|\,;\, \ell \in \mathbb{N}\} & \text{for } \ t \neq a\,,\ t \neq s_k\,. \end{cases} \tag{1.17}$$

Let $D \in \mathcal{F}^*_\delta(a,b)$, $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}$ be an arbitrary partition. Arguing as in Lemma 1.3 we obtain that $\tau_1 = t_0 = a$. Moreover, as D belongs to $\mathcal{F}^*_\delta(a,b)$, we have $t_1 \neq \tau_2$. Assuming that $t_1 = s_k$ for some $k \in \mathbb{N}$ we may use again the argument of Lemma 1.3 and conclude that $|\tau_2 - t_1| < \delta(\tau_2) \leq |\tau_2 - s_k|$ which is a contradiction. We therefore have $g(t_1) = 0$, hence $K_D(v\,\chi_{\{a\}}, g) = 0$. We thus proved that $(K^*) \int_a^b \big\langle v\,\chi_{\{a\}}(t), dg(t)\big\rangle = 0$, although $\langle v, g\rangle\,(a+)$ does not exist. We see in particular that $(K) \int_a^b \big\langle v\,\chi_{\{a\}}(t), dg(t)\big\rangle$ does not exist.

For our purposes, it is necessary to extend the notion of the Kurzweil integral beyond K^* by reducing further the sets of admissible partitions. We fix a system $\mathcal{N}$ of subsets of $[a,b]$ with the following properties:

$$\overline{[a,b] \setminus A} \ = \ [a,b] \qquad \forall A \in \mathcal{N}, \tag{1.18}$$

$$A, B \in \mathcal{N} \quad \Rightarrow \quad A \cup B \ \in \mathcal{N}. \tag{1.19}$$

Elements of $\mathcal{N}$ will be called *negligible sets*. Typically, $\mathcal{N}$ can be for instance the system of all countable subsets or the system of all subsets of Lebesgue measure zero in $[a,b]$.

Definition 1.7 *Let $\mathcal{N}$ be a system of negligible sets in $[a,b]$, let $\delta \in \Gamma(a,b)$ be a given gauge, and let $A \in \mathcal{N}$ be a given set. A partition D of the form (1.2) is said to be* (δ, A)-fine *if it is δ-fine* and*

$$t_j \in [a,b] \setminus A \qquad \forall j = 1, \dots, m-1\,. \tag{1.20}$$

The set of all (δ, A)-fine partitions is denoted by $\mathcal{F}_{\delta,A}(a,b)$.

Definition 1.8 *Let a system $\mathcal{N}$ of negligible sets and $f, g : [a,b] \to X$ be given. We say that $J \in \mathbb{R}$ is the* KN-integral *over $[a,b]$ of f with respect to g and denote*

$$J \ = \ (KN) \int_a^b \langle f(t), dg(t)\rangle\ , \tag{1.21}$$

if for every $\varepsilon > 0$ there exist $\delta \in \Gamma(a,b)$ and $A \in \mathcal{N}$ such that for every $D \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$|J \ - \ K_D(f,g)| \ \leq \ \varepsilon\,. \tag{1.22}$$

The definition is again meaningful. Note first that the set $\mathcal{F}_{\delta,A}(a,b)$ is nonempty for every $\delta \in \Gamma(a,b)$ and every $A \in \mathcal{N}$ by Lemma 1.2. Furthermore, if J satisfying (1.22) exists, then it is unique. Indeed, assume that there exist $J_1 \neq J_2$ such that for every $\varepsilon > 0$ there exist $\delta_1, \delta_2 \in \Gamma(a,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for each $D_i \in \mathcal{F}_{\delta_i,A_i}(a,b)$, $i = 1,2$, we have

$$|J_i \;-\; K_{D_i}(f,g)| \;\le\; \varepsilon\,. \tag{1.23}$$

Choosing $\varepsilon < |J_1 - J_2|/2$ and putting $\delta = \min\{\delta_1, \delta_2\}$, $A = A_1 \cup A_2$ we may choose any $D \in \mathcal{F}_{\delta,A}(a,b)$. Then $D \in \mathcal{F}_{\delta_1,A_1}(a,b) \cap \mathcal{F}_{\delta_2,A_2}(a,b)$, hence $|J_i \;-\; K_D(f,g)| \;\le\; \varepsilon$ for $i = 1,2$, which is a contradiction.

Obviously, if $(K^*)\int_a^b \langle f(t), dg(t)\rangle$ exists, then $(KN)\int_a^b \langle f(t), dg(t)\rangle$ exists and both integrals are equal. In the trivial case $\mathcal{N} = \{\emptyset\}$, the KN-integral and the K^*-integral coincide. Note also the result of [23] showing that it suffices to exclude all countable subsets of $[a,b]$ as negligible sets, and the Young integral (which we do not introduce here) becomes a special case of the KN-integral.

The main difference between the integrals (K^*) and (KN) consists in the following property.

Lemma 1.9 *Let $\mathcal{N}$ be a system of negligible sets, and let $A \in \mathcal{N}$ and $f, g : [a,b] \to X$ be such that $g(t) = 0$ for every $t \in [a,b] \setminus A$. Then we have*

$$(KN)\int_a^b \langle f(t), dg(t)\rangle \;=\; \langle f(b), g(b)\rangle \;-\; \langle f(a), g(a)\rangle\,. \tag{1.24}$$

The proof of Lemma 1.9 is obvious. Taking any $\delta \in \Gamma(a,b)$ such that $\delta(t) \le \min\{t-a, b-t\}$ for $t \in\,]a,b[$, we obtain $K_D(f,g) = \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle$ for all $D \in \mathcal{F}_{\delta,A}(a,b)$.

The identity (1.24) does not hold for the K^*-integral even if f is continuous and g regulated. The construction of the counterexample is however rather technical and details can be found in [25]. On the other hand, we have the following result which shows that all aforementioned Kurzweil-type integrals coincide if the function g is continuous.

Proposition 1.10 *Let $f, g : [a,b] \to X$ be such that $(KN)\int_a^b \langle f(t), dg(t)\rangle = J$ exists for some choice of $\mathcal{N}$, and let g be continuous in $[a,b]$. Then $(K)\int_a^b \langle f(t), dg(t)\rangle$ exists and equals J.*

The proof of Proposition 1.10 is based on the following two auxiliary results. For a finite set S, we denote by $\#S$ the number of its elements.

Lemma 1.11 *Let $\delta \in \Gamma(a,b)$ be a gauge, and let $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}_\delta(a,b)$ be an arbitrary partition. Let $\mathcal{R}(D)$ denote the set of all partitions*

$$D' \;=\; \{(\tau'_i, [t'_{i-1}, t'_i])\,;\, i = 1, \dots, m'\} \in \mathcal{F}_\delta(a,b) \tag{1.25}$$

such that

$$\bigcup_{i=1}^{m'} \{\tau'_i\} \;=\; \bigcup_{j=1}^{m} \{\tau_j\}\,, \tag{1.26}$$

$$\bigcup_{i=0}^{m'} \{t'_i\} \;\subset\; \bigcup_{j=1}^{m} \{t_j\}\,. \tag{1.27}$$

For $D' \in \mathcal{R}(D)$ of the form (1.25) set

$$\mu(D') \;=\; \#\{i = 1, \dots, m'-1\,;\ \tau'_i = \tau'_{i+1}\}\,, \tag{1.28}$$

and assume that $\mu(D') > 0$. Then there exists $D'' \in \mathcal{R}(D)$ such that $\mu(D'') = \mu(D') - 1$, and for every $f, g : [a,b] \to X$ we have $K_{D''}(f,g) = K_{D'}(f,g)$.

Proof of Lemma 1.11. Assume that $\tau'_i = \tau'_{i+1}$ for some $i = 1, \dots, m'-1$. It suffices to put

$$\tau''_k \;=\; \begin{cases} \tau'_k & \text{for} \quad k = 1, \dots, i\,, \\ \tau'_{k+1} & \text{for} \quad k = i+1, \dots, m'-1\,, \end{cases} \tag{1.29}$$

$$t''_k \;=\; \begin{cases} t'_k & \text{for} \quad k = 1, \dots, i-1\,, \\ t'_{k+1} & \text{for} \quad k = i, \dots, m'-1\,. \end{cases} \tag{1.30}$$

We have by hypothesis $\tau'_i = \tau'_{i+1} = t'_i$ and $[t'_{i-1}, t'_{i+1}] = [t'_{i-1}, t'_i] \cup [t'_i, t'_{i-1}] \subset I_\delta(\tau'_i)$, hence $D'' = \{(\tau''_k, [t''_{k-1}, t''_k])\,;\ k = 1, \dots, m'-1\}$ belongs to $\mathcal{F}_\delta(a,b)$, and therefore also to $\mathcal{R}(D)$. For every $f, g : [a,b] \to X$ we have

$$\begin{aligned} \sum_{k=1}^{m'} \big\langle f(\tau'_k), g(t'_k) - g(t'_{k-1}) \big\rangle \;&=\; \sum_{k=1}^{i-1} \big\langle f(\tau'_k), g(t'_k) - g(t'_{k-1}) \big\rangle + \big\langle f(\tau'_i), g(t'_{i+1}) - g(t'_{i-1}) \big\rangle \\ &\quad + \sum_{k=i+1}^{m'-1} \big\langle f(\tau'_{k+1}), g(t'_{k+1}) - g(t'_k) \big\rangle \\ &=\; \sum_{k=1}^{m'-1} \big\langle f(\tau''_k), g(t''_k) - g(t''_{k-1}) \big\rangle\,, \end{aligned}$$

and Lemma 1.11 is proved. ■

Lemma 1.12 *Let $\mathcal{N}$ be any system of negligible sets, and let $\delta \in \Gamma(a,b)$, $A \in \mathcal{N}$, and $D = \{(\tau_j, [t_{j-1}, t_j])\,;\ j = 1, \dots, m\} \in \mathcal{F}_\delta(a,b)$ be given. Assume that*

$$\tau_j < \tau_{j+1} \qquad \forall\, j = 1, \dots, m-1\,. \tag{1.31}$$

Then for every $\eta > 0$ there exists $D_\eta = \{(\tau_j, [t^*_{j-1}, t^*_j])\,;\ j = 1, \dots, m\} \in \mathcal{F}_{\delta,A}(a,b)$ such that

$$|t_j - t^*_j| \;<\; \eta \qquad \forall\, j = 0, \dots, m\,. \tag{1.32}$$

Proof of Lemma 1.12. Put $t^*_0 = t_0 = a$, $t^*_m = t_m = b$. For every $j = 1, \dots, m-1$ we have

$$t_j \in [\tau_j, \tau_{j+1}] \cap \,]\tau_{j+1} - \delta(\tau_{j+1}), \tau_j + \delta(\tau_j)[$$

by virtue of (1.5), hence for every $\eta > 0$ and every $j = 1, \dots, m-1$, the set

$$K^\eta_j \;=\;]\tau_j, \tau_{j+1}[\, \cap \,]\tau_{j+1} - \delta(\tau_{j+1}), \tau_j + \delta(\tau_j)[\, \cap \,]t_j - \eta, t_j + \eta[\tag{1.33}$$

is a nondegenerate open interval. We obtain the assertion by choosing arbitrarily $t^*_j \in K^\eta_j \setminus A$ for $j = 1, \dots, m-1$. ■

We are now ready to prove Proposition 1.10.

Proof of Proposition 1.10. Let $\varepsilon > 0$ be given. We find $\delta \in \Gamma(a,b)$ and $A \in \mathcal{N}$ such that for every $\tilde{D} \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$|K_{\tilde{D}}(f,g) - J| \le \frac{\varepsilon}{2}. \tag{1.34}$$

Let $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m\} \in \mathcal{F}_\delta(a,b)$ be arbitrary. We claim that

$$|K_D(f,g) - J| \le \varepsilon. \tag{1.35}$$

To check that (1.35) holds, we use Lemma 1.11 and find $D' \in \mathcal{R}(D)$ of the form (1.25) such that $\mu(D') = 0$ and

$$K_{D'}(f,g) = K_D(f,g). \tag{1.36}$$

Let now $\eta > 0$ be such that the implication

$$|t - s| < \eta \quad \Rightarrow \quad |g(t) - g(s)| \sum_{i=1}^{m'} |f(\tau_i')| \le \frac{\varepsilon}{4} \tag{1.37}$$

holds for every $t, s \in [a,b]$. By Lemma 1.12 we find $D_\eta = \{(\tau_i', [t_{i-1}^*, t_i^*])\,;\, i = 1,\dots,m'\} \in \mathcal{F}_{\delta,A}(a,b)$ such that $|t_i' - t_i^*| < \eta$ for all $i = 1,\dots,m'$. Then (1.37) yields

$$|K_{D_\eta}(f,g) - K_{D'}(f,g)| = \left| \sum_{i=1}^{m'} \left\langle f(\tau_i'), g(t_i') - g(t_{i-1}') - g(t_i^*) + g(t_{i-1}^*) \right\rangle \right| \le \frac{\varepsilon}{2}. \tag{1.38}$$

On the other hand, by (1.34) we have that

$$|K_{D_\eta}(f,g) - J| \le \frac{\varepsilon}{2}. \tag{1.39}$$

Combining (1.38) with (1.39) and (1.36) we obtain (1.35), and Proposition 1.10 follows. ■

1.3 Basic properties

The integrals $(K), (K^*), (KN)$ are linear with respect to both functions f and g. For the sake of completeness, we state this easy result explicitly.

Proposition 1.13 *Let $\int_a^b \langle f(t), dg(t)\rangle$ denote one of the integrals $(K), (K^*)$, or (KN).*

(i) *Let $\int_a^b \langle f_1(t), dg(t)\rangle$, $\int_a^b \langle f_2(t), dg(t)\rangle$ exist. Then we have*

$$\int_a^b \langle (f_1 + f_2)(t), dg(t)\rangle = \int_a^b \langle f_1(t), dg(t)\rangle + \int_a^b \langle f_2(t), dg(t)\rangle. \tag{1.40}$$

(ii) *Let $\int_a^b \langle f(t), dg_1(t)\rangle$, $\int_a^b \langle f(t), dg_2(t)\rangle$ exist. Then we have*

$$\int_a^b \langle f(t), d(g_1 + g_2)(t)\rangle = \int_a^b \langle f(t), dg_1(t)\rangle + \int_a^b \langle f(t), dg_2(t)\rangle. \tag{1.41}$$

(iii) *Let* $\int_a^b \langle f(t), dg(t)\rangle$ *exist. Then for every constant* $\lambda \in \mathbb{R}$ *we have*

$$\int_a^b \langle \lambda f(t), dg(t)\rangle \;=\; \int_a^b \langle f(t), d(\lambda g)(t)\rangle \;=\; \lambda \int_a^b \langle f(t), dg(t)\rangle\,. \tag{1.42}$$

Proof. Let us consider for instance the KN-integral with any system $\mathcal{N}$ of negligible sets, and let $\varepsilon > 0$ be given. We find δ_1, $\delta_2 \in \Gamma(a,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for all $D_i \in \mathcal{F}_{\delta_i, A_i}(a,b)$, $i = 1, 2$ we have

$$\left|(KN)\int_a^b \langle f_i(t), dg(t)\rangle \;-\; K_{D_i}(f_i, g)\right| \;<\; \frac{\varepsilon}{2}\,.$$

Put $\delta = \min\{\delta_1, \delta_2\}$, $A = A_1 \cup A_2$. From the implication (1.13) we infer that for every $D \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$\left|(KN)\int_a^b \langle f_1(t), dg(t)\rangle \;+\; (KN)\int_a^b \langle f_2(t), dg(t)\rangle \;-\; K_D(f_1 + f_2, g)\right| \;<\; \varepsilon\,,$$

and (1.40) follows. The same argument applies to the case (ii), while (iii) is trivial. ■

In order to analyze the behaviour of the Kurzweil integral with respect to the variation of the integration domain in Proposition 1.15 below, we derive the following Bolzano-Cauchy-type characterization analogous to [32, Proposition 7]. Indeed, corresponding statements hold for the integrals (K) and (K^*), too.

Lemma 1.14 *Let* $\mathcal{N}$ *be a system of negligible sets in* $[a,b]$, *and let* $f, g : [a,b] \to X$ *be given functions. Then* $(KN)\int_a^b \langle f(t), dg(t)\rangle$ *exists if and only if*

$$\forall \varepsilon > 0 \;\; \exists \delta \in \Gamma(a,b) \;\; \exists A \in \mathcal{N} \;\; \forall D, D' \in \mathcal{F}_{\delta,A}(a,b) \;:\; |K_D(f,g) - K_{D'}(f,g)| \le \varepsilon\,. \tag{1.43}$$

Proof. If $(KN)\int_a^b \langle f(t), dg(t)\rangle$ exists, then (1.43) trivially holds. Conversely, assume that (1.43) is satisfied. We find $\delta_0 \in \Gamma(a,b)$ and $A_0 \in \mathcal{N}$ such that (1.43) holds with $\varepsilon = 1$. For each $n \in \mathbb{N}$ we construct by induction $\delta_n \in \Gamma(a,b)$, $\delta_n \le \delta_{n-1}$, and $A_n \in \mathcal{N}$, $A_n \supset A_{n-1}$ such that for all $D, D' \in \mathcal{F}_{\delta_n, A_n}(a,b)$ we have

$$|K_D(f,g) - K_{D'}(f,g)| \le 2^{-n}\,. \tag{1.44}$$

We fix some $D_n \in \mathcal{F}_{\delta_n, A_n}(a,b)$ for each $n \in \mathbb{N}$, and set $J_n = K_{D_n}(f,g)$. For all $m \ge n$ we have by (1.44) that $|J_n - J_m| \le 2^{-n}$, hence $\{J_n\}$ is a Cauchy sequence, $J_n \to J$ as $n \to \infty$.

Let now $\varepsilon > 0$ be given. We fix $n \in \mathbb{N}$ such that $2^{-n} \le \varepsilon$, and put $\delta = \delta_n$, $A = A_n$. From (1.44) it follows that $|K_D(f,g) - J_m| \le \varepsilon$ for all $D \in \mathcal{F}_{\delta,A}(a,b)$ and all $m \ge n$, and letting $m \to \infty$ we obtain $J = (KN)\int_a^b \langle f(t), dg(t)\rangle$, which we wanted to prove. ■

Proposition 1.15 *Let* $\int_a^b \langle f(t), dg(t)\rangle$ *denote one of the integrals* $(K), (K^*)$, *or* (KN), *and let* $f, g : [a,b] \to X$ *be given functions. Let* $s \in\,]a,b[$ *be given.*

(i) *Let* $\int_a^b \langle f(t), dg(t)\rangle$ *exist. Then* $\int_a^s \langle f(t), dg(t)\rangle$, $\int_s^b \langle f(t), dg(t)\rangle$ *exist.*

(ii) *Let* $\int_a^s \langle f(t), dg(t)\rangle$, $\int_s^b \langle f(t), dg(t)\rangle$ *exist. Then we have*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \int_a^s \langle f(t), dg(t)\rangle + \int_s^b \langle f(t), dg(t)\rangle\,. \tag{1.45}$$

Proof. We restrict ourselves to the KN-integral, the rest is similar. Let $\mathcal{N}$ be a system of negligible sets in $[a,b]$.

(i) Assuming that $(KN)\int_a^b \langle f(t), dg(t)\rangle$ exists, we prove that

$$\forall \varepsilon > 0 \;\; \exists \delta \in \Gamma(a,s) \;\; \exists A \in \mathcal{N} \;\; \forall D, D' \in \mathcal{F}_{\delta,A}(a,s) \,:\, |K_D(f,g) - K_{D'}(f,g)| \le \varepsilon\,, \tag{1.46}$$

and then use Lemma 1.14 to conclude that $(KN)\int_a^s \langle f(t), dg(t)\rangle$ exists.

Let $\varepsilon > 0$ be given. We find $\delta_0 \in \Gamma(a,b)$ and $A \in \mathcal{N}$ such that for every $D_0, D_0' \in \mathcal{F}_{\delta_0,A}(a,b)$ we have

$$|K_{D_0}(f,g) - K_{D_0'}(f,g)| \le \varepsilon\,, \tag{1.47}$$

and for $t \in [a,b]$ set

$$\delta(t) \;=\; \begin{cases} \min\{\delta_0(t), |t-s|\} & \text{for} \quad t \in [a,b]\setminus\{s\}\,, \\ \delta_0(s) & \text{for} \quad t = s\,. \end{cases} \tag{1.48}$$

Let $D, D' \in \mathcal{F}_{\delta,A}(a,s)$ be arbitrary, and let $D^* \in \mathcal{F}_{\delta,A}(s,b)$ be fixed. Then

$$\begin{aligned} D &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m\}\,, \\ D' &= \{(\tau_k', [t_{k-1}', t_k'])\,;\, k = 1,\dots,m'\}\,, \\ D^* &= \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 1,\dots,m^*\}\,, \end{aligned}$$

and we have $\tau_m = t_m = \tau_{m'}' = t_{m'}' = \tau_1^* = t_0^* = s$ by virtue of (1.48) and Lemma 1.3. Set

$$\begin{aligned} D_0 &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m-1\} \cup \{(s, [t_{m-1}, t_1^*])\} \qquad (1.49) \\ &\quad \cup \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 2,\dots,m^*\}\,, \\ D_0' &= \{(\tau_k', [t_{k-1}', t_k'])\,;\, k = 1,\dots,m'-1\} \cup \{(s, [t_{m'-1}', t_1^*])\} \qquad (1.50) \\ &\quad \cup \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 2,\dots,m^*\}\,. \end{aligned}$$

Then $D_0, D_0' \in \mathcal{F}_{\delta,A}(a,b) \subset \mathcal{F}_{\delta_0,A}(a,b)$, hence (1.47) holds. Together with the identity

$$\begin{aligned} K_{D_0}(f,g) - K_{D_0'}(f,g) &= \sum_{j=1}^{m-1} \langle f(\tau_j), g(t_j) - g(t_{j-1})\rangle + \langle f(s), g(t_1^*) - g(t_{m-1})\rangle \\ &\quad - \sum_{k=1}^{m'-1} \left\langle f(\tau_k'), g(t_k') - g(t_{k-1}')\right\rangle - \left\langle f(s), g(t_1^*) - g(t_{m'-1}')\right\rangle \\ &= \sum_{j=1}^{m} \langle f(\tau_j), g(t_j) - g(t_{j-1})\rangle - \sum_{k=1}^{m'} \left\langle f(\tau_k'), g(t_k') - g(t_{k-1}')\right\rangle \\ &= K_D(f,g) - K_{D'}(f,g) \end{aligned}$$

this implies (1.46). We analogously check that $(KN)\int_s^b \langle f(t), dg(t)\rangle$ exists, and (i) is proved.

(ii) Put $J_1 = (KN)\int_a^s \langle f(t), dg(t)\rangle$, $J_2 = (KN)\int_s^b \langle f(t), dg(t)\rangle$. For $\varepsilon > 0$ we find $\delta_1 \in \Gamma(a,s)$, $\delta_2 \in \Gamma(s,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for every $D_1 \in \mathcal{F}_{\delta_1,A_1}(a,s)$, $D_2 \in \mathcal{F}_{\delta_2,A_2}(s,b)$ we have

$$|J_1 - K_{D_1}(f,g)| \le \varepsilon/2\,, \quad |J_2 - K_{D_2}(f,g)| \le \varepsilon/2\,. \tag{1.51}$$

Set $A = A_1 \cup A_2$, and

$$\delta(t) = \begin{cases} \min\{\delta_1(t), s-t\} & \text{for} \quad t \in [a,s[\,, \\ \min\{\delta_2(t), t-s\} & \text{for} \quad t \in]s,b]\,, \\ \min\{\delta_1(s), \delta_2(s)\} & \text{for} \quad t = s\,. \end{cases} \tag{1.52}$$

Let $D \in \mathcal{F}_{\delta,A}(a,b)$ be arbitrary, $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m\}$. We find $k \in \{1,\dots,m\}$ such that $s \in [t_{k-1}, t_k]$. Then $s = \tau_k$ by (1.52), hence $t_{k-1} < s < t_k$, and we may put

$$\begin{aligned} D_1 &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,k-1\} \cup \{(s, [t_{k-1}, s])\}\,, \\ D_2 &= \{(s, [s, t_k])\} \cup \{(\tau_j, [t_{j-1}, t_j])\,;\, j = k+1,\dots,m\}\,. \end{aligned}$$

We have $D_1 \in \mathcal{F}_{\delta_1,A_1}(a,s)$, $D_2 \in \mathcal{F}_{\delta_2,A_2}(s,b)$, and $K_D(f,g) = K_{D_1}(f,g) + K_{D_2}(f,g)$, hence

$$|J_1 + J_2 - K_D(f,g)| \le \varepsilon$$

as a consequence of (1.51), and the proof is complete. ■

In order to preserve the consistency of (1.45) also in the limit cases $s = a$ and $s = b$, we define for each of the integrals (K), (K^*), and (KN)

$$\int_s^s \langle f(t), dg(t)\rangle = 0 \qquad \forall s \in [a,b]\,,\ \forall\, f,g : [a,b] \to X\,. \tag{1.53}$$

We conclude this section by establishing some typical formulas.

Proposition 1.16 *For every $f : [a,b] \to X$, $a \le r \le s \le b$ and $v \in X$ we have*

$$\text{(i)}\ \ (K)\int_a^b \big\langle f(t), d\big(v\,\chi_{\{s\}}\big)(t)\big\rangle = \begin{cases} 0 & \text{if } s \in]a,b[\,, \\ -\langle f(a), v\rangle & \text{if } s = a\,, \\ \langle f(b), v\rangle & \text{if } s = b\,, \end{cases}$$

$$\text{(ii)}\ \ (K)\int_a^b \big\langle f(t), d\big(v\,\chi_{]r,s[}\big)(t)\big\rangle = \langle f(r) - f(s), v\rangle\,.$$

Proof. To check that (i) holds for $s = a$, we put $\delta(t) := t - a$ for $t \in]a,b]$, $\delta(a) := (b-a)/2$. Let $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m\} \in \mathcal{F}_\delta(a,b)$ be an arbitrary partition. By Lemma 1.3 we have $a = \tau_1$, hence $K_D\big(f, v\,\chi_{\{a\}}\big) = -\langle f(a), v\rangle$ and the assertion follows. An analogous argument yields the result for $s = b$. For $a < s < b$ it suffices to use the identity

$$(K)\int_a^b \big\langle f(t), d\big(v\,\chi_{\{s\}}\big)(t)\big\rangle = (K)\int_a^s \big\langle f(t), d\big(v\,\chi_{\{s\}}\big)(t)\big\rangle + (K)\int_s^b \big\langle f(t), d\big(v\,\chi_{\{s\}}\big)(t)\big\rangle$$

as a special case of (1.45). We next use a similar identity

$$(K)\int_a^b \left\langle f(t), d\left(v\chi_{\{\,]r,s[\,\}}\right)(t)\right\rangle \tag{1.54}$$

$$= (K)\int_a^r \langle f(t), dg_1(t)\rangle + (K)\int_r^s \langle f(t), dg_2(t)\rangle + (K)\int_s^b \langle f(t), dg_1(t)\rangle$$

$$- (K)\int_r^s \left\langle f(t), d\left(v\chi_{\{r\}}\right)(t)\right\rangle - (K)\int_r^s \left\langle f(t), d\left(v\chi_{\{s\}}\right)(t)\right\rangle$$

with $g_1(t) \equiv 0$, $g_2(t) \equiv v$ in their corresponding domains, hence

$$(K)\int_a^r \langle f(t), dg_1(t)\rangle = (K)\int_r^s \langle f(t), dg_2(t)\rangle = (K)\int_s^b \langle f(t), dg_1(t)\rangle = 0,$$

and (ii) is obtained as a consequence of (i) and (1.54). ■

Proposition 1.17 *For every* $g : [a,b] \to X$, $a \le r \le s \le b$ *and* $v \in X$ *we have*

(i) $(K)\int_a^b \left\langle v\,\chi_{\{s\}}(t), dg(t)\right\rangle = \langle v, g\rangle(s+) - \langle v, g\rangle(s-)$,

(ii) $(K)\int_a^b \left\langle v\,\chi_{]r,s[}(t), dg(t)\right\rangle = \langle v, g\rangle(s-) - \langle v, g\rangle(r+)$,

provided the limits on the right-hand sides exist, with the convention $\langle v, g\rangle(a-) = \langle v, g(a)\rangle$, $\langle v, g\rangle(b+) = \langle v, g(b)\rangle$.

Proof. We proceed by the same decomposition argument as in the proof of Proposition 1.16 using Lemma 1.5 and its counterpart for $s = b$, as well as the obvious fact that for $f_2(t) \equiv v$ for $t \in [r,s]$ we have $(K)\int_r^s \langle f_2(t), dg(t)\rangle = \langle v, g(s) - g(r)\rangle$. ■

Remark 1.18 Here and in the sequel, note that whenever we integrate functions f, g defined in $[a,b]$ over an interval $[r,s] \subset [a,b]$, we automatically consider their restrictions $f|_{[r,s]}, g|_{[r,s]}$. In particular, we have e. g. $\left\langle v, f|_{[r,s]}\right\rangle(s+) = \langle v, f(s)\rangle$, $\left\langle v, f|_{[r,s]}\right\rangle(r-) = \langle v, f(r)\rangle$.

2 Regulated functions and total variation

Let Y be a Banach space endowed with norm $\|\cdot\|$. For a given function $g:[a,b]\to Y$ and a given division $d\in\mathcal{D}_{a,b}$ of the form (1.1) we define the *variation* $\mathcal{V}_d(g)$ of g on d by the formula

$$\mathcal{V}_d(g) := \sum_{j=1}^{m} \|g(t_j)-g(t_{j-1})\|$$

and the *total variation* $\operatorname{Var}_{[a,b]} g$ of g by

$$\operatorname*{Var}_{[a,b]} g := \sup\{\mathcal{V}_d(g)\,;\, d\in\mathcal{D}_{a,b}\}\,.$$

In a standard way (cf. [7]) we denote the set of functions of bounded variation by

$$BV(a,b\,;Y) := \{g:[a,b]\to Y\,;\, \operatorname*{Var}_{[a,b]} g<\infty\}\,. \tag{2.1}$$

Let us further introduce the set $S(a,b\,;Y)$ of all *step functions* of the form

$$w(t) := \sum_{k=0}^{m} \hat c_k\,\chi_{\{t_k\}}(t) + \sum_{k=1}^{m} c_k\,\chi_{]t_{k-1},t_k[}(t)\,,\quad t\in[a,b]\,, \tag{2.2}$$

where $d=\{t_0,\dots,t_m\}\in\mathcal{D}_{a,b}$ is a given division, and $\hat c_0,\dots,\hat c_m,c_1,\dots,c_m$ are given elements from Y.

2.1 Regulated functions

It is well-known (see e.g. the Appendix of [7]) that every function of bounded variation with values in a Banach space admits one-sided limits at each point of its domain of definition. Following [2], we separate this property from the notion of total variation and introduce the following definition.

Definition 2.1 *We say that a function $f:[a,b]\to Y$ is* regulated *if for every $t\in[a,b]$ there exist both one-sided limits $f(t+),f(t-)\in Y$ with the convention $f(a-)=f(a)$, $f(b+)=f(b)$.*

According to [18, 36], we denote by $G(a,b\,;Y)$ the set of all regulated functions $f:[a,b]\to Y$, and by $G_L(a,b\,;Y)$ and $G_R(a,b\,;Y)$ the space of left-continuous and right-continuous regulated functions on $[a,b]$, respectively. We further set $BV_L(a,b\,;Y)=BV(a,b\,;Y)\cap G_L(a,b\,;Y)$, $BV_R(a,b\,;Y)=BV(a,b\,;Y)\cap G_R(a,b\,;Y)$, and $S_L(a,b\,;Y)=S(a,b\,;Y)\cap G_L(a,b\,;Y)$, $S_R(a,b\,;Y)=S(a,b\,;Y)\cap G_R(a,b\,;Y)$. Let us introduce in $G(a,b\,;Y)$ a system of seminorms

$$\|f\|_{[s,t]} := \sup\{\|f(\tau)\|\,;\,\tau\in[s,t]\} \tag{2.3}$$

for any subinterval $[s,t]\subset[a,b]$. Indeed, $\|\cdot\|_{[a,b]}$ is a norm.

For a given function $g\in G(a,b\,;Y)$ and a given division $d\in\mathcal{D}_{a,b}$ we define the *essential variation* $\overline{\mathcal{V}}_d(g)$ of g on d by the formula

$$\overline{\mathcal{V}}_d(g) := \sum_{j=1}^{m}\|g(t_j-)-g(t_{j-1}+)\| + \sum_{j=0}^{m}\|g(t_j+)-g(t_j-)\|$$

and the *total essential variation* $\overline{\mathrm{Var}}_{[a,b]}\, g$ of g by

$$\overline{\mathrm{Var}}_{[a,b]}\, g \ := \ \sup\{\overline{\mathcal{V}}_d(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

We denote the space of functions of *essentially bounded variation* by

$$\overline{BV}(a,b\,;\,Y) \ := \ \{g : [a,b] \to Y\,;\, \overline{\mathrm{Var}}_{[a,b]}\, g < \infty\}\,. \tag{2.4}$$

The terminology has been taken from [15], although we restrict ourselves a priori to regulated functions which makes the analysis easier. This however means here in particular that $\mathcal{V}_d(g)$ is defined for *every* function $g : [a,b] \to Y$, but $\overline{\mathcal{V}}_d(g)$ only for a regulated function g.

A function $f : [a,b] \to Y$ is called *absolutely continuous*, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the implication

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \quad \Longrightarrow \quad \sum_{k=1}^{n} \|u(b_k) - u(a_k)\| < \varepsilon \tag{2.5}$$

holds for every sequence of intervals $]a_k, b_k[\, \subset [a,b]$ such that $]a_k,b_k[\, \cap\,]a_j,b_j[\, = \emptyset$ for $k \neq j$. We denote by $AC(a,b\,;\,Y)$ the space of all absolutely continuous functions $f : [a,b] \to Y$.

We summarize some easy basic properties of the above spaces in Lemma 2.2 below the proof of which is left to the reader.

Lemma 2.2

(i) *Every regulated function is bounded.*

(ii) *We have* $\overline{\mathrm{Var}}_{[a,b]}\, g \le \mathrm{Var}_{[a,b]}\, g$ *for every* $g \in G(a,b\,;\,Y)$ *and* $\overline{\mathrm{Var}}_{[a,b]}\, g = \mathrm{Var}_{[a,b]}\, g$ *for every* $g \in G_L(a,b\,;\,Y) \cup G_R(a,b\,;\,Y)$.

(iii) *The sets* $AC(a,b\,;\,Y)$, $S(a,b\,;\,Y)$, $BV(a,b\,;\,Y)$, $\overline{BV}(a,b\,;\,Y)$, $G(a,b\,;\,Y)$ *are vector spaces satisfying the inclusions*

$$\Big(AC(a,b\,;\,Y) \cup S(a,b\,;\,Y)\Big) \subset BV(a,b\,;\,Y) \subset \overline{BV}(a,b\,;\,Y) \subset G(a,b\,;\,Y)\,.$$

(iv) *The space* $G(a,b\,;\,Y)$ *is complete and non-separable with respect to the norm* $\|\cdot\|_{[a,b]}$.

(v) *Given* $C > 0$, *the set* $V_C := \{g \in \overline{BV}(a,b\,;\,Y)\,;\, \overline{\mathrm{Var}}_{[a,b]}\, g \le C\}$ *is closed in* $G(a,b\,;\,Y)$.

(vi) *The space* $C(a,b\,;\,Y)$ *of* continuous functions $f : [a,b] \to Y$ *is a closed subspace of* $G(0,T\,;\,X)$, *and* $AC(a,b\,;\,Y)$ *is a dense subspace of* $C(a,b\,;\,Y)$ *with respect to the norm* $\|\cdot\|_{[a,b]}$.

Let us denote by $\mathbb{R}_+$ the interval $[0,\infty[$ and by Φ the set of all increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = \varphi(0+) = 0$, $\varphi(+\infty) = +\infty$. For $\varphi \in \Phi$, $g : [a,b] \to Y$ and a division $d \in \mathcal{D}_{a,b}$ of the form (1.1) we define the φ*-variation* $\mathcal{V}_d^{\varphi}(g)$ of g on d by the formula

$$\mathcal{V}_d^{\varphi}(g) \ := \ \sum_{j=1}^{m} \varphi\left(\|g(t_j) - g(t_{j-1})\|\right)$$

and the *total φ-variation* $\varphi\text{–Var}_{[a,b]}\, g$ of g by

$$\varphi\text{–}\underset{[a,b]}{\text{Var}}\, g \;:=\; \sup\{\mathcal{V}_d^{\varphi}(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

This concept can be used for an alternative characterization of regulated functions.

Proposition 2.3 *A function $f : [a,b] \to Y$ is regulated if and only if there exists $\varphi \in \Phi$ such that $\varphi\text{–Var}_{[a,b]}\, f \le 1$.*

Proof. Let f be regulated. Then for every $r > 0$, the number of pairwise disjoint intervals $]a_k, b_k[\subset [a,b]$ such that $\|f(b_k) - f(a_k)\| \ge r$ is bounded above by some $N(r) \in \mathbb{N}$. In particular, we may take $N(r) = 1$ for $r > 2\,\|f\|_{[a,b]}$ and assume that $N :]0,\infty[\to \mathbb{N}$ is non-increasing. Set $R = \|f\|_{[a,b]}$. We claim that the assertion holds provided we put

$$\varphi(r) \;=\; \frac{r}{2R\,N(r/2)} \qquad \text{for } r > 0\,. \tag{2.6}$$

Indeed, then $\varphi \in \Phi$, and putting

$$M_k \;=\; \left\{ j \in \{1,\dots,m\}\,;\, \|f(t_j) - f(t_{j-1})\| \in]2^{-k+1}R, 2^{-k+2}R] \right\}\,, \qquad k \in \mathbb{N}$$

for an arbitrary division $d = \{t_0,\dots,t_m\} \in \mathcal{D}_{a,b}$, we obtain that

$$\begin{aligned}\sum_{j=1}^{m} \varphi(\|f(t_j) - f(t_{j-1})\|) \;&=\; \sum_{k=1}^{\infty}\sum_{j\in M_k} \varphi(\|f(t_j) - f(t_{j-1})\|)\\ &\le\; \sum_{k=1}^{\infty} N(2^{-k+1}R)\,\varphi(2^{-k+2}R) \;=\; \sum_{k=1}^{\infty} 2^{-k+1} \;=\; 1\,.\end{aligned}$$

Conversely, let $\varphi\text{–Var}_{[a,b]}\, f \le 1$, and let $t \in]a,b]$ be arbitrary. Assume that the limit $f(t-)$ does not exist. Then there exists $\varepsilon > 0$ and a sequence $t_k \nearrow t$ such that $\|f(t_{k+1}) - f(t_k)\| \ge \varepsilon$ for all $k \in \mathbb{N}$. For every $m \in \mathbb{N}$ we have $1 \ge \sum_{k=1}^{m} \varphi(\|f(t_{k+1}) - f(t_k)\|) \ge m\,\varphi(\varepsilon)$ which is a contradiction. We similarly check that $f(t+)$ exists for all $t \in [a,b[$, and the proof is complete. ■

We now follow the lines of [18] and investigate some local properties of regulated functions.

Proposition 2.4

(i) *Let $f \in G(a,b\,;Y)$ and $\varepsilon > 0$ be given. Then there exists a division $d = \{t_0,\dots,t_m\} \in \mathcal{D}_{a,b}$ such that for every $j = 1,\dots,m$ we have*

$$t_{j-1} < \tau < t < t_j \quad\Longrightarrow\quad \|f(t) - f(\tau)\| < \varepsilon\,. \tag{2.7}$$

In particular, the set of discontinuity points of a regulated function is at most countable.

(ii) *For every $f \in G(a,b\,;Y)$ and $\varepsilon > 0$ there exists $w \in S(a,b\,;Y)$ such that $\|f - w\|_{[a,b]} \le \varepsilon$, $w(t) \in \cup_{\tau\in[a,b]}\{f(\tau)\}$ for every $t \in [a,b]$, $\text{Var}_{[a,b]}\, w \le \text{Var}_{[a,b]}\, f$ and $\overline{\text{Var}}_{[a,b]}\, w \le \overline{\text{Var}}_{[a,b]}\, f$.*

(iii) *Let $g \in G(a,b\,;Y)$ and $C > 0$ be given, and assume that $\text{Var}_{[a,s]}\, g \le C$ for every $s \in]a,b[$. Then $g \in BV(a,b\,;Y)$ and $\text{Var}_{[a,b]}\, g = \lim_{s\to b-} \text{Var}_{[a,s]}\, g + \|g(b) - g(b-)\|$.*

Proof.

(i) Put $S_f^\varepsilon = \{a\} \cup \{b\} \cup \{t \in [a,b]\,;\, \max\{\|f(t)-f(t-)\|, \|f(t+)-f(t)\|, \|f(t+)-f(t-)\|\} \ge \varepsilon\}$. The set S_f^ε is finite, $S_f^\varepsilon = \{s_0, s_1, \dots, s_\ell\}$, $a = s_0 < s_1 < \dots < s_\ell = b$, and the set $\cup_{n=1}^\infty S_f^{1/n}$ of all discontinuity points of f is at most countable. For $i = 1, \dots, \ell$ put

$$f_i(t) \;=\; \begin{cases} f(t) & \text{for } \; t \in]s_{i-1}, s_i[\,, \\ f(s_{i-1}+) & \text{for } \; t = s_{i-1} \,, \\ f(s_i-) & \text{for } \; t = s_i \,, \end{cases} \tag{2.8}$$

and

$$h_i \;=\; \inf\{t - \tau\,;\, s_{i-1} \le \tau < t \le s_i\,,\; \|f_i(t) - f_i(\tau)\| \ge \varepsilon\}\,. \tag{2.9}$$

By construction we have $h_i > 0$ for all i, and choosing any division $d_i = \{\tau_0, \tau_1, \dots, \tau_{m_i}\}$, $s_{i-1} = \tau_0 < \tau_1 < \dots < \tau_{m_i} = s_i$ such that $\tau_j - \tau_{j-1} < h_i$ for $j = 1, \dots, m_i$ and $i = 1, \dots, \ell$, we may simply put $d = d_1 \cup d_2 \cup \dots \cup d_\ell$.

(ii) Let $\varepsilon > 0$ be given and let $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ be as in (i). For each $j = 1, \dots, m$ we fix an arbitrary continuity point $\tau_j \in]t_{j-1}, t_j[$ of f, and put

$$\begin{aligned} w(t_j) \;&=\; f(t_j) \\ w(t) \;&=\; f(\tau_j) \quad \text{for } \; t \in]t_{j-1}, t_j[\,, \end{aligned}$$

for every $j = 1, \dots, m$. Then $w \in S(a,b\,;Y)$ and from (i) we immediately obtain that $\|f(t) - w(t)\| \le \varepsilon$ for every $t \in [a,b]$. Moreover, putting $\hat{d} = \{t_0, \tau_1, t_1, \tau_2, t_2, \dots, \tau_m, t_m\} \in \mathcal{D}_{a,b}$ we have $\mathrm{Var}_{[a,b]}\, w = \mathcal{V}_{\hat{d}}(f)$, $\overline{\mathrm{Var}}_{[a,b]}\, w \le \overline{\mathcal{V}}_{\hat{d}}(f)$, and the assertion follows.

(iii) Let $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ be an arbitrary division. For $t_{m-1} < s < b$ set $\tilde{d} = \{t_0, t_1, \dots, t_{m-1}, s\}$. Then

$$\mathcal{V}_d(g) \;=\; \mathcal{V}_{\tilde{d}}(g) + \|g(b) - g(s)\|\,, \tag{2.10}$$

hence $\mathcal{V}_d(g) \le \mathrm{Var}_{[a,s]}\, g + \|g(b) - g(s)\|$, and letting s tend to $b-$ we obtain

$$\operatorname*{Var}_{[a,b]} g \;\le\; \liminf_{s \to b-} \operatorname*{Var}_{[a,s]} g + \|g(b) - g(b-)\|\,.$$

Conversely, for an arbitrary division $\tilde{d} \in \mathcal{D}_{a,s}$ of the above form put $d = \{t_0, t_1, \dots, t_{m-1}, s, b\}$. From (2.10) we then obtain that $\mathrm{Var}_{[a,s]}\, g \le \mathrm{Var}_{[a,b]}\, g - \|g(b) - g(s)\|$, and the assertion follows. The proof of Proposition 2.4 is complete. ∎

2.2 Kurzweil integration of regulated functions

In the sequel we restrict ourselves to a separable Hilbert space X as in Definition 1.4 and investigate properties of the Kurzweil integration in the space of regulated functions. Our consideration will from now on focus on one type of the Kurzweil integral which is particularly suitable for applications to variational inequalities, namely we write simply

$$\int_a^b \langle f(t), dg(t)\rangle = (KN) \int_a^b \langle f(t), dg(t)\rangle \;, \quad \mathcal{N} \;=\; \{A \subset [a,b]\,;\, A \;\text{ at most countable }\}\,. \tag{2.11}$$

Propositions 1.13, 1.16 and 1.17 enable us to evaluate the integral $\int_a^b \langle f(t), dg(t)\rangle$ provided one of the functions f, g belongs to $S(a,b\,;X)$. The next strategy consists in exploiting the density of $S(a,b\,;X)$ in $G(a,b\,;X)$ established in Proposition 2.4 (ii). We first notice that for all functions $f,g:[a,b]\to X$ and every partition D of the form (1.2) we have

$$\begin{aligned} K_D(f,g) &= \sum_{j=1}^{m} \langle f(\tau_j), g(t_j)-g(t_{j-1})\rangle \\ &= \langle f(b),g(b)\rangle - \langle f(a),g(a)\rangle - \sum_{j=0}^{m} \langle f(\tau_{j+1})-f(\tau_j), g(t_j)\rangle \end{aligned} \tag{2.12}$$

where we put $\tau_0 := a$, $\tau_{m+1} := b$. Let now g belong to $G(a,b\,;X)$ and let A be the set of all discontinuity points of g. For an arbitrary $\delta\in\Gamma(a,b)$ and $D\in\mathcal{F}_{\delta,A}(a,b)$ we then obtain from (2.12) that

$$|K_D(f,g)| \;\le\; \min\left\{ \|f\|_{[a,b]} \overline{\operatorname*{Var}_{[a,b]}}\, g\,,\, \left(|f(a)|+|f(b)|+\operatorname*{Var}_{[a,b]} f\right) \|g\|_{[a,b]} \right\}. \tag{2.13}$$

Indeed, interesting cases are those where the right-hand side of (2.13) is bounded. The extension of the Kurzweil integral to $G(a,b\,;X)$ is based on the following convergence theorem.

Theorem 2.5 *Let $g,f,f_n:[a,b]\to\mathbb{R}$ be given for $n\in\mathbb{N}$ such that $\lim_{n\to\infty}\|f-f_n\|_{[a,b]}=0$. Then the following implications hold.*

(i) *If $g\in\overline{BV}(a,b\,;X)$ and $\int_a^b \langle f_n(t), dg(t)\rangle$ exists for each $n\in\mathbb{N}$, then $\int_a^b \langle f(t), dg(t)\rangle$ exists and we have*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \lim_{n\to\infty}\int_a^b \langle f_n(t), dg(t)\rangle$$

(ii) *If $g\in BV(a,b\,;X)$ and $\int_a^b \langle g(t), df_n(t)\rangle$ exists for each $n\in\mathbb{N}$, then $\int_a^b \langle g(t), df(t)\rangle$ exists and we have*

$$\int_a^b \langle g(t), df(t)\rangle \;=\; \lim_{n\to\infty}\int_a^b \langle g(t), df_n(t)\rangle$$

Proof.

(i) For $n\in\mathbb{N}$ put $J_n := \int_a^b \langle f_n(t), dg(t)\rangle$. Let A be the set of all discontinuity points of g. For each n we find $\delta_n\in\Gamma(a,b)$ and $A\subset A_n\in\mathcal{N}$ such that for every $D\in\mathcal{F}_{\delta_n,A_n}(a,b)$ we have

$$|K_D(f_n,g) \,-\, J_n| \;<\; \frac{1}{n}\,.$$

For $m,n\in\mathbb{N}$ put $\delta_{mn} := \min\{\delta_m,\delta_n\}$, $A_{mn}=A_m\cup A_n$. For every $D\in\mathcal{F}_{\delta_{mn},A_{mn}}(a,b)$ we have

$$|K_D(f_n,g) \,-\, J_n| \;<\; \frac{1}{n}\,,\quad |K_D(f_m,g) \,-\, J_m| \;<\; \frac{1}{m}\,,$$

and (2.13) implies that

$$\begin{aligned}|J_n - J_m| \ &\le\ |K_D(f_n, g) \,-\, J_n \,-\, K_D(f_m, g) \,+\, J_m| \ +\ |K_D(f_n - f_m, g)|\\ &\le\ \frac{1}{m} + \frac{1}{n} + \|f_n - f_m\|_{[a,b]} \operatorname*{\overline{Var}}_{[a,b]} g\,,\end{aligned}$$

hence $\{J_n\}$ is a Cauchy sequence and we may put $J := \lim_{n\to\infty} J_n$. For each $D \in \mathcal{F}_{\delta_n, A_n}(a,b)$ we then have

$$\begin{aligned}|K_D(f,g) \,-\, J| \ &\le\ |K_D(f - f_n, g)| \,+\, |K_D(f_n, g) \,-\, J_n| \,+\, |J_n - J|\\ &\le\ \|f - f_n\|_{[a,b]} \operatorname*{\overline{Var}}_{[a,b]} g \,+\, 1/n \,+\, |J_n - J|\,,\end{aligned}$$

hence $\int_a^b \langle f(t), dg(t)\rangle = J$ and **(i)** is proved. The same argument based on (2.13) with $f := g$, $g := f_n - f_m$ yields **(ii)**. ■

Corollary 2.6 *If* $f \in G(a,b\,;\,X)$ *and* $g \in \overline{BV}(a,b\,;\,X)$, *then* $\int_a^b \langle f(t), dg(t)\rangle$ *exists and satisfies the estimate*

$$\left|\int_a^b \langle f(t), dg(t)\rangle\right| \ \le\ \|f\|_{[a,b]} \operatorname*{\overline{Var}}_{[a,b]} g\,. \tag{2.14}$$

Moreover, for every $g \in \overline{BV}(a,b\,;\,X)$ *we have*

$$\operatorname*{\overline{Var}}_{[a,b]} g \ =\ \sup\left\{\int_a^b \langle f(t), dg(t)\rangle\ ;\ f \in S(a,b\,;\,B_1(0))\right\}\,, \tag{2.15}$$

where $B_r(x_0)$ *for* $r > 0$ *and* $x_0 \in X$ *denotes the ball* $\{x \in X\,;\,|x - x_0| \le r\}$ *centered at* x_0 *with radius* r.

Proof. Using Proposition 2.4 (ii) we approximate f uniformly by step functions. Passing to the limit we obtain the existence of $\int_a^b \langle f(t), dg(t)\rangle$ and inequality (2.14) directly from Theorem 2.5 and identity (2.13). To prove (2.15), we consider an arbitrary $\varepsilon > 0$, and find $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that

$$\sum_{j=1}^m |g(t_j-) - g(t_{j-1}+)| + \sum_{j=0}^m |g(t_j+) - g(t_j-)| \ \ge\ \operatorname*{\overline{Var}}_{[a,b]} g \,-\, \varepsilon\,. \tag{2.16}$$

Let $\sigma : X \to X$ be the function

$$\sigma(x) = x/|x| \quad \text{for}\ \ x \ne 0\,, \quad \sigma(0) = 0\,, \tag{2.17}$$

and put

$$f(t) \ =\ \sum_{j=1}^m \sigma(g(t_j-) - g(t_{j-1}+))\,\chi_{\,]t_{j-1},t_j[}\,(t) + \sum_{j=0}^m \sigma(g(t_j+) - g(t_j-))\,\chi_{\{t_j\}}\,(t)\,. \tag{2.18}$$

Then $f \in S(a,b;B_1(0))$. From Proposition 1.17 it follows that

$$\int_a^b \langle f(t), dg(t)\rangle \ =\ \sum_{j=1}^m |g(t_j-) - g(t_{j-1}+)| + \sum_{j=0}^m |g(t_j+) - g(t_j-)|\,, \tag{2.19}$$

and we obtain the assertion from (2.16). ■

Corollary 2.7 *If* $f \in BV(0,T\,;\,X)$ *and* $g \in G(0,T\,;\,X)$, *then* $\int_a^b \langle f(t), dg(t)\rangle$ *exists and satisfies the estimates*

$$\left|\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle\right| \;\le\; \Big(|f(b)| + \operatorname*{Var}_{[a,b]} f\Big)\, \|g\|_{[a,b]}\,, \tag{2.20}$$

$$\left|\langle f(b), g(b)\rangle - \int_a^b \langle f(t), dg(t)\rangle\right| \;\le\; \Big(|f(a)| + \operatorname*{Var}_{[a,b]} f\Big)\, \|g\|_{[a,b]}\,. \tag{2.21}$$

Moreover, for every $f \in BV(0,T\,;\,X)$ *we have*

$$|f(b)| + \operatorname*{Var}_{[a,b]} f \;=\; \sup\left\{\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle\ ;\ g \in S_R(a,b\,;\,B_1(0))\right\}, \tag{2.22}$$

$$|f(a)| + \operatorname*{Var}_{[a,b]} f \;=\; \sup\left\{\langle f(b), g(b)\rangle - \int_a^b \langle f(t), dg(t)\rangle\ ;\ g \in S_L(a,b\,;\,B_1(0))\right\}. \tag{2.23}$$

Proof. The existence of $\int_a^b \langle f(t), dg(t)\rangle$ is obtained similarly as in Corollary 2.6, and inequalities (2.20) – (2.21) follow from (2.12). To prove (2.22) – (2.23), we fix $\varepsilon > 0$ and find a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that

$$\sum_{j=1}^{m} |f(t_j) - f(t_{j-1}| \;\ge\; \operatorname*{Var}_{[a,b]} f \,-\, \varepsilon\,. \tag{2.24}$$

Let $\sigma : X \to X$ be as in (2.17, and for $t \in [a,b]$ put

$$g(t) \;=\; \sigma(f(b))\,\chi_{\{b\}}(t) - \sum_{j=1}^{m} \sigma(f(t_j) - f(t_{j-1}))\,\chi_{[t_{j-1},t_j[}(t)\,. \tag{2.25}$$

We then infer from (2.24), Propositions 1.13 and 1.16 that

$$\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \;=\; |f(b)| + \sum_{j=1}^{m} |f(t_j) - f(t_{j-1}| \;\ge\; |f(b)| + \operatorname*{Var}_{[a,b]} f - \varepsilon\,, \tag{2.26}$$

and (2.22) follows from (2.20) and (2.26). The proof of (2.23) is analogous. ■

The relation between Kurzweil's integral and other integration concepts has been discussed in detail in [33]. We mention in Proposition 2.9 below only one result which is directly related to variational inequalities.

Let us denote by $L^1(a,b\,;\,X)$ the space of Bochner integrable functions $u : [a,b] \to X$ endowed with norm $|u|_1 = (L)\int_a^b |u(t)|\,dt$, where (L) stands for the Lebesgue integration (for more information about the Bochner integral, see e. g. [40] or [7, Appendix]). We cite without proof the following result which is a special case of [7, Corollary A.2].

Theorem 2.8 *For every function* $g \in AC([a,b]\,;X)$ *there exists an element* $\dot g \in L^1(a,b\,;X)$ *such that*

(i) $\dot g(t) = \lim_{h\to 0} \dfrac{g(t+h) - g(t)}{h}$ *a. e.*,

(ii) $g(t) - g(s) = (B)\displaystyle\int_s^t \dot g(\tau)\,d\tau$ *for all* $a \le s < t \le b$, *where (B) denotes the Bochner integration.*

Note that this representation theorem does not hold for a general Banach space X, and a discussion on this subject can be found in the Appendix of [7].

According to Theorem 2.8, it is justified to denote similarly as in the scalar-valued case by $W^{1,1}(a,b\,;X)$ the space of absolutely continuous functions with values in a separable Hilbert space X. The space $W^{1,1}(a,b\,;X)$ is a Banach space endowed with norm $|g|_{1,1} = |g(a)|+|\dot g|_1$. For $1 \le p < \infty$ we similarly introduce in a standard way the Banach spaces $W^{1,p}(a,b\,;X)$ endowed with norm $|g|_{1,p} = |g(a)| + |\dot g|_p$, where $|u|_p = (L)\left(\int_a^b |u(t)|^p\,dt\right)^{(1/p)}$.

In the next sections, we will make use of the following identity.

Proposition 2.9 *If $f \in G(a,b\,;X)$ and $g \in W^{1,1}(a,b\,;X)$, then*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; (L)\int_a^b \langle f(t), \dot g(t)\rangle\,dt\,.$$

Proof. The assertion holds for each function $f \in S(a,b\,;X)$ by virtue of Proposition 1.17 and Theorem 2.8. Approximating an arbitrary function $f \in G(a,b\,;X)$ uniformly by step functions, we may use Theorem 2.5 to complete the proof. ■

As an easy consequence of Corollaries 2.6, 2.7, we have the following convergence result.

Proposition 2.10 *Consider $f, f_n \in G(0,T\,;X)$, $g, g_n \in \overline{BV}(0,T\,;X)$, $n \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \|f-f_n\|_{[a,b]} = 0\,,\quad \lim_{n\to\infty}\|g-g_n\|_{[a,b]} = 0\,,\quad \sup_{n\in\mathbb{N}} \overline{\operatorname*{Var}_{[a,b]}}\, g_n \;=\; C \;<\; \infty\,.$$

Then

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \lim_{n\to\infty}\int_a^b \langle f_n(t), dg_n(t)\rangle\,. \tag{2.27}$$

Proof. For any $w \in S(a,b\,;X)$ we have by Corollaries 2.6 and 2.7 that

$$\begin{aligned}\left|\int_a^b \langle f(t), dg(t)\rangle - \int_a^b \langle f_n(t), dg_n(t)\rangle\right| &\le \left|\int_a^b \langle (f-f_n)(t), dg_n(t)\rangle\right| \\ &+ \left|\int_a^b \langle (f-w)(t), d(g-g_n)(t)\rangle\right| + \left|\int_a^b \langle w(t), d(g-g_n)(t)\rangle\right| \\ &\le C\,\|f-f_n\|_{[a,b]} + 2C\,\|f-w\|_{[a,b]} + \left(2\,\|w\|_{[a,b]} + \operatorname*{Var}_{[a,b]} w\right)\|g-g_n\|_{[a,b]}\end{aligned}$$

and the assertion follows from Proposition 2.4 (ii). ■

Notice that the pointwise convergence $g_n(t) \to g(t)$ for every $t \in [a,b]$ is not sufficient in Proposition 2.10 as in the case of the Riemann-Stieltjes integral. In the example $X = \mathbb{R}$,

$$f_n(t) = f(t) = \chi_{\{0\}}(t)\,,\quad g(t) \equiv 0\,,\quad g_n(t) = \chi_{]0,1/n[}(t)\qquad \text{for } t \in [0,1] \tag{2.28}$$

we have $\int_0^1 f_n(t)\,dg_n(t) = 1$ for every $n \in \mathbb{N}$, $\int_0^1 f(t)\,dg(t) = 0$, hence the assertion of Proposition 2.10 does not hold. Similarly, putting

$$f_n(t) \;:=\; \sum_{k=1}^{n} (-1)^{k-1} \chi_{\{k/n^2\}}(t) \qquad \text{for } t \in [0,1]\,, \tag{2.29}$$

$$g_n(t) \;:=\; \begin{cases} \frac{1}{2n}\left((-1)^k + 1\right) & \text{for } t \in \left[\frac{k-1}{n^2}, \frac{k}{n^2}\right[\;, \quad k = 1,\dots,n\,, \\ 0 & \text{for } t \in \left[\frac{1}{n}, 1\right]\,, \end{cases} \tag{2.30}$$

for $n \in \mathbb{N}$, we see that $f_n, g_n \in S(a,b\,;X)$, $\|f_n\|_{[a,b]} \le 1$, $\mathrm{Var}_{[a,b]}\, g_n \le 1$, $\|g_n\|_{[a,b]} \to 0$ and $f_n(t) \to 0$ for every $t \in [a,b]$ as $n \to \infty$, while $\int_a^b f_n(t)\,dg_n(t) \to 1$.

Below in Section 5 we will consider different types of weak convergence in $G(a,b\,;X)$ which lead to more general convergence theorems. We conclude this section with two integration-by-parts formulas.

Proposition 2.11 *For every $f \in G(a,b\,;X)$, $g \in BV(a,b\,;X)$ we have*

$$\begin{aligned} \int_a^b \langle f(t), dg(t)\rangle + \int_a^b \langle g(t), df(t)\rangle \;&=\; \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &+ \sum_{t\in[a,b]} \Big(\langle f(t) - f(t-)\,,\, g(t) - g(t-)\rangle - \langle f(t+) - f(t)\,,\, g(t+) - g(t)\rangle \Big)\,. \end{aligned} \tag{2.31}$$

Proof. From Proposition 2.4 (i) it follows that the sum on the right-hand side of (2.31) is at most countable, hence the formula is meaningful. Using Proposition 1.16 we check in a straightforward way that (2.31) holds for every $g \in BV(a,b\,;X)$ whenever f is of the form $v\,\chi_{\{r\}}$ or $v\,\chi_{]r,s[}$, hence also for every $f \in S(a,b\,;X)$ by Proposition 1.13. For $f \in G(a,b\,;X)$ and $n \in \mathbb{N}$ we find $f_n \in S(a,b\,;X)$ such that $\|f - f_n\|_{[a,b]} \to 0$ as $n \to \infty$. Using Proposition 2.10 and the obvious inequality $\sum_{t\in[a,b]}(|g(t) - g(t-)| + |g(t+) - g(t)|) \;\le\; \mathrm{Var}_{[a,b]}\, g$ we pass to the limit. ∎

Corollary 2.12 *For every $g \in BV(a,b\,;X)$ we have*

$$\int_a^b \langle g(t+), dg(t)\rangle \;=\; \frac{1}{2}\left(|g(b)|^2 - |g(a)|^2\right) + \frac{1}{2}\sum_{t\in[a,b]} |g(t+) - g(t-)|^2\,. \tag{2.32}$$

Proof. The function $g_+(t) := g(t+)$ satisfies $g_+(t+) = g(t+) = g_+(t)$ for every $t \in [a,b]$ and $g_+(t-) = g(t-)$ for every $t \in\,]a,b]$, and by Proposition 2.11 we have

$$\int_a^b \langle g_+(t), dg_+(t)\rangle \;=\; \frac{1}{2}\left(|g(b)|^2 - |g(a+)|^2\right) + \frac{1}{2}\sum_{t\in\,]a,b]} |g(t+) - g(t-)|^2$$

(note that the sum is taken over the semi-open interval $]a,b]$), while (1.24) yields that

$$\int_a^b \langle g_+(t), d(g - g_+)(t)\rangle \;=\; \langle g(a+), g(a+) - g(a)\rangle\,.$$

Combining the above identities we obtain the assertion. ∎

3 An abstract variational inequality

In this section we explain a method of solving an abstract evolution variational problem which we denote as Problem $\mathcal{P}(u,\xi^0)$. In [28, 29] it is called a *sweeping process*, and we proceed in principle in the same spirit with arguments adapted to the Kurzweil integral setting for regulated functions with values in a separable Hilbert space X. We consider similarly as in Section A.6 a family $Z(v) \subset X$ of convex closed sets parametrized by elements v of a closed subset V of a Banach space Y. The scalar product and norm in X as denoted as above by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, for the norm in Y we use the symbol $\|\cdot\|$. We first state the problem for a general convex set evolution restricting ourselves to right-continuous regulated inputs, and then extend the results to general regulated inputs under more specific assumptions on $Z(v)$. We also show that in this case, the solutions are absolutely continuous if the inputs are absolutely continuous, and the input-output mapping is continuous in $W^{1,p}(0,T\,;X)$ for $1 \le p < \infty$.

3.1 Statement of the problem

In a general setting, our model problem reads as follows.

Definition 3.1 *Let $T > 0$, $u \in G_R(0,T\,;V)$, and $\xi^0 \in Z(u(0))$ be given. We say that $\xi \in BV_R(0,T\,;X)$ is a solution to Problem $\mathcal{P}(u,\xi^0)$ if*

(i) $\xi(t) \in Z(u(t)) \qquad \forall t \in [0,T]$,

(ii) $\xi(0) = \xi^0$,

(iii) $\int_0^T \langle w(t)-\xi(t), d\xi(t)\rangle \;\ge\; 0 \qquad \forall w \in \mathcal{T}(u)$,

where $\mathcal{T}(u)$ is the set of admissible test functions

$$\mathcal{T}(u) \;=\; \{w \in G(0,T\,;X)\,;\, w(t) \in Z(u(t)) \;\; \forall t \in [0,T]\}\,. \tag{3.1}$$

By $\mathrm{Dom}\,(\mathcal{P})$ we denote the set of all pairs (u,ξ^0) with $u \in G_R(0,T\,;V)$, $\xi^0 \in Z(u(0))$, for which Problem $\mathcal{P}(u,\xi^0)$ has a solution.

We will assume that Hypothesis A.26 holds, and denote for simplicity as in Subsection A.6 by (P_v,Q_v) the projection pair $(P_{Z(v)},Q_{Z(v)})$ for every $v \in V$. Then $Q_{u(t)}x \in \mathcal{T}(u)$ for every $x \in X$ and $u \in G_R(0,T\,;V)$, hence $\mathcal{T}(u)$ is non-empty.

Lemma 3.2 *Let Hypothesis A.26 hold, and let $\xi \in BV_R(0,T\,;X)$ be a solution to Problem $\mathcal{P}(u,\xi^0)$ for some given $(u,\xi^0) \in \mathrm{Dom}\,(\mathcal{P})$. Then we have*

(i) $\displaystyle\int_s^t \langle w(\tau)-\xi(\tau), d\xi(\tau)\rangle \;\ge\; 0$ *for all* $0 \le s < t \le T$ *and* $w \in \mathcal{T}(u)$,

(ii) $\xi(t) \;=\; Q_{u(t)}\,\xi(t-)$ *for every* $t \in [0,T]$,

(iii) *If $\xi \in W^{1,1}(0,T\,;X)$, then there exists a set $A \subset\,]0,T[$ of zero Lebesgue measure such that $\left\langle z-\xi(t), \dot\xi(t)\right\rangle \ge 0$ for all $z \in Z(u(t))$ and $t \in\,]0,T[\,\setminus A$.*

Condition (iii) can also be rewritten in the form

$$-\dot{\xi}(t) \;\in\; N_{Z(u(t))}(\xi(t)) \quad a.\,e.\,, \tag{3.2}$$

where $N_{Z(u(t))}(\xi(t))$ is the normal cone to $Z(u(t))$ at the point $\xi(t)$ as in (A.2.1), or alternatively

$$-\dot{\xi}(t) \;\in\; \partial I_{Z(u(t))}(\xi(t)) \quad a.\,e.\,, \tag{3.3}$$

where $\partial I_Z(x)$ denotes the subdifferential of the indicator function I_Z of a closed convex set Z at the point x. Recall that $I_Z(x)$ is defined to be 0 if $x \in Z$ and $+\infty$ otherwise.

Proof of Lemma 3.2.

(i) Let $w \in \mathcal{T}(u)$ and $0 \le s < t \le T$ be given. For $\tau \in [0,T]$ put

$$\tilde{w}(\tau) \;=\; \big(\chi_{[0,s]}(\tau) + \chi_{]t,T]}(\tau)\big)\,\xi(\tau) + \chi_{]s,t]}(\tau) w(\tau)\,.$$

Then $\tilde{w} \in \mathcal{T}(u)$. Using Propositions 1.13, 1.15, 1.17 and the fact that ξ is right-continuous, we obtain that

$$\begin{aligned} 0 \;\le\; & \int_0^T \langle \tilde{w}(\tau) - \xi(\tau), d\xi(\tau)\rangle = \int_s^t \langle w(\tau) - \xi(\tau), d\xi(\tau)\rangle - \int_s^t \big\langle (w(s) - \xi(s))\chi_{\{s\}}(\tau), d\xi(\tau)\big\rangle \\ & + \int_t^T \big\langle (w(t) - \xi(t))\chi_{\{t\}}(\tau), d\xi(\tau)\big\rangle \;=\; \int_s^t \langle w(\tau) - \xi(\tau), d\xi(\tau)\rangle\,, \end{aligned}$$

which we wanted to prove.

(ii) We proceed similarly as above and for an arbitrary $z \in Z(u(t))$ put

$$w(\tau) \;=\; \big(\chi_{[0,t[}(\tau) + \chi_{]t,T]}(\tau)\big)\,\xi(\tau) + \chi_{\{t\}}(\tau) z\,.$$

Then Proposition 1.17 yields

$$0 \;\le\; \int_0^T \langle w(\tau) - \xi(\tau), d\xi(\tau)\rangle \;=\; \langle z - \xi(t), \xi(t) - \xi(t-)\rangle\,,$$

and the assertion follows from Lemma A.2 (iv).

(iii) For every $x \in X$ and $0 \le s < t \le T$ we have by Proposition 2.9

$$0 \;\le\; \int_s^t \big\langle Q_{u(\tau)}x - \xi(\tau), d\xi(\tau)\big\rangle \;=\; \int_s^t \Big\langle Q_{u(\tau)}x - \xi(\tau), \dot{\xi}(\tau)\Big\rangle\, d\tau\,.$$

Let $S \subset X$ be a countable set which is dense in X. We find a set $A \subset]0,T[$ of measure 0 such that

$$\Big\langle Q_{u(t)}x - \xi(t), \dot{\xi}(t)\Big\rangle \;\ge\; 0 \qquad \forall t \in]0,T[\setminus A \;\; \forall x \in S\,. \tag{3.4}$$

By density, we see that (3.4) holds for all $x \in X$. For $z \in Z(u(t))$ have $Q_{u(t)}z = z$, and the proof is complete. ∎

From (A.6.3) it follows that $\xi(t-) \in Z(u(t-))$ for all $t \in [0,T]$. The trajectory

$$K(u) \;=\; \{u(t), u(t-)\,;\, t \in [0,T]\} \tag{3.5}$$

of an arbitrary function $u \in G_R(0,T\,;\,V)$ is a compact subset of V, and using Lemma 3.2 we obtain from (A.6.2) and (A.6.5) that

$$|\xi(t) - \xi(t-)| \;=\; |P_{u(t)}\xi(t-)| \;\le\; \mu_{K(u)}(\|u(t) - u(t-)\|) \quad \forall t \in [0,T]\,. \tag{3.6}$$

In particular, if u is continuous at a point $t \in [0,T]$, then so is ξ.

3.2 Construction of the solution

We derive an explicit formula for the solution if the input u is a step function, and then extend the existence result using general convergence theorems for the Kurzweil integral from Section 1. Before, we show that the solution is uniquely determined by the data.

Lemma 3.3 *Let Hypothesis A.26 hold. Then for every $(u,\xi^0) \in \mathrm{Dom}\,(\mathcal{P})$ the solution ξ to Problem $\mathcal{P}(u,\xi^0)$ is unique.*

Proof. Let $\xi, \eta \in BV_R(0,T\,;\,X)$ be two solutions of Problem $\mathcal{P}(u,\xi^0)$. Then we have for all $t \in [0,T]$ that $\int_0^t \langle \eta(\tau) - \xi(\tau), d\xi(\tau)\rangle \geq 0$, $\int_0^t \langle \xi(\tau) - \eta(\tau), d\eta(\tau)\rangle \geq 0$, and Corollary 2.12 yields that

$$0 \geq \int_0^t \langle \eta(\tau) - \xi(\tau), d(\eta - \xi)(\tau)\rangle \geq \frac{1}{2}|\eta(t) - \xi(t)|^2\,,$$

and the assertion follows. ■

Lemma 3.3 enables us to define the input-output operator $\mathfrak{p} : \mathrm{Dom}\,(\mathcal{P}) \to BV_R(0,T\,;\,X)$ which with each $(u,\xi^0) \in \mathrm{Dom}\,(\mathcal{P})$ associates the solution $\xi = \mathfrak{p}[u,\xi^0]$ of Problem $\mathcal{P}(u,\xi^0)$.

Proposition 3.4 *Let Hypothesis A.26 hold, and let $u \in G_R(0,T\,;\,V)$ be a step function of the form*

$$u(t) \;=\; \sum_{k=1}^{m} u_{k-1}\,\chi_{[t_{k-1},t_k[}\,(t) + u_m\,\chi_{\{T\}}\,(t)\,, \tag{3.7}$$

where $0 = t_0 < t_1 < \cdots < t_m = T$ is a division of $[0,T]$ and $u_0, u_1, \ldots, u_m$ are elements of V. Then $(u,\xi^0) \in \mathrm{Dom}\,(\mathcal{P})$ for every $\xi^0 \in Z(u_0)$, and $\xi = \mathfrak{p}[u,\xi^0]$ has the form

$$\xi(t) \;=\; \sum_{k=1}^{m} \xi_{k-1}\,\chi_{[t_{k-1},t_k[}\,(t) + \xi_m\,\chi_{\{T\}}\,(t)\,, \tag{3.8}$$

where $\xi_0 = \xi^0$ and

$$\xi_k \;=\; Q_{u_k}\xi_{k-1} \qquad \text{for } \; k = 1,\ldots,m\,. \tag{3.9}$$

With the notation from (3.5) we moreover have

$$|\xi_k - \xi_{k-1}| \;\leq\; \mu_{K(u)}(\|u_k - u_{k-1}\|) \quad \text{for } \; k = 1,\ldots,m\,. \tag{3.10}$$

Proof. Let $w \in \mathcal{T}(u)$ be arbitrary. By Proposition 1.16 we have

$$\int_0^T \langle w(t) - \xi(t), d\xi(t)\rangle \;=\; \sum_{k=1}^{m} \langle w(t_k) - \xi_k, \xi_k - \xi_{k-1}\rangle\,.$$

For all $k = 1,\ldots,m$ we have $\xi_k = Q_{u_k}\xi_{k-1}$, $\xi_k - \xi_{k-1} = -P_{u_k}\xi_{k-1}$, and $w(t_k) \in Z(u_k)$, hence $\langle w(t_k) - \xi_k, \xi_k - \xi_{k-1}\rangle \geq 0$ by virtue of Lemma A.2 (i). We have thus checked that $\xi = \mathfrak{p}[u,\xi^0]$. Inequality (3.10) follows from (A.6.2) and (A.6.5). ■

Further extensions of $\mathrm{Dom}\,(\mathcal{P})$ are based on the following estimate.

Lemma 3.5 *Let Hypothesis A.26 hold, and let* $(u,\xi^0),(\tilde u,\tilde\xi^0) \in \mathrm{Dom}\,(\mathcal{P})$ *be given,* $\xi = \mathfrak{p}[u,\xi^0]$, $\tilde\xi = \mathfrak{p}[\tilde u,\tilde\xi^0]$. *Then for all* $t\in[0,T]$ *we have*

$$|\xi(t)-\tilde\xi(t)|^2 \;\le\; |\xi^0-\tilde\xi^0|^2 + 2\left(\operatorname*{Var}_{[0,t]}\xi + \operatorname*{Var}_{[0,t]}\tilde\xi\right)\mu_{K(u)\cup K(\tilde u)}(\|u-\tilde u\|_{[0,t]})\,. \tag{3.11}$$

Proof. The functions $t\mapsto Q_{u(t)}\tilde\xi(t)$, $t\mapsto Q_{\tilde u(t)}\xi(t)$ are regulated by virtue of Lemma A.28 and belong to $\mathcal{T}(u)$, $\mathcal{T}(\tilde u)$, respectively. Then Lemma 3.2 yields that

$$\int_0^t \left\langle Q_{u(\tau)}\tilde\xi(\tau)-\xi(\tau), d\xi(\tau)\right\rangle \ge 0\,, \quad \int_0^t \left\langle Q_{\tilde u(\tau)}\xi(\tau)-\tilde\xi(\tau), d\tilde\xi(\tau)\right\rangle \ge 0\,,$$

and from (A.6.2), (A.6.5) and Corollary 2.6 it follows that

$$\begin{aligned}\int_0^t \left\langle \tilde\xi(\tau)-\xi(\tau), d(\xi-\tilde\xi)(\tau)\right\rangle \;&\ge\; \int_0^t \left\langle P_{u(\tau)}\tilde\xi(\tau), d\xi(t)\right\rangle + \int_0^t \left\langle P_{\tilde u(\tau)}\xi(\tau), d\tilde\xi(\tau)\right\rangle \\ &\ge\; -\mu_{K(u)\cup K(\tilde u)}(\|u-\tilde u\|_{[0,t]})\left(\operatorname*{Var}_{[0,t]}\xi+\operatorname*{Var}_{[0,t]}\tilde\xi\right).\end{aligned}$$

Using Corollary 2.12 we easily complete the proof. ∎

Proposition 3.6 *Let Hypothesis A.26 be fulfilled, let* $\{(u_n,\xi_n^0)\,;\,n\in\mathbb{N}\}$ *be a sequence in* $\mathrm{Dom}\,(\mathcal{P})$ *such that* $\|u_n-u\|_{[0,T]}\to 0$, $|\xi_n^0-\xi^0|\to 0$ *as* $n\to\infty$, *and let there exist a constant* $C>0$ *such that the functions* $\xi_n = \mathfrak{p}[u_n,\xi_n^0]$ *satisfy the inequality* $\mathrm{Var}_{[0,T]}\,\xi_n \le C$. *Then* $(u,\xi^0)\in\mathrm{Dom}\,(\mathcal{P})$, *and putting* $\xi=\mathfrak{p}[u,\xi^0]$ *we have*

$$\lim_{n\to\infty}\|\xi_n-\xi\|_{[0,T]} \;=\; 0\,. \tag{3.12}$$

Proof. Put $\tilde K = K(u)\cup\bigcup_{n=1}^\infty K(u_n)$. Then $\tilde K$ is compact, and from (3.11) we infer for all $n,m\in\mathbb{N}$ and $t\in[0,T]$ that

$$|\xi_n(t)-\xi_m(t)|^2 \;\le\; |\xi_n^0-\xi_m^0|^2 + 4C\,\mu_{\tilde K}\left(\|u_n-u_m\|_{[0,t]}\right). \tag{3.13}$$

We see that $\{\xi_n\}$ is a Cauchy sequence in $G_R(0,T\,;\,X)$ with uniformly bounded variation, hence $\xi=\lim_{n\to\infty}\xi_n$ belongs to $BV_R(0,T\,;\,X)$, and from (A.6.3) it follows that $\xi(t)\in Z(u(t))$ for all $t\in[0,T]$.

It remains to check that inequality (iii) in Definition 3.1 is fulfilled. For every $n\in\mathbb{N}$ and $w\in\mathcal{T}(u)$ we have

$$\int_0^T \left\langle Q_{u_n(t)}w(t)-\xi_n(t), d\xi_n(t)\right\rangle \;\ge\; 0\,, \tag{3.14}$$

where, as a consequence of (A.6.2) and (A.6.5), we have

$$|w(t)-Q_{u_n(t)}w(t)| \;=\; |P_{u_n(t)}w(t)| \;\le\; \mu_{\tilde K}(\|u_n(t)-u(t)\|) \;\le\; \mu_{\tilde K}(\|u_n-u\|_{[0,T]})\,. \tag{3.15}$$

By Proposition 2.10 we pass to the limit as $n\to\infty$ in (3.14) and obtain the assertion. ∎

Under slightly more restrictive assumptions we now show that $\mathrm{Dom}\,(\mathcal{P})$ contains all right-continuous input functions with bounded variation.

Proposition 3.7 *Assume in addition to Hypothesis A.26 that for every compact set $K \subset V$ there exists a constant $\lambda_K > 0$ such that*

$$\Delta(v,w) \ \leq\ \lambda_K \,\|v-w\| \qquad \forall v,w \in K\,. \tag{3.16}$$

Then every (u,ξ^0) with $u \in BV_R(0,T\,;\,V)$ and $\xi^0 \in Z(u(0))$ belongs to $\mathrm{Dom}\,(\mathcal{P})$ *and $\xi = \mathfrak{p}[u,\xi^0]$ satisfies the estimate*

$$\operatorname*{Var}_{[s,t]} \xi \ \leq\ \lambda_{K(u)} \operatorname*{Var}_{[s,t]} u \qquad \textit{for every}\ \ 0 \leq s < t \leq T\,. \tag{3.17}$$

Proof. By Proposition 2.4 (ii) we find a sequence $\{u^{(n)}\}$ of step functions of the form

$$u^{(n)}(\tau) \ =\ \sum_{k=1}^{m_n} u^{(n)}_{k-1}\, \chi_{[t^{(n)}_{k-1},t^{(n)}_k[}(\tau) + u^{(n)}_{m_n}\, \chi_{\{T\}}(\tau) \quad \text{ for } \ \tau \in [0,T]\,, \tag{3.18}$$

where $0 = t^{(n)}_0 < t^{(n)}_1 < \cdots < t^{(n)}_{m_n} = T$ is a division of $[0,T]$ and $u^{(n)}_0, u^{(n)}_1, \ldots, u^{(n)}_{m_n}$ are elements of V such that $\left\|u^{(n)} - u\right\|_{[0,T]} \to 0$ as $n \to \infty$, $K(u^{(n)}) \subset K(u)$, $\mathrm{Var}_{\,[0,T]}\, u^{(n)} \leq \mathrm{Var}_{\,[0,T]}\, u$. For given $0 \leq s < t \leq T$ we may assume that both s,t belong to $\{t^{(n)}_0, t^{(n)}_1, \ldots, t^{(n)}_{m_n}\}$ and $u^{(n)}(s) = u(s)$, $u^{(n)}(t) = u(t)$ for all $n \in \mathbb{N}$. By construction we then have $\mathrm{Var}_{\,[s,t]}\, u^{(n)} \leq \mathrm{Var}_{\,[s,t]}\, u$. The solutions $\xi^{(n)} = \mathfrak{p}[u^{(n)},\xi^0]$ constructed according to Proposition 3.4 fulfil the inequalities

$$\operatorname*{Var}_{[0,T]} \xi^{(n)} \ \leq\ \lambda_{K(u)} \operatorname*{Var}_{[0,T]} u\,, \quad \operatorname*{Var}_{[s,t]} \xi^{(n)} \ \leq\ \lambda_{K(u)} \operatorname*{Var}_{[s,t]} u \tag{3.19}$$

as a consequence of (3.10), and it suffices to use Proposition 3.6. ■

Similarly as in [20, 26, 28, 29], the non-empty interior condition in Hypothesis A.27 enables us to extend $\mathrm{Dom}\,(\mathcal{P})$ to all input functions $u \in G_R(0,T\,;\,V)$. The main result of this section reads as follows.

Theorem 3.8 *Let Hypotheses A.26 and A.27 be fulfilled, and let $\{(u_n,\xi^0_n)\,;\, n \in \mathbb{N}\}$ be a sequence in* $\mathrm{Dom}\,(\mathcal{P})$ *such that $\|u_n - u\|_{[0,T]} \to 0$, $|\xi^0_n - \xi^0| \to 0$ as $n \to \infty$. Then $(u,\xi^0) \in$* $\mathrm{Dom}\,(\mathcal{P})$ *and putting $\xi_n = \mathfrak{p}[u_n,\xi^0_n]$, $\xi = \mathfrak{p}[u,\xi^0]$ we have*

$$\lim_{n\to\infty} \|\xi_n - \xi\|_{[0,T]} \ =\ 0\,, \quad \operatorname*{Var}_{[0,T]} \xi \ \leq\ \liminf_{n\to\infty} \operatorname*{Var}_{[0,T]} \xi_n \ <\ \infty\,. \tag{3.20}$$

We first prove the following "selection lemma".

Lemma 3.9 *Let Hypotheses A.26 and A.27 hold, and let $u \in G_R(0,T\,;\,V)$ be given. Then there exists $r > 0$ and $z \in BV_R(0,T\,;\,X)$ such that*

$$B_{2r}(z(t)) \ \subset\ Z(u(t)) \qquad \forall t \in [0,T]\,. \tag{3.21}$$

Proof of Lemma 3.9. We use Proposition A.31 with $K = K(u)$ to find $\tilde{\varrho} > 0$ such that for every $t \in [0,T]$ there exists $x(t) \in X$ satisfying the inclusion

$$B_{\tilde{\varrho}}(x(t)) \ \subset\ Z(u(t)) \qquad \forall t \in [0,T]\,.$$

By Proposition 2.4 (i) we find a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that for every $j = 1, \dots, m$ and $t \in [t_{j-1}, t_j[$ we have

$$\mu_{K(u)}\left(\|u(t) - u(t_{j-1})\|\right) \;<\; \frac{\tilde{\varrho}}{2}\,,$$

and for $t \in [0,T]$ put

$$z(t) \;=\; \sum_{j=1}^{m} x(t_{j-1})\,\chi_{[t_{j-1},t_j[}\,(t) + x(t_m)\,\chi_{\{t_m\}}\,(t)\,. \tag{3.22}$$

Then $z \in BV_R(0,T\,;\,X)$, $B_{\tilde{\varrho}}(z(t)) \;\subset\; Z(u(t_{j-1}))$ for all $j = 1, \dots, m$ and $t \in [t_{j-1}, t_j[$, $B_{\tilde{\varrho}}(z(t_m)) \;\subset\; Z(u(t_m))$. By Lemma A.30 we then obtain the assertion for $r = \tilde{\varrho}/4$. ■

Proof of Theorem 3.8. Let r and z be as in Lemma 3.9, and let $\tilde{K} = K(u) \cup \bigcup_{n=1}^{\infty} K(u_n)$. Then $\tilde{K}$ is compact, and we find $n_0 \in \mathbb{N}$ sufficiently large such that

$$\mu_{\tilde{K}}\left(\|u_n - u\|_{[0,T]}\right) \;\le\; r \qquad \text{for } \; n \ge n_0\,.$$

From Lemma A.30 we then obtain for all $n \ge n_0$ and $t \in [0,T]$ that

$$B_r(z(t)) \;\subset\; Z(u_n(t))\,. \tag{3.23}$$

Let $f \in G(0,T\,;\,X)$ be an arbitrary function such that $\|f\|_{[0,T]} \le 1$. Then the function $w(t) = z(t) - r\,f(t)$ belongs to $\mathcal{T}(u_n)$ for all $n \ge n_0$, and from Lemma 3.2 it follows for all $t \in [0,T]$ that

$$\int_0^t \langle z(\tau) - r\,f(\tau) - \xi_n(\tau), d\xi_n(\tau)\rangle \;\ge\; 0\,. \tag{3.24}$$

Using Corollaries 2.6, 2.7, and 2.12 we obtain

$$r \operatorname*{Var}_{[0,t]} \xi_n + \frac{1}{2}|\xi_n(t)|^2 \;\le\; \frac{1}{2}|\xi_n^0|^2 + z^*\;\|\xi_n\|_{[0,t]}\;, \tag{3.25}$$

where we put $z^* = 2\,\|z\|_{[0,T]} + \operatorname{Var}_{[0,T]} z$. There exists therefore a constant $C > 0$ independent of n such that $\|\xi_n\|_{[0,T]} \le C$, $\operatorname{Var}_{[0,T]} \xi_n \le C$, and from Lemma 3.5 we conclude that $\{\xi_n\}$ is a Cauchy sequence in $G_R(0,T\,;\,X)$ and there exists $\xi \in BV_R(0,T\,;\,X)$ such that (3.20) holds. To check that $\xi = \mathfrak{p}[u, \xi^0]$ we use Proposition 3.6. ■

Corollary 3.10 *Let Hypotheses A.26, A.27 hold. Then* Dom$\,(\mathcal{P})$ *contains all* (u, ξ^0) *with* $u \in G_R(0,T\,;\,V)$ *and* $\xi^0 \in Z(u(0))$*, and the mapping* $\mathfrak{p} : \text{Dom}\,(\mathcal{P}) \to G_R(0,T\,;\,X)$ *is continuous.*

3.3 The play operator in $G(0,T\,;\,X)$

Let us consider now the special case

$$V = Y = X\,, \quad Z(v) = v - Z\,, \tag{3.26}$$

where $Z \subset X$ is a fixed convex set, $0 \in Z$. This is the most frequent situation in applications to elastoplasticity and hysteresis, see [9, 22, 38]. The operator $\mathfrak{p} : (u, \xi^0) \mapsto \xi$ is then called the (multidimensional) play and its properties have been extensively studied also in [13, 20]. We show here how it can be extended to arbitrary regulated (not necessarily right-continuous) inputs.

Theorem 3.11 *Let there exist $r > 0$ such that $B_r(0) \subset Z$. Then for every $u \in G(0,T\,;\,X)$ and $\xi^0 \in u(0) - Z$ there exists a unique $\xi \in \overline{BV}(0,T\,;\,X)$ such that*

(i) $\xi(0) = \xi^0$,

(ii) $u(t) - \xi(t) \in Z$ *for every* $t \in [0,T]$,

(iii) $\displaystyle\int_0^t \langle u(\tau+) - \xi(\tau+) - w(\tau), d\xi(\tau)\rangle \ge 0$ *for every* $w \in G(0,T\,;\,Z)$ *and* $t \in [0,T]$.

Proof. Put $\tilde u(\tau) = u(\tau+)$, $\tilde\xi^0 = \xi^0 + P_Z(\tilde u(0) - \xi^0)$, $\tilde\xi(\tau) = \mathfrak{p}[\tilde u, \tilde\xi^0](\tau)$ for $\tau \in [0,T]$, and $\xi(\tau) = \tilde\xi(\tau-) + P_Z(u(\tau) - \tilde\xi(\tau-))$ for $\tau \in]0,T]$, $\xi(0) = \xi^0$. We then have $u(\tau) - \xi(\tau) \in Z$ and $\xi(\tau+) = \tilde\xi(\tau) + P_Z(u(\tau+) - \tilde\xi(\tau)) = \tilde\xi(\tau)$ for all $\tau \in [0,T]$. For an arbitrary $w \in G(0,T\,;\,Z)$ and a fixed $t \in [0,T]$ put

$$\tilde w(\tau) \;=\; w(\tau) + (\tilde u(t) - \tilde\xi(t) - w(t))\;\chi_{\{t\}}(\tau) \quad \text{for } \tau \in [0,T]\,.$$

We then have

$$\int_0^t \langle u(\tau+) - \xi(\tau+) - w(\tau), d\xi(\tau)\rangle \;=\; \int_0^t \left\langle \tilde u(\tau) - \tilde\xi(\tau) - \tilde w(\tau), d\tilde\xi(\tau)\right\rangle$$
$$+\;\int_0^t \left\langle u(\tau+) - \xi(\tau+) - w(\tau) - \tilde u(\tau) + \tilde\xi(\tau) + \tilde w(\tau), d\tilde\xi(\tau)\right\rangle$$
$$+\;\int_0^t \langle u(\tau+) - \xi(\tau+) - w(\tau), d(\xi - \eta)(\tau)\rangle\;,$$

where by Proposition 1.17 and Remark 1.18 we have

$$\int_0^t \left\langle u(\tau+) - \xi(\tau+) - w(\tau) - \tilde u(\tau) + \tilde\xi(\tau) + \tilde w(\tau), d\tilde\xi(\tau)\right\rangle$$
$$=\;\int_0^t \left\langle (u(t) - \xi(t) - w(t))\chi_{\{t\}}(\tau), d\tilde\xi(\tau)\right\rangle \;=\; \left\langle u(t) - \xi(t) - w(t), \tilde\xi(t) - \tilde\xi(t-)\right\rangle\,.$$

From Lemma 1.9 we further obtain that

$$\int_0^t \langle u(\tau+) - \xi(\tau+) - w(\tau), d(\xi - \eta)(\tau)\rangle$$
$$=\;\left\langle u(t) - \xi(t) - w(t), \xi(t) - \tilde\xi(t)\right\rangle - \left\langle u(0+) - \xi(0+) - w(0), \xi(0) - \tilde\xi(0)\right\rangle$$
$$=\;\left\langle u(t) - \xi(t) - w(t), \xi(t) - \tilde\xi(t)\right\rangle + \left\langle \tilde u(0) - \tilde\xi(0) - w(0), \tilde\xi(0) - \xi(0)\right\rangle\,,$$

hence

$$\int_0^t \langle u(\tau+) - \xi(\tau+) - w(\tau), d\xi(\tau)\rangle \;=\; \int_0^t \left\langle \tilde u(\tau) - \tilde\xi(\tau) - \tilde w(\tau), d\tilde\xi(\tau)\right\rangle \tag{3.27}$$
$$+\;\left\langle u(t) - \xi(t) - w(t), \xi(t) - \tilde\xi(t-)\right\rangle + \left\langle \tilde u(0) - \tilde\xi(0) - w(0), \tilde\xi(0) - \xi(0)\right\rangle\,.$$

All three terms on the right-hand side of (3.27) are non-negative by hypothesis and by Lemma A.2, hence (iii) holds. Uniqueness follows from Corollary 2.12 similarly as in Lemma 3.3. ■

Remark 3.12 We cannot write the variational inequality (iii) in Theorem 3.11 in the form

$$\text{(iii)'} \quad \int_0^t \langle u(\tau) - \xi(\tau) - y(\tau), d\xi(\tau)\rangle \ \geq\ 0 \quad \forall y \in G(0,T;Z) \quad \forall t \in [0,T]$$

analogous to Definition 3.1. It suffices to consider the scalar case $X = \mathbb{R}$, $Z = [-r, r]$ for some $r > 0$, $u(\tau) = \bar{u}\, \chi_{]0,T]}(\tau)$ with some $\bar{u} > r$. Assume that there exists $\xi \in \overline{BV}(0,T\,;X)$ satisfying (iii)', $\|u - \xi\|_{[0,T]} \leq r$, $\xi(0) = 0$. Putting $y(\tau) = r\, \chi_{\{0\}}(\tau) + (u(\tau) - \xi(\tau))\ \chi_{]0,T]}(\tau)$ we obtain from (iii)' and Proposition 1.17 that

$$0 \leq \int_0^t (u(\tau) - \xi(\tau) - r)\, \chi_{\{0\}}(\tau)\, d\xi(\tau) \ =\ -r\, \xi(0+)\,,$$

hence $\xi(0+) \leq 0$ and $u(0+) - \xi(0+) \geq \bar{u} > r$, which is a contradiction.

3.4 The play and stop in $W^{1,1}(0,T\,;X)$

Let us now replace the condition $B_r(0) \subset Z$ in Theorem 3.11 by a weaker one $0 \in Z$, and assume that $u \in W^{1,1}(0,T\,;X)$. The hypotheses of Proposition 3.7 are then satisfied with $\lambda_K \equiv 1$ independently of K, and from (3.17) we infer that $\xi \in W^{1,1}(0,T\,;X)$ and $|\dot{\xi}(t)| \leq |\dot{u}(t)|$ a. e. Below we prove even more, namely that the input-output mapping is continuous with respect to the norm in $W^{1,1}(0,T\,;X)$. We proceed in several steps.

Proposition 3.13 *Let $Z \subset X$ be a convex closed set with $0 \in Z$, let $u \in W^{1,1}(0,T\,;X)$ be given, let $Z(u(t)) = u(t) - Z$ for all $t \in [0,T]$, and let $\xi^0 \in u(0) - Z$ be a given initial value. Then $\xi := \mathfrak{p}[u, \xi^0]$ belongs to $W^{1,1}(0,T\,;X)$ and has the properties*

(i) $\Big\langle \dot{\xi}(t), u(t) - \xi(t) - z \Big\rangle \geq 0$ *a. e.* $\quad \forall z \in Z$,

(ii) $\Big\langle \dot{\xi}(t), \dot{u}(t) - \dot{\xi}(t) \Big\rangle = 0$ $\quad$ *a. e.*

Proof. Statement (i) follows from Lemma 3.2 (iii). To prove (ii), it suffices to put $z = (u - \xi)(t \pm h)$ in (i) and let h tend to $0+$ using Theorem 2.8. ■

Let us denote $x^0 = u(0) - \xi^0 \in Z$, $x(t) = u(t) - \xi(t)$. The mapping $\mathfrak{s} : W^{1,1}(0,T\,;X) \times Z \to W^{1,1}(0,T\,;X) : (u, x^0) \mapsto x$ is called the *stop* which is related to the play through the formula

$$\mathfrak{p}[u, \xi^0](t) + \mathfrak{s}[u, x^0](t) \ =\ u(t) \qquad \forall t \in [0,T] \tag{3.28}$$

for every $u \in W^{1,1}(0,T\,;X)$ and every ξ^0, x^0 such that $x^0 \in Z$ and $\xi^0 + x^0 = u(0)$. It is easy to see that $\mathfrak{s}$ as operator from $W^{1,1}(0,T\,;X) \times Z$ to $C(0,T\,;X)$ is Lipschitz continuous. Indeed, putting $x_1 = \mathfrak{s}[u_1, x_1^0]$, $x_2 = \mathfrak{s}[u_2, x_2^0]$ for given $x_1^0, x_2^0 \in Z$, $u_1, u_2 \in W^{1,1}(0,T\,;X)$ we immediately obtain from Proposition 3.13 (i) that

$$\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 \ \leq\ \langle x_1(t) - x_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \quad \text{a. e.}, \tag{3.29}$$

and dividing by $|x_1(t) - x_2(t)|$ we obtain after integration that

$$|x_1(t) - x_2(t)| \le |x_1^0 - x_2^0| + \int_0^t |\dot u_1(\tau) - \dot u_2(\tau)|\, d\tau \qquad \forall t \in [0,T]\,. \tag{3.30}$$

The continuity of $\mathfrak{s}$ in $W^{1,p}(a,b\,;\,X) \times Z \to W^{1,p}(a,b\,;\,X)$ for $1 \le p < \infty$ is established in Theorem 3.14 below.

Theorem 3.14 *Let $Z \subset X$ be a convex closed set with $0 \in Z$, let $\{u_n\}$ be a given sequence in $W^{1,p}(a,b\,;\,X)$ for some $p \in [1,\infty[$ such that $\lim_{n\to\infty} |u_n - u|_{1,p} = 0$, and let $x_n^0 \in Z$ be given initial values, $\lim_{n\to\infty} |x_n^0 - x^0| = 0$. Put $x_n = \mathfrak{s}[u_n, x_n^0]$ for $n \in \mathbb{N}$, $x = \mathfrak{s}[u, x^0]$. Then $\lim_{n\to\infty} |x_n - x|_{1,p} = 0$.*

Before passing to the proof, we mention two results from the Lebesgue and Bochner integration theory.

Theorem 3.15 (Lebesgue Dominated Convergence Theorem) *Let $p \in [1,\infty[$ be given and let $v_n \in L^p(a,b\,;\,X)$, $g_n \in L^p(a,b\,;\,\mathbb{R})$ be given sequences for $n \in \mathbb{N} \cup \{0\}$ such that*

(i) $\lim_{n\to\infty} \int_a^b |g_n(t) - g_0(t)|^p\, dt = 0$,

(ii) $\lim_{n\to\infty} |v_n(t) - v_0(t)| = 0$ *a. e.*,

(iii) $|v_n(t)| \le g_n(t)$ *a. e. for all* $n \in \mathbb{N} \cup \{0\}$.

Then $\lim_{n\to\infty} |v_n - v_0|_p = 0$.

We recall this well-known result in order to show the contrast to the following Theorem 3.16. Notice that it does not follow from Theorem 3.15, since *we do not assume the pointwise convergence* here. It was proved in [21] and further results in this direction can also be found in [8, 39].

Theorem 3.16 *Let $v_n \in L^1(a,b\,;\,X)$, $g_n \in L^1(a,b\,;\,\mathbb{R})$ be given sequences for $n \in \mathbb{N} \cup \{0\}$ such that*

(i) $\lim_{n\to\infty} \int_a^b \langle v_n(t), \varphi(t)\rangle\, dt = \int_a^b \langle v_0(t), \varphi(t)\rangle\, dt \quad \forall \varphi \in C(0,T\,;\,X)$,

(ii) $\lim_{n\to\infty} \int_a^b |g_n(t) - g_0(t)|\, dt = 0$,

(iii) $|v_n(t)| \le g_n(t)$ *a. e.* $\forall n \in \mathbb{N}$,

(iv) $|v_0(t)| = g_0(t)$ *a. e.*

Then $\lim_{n\to\infty} |v_n - v_0|_1 = 0$.

Proof. Using Lusin's theorem, we check that property (i) is satisfied for every $\varphi \in L^\infty(a,b;X)$. For $t \in [a,b]$ put

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad v_0(t) = 0\,, \\ v_0(t)\,/\,g_0(t) & \text{if} \quad v_0(t) \neq 0\,. \end{cases}$$

Then $\varphi \in L^\infty(a,b;X)$ and the inequality

$$\begin{aligned} |v_n(t) - v_0(t)|^2 \ &\leq \ g_n^2(t) - 2\,\langle v_n(t), v_0(t)\rangle + g_0^2(t) \\ &= \ |g_n(t) - g_0(t)|^2 + 2g_0(t)\ (g_n(t) - g_0(t) + \langle v_0(t), \varphi(t)\rangle - \langle v_n(t), \varphi(t)\rangle) \end{aligned}$$

holds for a. e. $t \in [a,b]$. By Hölder's inequality we have

$$\begin{aligned} \int_a^b |v_n(t) - v_0(t)|\,dt \ &\leq \ \int_a^b |g_n(t) - g_0(t)|\,dt \\ &+ \ \left(\int_a^b 2g_0(t)\,dt\right)^{1/2} \left(\int_a^b (g_n(t) - g_0(t) + \langle v_0(t), \varphi(t)\rangle - \langle v_n(t), \varphi(t)\rangle)\ dt\right)^{1/2}, \end{aligned}$$

and we can pass to the limit as $n \to \infty$. ■

We are now ready to prove Theorem 3.14.

Proof of Theorem 3.14. For $n \in \mathbb{N} \cup \{0\}$ put $\xi_n = u_n - x_n$, $y_n = x_n - \xi_n$. From (3.30) we infer that $|\xi_n - \xi_0|_\infty \to 0$, $|x_n - x_0|_\infty \to 0$, $|y_n - y_0|_\infty \to 0$. By Proposition 3.13 (ii) we have

$$|\dot y_n| \ = \ |\dot u_n| \qquad \text{a. e.} \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,. \tag{3.31}$$

Theorem 3.16 for $v_n := \dot y_n$, $g_n := |\dot u_n|$ yields $\lim_{n\to\infty} |y_n - y_0|_{1,1} = 0$. There exists therefore a subsequence $\{y_{n_k}\}$ such that $\lim_{k\to\infty} |\dot y_{n_k}(t) - \dot y_0(t)| = 0$ a. e. and from Theorem 3.15 we conclude

$$\lim_{k\to\infty} |y_{n_k} - y_0|_{1,p} = 0\,. \tag{3.32}$$

Since every subsequence of $\{y_n\}$ contains a subsequence satisfying (3.32), the proof is complete if we take into account the relations $x_n = (u_n + y_n)/2$, $\xi_n = (u_n - y_n)/2$. ■

In [22, Theorem I.3.12] it is proved that the stop depends continuously also on Z in terms of the Hausdorff distance d_H defined in (A.5.1). We cite this result without proof.

Theorem 3.17 *Let $\{Z_n\,;\, n \in \mathbb{N} \cup \{0\}\}$ be a sequence of convex closed sets in X such that $0 \in \cap_{n=0}^\infty Z_n$, $\lim_{n\to\infty} d_H(Z_0, Z_n) = 0$, and let $\{x_n^0 \in Z_n\}$ be a sequence of initial values such that $\lim_{n\to\infty} |x_n^0 - x_0^0| = 0$. Let $\{u_n\,;\, n \in \mathbb{N} \cup \{0\}\}$ be a sequence in $W^{1,p}(0,T\,;X)$ such that $\lim_{n\to\infty} |u_n - u_0|_{1,p} = 0$ for some $p \in [1,+\infty[$. Put $x_n = \mathfrak{s}_n[u_n, x_n^0]$ for $n \in \mathbb{N} \cup \{0\}$, where $\mathfrak{s}_n$ is the stop associated with the convex set Z_n. Then $\lim_{n\to\infty} |x_n - x_0|_{1,p} = 0$.*

Remark 3.18 A counterpart of Theorem 3.14 with $p = \infty$ does not hold even if $\dim X = 1$ with respect to the norm in $W^{1,\infty}(0,T)$ defined as $|u|_{1,\infty} = |u(0)| + \operatorname{sup\,ess}\{|\dot u(t)|\,;\, t \in [0,T]\}$. It suffices to consider $Z = [-1,1]$, $T = 1$ and the sequence $u_n(t) = (1 + 1/n)\,t$ for $t \in [0,1]$, $n \in \mathbb{N}$ with $u_0(t) = t$, $x_n^0 = 0$. For $x_n = \mathfrak{s}[u_n, x_n^0]$, $\xi_n = u_n - x_n$ we then have

$$\xi_0(t) \equiv 0, \quad \xi_n(t) = \begin{cases} 0 \quad \text{for} \quad t \in \left[0, \frac{n}{n+1}\right], \\ \left(1 + \frac{1}{n}\right) t - 1 \quad \text{for} \quad t \in \left]\frac{n}{n+1}, 1\right] \end{cases} \qquad \text{for} \quad n \in \mathbb{N}\,,$$

hence $|u_n - u_0|_{1,\infty} \to 0$, $|\xi_n - \xi_0|_{1,\infty} \geq 1$.

4 The *wbo*-convergence

Most of the contents of this section are recent results from [10, 26]. We introduce a particular "weak" convergence concept (the so-called *wbo*-convergence; the abbreviation 'wbo' stands for 'weak bounded oscillation') which will be shown in Example 5.3 to be independent of the usual weak convergence, and derive equivalent criteria for the "*wbo*-sequential compactness" inspired by Fraňková's generalization of the Helly Selection Principle. The methods of proof are based on properties of initial-value problems for variational inequalities established in the previous section. This is why all results are stated on the interval $[0,T]$, but they can be extended by an easy linear time transformation to any interval $[a,b]$.

4.1 Uniformly bounded oscillation

We start with some definitions.

Definition 4.1 *Let $U \subset G(0,T\,;\,X)$ be an arbitrary set, and assume with the notation of Proposition 2.3 that $\varphi \in \Phi$ is an arbitrary function. Then U is said to have*

(i) uniformly bounded ε-variation *if there exists a non-increasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\forall \varepsilon > 0 \quad \forall u \in U \quad \exists \psi \in BV(0,T\,;\,X)\,: \quad \|u-\psi\|_{[0,T]} \le \varepsilon\,, \quad \operatorname*{Var}_{[0,T]} \psi \le L(\varepsilon)\,. \tag{4.1}$$

(ii) uniformly bounded oscillation *if*

– *there exists a constant $R > 0$ such that*

$$|u(t) - u(s)| \ \le\ R \qquad \forall u \in U \quad \forall s,t \in [0,T]\,, \tag{4.2}$$

– *there exists a non-increasing function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $r > 0$ and every system $\{\,]a_k, b_k[\ ;\, k = 1,\dots,m\}$ of pairwise disjoint intervals $]a_k, b_k[\subset [0,T]$ the implication*

$$\Big(|u(b_k) - u(a_k)| \ge r \quad \forall k = 1,\dots,m\Big) \quad \Rightarrow \quad m \le N(r) \tag{4.3}$$

holds for every $u \in U$.

(iii) uniformly bounded φ-variation *if for every division $0 = t_0 < t_1 < \cdots < t_m = T$ we have*

$$\sum_{j=1}^{m} \varphi(|u(t_j) - u(t_{j-1})|) \ \le\ 1 \qquad \forall u \in U\,.$$

Below we prove in detail the following equivalence result.

Theorem 4.2 *Let $U \subset G(0,T\,;\,X)$ be a given set. Then the following three conditions are equivalent.*

(i) *U has uniformly bounded ε-variation.*

(ii) *U has uniformly bounded oscillation.*

(iii) *There exists $\varphi \in \Phi$ such that U has uniformly bounded φ-variation.*

The equivalence (i) $\Leftrightarrow$ (iii) restricted to continuous functions $[0,T] \to X$ along with quantitative estimates has been established in [35, Theorem 17] in the power law case $\varphi(r) = Cr^p$ with $C > 0$ and $p \geq 1$.

Before passing to the proof of Theorem 4.2, we need the following auxiliary result which uses the concept of play operator defined in the previous section and generalizes the BV-estimate in Theorem 3.11.

Lemma 4.3 *Let $Z \subset X$ be a convex closed set with $B_r(0) \subset Z$ for some $r > 0$, and let $c_0 > 0$ be given. Let $\mathfrak{p}$ be the operator which with each $u \in G_R(0,T\,;\,X)$ and ξ^0 of the form $u(0) - x^0$ with $x^0 \in Z$ associates the solution ξ of the problem*

(i) $\xi(0) = \xi^0$,

(ii) $u(t) - \xi(t) \in Z$ *for every* $t \in [0,T]$,

(iii) $\displaystyle\int_0^T \langle u(t) - \xi(t) - w(t), d\xi(t)\rangle \;\geq\; 0$ *for every* $w \in G(0,T\,;\,Z)$.

Let $U \subset G_R(0,T\,;\,X)$ be a set with uniformly bounded oscillation. Then there exists $C > 0$ such that

$$\operatorname*{Var}_{[0,T]} \mathfrak{p}[u,\xi^0] \;\leq\; C \tag{4.4}$$

for every $x^0 \in Z \cap B_{c_0}(0)$ and every $u \in U$.

Proof of Lemma 4.3. Let $x^0 \in Z \cap B_{c_0}(0)$ and $u \in U$ be arbitrary. We define the set

$$S \;=\; \{t \in [0,T[\;;\; |u(t) - u(t-)| > r\}\,. \tag{4.5}$$

Then $S = \{s_i\}_{i=1}^p$ with

$$0 < s_1 < \cdots < s_p < T\,, \quad 0 \leq p \leq N(r) \tag{4.6}$$

by Definition 4.1 (ii). We put $s_0 = 0$, $s_{p+1} = T$, and for $i = 1, \ldots, p+1$ define recursively the sequences

$$t_0^i \;=\; s_{i-1}\,, \tag{4.7}$$

$$t_k^i \;=\; \sup\{t \in]t_{k-1}^i, T]\,;\, |u(\tau) - u(t_{k-1}^i)| \leq r/2 \;\; \forall \tau \in]t_{k-1}^i, t[\,\} \tag{4.8}$$

as long as $t_{k-1}^i < s_i$. It follows from (4.8) that $t_k^i \leq s_i$. Indeed, assuming $t_k^i > s_i$ would imply $|u(s_i) - u(t_{k-1}^i)| \leq r/2$, $|u(s_i-) - u(t_{k-1}^i)| \leq r/2$, hence $|u(s_i) - u(s_i-)| \leq r$ in contradiction with (4.5).

Let us consider now a fixed $i \in \{1,\dots,p+1\}$, and assume that for some $q \in \mathbb{N}$ we have $s_{i-1} = t_0^i < t_1^i < \cdots < t_q^i < s_i$. For each $k = 1,\dots,q$ we have by (4.8) that $|u(t_k^i) - u(t_{k-1}^i)| \ge r/2$, and the uniformly bounded oscillation hypothesis then implies that $q \le N(r/2)$. For each $i = 1,\dots,p+1$ there exists therefore $q_i \in \mathbb{N}$ such that

$$s_{i-1} = t_0^i < t_1^i < \cdots < t_{q_i}^i = s_i\,, \qquad q_i \le q^* := N(r/2) + 1\,. \tag{4.9}$$

Consider now a fixed $k \in \{1,\dots,q_i\}$. By Lemma 3.2 (i) we have for every $y \in G(0,T\,;\,Z)$, and $s \in]t_{k-1}^i, t_k^i[$ that

$$\int_{t_{k-1}^i}^{s} \langle u(\tau) - \xi(\tau) - y(\tau), d\xi(\tau)\rangle \;\ge\; 0\,. \tag{4.10}$$

In particular, we may choose an arbitrary $w \in S(0,T\,;\,B_1(0))$ and put in (4.10)

$$y(\tau) \;=\; \Big(u(\tau) - u(t_{k-1}) + \frac{r}{2}\,w(\tau)\Big)\;\chi_{[t_{k-1}^i, t_k^i[}(\tau)$$

Indeed, then $\|y\|_{[0,T]} \le r$ according to (4.8), hence $y(\tau) \in Z$ for all $\tau \in [0,T]$. Then (4.10) yields

$$\frac{r}{2}\int_{t_{k-1}^i}^{s} \langle w(\tau), d\xi(\tau)\rangle \;\le\; \int_{t_{k-1}^i}^{s} \big\langle u(t_{k-1}^i) - \xi(\tau), d\xi(\tau)\big\rangle$$

and from Corollaries 2.6, 2.12, and Lemma 2.2 (ii) it follows that

$$r \operatorname*{Var}_{[t_{k-1}^i, s]} \xi \;\le\; |u(t_{k-1}^i) - \xi(t_{k-1}^i)|^2 - |u(t_{k-1}^i) - \xi(s)|^2\,.$$

Letting $s \to t_k^i-$ we obtain on the one hand that

$$|u(t_{k-1}^i) - \xi(t_{k-1}^i)| \;\ge\; |u(t_{k-1}^i) - \xi(t_k^i-)|\,, \tag{4.11}$$

and Proposition 2.4 (iii) yields on the other hand that

$$r \operatorname*{Var}_{[t_{k-1}^i, t_k^i]} \xi \;\le\; r\,|\xi(t_k^i) - \xi(t_k^i-)| + |u(t_{k-1}^i) - \xi(t_{k-1}^i)|^2 \tag{4.12}$$

for every $i = 1,\dots,p+1$ and $k = 1,\dots,q_i$.

We now denote $x_k^i = u(t_k^i) - \xi(t_k^i) \in Z$, $x_k^{i-} = u(t_k^i-) - \xi(t_k^i-) \in Z$ whenever it makes sense. Then it follows from (4.11) that

$$|x_k^{i-}| \;\le\; |x_{k-1}^i| + |u(t_k^i-) - u(t_{k-1}^i)| \;\le\; |x_{k-1}^i| + R \qquad \text{for } \; k = 1,\dots,q_i\,. \tag{4.13}$$

On the other hand, inequality (3.6) yields that

$$|\xi(t_{k-1}^i) - \xi(t_{k-1}^i-)| \le |u(t_{k-1}^i) - u(t_{k-1}^i-)|\,, \tag{4.14}$$

hence

$$|x_{k-1}^i| \;\le\; |x_{k-1}^{i-}| + 2\,|u(t_{k-1}^i) - u(t_{k-1}^i-)| \;\le\; |x_{k-1}^{i-}| + 2R \qquad \text{for } \; k = 1,\dots,q_i\,. \tag{4.15}$$

Summing up the inequalities (4.13) and (4.15) over k, we obtain that $|x_k^i| \le |x_0^i| + 3Rq^*$ for all $i = 1,\dots,p+1$ and $k = 1,\dots,q_i$, hence $|x_0^{i+1}| \;=\; |x_{q_i}^i| \;\le\; |x_0^i| + 3Rq^*$ for $i = 1,\dots,p$, and

$$|x_k^i| \;\le\; |x^0| + 3(p+1)Rq^* \le c_0 + 3(p+1)Rq^* \tag{4.16}$$

for all $i = 1, \dots, p+1$ and $k = 1, \dots, q_i$. From (4.12), (4.14), and (4.16) it further follows that

$$\operatorname*{Var}_{[t^i_{k-1}, t^i_k]} \xi \;\le\; |u(t^i_{k-1}) - u(t^i_{k-1}-)| + \frac{1}{r}|x^i_{k-1}|^2 \;\le\; R + \frac{1}{r}(c_0 + 3(p+1)Rq^*)^2 \tag{4.17}$$

independently of k and i. Summing up the above inequality over k and i we thus obtain

$$\operatorname*{Var}_{[a,b]} \xi \;\le\; (1+p)q^* \left(R + \frac{1}{r}(c_0 + 3(p+1)Rq^*)^2 \right) \tag{4.18}$$

which together with (4.6) and (4.9) completes the proof. ■

We now finish the proof of Theorem 4.2.

Proof of Theorem 4.2.

(i) ⇒ (ii) Let U have uniformly bounded ε-variation, and let $]a_k, b_k[$, $k = 1, \dots, m$ be pairwise disjoint subintervals of $[0,T]$ such that for some $u \in U$ and $r > 0$ we have $|u(b_k) - u(a_k)| \ge r$. For $\varepsilon = r/4$ we find $\psi_\varepsilon \in BV(0,T\,;\,X)$ such that $\|u - \psi_\varepsilon\|_{[0,T]} \le r/4$, $\operatorname{Var}_{[0,T]} \psi_\varepsilon \le L(r/4)$. Then

$$L(r/4) \;\ge\; \sum_{k=1}^{m} |\psi_\varepsilon(b_k) - \psi_\varepsilon(a_k)| \;\ge\; \sum_{k=1}^{m} (|u(b_k) - u(a_k)| - r/2) \;\ge\; \frac{mr}{2},$$

hence $m \le 2/rL(r/4)$. We now repeat the argument with $\varepsilon = 1$, and for each $0 \le s < t \le T$ we obtain

$$|u(t) - u(s)| \;\le\; 2 + |\psi_1(t) - \psi_1(s)| \;\le\; 2 + \operatorname*{Var}_{[0,T]} \psi_1 \;\le\; 2 + L(1)\,,$$

and the assertion is proved.

(ii) ⇒ (i) Assume that U has uniformly bounded oscillation, consider an arbitrary $\varepsilon > 0$, and put $Z = B_{\varepsilon/2}(0)$, $x^0 = 0$. Let $\mathfrak{p}$ and ξ^0 be as in Lemma 4.3. For $t \in [0,T]$ and $u \in U$ put $u_+(t) = u(t+)$, $\xi(t) = \mathfrak{p}[u_+, \xi^0](t)$. The set $U_+ = \{u_+\,;\, u \in U\}$ has uniformly bounded oscillation, and by Lemma 4.3 there exists $C_\varepsilon > 0$ independent of u such that

$$\operatorname*{Var}_{[0,T]} \xi \le C_\varepsilon\,, \qquad \|u_+ - \xi\|_{[0,T]} \le \varepsilon/2\,. \tag{4.19}$$

For $u \in U$ we introduce the sets

$$S(u) \;=\; \{t \in [0,T]\,;\, u(t) \ne u_+(t)\}\,, \quad S_\varepsilon(u) \;=\; \{t \in [0,T]\,;\, |u_+(t) - u(t)| \ge \varepsilon/2\}\,, \tag{4.20}$$

and define the function

$$\psi(t) \;=\; \xi(t) + (u(t) - u_+(t))\,\chi_{S_\varepsilon(u)}(t)\,. \tag{4.21}$$

By hypothesis of uniformly bounded oscillation, the number of elements $\#S_\varepsilon(u)$ of $S_\varepsilon(u)$ is bounded above by a constant independent of u, say,

$$\#S_\varepsilon(u) \;\le\; N(\varepsilon/4)\,, \tag{4.22}$$

hence

$$\operatorname*{Var}_{[0,T]} \psi \;\le\; \operatorname*{Var}_{[0,T]} \xi + 2 \sum_{t \in S_\varepsilon(u)} |u_+(t) - u(t)| \;\le\; C_\varepsilon + 2RN(\varepsilon/4)\,. \tag{4.23}$$

We moreover have

$$|u(t) - \psi(t)| \;=\; \begin{cases} |u_+(t) - \xi(t)| \le \varepsilon/2 & \text{for } \; t \in [0,T] \setminus (S(u) \setminus S_\varepsilon(u))\,, \\ |u(t) - u_+(t) + u_+(t) - \xi(t)| \le \varepsilon & \text{for } \; t \in S(u) \setminus S_\varepsilon(u)\,, \end{cases} \tag{4.24}$$

hence U has uniformly bounded ε-variation, and the implication follows.

(ii) $\Rightarrow$ (iii) We can basically repeat the argument of Proposition 2.3. Let U have uniformly bounded oscillation with a function $N(r)$ and bound R. We set $N(r) = 1$ for $r > R$ and define $\varphi \in \Phi$ by the formula

$$\varphi(r) \;=\; \frac{r}{RN(r/2)} \quad \text{for } \; r > 0\,. \tag{4.25}$$

Let $u \in U$ and $0 = t_0 < \cdots < t_m = T$ be arbitrary, and let M_k be the sets

$$M_k \;=\; \big\{ j \in \{1,\dots,m\}\,;\, |u(t_j) - u(t_{j-1})| \in]2^{-k}R, 2^{-k+1}R] \big\} \; \text{ for } \; k \in \mathbb{N}\,. \tag{4.26}$$

The number of elements of M_k does not exceed $N(2^{-k}R)$, and we have

$$\sum_{j=1}^{m} \varphi(|u(t_j) - u(t_{j-1})|) \;=\; \sum_{k=1}^{\infty} \sum_{j\in M_k} \varphi(|u(t_j) - u(t_{j-1})|) \;\le\; \sum_{k=1}^{\infty} N(2^{-k}R)\varphi(2^{-k+1}R) \;=\; 1\,, \tag{4.27}$$

which we wanted to prove.

(iii) $\Rightarrow$ (ii) This part is easy. Let $]a_k, b_k[$, $k = 1,\dots,m$ be pairwise disjoint subintervals of $[0,T]$ such that for some $u \in U$ and $r > 0$ we have $|u(b_k) - u(a_k)| \ge r$. Then

$$1 \;\ge\; \sum_{k=1}^{m} \varphi(|u(b_k) - u(a_k)|) \;\ge\; m\,\varphi(r)\,, \tag{4.28}$$

hence $m \le 1/\varphi(r)$. Furthermore, for each $0 \le s < t \le T$ we have $\varphi(|u(t) - u(s)|) \le 1$, hence $|u(t) - u(s)| \le \sup\{r \ge 0\,;\, \varphi(r) \le 1\}$, and the proof is complete. ∎

4.2 Convergent subsequences

The concept of uniformly bounded ε-variation has been introduced in [16, Definition 3.3] with the intention to extend the Helly Selection Principle to regulated functions. For our purposes, we state this result in the following form.

Proposition 4.4 *Let $\{f_n\,;\, n \in \mathbb{N}\}$ be a bounded sequence of functions from $G(0,T\,;\,X)$ which has uniformly bounded ε-variation. Then there exist $f \in G(0,T\,;\,X)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(t)$ converge weakly to $f(t)$ as $k \to \infty$ for every $t \in [0,T]$.*

The original reference goes back to [16, Theorem 3.8] in the case $\dim X < \infty$. The extension to a general separable Hilbert space X has been done in [26, Theorem 5.2]. The proof of Theorem 4.4 consists in a gradual selection of subsequences similar to the proof of the classical

Helly Selection Principle (see e. g. [19], pp. 372 – 374). In order to make the diagonalization argument more transparent, we introduce the following notation.

By $\mathcal{G}(\mathbb{N})$ we denote the set of all infinite subsets $M \subset \mathbb{N}$. We say that a sequence $\{x_n\,;\, n \in \mathbb{N}\}$ of elements of a topological space M*-converges to* x if for every neighborhood $\mathcal{U}(x)$ of x there exists n_0 such that $x_n \in \mathcal{U}(x)$ for every $n \in M$, $n \ge n_0$.

We start with the following Lemma as the Hilbert-space version of [5], Theorem I.3.5.

Lemma 4.5 *Let* $\{\psi_n\,;\, n \in \mathbb{N}\}$ *be a bounded sequence in* $BV(0,T\,;\,X)$ *such that* $\operatorname{Var}_{[0,T]} \psi_n \le C$ *for every* $n \in \mathbb{N}$. *Then there exist* $\psi \in BV(0,T\,;\,X)$ *and a set* $M \in \mathcal{G}(\mathbb{N})$ *such that* $\operatorname{Var}_{[0,T]} \psi \le C$ *and the sequence* $\psi_n(t)$ *weakly* M*-converges in* X *to* $\psi(t)$ *for every* $t \in [0,T]$.

Proof. Let $\{w_j\,;\, j \in \mathbb{N}\}$ be a countable dense subset of X. The functions $t \mapsto \langle \psi_n(t), w_1\rangle$ have uniformly bounded variation, and by virtue of the one-dimensional Helly Selection Principle we find $N_1 \in \mathcal{G}(\mathbb{N})$ such that the sequence $\{\langle \psi_n(t), w_1\rangle\}$ N_1-converges to a limit $v_1(t)$ for every $t \in [0,T]$. By induction we construct a sequence $\{N_k\,;\, k \in \mathbb{N}\}$ of sets in $\mathcal{G}(\mathbb{N})$, $N_1 \supset N_2 \supset \dots$, such that the sequence $\{\langle \psi_n(t), w_j\rangle\}$ N_j-converges to a limit $v_j(t)$ for every $t \in [0,T]$. We now put $n_1 := \min N_1$, $n_k := \min\{n \in N_k\,;\, n > n_{k-1}\}$ for $k = 2, 3, \dots$, and define the set $M := \{n_k\,;\, k \in \mathbb{N}\} \in \mathcal{G}(\mathbb{N})$. By construction, every N_j-convergent sequence is M-convergent, hence $\{\langle \psi_n(t), w_j\rangle\}$ M-converges to $v_j(t)$ for every $t \in [0,T]$ and $j \in \mathbb{N}$.

For a fixed $t \in [0,T]$, the mapping $w_j \mapsto v_j(t)$ can be extended in a unique way to a bounded linear functional on X. By the Riesz Representation Theorem, there exists an element $\psi(t) \in X$ such that $v_j(t) = \langle \psi(t), w_j\rangle$ for every $j \in \mathbb{N}$. Since the system $\{w_j\}$ is dense in X, we obtain that

$$\lim_{k\to\infty} \langle \psi_{n_k}(t), w\rangle \;=\; \langle \psi(t), w\rangle$$

for every $w \in X$ and $t \in [0,T]$. Moreover, for any fixed division $0 = t_0 < t_1 < \dots < t_m = T$ we have

$$\sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})| \;\le\; \liminf_{k\to\infty} \sum_{i=1}^{m} |\psi_{n_k}(t_i) - \psi_{n_k}(t_{i-1})| \;\le\; C\,,$$

and the assertion follows. ■

We now use Lemma 4.5 to prove Theorem 4.4 by an argument similar to the one used in [16] in the case $\dim X < \infty$.

Proof of Theorem 4.4. We fix a sequence $\varepsilon_i \to 0$ and for every $n, i \in \mathbb{N}$ we find $\psi_n^i \in BV(0,T\,;\,X)$ such that $\|\psi_n^i - f_n\|_{[0,T]} < \varepsilon_i$, $\operatorname{Var}_{[0,T]} \psi_n^i \le L(\varepsilon_i)$. We now apply Lemma 4.5 to find $M_1 \in \mathcal{G}(\mathbb{N})$ and $\psi^1 \in BV(0,T\,;\,X)$ such that $\operatorname{Var}_{[0,T]} \psi^1 \le L(\varepsilon_1)$ and $\psi_n^1(t)$ weakly M_1-converges to $\psi^1(t)$ for every $t \in [0,T]$. We continue by induction and construct a sequence $\{M_i\}$ of sets in $\mathcal{G}(\mathbb{N})$, $M_1 \supset M_2 \supset \dots$, such that the sequence $\{\psi_n^i(t)\}$ weakly M_i-converges to $\psi^i(t)$ for every $t \in [0,T]$ and $i \in \mathbb{N}$, $\psi^i \in BV(0,T\,;\,X)$, $\operatorname{Var}_{[0,T]} \psi^i \le L(\varepsilon_i)$. Putting $n_1 := \min M_1$, $n_k := \min\{n \in M_k\,;\, n > n_{k-1}\}$ for $k = 2, 3, \dots$, $M^* := \{n_k\,;\, k \in \mathbb{N}\}$ we argue as in the proof of Lemma 4.5 to obtain that $\psi_n^i(t)$ weakly M^*-converges to $\psi^i(t)$ for every $t \in [0,T]$ and $i \in \mathbb{N}$.

We now check that $\{\psi^i\,;\, i \in \mathbb{N}\}$ is a Cauchy sequence in $G(0,T\,;\,X)$. For $i, j, n \in \mathbb{N}$ we have

$$\left\|\psi_n^i - \psi_n^j\right\|_{[0,T]} \le \left\|\psi_n^i - f_n\right\|_{[0,T]} + \left\|f_n - \psi_n^j\right\|_{[0,T]} \le \varepsilon_i + \varepsilon_j.$$

Consequently we have for $t \in [0,T]$ that

$$\left|\psi^i(t) - \psi^j(t)\right| \le \liminf_{k\to+\infty} \left|\psi^i_{n_k}(t) - \psi^j_{n_k}(t)\right| \le \varepsilon_i + \varepsilon_j,$$

from which readily follows that $\{\psi^i\}$ is a Cauchy sequence in $G(0,T\,;\,X)$. We denote by $f \in G(0,T\,;\,X)$ its limit. For each $t \in [0,T]$, $w \in X$ and $k \in \mathbb{N}$ we then have

$$\langle f(t) - f_{n_k}(t), w\rangle \;=\; \left\langle f(t) - \psi^i(t), w\right\rangle + \left\langle \psi^i(t) - \psi^i_{n_k}(t), w\right\rangle + \left\langle \psi^i_{n_k}(t) - f_{n_k}(t), w\right\rangle$$

for a suitably chosen i, and we easily conclude that $f_{n_k}(t)$ weakly converges to $f(t)$ for every $t \in [0,T]$. Theorem 4.4 is proved. ■

4.3 The *wbo*-convergence

The concepts presented here were applied to the limit passage in relaxation oscillation problems in [24]. Here, we focus on the relationship with the Kurzweil integral.

Definition 4.6 *A sequence $\{f_n\}$ in $G(0,T\,;\,X)$ is said to wbo-*converge *to a function $f \in G(0,T\,;\,X)$, if the set $\{f_n\,;\, n \in \mathbb{N}\}$ has uniformly bounded oscillation and $f_n(t) \to f(t)$ as $n \to \infty$ weakly in X for every $t \in [a,b]$.*

The following generalization of Proposition 2.10 reflects the "weak" character of the *wbo*-convergence.

Theorem 4.7 *Let $f \in G(0,T\,;\,X)$ be given, and let $\{f_n\,;\, n \in \mathbb{N}\}$ be a sequence in $G(0,T\,;\,X)$ such that $f_n(t) \to f(t)$ weakly in X for every $t \in [0,T]$. Then the following two conditions are equivalent.*

(i) *f_n have uniformly bounded oscillation;*

(ii) *For every sequence $\{g_n\}$ in $\overline{BV}(0,T\,;\,X)$ such that $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$ and $\overline{\mathrm{Var}}\; g_n \le C$ independently of n we have*

$$\int_0^T \langle f_n(t), dg_n(t)\rangle \;\to\; \int_0^T \langle f(t), dg(t)\rangle\;.$$

For the proof of Theorem 4.7 we need the following lemma.

Lemma 4.8 *Consider $w \in S(0,T\,;\,X)$ and $\tilde{f}_n : [0,T] \to X$, $\tilde{f}_n(t) \to 0$ weakly for every $t \in [0,T]$. Then*

$$\lim_{n\to\infty} \int_0^T \left\langle \tilde{f}_n(t), dw(t)\right\rangle \;=\; 0\,.$$

Proof of Lemma 4.8. For a function w of the form (2.2) we have by Proposition 1.16

$$\int_0^T \left\langle \tilde{f}_n(t), dw(t) \right\rangle = \sum_{k=0}^{m} \left\langle \tilde{f}_n(t_k), c_{k+1} - c_k \right\rangle,$$

where we put $c_0 := \hat{c}_0$, $c_{m+1} := \hat{c}_m$, and it suffices to pass to the limit as $n \to \infty$. ■

Proof of Theorem 4.7.

(i) ⇒ (ii) The sequence $\{f_n\}$ is obviously bounded in $G(0,T\,;\,X)$. Indeed, as $\{f_n(0)\}$ is weakly convergent, it is necessarily bounded and we have for every n and t that

$$|f_n(t)| \le |f_n(t) - f_n(0)| + |f_n(0)| \le R + |f_n(0)|$$

which for a fixed ε yields an upper bound for $\|f_n\|_{[0,T]}$ independent of n, say $\|f_n\|_{[0,T]} \le R'$. Let now $\varepsilon > 0$ be arbitrarily chosen. By Theorem 4.2 we find for each $n \in \mathbb{N}$ functions $\{\psi^\varepsilon\}$, $\{\psi_n^\varepsilon\}$ in $BV(0,T\,;\,X)$ such that $\|f_n - \psi_n^\varepsilon\|_{[0,T]} \le \varepsilon$, $\|f - \psi^\varepsilon\|_{[0,T]} \le \varepsilon$, $\mathrm{Var}_{[0,T]}\, \psi_n^\varepsilon \le L(\varepsilon)$, $\mathrm{Var}_{[0,T]}\, \psi^\varepsilon \le L(\varepsilon)$. By Proposition 2.4 (ii) there exists a step function $w \in S(0,T\,;\,X)$ such that $\|g - w\|_{[0,T]} \le \varepsilon / L(\varepsilon)$, $\overline{\mathrm{Var}}_{[0,T]}\, w \le C$. Using Lemma 4.8 and the uniform convergence of $\{g_n\}$, we find n_0 such that for $n \ge n_0$ we have $|\int_0^T \langle (f - f_n)(t), dw(t) \rangle| \le \varepsilon$, $\|g - g_n\|_{[0,T]} \le \varepsilon / L(\varepsilon)$. Then Corollaries 2.6, 2.7 yield

$$\begin{aligned}
\left| \int_0^T \langle f(t), dg(t) \rangle - \int_0^T \langle f_n(t), dg_n(t) \rangle \right| &\le \left| \int_0^T \langle (f - \psi^\varepsilon - f_n + \psi_n^\varepsilon)(t), d(g - w)(t) \rangle \right| \\
&+ \left| \int_0^T \langle (f - f_n)(t), dw(t) \rangle \right| + \left| \int_0^T \langle (\psi^\varepsilon - \psi_n^\varepsilon)(t), d(g - w)(t) \rangle \right| \\
&+ \left| \int_0^T \langle (f_n - \psi_n^\varepsilon)(t), d(g - g_n)(t) \rangle \right| + \left| \int_0^T \langle \psi_n^\varepsilon(t), d(g - g_n)(t) \rangle \right| \\
&\le 2\,C\, \|f - \psi^\varepsilon - f_n + \psi_n^\varepsilon\|_{[0,T]} + \varepsilon + \left(4\,(R' + \varepsilon) + 2\,L(\varepsilon)\right) \|g - w\|_{[0,T]} \\
&\quad + 2\,C\, \|f_n - \psi_n^\varepsilon\|_{[0,T]} + \left(2\,(R' + \varepsilon) + L(\varepsilon)\right) \|g - g_n\|_{[0,T]} \\
&\le M\,\varepsilon
\end{aligned}$$

for $n \ge n_0$, where M is a constant independent of n and ε, hence (2.27) holds.

(ii) ⇒ (i) Let us consider a sequence of pairwise disjoint intervals $]a_k^n, b_k^n[$, $k = 1, \dots, m(n)$ in $[0,T]$ such that for some $r > 0$ we have $|f_n(b_k^n) - f_n(a_k^n)| \ge r$ for all n and k, and assume that $\limsup_{n\to\infty} m(n) = \infty$. We find $v_k^n \in B_1(0)$ such that $\langle v_k^n, f_n(b_k^n) - f_n(a_k^n) \rangle = |f_n(b_k^n) - f_n(a_k^n)|$ for all n and k, and set

$$g_n(t) = \frac{1}{m(n)} \sum_{k=1}^{m(n)} v_k^n \; \chi_{]a_k^n, b_k^n[}\,(t)$$

for $n \in \mathbb{N}$ and $t \in [0,T]$. Then $\mathrm{Var}_{[0,T]}\, g_n \le 2$, there exists a subsequence of $\{g_n\}$ which converges uniformly to 0, and

$$\int_0^T \langle f_n(t), dg_n(t) \rangle = \frac{1}{m(n)} \sum_{k=1}^{m(n)} |f_n(b_k^n) - f_n(a_k^n)| \ge r$$

which is a contradiction. ■

5 Topology of the space of regulated functions

The topological structure of the space $G(a,b\,;\,X)$ is very rich. We now compare the *wbo*-convergence introduced in the previous section with the usual weak convergence and with the convergence in the so-called *Skorokhod metric*, and use the concept of uniformly bounded oscillation for the characterization of compact sets in $G(a,b\,;\,X)$ if $\dim X < \infty$.

5.1 Representation of bounded linear functionals

Representation formulas for bounded linear functionals on $G_L(a,b\,;\,X)$ using the Dushnik integral were derived for the first time in [18], and a reformulation in terms of the Kurzweil integral was published in [36]. The extension to the whole space $G(a,b\,;\,X)$ was done in [10]. We denote by $BV_0(a,b\,;\,X)$ the closed subspace of $BV(a,b\,;\,X)$ consisting of all functions $\mu \in BV(a,b\,;\,X)$ which vanish everywhere except on a countable set.

Theorem 5.1 *Let* $P : G(a,b\,;\,X) \to \mathbb{R}$ *be a bounded linear functional. Then there exist uniquely determined functions* $\mu \in BV_0(a,b\,;\,X)$ *and* $f, \hat{f} \in BV(a,b\,;\,X)$ *such that* $\mu(b) = f(b)$, $\mu(a) = \hat{f}(a)$, *and for every* $g \in G(a,b\,;\,X)$ *we have*

$$P(g) \;=\; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t)\rangle\,, \tag{5.1}$$

$$P(g) \;=\; \left\langle \hat{f}(b), g(b)\right\rangle - \int_a^b \left\langle \hat{f}(t), dg(t)\right\rangle + \sum_{t\in[a,b]} \langle \mu(t), g(t) - g(t-)\rangle\,. \tag{5.2}$$

Moreover, the norm

$$\|P\| \;=\; \sup\{P(g)\,;\, g \in G(a,b\,;\,X)\,,\ \|g\|_{[a,b]} \le 1\} \tag{5.3}$$

of P *satisfies the estimates*

$$\|P\| \;\le\; |\mu(a)| + |f(b)| + \mathrm{Var}_{[a,b]}\, f + \mathrm{Var}_{[a,b]}\, \mu \;\; \le 3\,\|P\|\,, \tag{5.4}$$

$$\|P\| \;\le\; |\hat{f}(a)| + |\mu(b)| + \mathrm{Var}_{[a,b]}\, \hat{f} + \mathrm{Var}_{[a,b]}\, \mu \;\; \le 3\,\|P\|\,. \tag{5.5}$$

In particular, the dual $G(a,b\,;\,X)'$ *to* $G(a,b\,;\,X)$ *is isomorphic to each of the two spaces* $V_b = \{(f,\mu) \in BV(a,b\,;\,X) \times BV_0(a,b\,;\,X)\,;\, f(b) = \mu(b)\}$ *and* $V_a = \{(\hat{f},\mu) \in BV(a,b\,;\,X) \times BV_0(a,b\,;\,X)\,;\, \hat{f}(a) = \mu(a)\}$.

The set $M = \{t \in [a,b]\,;\, \mu(t) \neq 0\}$ is countable and $\sum_{t\in M} |\mu(t)| \le \mathrm{Var}_{[a,b]}\, \mu < \infty$, hence formulas (5.1) – (5.2) are meaningful. Let us recall also the Riesz Representation Theorem (see [19, Chapter 4, §6] for real-valued functions) which states that every bounded linear functional P_C on $C(a,b\,;\,X)$ can be represented in a unique way as

$$P_C(g) \;=\; \int_a^b \langle g(t), df^*(t)\rangle \qquad \forall g \in C(a,b\,;\,X)\,, \qquad \|P_C\| \;=\; \mathop{\mathrm{Var}}_{[a,b]} f^* \tag{5.6}$$

with some $f^* \in BV(0,T\,;\,X)$ such that $f^*(0) = 0$ and $f^*(t) = (f^*(t+) + f^*(t-))/2$ for all $t \in]a,b[$. This also follows from (5.2) if we put $f_*(t) = \hat{f}(t) - \hat{f}(a)\,\chi_{\{a\}}(t)$, $f^*(t) =$

$(f_*(t+) + f_*(t-))/2$ for $t \in]a,b[$, $f^*(t) = f_*(t)$ for $t = a, b$, and integrate by parts using Proposition 2.11. The main difference between, say, (5.1) and (5.6) consists in the fact that f^* in (5.6) generates a measure, so that the integral can also be interpreted in the Lebesgue sense, while dg on the right-hand side of (5.1) or (5.2) is not a measure in general.

Proof of Theorem 5.1. It suffices to prove (5.1). Indeed, putting

$$\hat{f}(t) = f(a) - f(t) + \mu(t) \tag{5.7}$$

we have $\hat{f}(b) = f(a)$, $\hat{f}(a) = \mu(a)$, and from the identity

$$\int_a^b \langle \mu(t), dg(t)\rangle \; = \; \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t-)\rangle \tag{5.8}$$

for every $g \in G(a,b\,;X)$ we obtain (5.2). In fact, formula (5.8) can be verified using the results of Subsection 2.2. For each $n \in \mathbb{N}$ we find a finite set $M_n \subset M$ such that $\sum_{M\setminus M_n} |\mu(t)| < 1/n$, and define $\mu_n(t) = \sum_{s\in M_n} \mu(s)\ \chi_{\{s\}}(t)$. Identity (5.8) holds with μ replaced by μ_n for every n as a consequence of Proposition 1.17 (i), and using the estimate in Corollary 2.7 we pass to the limit as $n \to \infty$.

In the proof of (5.1) we proceed analogously as in the Riesz Representation Theorem in [19]. For each fixed $t \in [a,b]$, the mappings $v \mapsto P(v\,\chi_{[t,b]})$, $v \mapsto P(v\,\chi_{\{t\}})$ are bounded linear functionals on X. There exist therefore elements denoted by $f(t)$, $\mu(t)$, respectively, such that

$$\langle f(t), v\rangle \; = \; P(v\,\chi_{[t,b]})\,, \quad \langle \mu(t), v\rangle \; = \; P(v\,\chi_{\{t\}}) \qquad \forall v \in X\,. \tag{5.9}$$

For an arbitrary division $a = t_0 < t_1 < \cdots < t_m = b$ and an arbitrary sequence $c_0, c_1, \ldots, c_m \in B_1(0)$ we have

$$\left|\sum_{j=1}^m \langle f(t_j) - f(t_{j-1}), c_j\rangle\right| \; = \; \left|P\left(\sum_{j=1}^m c_j \chi_{[t_{j-1},t_j[}\right)\right| \; \le \; \|P\|\,, \tag{5.10}$$

$$\left|\sum_{j=0}^m \langle \mu(t_j), c_j\rangle\right| \; = \; \left|P\left(\sum_{j=0}^m c_j \chi_{\{t_j\}}\right)\right| \; \le \; \|P\|\,, \tag{5.11}$$

hence $f \in BV(a,b\,;X)$, $\mu \in BV(a,b\,;X)$, $\mu(t) \ne 0$ in at most countably many points. Furthermore, for every function $g \in S(a,b\,;X)$ of the form

$$g = \sum_{j=0}^m \hat{c}_j\,\chi_{\{t_j\}} + \sum_{j=1}^m c_j\,\chi_{\,]t_{j-1},t_j[} \tag{5.12}$$

with any choice of the division $a = t_0 < t_1 < \cdots < t_m = b$ and of the sequences $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_m$, $c_1, \ldots, c_m \in X$ we have $P(c_j\,\chi_{\,]t_{j-1},t_j[}) = \langle f(t_{j-1}) - f(t_j) - \mu(t_{j-1}), c_j\rangle$, hence

$$P(g) \; = \; \langle \mu(b), \hat{c}_m\rangle + \sum_{j=0}^{m-1} \langle \mu(t_j), \hat{c}_j - c_{j+1}\rangle - \sum_{j=1}^m \langle f(t_j) - f(t_{j-1}), c_j\rangle\ . \tag{5.13}$$

On the other hand, Proposition 1.16 yields

$$\int_a^b \langle f(t), dg(t)\rangle \; = \; \langle f(b), \hat{c}_m\rangle - \langle f(a), \hat{c}_0\rangle - \sum_{j=1}^m \langle f(t_j) - f(t_{j-1}), c_j\rangle\ , \tag{5.14}$$

hence (note that $f(b) = \mu(b)$)

$$P(g) \;=\; \langle f(a), \hat{c}_0\rangle + \int_a^b \langle f(t), dg(t)\rangle + \sum_{j=0}^{m-1} \langle \mu(t_j), \hat{c}_j - c_{j+1}\rangle \tag{5.15}$$

which is nothing but formula (5.1) for every $g \in S(a,b\,;\,X)$ of the form (5.12). Every function $g \in G(a,b\,;\,X)$ can be approximated by a uniformly convergent sequence of functions from $S(a,b\,;\,X)$, hence, by Corollary 2.7, identity (5.1) holds for every $g \in G(a,b\,;\,X)$.

The uniqueness of f, $\hat{f}$, μ is easy. Assume for instance that

$$\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t)\rangle \;=\; 0 \qquad \forall g \in G(a,b\,;\,X)\,. \tag{5.16}$$

Choosing in (5.16) $g(t) = g_s(t) := \mu(s)\,\chi_{\{s\}}(t)$ for $s \in [a,b[$ and $t \in [a,b]$ we obtain from Proposition 1.16 that $\mu(s) = 0$ for all $s \in [a,b[$. Similarly, the choice $g(t) = \tilde{g}_s(t) := f(s)\,\chi_{[s,b]}(t)$ for $s \in [a,b]$ and $t \in [a,b]$ yields that $f(s) = 0$ for all $s \in [a,b]$, hence also $\mu(b) = f(b) = 0$. The uniqueness of $\hat{f}$ can be obtained similarly.

We now pass to the proof of (5.4). From Corollary 2.7 it immediately follows that

$$\|P\| \;\le\; |f(b)| + \operatorname*{Var}_{[a,b]} f + 2\sum_{t\in[a,b[} |\mu(t)| \;\le\; |\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu\,. \tag{5.17}$$

Conversely, for every $\varepsilon > 0$ we use Corollary 2.7 and find $g_\varepsilon \in G(a,b\,;\,X)$, $\|g_\varepsilon\|_{[a,b]} \le 1$, such that

$$\langle f(a), g_\varepsilon(a)\rangle + \int_a^b \langle f(t), dg_\varepsilon(t)\rangle \;>\; |f(b)| + \operatorname*{Var}_{[a,b]} f - \varepsilon\,. \tag{5.18}$$

We construct a finite set $D_\varepsilon \subset [a,b]$, $D_\varepsilon = \{t_1^\varepsilon, \dots, t_{m(\varepsilon)}^\varepsilon\}$, such that $a \in D_\varepsilon$ if $\mu(a) \neq 0$, $b \in D_\varepsilon$ if $\mu(b) \neq 0$, $\mu(t_i^\varepsilon) \neq 0$ for $i = 1,\dots,m(\varepsilon)$ and

$$\sum_{t\in[a,b]\setminus D_\varepsilon} |\mu(t)| \;<\; \frac{\varepsilon}{2}\,. \tag{5.19}$$

Let λ_ε be the function

$$\lambda_\varepsilon(t) \;=\; \sum_{i=1}^{m(\varepsilon)} \left(g_\varepsilon(t_i^\varepsilon+) - g_\varepsilon(t_i^\varepsilon) + 2\frac{\mu(t_i^\varepsilon)}{|\mu(t_i^\varepsilon)|}\right) \chi_{\{t_i^\varepsilon\}}(t) \qquad \text{for } \ t \in [a,b]\,, \tag{5.20}$$

and set

$$g(t) \;=\; g_\varepsilon(t) + \lambda_\varepsilon(t) \qquad \text{for } \ t \in [a,b]\,. \tag{5.21}$$

Proposition 1.16 yields (note that $\langle f(b), \lambda_\varepsilon(b)\rangle = 2|\mu(b)| = 2|f(b)|$)

$$\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \;=\; \langle f(a), g_\varepsilon(a)\rangle + \int_a^b \langle f(t), dg_\varepsilon(t)\rangle + 2|f(b)|\,. \tag{5.22}$$

Furthermore, for $i = 1,\dots,m(\varepsilon)$, $t_i^\varepsilon \neq b$ we have

$$\langle \mu(t_i^\varepsilon), g(t_i^\varepsilon+) - g(t_i^\varepsilon)\rangle \;=\; \langle \mu(t_i^\varepsilon), g_\varepsilon(t_i^\varepsilon+) - g_\varepsilon(t_i^\varepsilon) - \lambda_\varepsilon(t_i^\varepsilon)\rangle \;=\; -2|\mu(t_i^\varepsilon)|\,, \tag{5.23}$$

hence, by virtue of (5.19) – (5.20),

$$-\sum_{t\in D_\varepsilon} \langle \mu(t), g(t+) - g(t)\rangle \;=\; 2\sum_{t\in D_\varepsilon\setminus\{b\}} |\mu(t)| \;\geq\; |\mu(a)| + \operatorname*{Var}_{[a,b]} \mu - |\mu(b)| - \varepsilon\,, \tag{5.24}$$

$$\left|\sum_{t\in[a,b]\setminus D_\varepsilon} \langle \mu(t), g(t+) - g(t)\rangle\right| \;\leq\; 2\sum_{t\in[a,b[\setminus D_\varepsilon} |\mu(t)| \;\leq\; \varepsilon\,. \tag{5.25}$$

From (5.18), (5.22), and (5.24) – (5.25) it follows that

$$\begin{aligned} P(g) \;&=\; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t)\rangle \\ &\geq\; |\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu - 3\varepsilon\,, \end{aligned} \tag{5.26}$$

hence

$$|\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu \;\leq\; \|P\|\,\|g\|_{[a,b]} + 3\varepsilon \;\leq\; 3\,(\|P\| + \varepsilon)\,. \tag{5.27}$$

Since ε is arbitrary, we obtain (5.4) from (5.17) and (5.27). The proof of (5.5) is analogous. ∎

Representations formulas have a particularly simple form if we restrict ourselves to left-continuous or right-continuous functions.

Corollary 5.2 *For every functionals* $P_R \in G_R(a,b\,;X)'$, $P_L \in G_L(a,b\,;X)'$ *there exist uniquely determined functions* $f, \hat{f} \in BV(a,b\,;X)$ *such that*

$$P_R(g) \;=\; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \qquad \forall g \in G_R(a,b\,;X)\,, \tag{5.28}$$

$$P_L(g) \;=\; \left\langle \hat{f}(b), g(b)\right\rangle - \int_a^b \left\langle \hat{f}(t), dg(t)\right\rangle \qquad \forall g \in G_L(a,b\,;X)\,, \tag{5.29}$$

and we have

$$\|P_R\| \;=\; |f(b)| + \operatorname*{Var}_{[a,b]} f\,, \qquad \|P_L\| \;=\; |\hat{f}(a)| + \operatorname*{Var}_{[a,b]} \hat{f}\,. \tag{5.30}$$

In particular, both $G_R(a,b\,;X)'$, $G_L(a,b\,;X)'$ *are isometrically isomorphic to* $BV(a,b\,;X)$.

This is a slight improvement with respect to Theorem 3.8 of [36], where bounded linear functionals P_L on $G_L(a,b\,;\mathbb{R})$ (with an obvious extension to X-valued functions) are represented in the form

$$P_L(g) \;=\; \langle c, g(a)\rangle + \int_a^b \langle p(t), dg(t)\rangle \tag{5.31}$$

with a vector $c \in X$ and a function $p \in BV(a,b\,;X)$. We thus "save" one component, although (5.31) is indeed (note that g is left-continuous!) equivalent to (5.29) with

$$\hat{f}(t) \;=\; c - p(t)\;\chi_{[a,b[}\,(t)\,.$$

Proof of Corollary 5.2. To prove (5.28) – (5.29), we just repeat the argument the proof of Theorem 5.1 with

$$\langle f(t), v\rangle \;=\; P(v\,\chi_{[t,b]}\,)\,, \quad \left\langle \hat{f}(t), v\right\rangle \;=\; P(v\,\chi_{[a,t]}\,) \qquad \forall v \in X\,. \tag{5.32}$$

Identities (5.30) follow from Corollary 2.7. ∎

5.2 Weak and *wbo*-convergences

A natural question is whether a functional P on $G_L(a,b;X)$ or $G_R(a,b;X)$ can also be represented in the form

$$P(g) = \langle f(b), g(b)\rangle - \int_a^b \langle g(t), df(t)\rangle , \tag{5.33}$$

$$P(g) = \left\langle \hat{f}(a), g(a)\right\rangle + \int_a^b \left\langle g(t), d\hat{f}(t)\right\rangle \tag{5.34}$$

analogous to (5.6) with functions $f, \hat{f} \in \overline{BV}(a,b;X)$. The following example shows that the answer is in general negative.

Example 5.3 Consider the functional $P_R(g) = \langle g(s-), v\rangle$ for some fixed $s \in]a,b[$ and $0 \neq v \in X$, which satisfies (5.1) with $\mu \equiv 0$ and $f = v\,\chi_{[a,s[}$. On the other hand, the sequence $\{g_n\}$, $g_n(t) = v\,\chi_{[s_n,s[}(t)$, $s_n \nearrow s$ has uniformly bounded oscillation and $g_n(t) \to 0$ for every $t \in [a,b]$, hence $\int_a^b \langle g_n(t), df(t)\rangle \to 0$ for every $f \in \overline{BV}(a,b;X)$ by virtue of Theorem 4.7, but $P_R(g_n) = |v|^2$ for every $n \in \mathbb{N}$ in contradiction with (5.33), (5.34). The same argument applies to the functional $P_L(g) = \langle g(s+), v\rangle$, with $g_n(t) = v\,\chi_{]s,s_n]}(t)$ and $s_n \searrow s$.

We see that the functions g_n *wbo*-converge to 0, but do not converge weakly in the usual sense. On the other hand, the functions

$$\hat{g}_n = \sum_{k=2^{n-1}+1}^{2^n-1} v\,\chi_{[a+(1/k)-2^{-2n},\, a+(1/k)[} \tag{5.35}$$

for $n \geq n_0$ sufficiently large with a fixed vector $0 \neq v \in X$ provide an example of a sequence in $G_R(a,b;X)$ which does not have uniformly bounded oscillation, but $\hat{g}_n(t) \to 0$ as $n \to \infty$ for every $t \in [a,b]$ and $P_R(\hat{g}_n) \to 0$ for every $P_R \in G_R(a,b;X)'$. Indeed, for every functional P_R of the form (5.28) we have

$$|P_R(\hat{g}_n)| \leq |v| \sum_{k=2^{n-1}+1}^{2^n-1} \left| f\left(a+\frac{1}{k}\right) - f\left(a+\frac{1}{k}-2^{-2n}\right)\right| \leq |v| \operatorname*{Var}_{[a+2^{-n},\, a+2^{1-n}]} f\,. \tag{5.36}$$

This yields that

$$\sum_{n=n_0}^{\infty} |P_R(\hat{g}_n)| \leq |v| \operatorname*{Var}_{[a,b]} f < \infty, \tag{5.37}$$

hence $P_R(\hat{g}_n) \to 0$ as $n \to \infty$. We see that there is no direct implication between the weak and the *wbo*-convergence in $G_R(a,b;X)$ or $G_L(a,b;X)$.

We conclude the considerations on representation of functionals by two statements illustrating in a different way the duality between $G(a,b;X)$ and $BV(a,b;X)$ with respect to convergences in Theorem 4.7.

Theorem 5.4 *Let $P : G(a,b;X) \to \mathbb{R}$ be a bounded linear functional. Then the following two conditions are equivalent.*

(i) *$P(g_n) \to P(g)$ for every wbo-convergent sequence $g_n \to g$ in $G(a,b\,;\,X)$;*

(ii) *There exist $f, \hat f \in BV(a,b\,;\,X)$ such that for every $g \in G(a,b\,;\,X)$ the identities (5.33), (5.34) hold.*

Proof. The implication (ii) $\Rightarrow$ (i) follows from Theorem 4.7 with $g_n \equiv g$. To prove the converse, we define the functions f, $\hat f$, and μ as in (5.9), (5.7). For every $v \in X$ and every $t \in [a,b[$ we have by hypothesis $P(v\,\chi_{]t,b]}) = \lim_{s\to t+} P(v\,\chi_{[s,b]})$, hence $f(t) = f(t+) + \mu(t)$, and similarly for $t \in]a,b]$ we have $P(v\,\chi_{[t,b]}) = \lim_{s\to t-} P(v\,\chi_{[s,b]})$, hence $f(t) = f(t-)$. We now invoke the integration-by-parts formula (2.31) which in combination with (5.1) yields (5.33). Using the fact that $\mu(t+) = 0$ for $t \in [a,b[\,$, $\mu(t-) = 0$ for $t \in]a,b]$, we obtain from (5.7) that $\hat f(t) = \hat f(t-) + \mu(t)$, $\hat f(t) = \hat f(t+)$ for every $t \in [a,b]$, and argue as above to obtain (5.34). ∎

Theorem 5.5 *Let $Q_L : BV_L(a,b\,;\,X) \to \mathbb{R}$ be a bounded linear functional. Then the following two conditions are equivalent.*

(i) *$Q_L(g_n) \to Q_L(g)$ for every sequence $\{g_n\}$ in $BV_L(a,b\,;\,X)$ such that $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$ and $\mathrm{Var}_{[a,b]}\, g_n \le C$ independently of n;*

(ii) *There exists $\hat f \in G(a,b\,;\,X)$ such that*

$$Q_L(g) \;=\; \Big\langle \hat f(b), g(b) \Big\rangle - \int_a^b \Big\langle \hat f(t), dg(t) \Big\rangle \qquad \forall g \in BV_L(a,b\,;\,X)\,. \tag{5.38}$$

Proof. The implication (ii) $\Rightarrow$ (i) follows again from Theorem 4.7 with $f_n \equiv \hat f$. To prove the converse, we argue as in the proof of Theorem 5.1 and define the function $\hat f : [0,T] \to X$ by the formula

$$\Big\langle \hat f(t), v \Big\rangle \;=\; Q_L(v\,\chi_{[a,t]}) \quad \forall v \in X \ \ \forall t \in [a,b]\,. \tag{5.39}$$

We check by contradiction that $\hat f \in G(a,b\,;\,X)$. Assuming that there exist $t \in]a,b]$, $\delta > 0$, and sequences $s_j \nearrow t$, $t_j \nearrow t$, $s_j < t_j < s_{j+1} < t_{j+1}$ for all $j \in \mathbb{N}$, such that $|\hat f(t_j) - \hat f(s_j)| \ge \delta$, it suffices to choose $v_j \in X$ in such a way that

$$\Big\langle \hat f(t_j) - \hat f(s_j), v_j \Big\rangle \;=\; |\hat f(t_j) - \hat f(s_j)|\,, \quad |v_j| = 1 \qquad \forall j \in \mathbb{N}\,,$$

and put

$$g_n \;=\; 2^{-n} \sum_{j=2^{n-1}+1}^{2^n} v_j\,\chi_{]s_j,t_j]} \qquad \text{for } \ n \in \mathbb{N}\,. \tag{5.40}$$

Then $\|g_n\|_{[a,b]} \to 0$, $\mathrm{Var}_{[a,b]}\, g_n = 1$ for $n \in \mathbb{N}$, and

$$Q_L(g_n) \;=\; 2^{-n} \sum_{j=2^{n-1}+1}^{2^n} \Big\langle v_j, \hat f(t_j) - \hat f(s_j) \Big\rangle \;\ge\; \frac{\delta}{2}$$

which is a contradiction. This implies that $\hat f(t-)$ exists for all $t \in]a,b]$. We analogously prove that $\hat f(t+)$ exists for all $t \in [a,b[$ by considering sequences $s_j \searrow t$, $t_j \searrow t$, $s_j > t_j > s_{j+1} > t_{j+1}$ for all $j \in \mathbb{N}$, hence $\hat f \in G(a,b\,;\,X)$.

For each function g of the form

$$g \;=\; c_0\,\chi_{\{a\}} + \sum_{j=1}^{m} c_j\,\chi_{\,]t_{j-1},t_j]}\,, \quad a = t_0 < t_1 < \cdots < t_m = b \tag{5.41}$$

we have by Proposition 1.16 that

$$Q_L(g) \;=\; \left\langle \hat f(a), c_0 \right\rangle + \sum_{j=1}^{m} \left\langle \hat f(t_j) - \hat f(t_{j-1}), c_j \right\rangle \;=\; \left\langle \hat f(b), g(b) \right\rangle - \int_a^b \left\langle \hat f(t), dg(t) \right\rangle . \tag{5.42}$$

Each function $g \in BV_L(a,b\,;\,X)$ can be uniformly approximated by step functions g_n of the form (5.41) and such that $\mathrm{Var}_{[a,b]}\, g_n \le \mathrm{Var}_{[a,b]}\, g$ by virtue of Proposition 2.4 (ii). Passing to the limit as $n \to \infty$ and using Proposition 2.10 we obtain the assertion. ■

5.3 Compact sets in $G(a,b\,;\,X)$

Compact sets in $G(a,b\,;\,X)$ obviously have uniformly bounded oscillation. We now use the above results to prove the following variant of the Arzelà-Ascoli compactness criterion referring to Proposition 2.4 (i). More can be found in [16, Section 2].

Theorem 5.6 *Let $\dim X < \infty$. Then a set $U \subset G(a,b\,;\,X)$ is relatively compact if and only if it is bounded and for every $\varepsilon > 0$ there exists a division $d = \{t_0,\dots,t_m\} \in \mathcal{D}_{a,b}$ such that for every $f \in U$ and every $j = 1,\dots,m$ we have*

$$t_{j-1} < \tau < t < t_j \quad \Longrightarrow \quad |f(t) - f(\tau)| < \varepsilon\,. \tag{5.43}$$

Proof. Let U be relatively compact and let $\varepsilon > 0$ be given. We find $v_1,\dots,v_n \in G(a,b\,;\,X)$ such that for every $f \in U$ there exists $i \in \{1,\dots,n\}$ for which $\|f - v_i\|_{[a,b]} < \varepsilon/3$. For each $i = 1,\dots,n$ we use Proposition 2.4 (i) and find a division $a = t_0^i < t_1^i < \cdots < t_{m_i}^i = b$ such that

$$t_{k-1}^i < \tau < t < t_k^i \quad \Longrightarrow \quad |v_i(t) - v_i(\tau)| < \varepsilon/3\,. \tag{5.44}$$

We now obtain (5.43) by putting $\{t_0,\dots,t_m\} = \bigcup_{i-1}^{n} \{t_0^i, t_1^i, \dots, t_{m_i}^i\}$.

Conversely, let U be a bounded set and let condition (5.43) hold. We first show that U has uniformly bounded oscillation. Indeed, let $\varepsilon > 0$ be given, and let $\{t_0,\dots,t_m\}$ be a division such that (5.43) holds. Assume that there exists pairwise disjoint intervals $]a_1,b_1[\,,\dots,\,]a_p,b_p[$ such that for some $f \in U$ we have $|f(b_\ell) - f(a_\ell)| \ge \varepsilon$ for $\ell = 1,\dots,p$. Then each interval $[a_\ell, b_\ell]$ contains at least one t_j, hence $p \le 2m$. We thus proved that U is a bounded set with uniformly bounded oscillation. There is nothing to prove if U is finite. Otherwise, we can use Theorem 4.4 and from each sequence in U select a subsequence $\{f_n\}$ which converges pointwise to an element $f \in G(a,b\,;\,X)$. The proof will be complete if we show that the convergence is uniform.

To this end, we consider again any $\varepsilon > 0$ and use condition (5.43) to find a suitable division $\{t_0,\dots,t_m\} \in \mathcal{D}_{a,b}$ such that for every $n \in \mathbb{N}$ and every $j = 1,\dots,m$ we have

$$t_{j-1} < \tau < t < t_j \quad \Longrightarrow \quad |f_n(t) - f_n(\tau)| < \varepsilon/3\,. \tag{5.45}$$

Let n_0 be such that

$$n \geq n_0 \implies \begin{cases} |f_n(t_j) - f(t_j)| < \varepsilon\,, \\ \left|f_n\left(\frac{t_j+t_{j-1}}{2}\right) - f\left(\frac{t_j+t_{j-1}}{2}\right)\right| < \varepsilon/3\,, \end{cases} \quad \text{for} \quad j = 0, \dots, m\,.$$

For each $t \in \,]t_{j-1}, t_j[$ and $n \geq n_0$ we then have

$$\begin{aligned} |f_n(t) - f(t)| \;&\leq\; \left|f_n(t) - f_n\left(\frac{t_j+t_{j-1}}{2}\right)\right| + \left|f_n\left(\frac{t_j+t_{j-1}}{2}\right) - f\left(\frac{t_j+t_{j-1}}{2}\right)\right| \\ &\quad + \left|f\left(\frac{t_j+t_{j-1}}{2}\right) - f(t)\right| \;<\; \varepsilon\,, \end{aligned}$$

and the assertion follows. ■

5.4 The Skorokhod metric

It is interesting to compare the above convergence concepts in the case $X = \mathbb{R}$ with the Skorokhod metric in the form presented in [6]. Let us briefly recall its definition.

Let us define the set

$$H \;=\; \{h \in W^{1,1}(a,b)\,;\, h(a) = a\,,\; h(b) = b\,,\; 0 < \dot{h}(t) < \infty \;\; a.\,e.\,\}$$

of increasing absolutely continuous homeomorphisms of $[a,b]$, and for $h \in H$, $r \geq 0$ put

$$M(h) \;=\; \operatorname*{sup\,ess}_{t \in \,]a,b[} \left|\log \dot{h}(t)\right|\,, \quad H_r \;=\{h \in H\,;\, M(h) \leq r\}\,. \tag{5.46}$$

The Skorokhod distance of two functions $f, g \in G_R(a,b\,;\mathbb{R})$ is defined by the formula

$$d_S(f,g) \;=\; \inf\{r \geq 0\,;\, \exists h \in H_r \,:\, \|f - g \circ h\|_{[a,b]} \leq r\}\,. \tag{5.47}$$

It is shown in [6] that d_S is a metric satisfying the inequality

$$d_S(f,g) \;\leq\; \|f - g\|_{[a,b]} \qquad \forall f, g \in G_R(a,b\,;\mathbb{R})\,,$$

and transforms $(G_R(a,b\,;\mathbb{R}), d_S)$ into a complete separable metric space. This is in fact the purpose of the construction, as $(G_R(a,b\,;\mathbb{R}), \|\cdot\|)$ is not separable. It is also easy to see that if $d_S(f_n, f) \to 0$ as $n \to \infty$, then f_n have uniformly bounded oscillation. Indeed, there exist $r_n \to 0$ and homeomorphisms h_n such that

$$e^{-r_n} \;\leq\; \dot{h}_n(t) \;\leq\; e^{r_n} \quad \text{a.\,e.}\,, \quad \|f_n - f \circ h_n\|_{[a,b]} \;\leq\; r_n\,.$$

The functions $f \circ h_n$ have uniformly bounded oscillation, hence the same holds for f_n. We now construct an example which shows that the complete metric space $(G_R(a,b\,;\mathbb{R}), d_S)$ is not a metric linear space, as the addition is not continuous with respect to d_S. Furthermore, the convergence with respect to the metric d_S does not imply weak convergence.

Example 5.7 Consider the functions $f = -g = \chi_{[(1/2),1]}$ in $G_R(0,1\,;\mathbb{R})$. For $\varepsilon \in]0,1[$ set

$$h_\varepsilon(t) = \begin{cases} (1+\varepsilon)t & \text{for } t \in [0,1/2]\,, \\ (1+\varepsilon)/2 + (1-\varepsilon)(t-1/2) & \text{for } t \in]1/2,1]\,. \end{cases}$$

Then $1-\varepsilon \le \dot{h}(t) \le 1+\varepsilon$ a. e., hence $h_\varepsilon \in H_{r_\varepsilon}$ with $r_\varepsilon = \log(1+\varepsilon/(1-\varepsilon))$. The functions $f_\varepsilon := f \circ h_\varepsilon$ satisfy $d(f_\varepsilon, f) \le r_\varepsilon$. On the other hand, for every $u \in G_R(0,1\,;\mathbb{R})$ we have $d_S(u,0) = \|u\|_{[0,1]}$, hence $d(f_\varepsilon + g, f+g) = d(f_\varepsilon + g, 0) = 1$, and we see that the mapping $(f,g) \mapsto f+g$ is discontinuous. If we now define a bounded linear functional P on $G_R(0,1\,;\mathbb{R})$ by the formula $P(f) = f((1/2)-)$ for $f \in G_R(0,1\,;\mathbb{R})$, we obtain $P(f_\varepsilon) = 1$ for all $\varepsilon > 0$, $P(f) = 0$.

6 Implicit problems

Applications in continuum mechanics (see e. g. [3, 4]) often lead to variational inequalities stated in Definition 3.1 with inputs u depending on the solution ξ in the form

$$u(t) \;=\; g(t,\xi(t))\,, \tag{6.1}$$

where $g:[0,T]\times X\to V$ is a given mapping. Such a problem is also called a *quasivariational inequality* and we present here two different solution methods. The classical one consists in using the Schauder-Tikhonov fixed point theorem in $G_R(0,T\,;\,X)$ under fairly general hypotheses on the convex constraint $Z(v)$ provided both X and Y are finite-dimensional. In this way, we establish the existence of a solution and show that uniqueness cannot be expected in general. Assuming some smoothness of each $Z(v)$ as well as a smooth dependence of $Z(v)$ on v, we explain in detail the existence and uniqueness argument of [11] based on the Banach Contraction Principle in the space of absolutely continuous functions with values in any separable Hilbert space X.

6.1 Existence

We will assume that $\dim X<\infty$, $\dim Y<\infty$, and that there exist $\alpha\in BV(0,T\,;\,\mathbb{R})$ and $\kappa>0$ such that the function g in (6.1) satisfies the inequality

$$\|g(t,x)-g(\tau,\tilde x)\| \;\le\; \operatorname*{Var}_{[\tau,t]}\alpha \,+\, \kappa\,|x-\tilde x| \tag{6.2}$$

for all $0\le\tau<t\le T$ and $x,\tilde x\in X$. Let $u^0\in V$ be given. For $C>1$ we consider the sets

$$U_C \;=\; \{u\in BV_R(0,T\,;\,V)\,;\, u(0)=u^0\,,\ \|u(t)-u(\tau)\|\le C\operatorname*{Var}_{[\tau,t]}\alpha \ \text{ for }\ 0\le\tau<t\le T\}\,. \tag{6.3}$$

The set U_C is a compact subset of $G_R(0,T\,;\,V)$. Indeed, it is closed by Lemma 2.2 (v). For the function $W(t)=\operatorname{Var}_{[0,t]}\alpha$ for $t\in[0,T]$ we may use Proposition 2.4 (i) and find a division $\{t_0,\dots,t_m\}$ such that for every $j=1,\dots,m$ we have

$$t_{j-1}<\tau<t<t_j \quad\Longrightarrow\quad W(t)-W(\tau)<\varepsilon/C\,, \tag{6.4}$$

and from Theorem 5.6 (applied to an equivalent Hilbert norm in Y) it follows that U_C is compact. Consequently, the set

$$K_C \;=\; \bigcup_{u\in U_C} K(u) \tag{6.5}$$

with $K(u)$ defined in (3.5) is a compact subset of V.

Theorem 6.1 *Let the hypotheses of Proposition 3.7 be fulfilled, and let ξ^0 be such that $\xi^0\in Z(g(0,\xi^0))$. Put $u^0=g(0,\xi^0)$ and assume that there exists $C>1$ such that the numbers K_C, κ and λ_{K_C} in (3.16), (6.2) and (6.5) satisfy the inequality*

$$\kappa\,\lambda_{K_C} \;\le\; \frac{C-1}{C}\,. \tag{6.6}$$

Then there exists $\xi\in BV_R(0,T\,;\,X)$ such that $\xi(t)=\mathfrak{p}[g(\cdot,\xi),\xi^0](t)$ for all $t\in[0,T]$.

Proof. We define the set Ω of all functions $\eta \in BV_R(0,T\,;\,X)$ such that

$$\eta(0) = \xi^0\,, \quad |\eta(t) - \eta(\tau)| \le \frac{\lambda_{K_C}}{1 - \kappa\,\lambda_{K_C}} \operatorname*{Var}_{[\tau,t]} \alpha \quad \text{for} \quad 0 \le \tau < t \le T\,. \tag{6.7}$$

For each $\eta \in \Omega$ and $t \in [0,T]$ put $u(t) = g(t,\eta(t))$, $\xi(t) = \mathfrak{p}[u,\xi^0](t)$. Then for $0 \le \tau < t \le T$ we have

$$\|u(t) - u(\tau)\| \;\le\; \operatorname*{Var}_{[\tau,t]} \alpha + \kappa\,|\eta(t) - \eta(\tau)| \;\le\; \frac{1}{1 - \kappa\,\lambda_{K_C}} \operatorname*{Var}_{[\tau,t]} \alpha \;\le\; C \operatorname*{Var}_{[\tau,t]} \alpha\,,$$

hence $u \in U_C$, and Proposition 3.7 yields that $\xi \in \Omega$. The set Ω is compact in $G_R(0,T\,;\,X)$ and convex, and the mapping $\Sigma : \Omega \to \Omega : \eta \mapsto \xi$ is continuous by Proposition 3.6, hence Σ admits a fixed point $\xi \in \Omega$ by virtue of the Schauder-Tikhonov fixed point theorem, see [14, Theorem 3.6.1]. ∎

6.2 Example of non-uniqueness

The solution of the implicit problem is in general non-unique even if the constant κ is arbitrarily small and we give a simple example illustrating this fact. More sophisticated examples can be found in [4, 11]. Especially the example in [11] suggests that the sufficient conditions for uniqueness in Theorem 6.10 below are optimal.

Example 6.2 Consider a family $Z(v) \subset \mathbb{R}^2$ of convex sets parametrized by $v = (v_1, v_2)$, $v_1 \le 0$, $v_2 \ge 0$, and defined as follows.

$$z \;=\; \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \;\in\; Z(v) \quad\Longleftrightarrow\quad \begin{cases} z_1 \;\le\; v_1\,, \\ z_1 \;\le\; v_1 - v_2 + \psi(v_2)\,z_2\,, \\ z_2 \;\ge\; 0\,, \end{cases} \tag{6.8}$$

where $\psi : [0,1] \to [0,1]$ is an increasing concave function such that $\psi(0) = 0$ and

$$\int_0^1 \frac{dx}{\psi(x)} \;<\; \infty\,. \tag{6.9}$$

Let $v = (v_1, v_2)$, $\tilde v = (\tilde v_1, \tilde v_2)$ be two admissible parameters. To estimate the Hausdorff distance of $Z(v)$ and $Z(\tilde v)$, we first notice that

$$z \;=\; \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z(v) \quad\Longleftrightarrow\quad \tilde z \;=\; \begin{pmatrix} z_1 - v_1 + \tilde v_1 \\ z_2 \end{pmatrix} \in Z(\tilde v_1, v_2)\,,$$

hence

$$d_H(Z(v), Z(\tilde v_1, v_2)) \;\le\; |v_1 - \tilde v_1|\,. \tag{6.10}$$

We may assume $v_2 > \tilde v_2$. Then

$$Z(\tilde v_1, v_2) \;\subset\; Z(\tilde v)\,. \tag{6.11}$$

Indeed, let there exist $z \in Z(\tilde v_1, v_2) \setminus Z(\tilde v)$. Then

$$\psi(v_2)\,z_2 - v_2 \;\ge\; z_1 - \tilde v_1 \;>\; \psi(\tilde v_2)\,z_2 - \tilde v_2\,, \tag{6.12}$$

hence $\psi(\tilde{v}_2)\, z_2 \;<\; \tilde{v}_2\,,\; (\psi(v_2) - \psi(\tilde{v}_2)\, z_2 \;>\; v_2 - \tilde{v}_2\,.$ In other words, we have

$$\frac{\psi(\tilde{v}_2)}{\tilde{v}_2} \;<\; \frac{\psi(v_2) - \psi(\tilde{v}_2)}{v_2 - \tilde{v}_2}$$

which contradicts the concavity of ψ, hence (6.11) holds. Let now $z \in Z(\tilde{v}) \setminus Z(\tilde{v}_1, v_2)$ be arbitrary. We have

$$\psi(v_2)\, z_2 - v_2 \;<\; z_1 - \tilde{v}_1 \;\le\; \psi(\tilde{v}_2)\, z_2 - \tilde{v}_2\,,$$

hence there exists $v_2^* \in [\tilde{v}_2, v_2[$ such that

$$z_1 - \tilde{v}_1 \;=\; \psi(v_2^*)\, z_2 - v_2^*\,. \tag{6.13}$$

We thus have $z \in Z(\tilde{v}_1, v_2^*)$. Set

$$z^* \;=\; \begin{pmatrix} z_1 + v_2^* - v_2 \\ z_2 \end{pmatrix} \;=\; \begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix}.$$

Then $z_1^* - \psi(v_2)\, z_2^* \;=\; \tilde{v}_1 - v_2 + (\psi(v_2^*) - \psi(v_2))\, z_2 \;\le\; \tilde{v}_1 - v_2\,,$ hence $z^* \in Z(\tilde{v}_1, v_2)$, and $|z - z^*| \;\le\; v_2 - v_2^*$. This yields in particular that

$$d_H(Z(\tilde{v}_1, v_2), Z(\tilde{v})) \;\le\; |v_2 - \tilde{v}_2|\,. \tag{6.14}$$

Combining (6.10) with 6.14) we obtain

$$d_H(Z(v), Z(\tilde{v})) \;\le\; |v_1 - \tilde{v}_1| + |v_2 - \tilde{v}_2|\,. \tag{6.15}$$

Let $x : [0, t_0] \to \mathbb{R}_+$ be the increasing solution to the problem

$$\dot{x}(t) \;=\; \kappa\, \psi(x(t))\,, \quad x(0) = 0\,, \tag{6.16}$$

where $\kappa \in]0,1[$ is a fixed constant and t_0 is chosen in such a way that

$$\psi(x(t_0)) \;\le\; \kappa\,. \tag{6.17}$$

For $t \in [0, t_0]$ set

$$\alpha(t) \;=\; x(t) \left(1 - \frac{1}{\kappa}\, \psi(x(t))\right) - t\,. \tag{6.18}$$

Then $\alpha(0) = 0$ and

$$\dot{\alpha}(t) \;\le\; \dot{x}(t) - 1 \;\le\; \kappa^2 - 1 \;<\; 0 \qquad \forall t \in]0, t_0[\,. \tag{6.19}$$

We now define the function $g(t,\eta) \;=\; (g_1(t,\eta), g_2(t,\eta))$ by the formula

$$\left.\begin{array}{rcl} g_1(t,\eta) & = & \alpha(t) \\ g_2(t,\eta) & = & \kappa\, \eta_2 \end{array}\right\} \quad \text{for} \quad t \in [0, t_0]\,,\; \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},\; \eta_1 \le 0\,,\; \eta_2 \ge 0\,. \tag{6.20}$$

The assumptions of Theorem 6.1 are fulfilled with $Y = \mathbb{R}^2$, $V = \{v = (v_1, v_2)\,;\; v_1 \le 0\,,\; v_2 \ge 0\}$ endowed with the norm $\|v\| = |v_1| + |v_2|$, $C = 1/(1-\kappa)$, $\lambda_K = 1$ for every K, hence the implicit problem has a solution. We now check that both

$$\xi^{(1)}(t) \;=\; \begin{pmatrix} -t \\ \frac{1}{\kappa}\, x(t) \end{pmatrix}, \qquad \xi^{(2)}(t) \;=\; \begin{pmatrix} \alpha(t) \\ 0 \end{pmatrix} \tag{6.21}$$

satisfy in $[0,t_0]$ the implicit problem in differential form

$$\begin{cases} \xi(t) \ \in \ Z(g(t,\xi(t)))\,, \\ \xi(0) \ = \ 0\,, \\ \Big\langle \dot\xi(t), z-\xi(t) \Big\rangle \ \geq \ 0 \qquad \forall t \in]0,t_0[\quad \forall z \in Z(g(t,\xi(t)))\,. \end{cases} \tag{6.22}$$

Notice first that for $v_1 \leq 0$, $v_2 \geq 0$ and $t \in [0,t_0]$ we have

$$z \ \in \ Z(g(t,\eta)) \quad \Longleftrightarrow \quad \begin{cases} z_1 \ \leq \ \alpha(t)\,, \\ z_1 \ \leq \ \alpha(t) - \kappa\,\eta_2 + \psi(\kappa\eta_2)\,z_2\,, \\ z_2 \ \geq \ 0\,, \end{cases} \tag{6.23}$$

hence

$$z \ \in \ Z(g(t,\xi^{(1)}(t))) \quad \Longleftrightarrow \quad \begin{cases} z_1 \ \leq \ \alpha(t)\,, \\ z_1 \ \leq \ \psi(x(t))\,z_2 - \big(t + \frac{1}{\kappa}\,x(t)\,\psi(x(t))\big)\,, \\ z_2 \ \geq \ 0\,, \end{cases} \tag{6.24}$$

$$z \ \in \ Z(g(t,\xi^{(2)}(t))) \quad \Longleftrightarrow \quad \begin{cases} z_1 \ \leq \ \alpha(t)\,, \\ z_2 \ \geq \ 0\,. \end{cases} \tag{6.25}$$

By (6.18) – (6.19) we have $0 \geq \alpha(t) \geq -t$, hence $\xi^{(i)}(t) \in Z(g(t,\xi^{(i)}(t)))$ for $t \in [0,t_0]$ and $i = 1,2$. Furthermore,

$$\dot\xi^{(1)}(t) \ = \ \begin{pmatrix} -1 \\ \psi(x(t)) \end{pmatrix}, \quad \dot\xi^{(2)}(t) \ = \ \begin{pmatrix} \dot\alpha(t) \\ 0 \end{pmatrix},$$

hence for $z \in Z(g(t,\xi^{(1)}(t)))$ we have by (6.24) that

$$\Big\langle \dot\xi^{(1)}(t), z \Big\rangle \ = \ -z_1 + \psi(x(t))\,z_2 \ \geq \ t + \frac{1}{\kappa}\,x(t)\,\psi(x(t)) \ = \ \Big\langle \dot\xi^{(1)}(t), \xi^{(1)}(t) \Big\rangle\,.$$

Similarly we obtain from (6.25) for $z \in Z(g(t,\xi^{(2)}(t)))$ that

$$\Big\langle \dot\xi^{(2)}(t), z \Big\rangle \ = \ \dot\alpha(t)\,z_1 \ \geq \ \dot\alpha(t)\,\alpha(t) \ = \ \Big\langle \dot\xi^{(2)}(t), \xi^{(2)}(t) \Big\rangle$$

using the fact that $\dot\alpha(t) < 0$. We thus verified that the implicit problem admits multiple solutions independently of how small the Lipschitz constant κ is.

6.3 The smooth explicit case

We now explain in detail how uniqueness can be obtained under additional smoothness assumptions using the argument from [11]. In particular, we will require the differentiability of the data with respect to the parameter $v \in V \subset Y$. For this reason we assume that Y is now a reflexive Banach space endowed with a norm $\|\cdot\|$ and V is a convex closed set with non-empty interior V°, but we impose no restriction on the dimension of the Hilbert space X any more. By Y' we denote the dual of Y, $(\!(\cdot,\cdot)\!)$ is the duality pairing between Y and Y', and $|\cdot|_{Y'}$, $|\cdot|_{\mathcal{L}(X,Y)}$ denote natural norms in the corresponding spaces.

In order to exploit properties of the Minkowski functional summarized in the Appendix, we slightly reformulate the problem and assume that there exist $0 < c < C$ such that

$$B_c(0) \subset Z(v) \subset B_C(0) \qquad \forall v \in V\,. \tag{6.26}$$

As before, we consider separately two problems, namely an explicit parameter-dependent problem (Problem $(\boldsymbol{\mathcal{P}})$) for which we derive additional estimates, and an implicit quasivariational inequality (Problem $(\boldsymbol{\mathcal{I}})$ stated in the next subsection).

For given functions $u \in W^{1,1}(0,T\,;\,X)$, $v \in W^{1,1}(0,T\,;\,V)$ and an initial condition $x^0 \in Z(v(0))$ we look for a function $\xi \in W^{1,1}(0,T\,;\,X)$ such that

$$
\begin{aligned}
(\boldsymbol{\mathcal{P}})\ \ &\text{(i)} \quad u(t) - \xi(t) \in Z(v(t)) \quad \forall t \in [0,T]\,,\\
&\text{(ii)} \quad u(0) - \xi(0) = x^0\,,\\
&\text{(iii)} \quad \left\langle \dot\xi(t), u(t) - \xi(t) - z \right\rangle \ \geq\ 0 \quad \forall z \in Z(v(t)) \quad \text{for a. e. } t \in\,]0,T[\,.
\end{aligned}
$$

Under the assumption (6.26), we denote by $Z^*(v)$ the polar set to $Z(v)$ defined in (A.3.1) for $v \in V$, and by $M^*(v,\cdot)$ its Minkowski functional. By Proposition A.11 we have $B_{1/C}(0) \subset Z^*(v) \subset B_{1/c}(0)$ for every $v \in V$, and the inequalities

$$\frac{|x|}{C} \ \leq\ M(v,x) \ \leq\ \frac{|x|}{c}\,, \tag{6.27}$$

$$c\,|x| \ \leq\ M^*(v,x) \ \leq\ C\,|x| \tag{6.28}$$

hold for every $x \in X$ and $v \in V$. We make the following hypothesis.

Hypothesis 6.3

(i) The partial derivatives $\partial_v M(v,x) \in Y'$, $\partial_x M(v,x) \in X$ exist for every $x \in X \setminus \{0\}$ and $v \in V^\circ$, the mappings

$$J(v,x) \ =\ M(v,x)\,\partial_x M(v,x) \,:\, V^\circ \times X \setminus \{0\} \to X\,, \tag{6.29}$$

$$K(v,x) \ =\ M(v,x)\,\partial_v M(v,x) \,:\, V^\circ \times X \setminus \{0\} \to Y' \tag{6.30}$$

admit continuous extensions to $x = 0$ and $v \in V$, and there exists a constant $K_0 > 0$ such that

$$|K(v,x)|_{Y'} \ \leq\ K_0 \qquad \forall x \in B_C(0) \ \ \forall v \in V\,. \tag{6.31}$$

(ii) For every $x, x' \in B_C(0)$ and $v, v' \in V$ we have

$$|J(v,x) - J(v',x')| \ \leq\ C_J\,(\|v - v'\| + |x - x'|)\,, \tag{6.32}$$

$$|K(v,x) - K(v',x')|_{Y'} \ \leq\ C_K\,(\|v - v'\| + |x - x'|) \tag{6.33}$$

with some fixed constants $C_J, C_K > 0$.

Note that by (A.3.25) we have $K(v,x) = \frac{1}{2}\partial_v \langle J(v,x), x\rangle$ for all $(v,x) \in V \times X$. By virtue of (A.4.15), inequality (6.32) can also be restated in terms of the Lipschitz dependence on both variables of the unit outward normal vector at points $x/M(v,x)$ with $|x| = 1$. According to Theorem A.20 it also implies a positive lower bound for the constant $c > 0$ in (6.26). The example in [11] shows that these conditions cannot be relaxed.

Proposition 6.4 *Let Hypotheses (6.26) and 6.3 (i) hold. Then Problem* $(\boldsymbol{\mathcal{P}})$ *admits a unique solution* $\xi \in W^{1,1}(0,T\,;\,X)$ *for every given functions* $u \in W^{1,1}(0,T\,;\,X)$, $v \in W^{1,1}(0,T\,;\,V)$ *and every initial condition* $x^0 \in Z(v(0))$.

Proof. Problem $(\boldsymbol{\mathcal{P}})$ has the form as in Definition 3.1, with parameters $\tilde{v} = (v,u) \in \tilde{V} := V \times X$ and convex sets $\tilde{Z}(\tilde{v}) := u - Z(v)$. We now prove the inequality

$$d_H(Z(v), Z(v')) \ \leq\ C\, K_0\, \|v - v'\| \qquad \forall v, v' \in V\,, \tag{6.34}$$

which will enable us afterwards to obtain the assertion directly from Proposition 3.7 and Lemma 3.2 (iii). To verify that (6.34) holds, we use Lemma A.24 to obtain that

$$d_H(Z(v), Z(v')) \ \leq\ C \sup_{|x|=C} |M(v,x) - M(v',x)|\,,$$

where by (6.27) we have for $|x| = C$ that $M(v,x) \geq 1$, $M(v',x) \geq 1$, hence

$$\left|\frac{1}{2}M^2(v,x) - \frac{1}{2}M^2(v',x)\right| \geq \frac{1}{2}|M(v,x) - M(v',x)|\,(M(v,x) + M(v',x)) \geq |M(v,x) - M(v',x)|$$

and (6.34) follows from (6.31). ■

In the following two lemmas we derive some useful formulas.

Lemma 6.5 *Let Hypothesis 6.3 (i) hold, let* $(v,u) \in W^{1,1}(0,T\,;\,V) \times W^{1,1}(0,T\,;\,X)$ *and* $x^0 \in Z(v(0))$ *be given, and let* $\xi \in W^{1,1}(0,T\,;\,X)$ *solve Problem* $(\boldsymbol{\mathcal{P}})$. *For* $t \in]0,T[$ *set*

$$\begin{aligned}
A[v,u](t) &= \left\langle \dot{\xi}(t), J(v(t), x(t)) \right\rangle\,,\\
B[v,u](t) &= \frac{1}{2}\, M^2(v(t), x(t))\,,\\
G[v,u](t) &= \langle \dot{u}(t), J(v(t), x(t))\rangle + \langle\!\langle K(v(t), x(t)), \dot{v}(t)\rangle\!\rangle\,,
\end{aligned}$$

with $x(t) = u(t) - \xi(t)$. *Then for a. e.* $t \in]0,T[$ *we have either*

(i) $\dot{\xi}(t) = 0$, $\frac{d}{dt}B[v,u](t) = G[v,u](t)$

or

(ii) $\dot{\xi}(t) \neq 0$, $x(t) \in \partial Z(v(t))$, $A[v,u](t) = G[v,u](t) > 0$, $B[v,u](t) = \max_{[0,T]} B[v,u] = 1/2$, $\frac{d}{dt}B[v,u](t) = 0$, *and*

$$\dot{\xi}(t) \ =\ \frac{A[v,u](t)}{|J(v(t), x(t))|^2}\, J(v(t), x(t))\,. \tag{6.35}$$

Proof. Let $L \subset]0,T[$ be the set of Lebesgue points of all functions $\dot u$, $\dot v$, $\dot\xi$, $\frac{d}{dt}B[v,u]$. Then L has full measure in $[0,T]$, and for $t \in L$ we have

$$\frac{d}{dt}B[v,u](t) \;=\; \langle \dot x(t), J(v(t),x(t))\rangle + (\!(K(v(t),x(t)), \dot v(t))\!)\ . \tag{6.36}$$

If $\dot\xi(t) = 0$, then $\dot x(t) = \dot u(t)$, and (i) follows from (6.36). If $\dot\xi(t) \neq 0$, then $x(t) \in \partial Z(v(t))$, hence $M(v(t),x(t)) = 1 = \max_{s\in[0,T]} M(v(s),x(s))$. We therefore have $B[r,u](t) = 1/2 = \max_{[0,T]} B[r,u]$, $\frac{d}{dt}B[r,u](t) = 0$. As a consequence of $(\mathcal{P})$ (iii) we have $\dot\xi(t) = k\, n(v(t),x(t))$ with a constant $k > 0$, where $n(v(t),x(t))$ is the unit outward normal to $Z(v(t))$ at the point $x(t)$, hence $k = \left\langle \dot\xi(t), n(v(t),x(t))\right\rangle$, and (6.35) follows from (A.4.15). Furthermore, (6.36) yields $\langle \dot x(t), J(v(t),x(t))\rangle = -\,(\!(K(v(t),x(t)), \dot v(t))\!)$, hence

$$\begin{aligned}\left\langle \dot\xi(t), J(v(t),x(t))\right\rangle &= \langle \dot u(t), J(v(t),x(t))\rangle - \langle \dot x(t), J(v(t),x(t))\rangle\\ &= \langle \dot u(t), J(v(t),x(t))\rangle + (\!(K(v(t),x(t)), \dot v(t))\!)\ ,\end{aligned}$$

and the proof is complete. ∎

In the situation of Lemma 6.5, we always have

$$|G[v,u](t)| \;\le\; |\dot u(t)||J(v(t),x(t))| + K_0\|\dot v(t)\|\,, \tag{6.37}$$

$$|\dot\xi(t)| \;\le\; |\dot u(t)| + CK_0\|\dot v(t)\|\,. \tag{6.38}$$

Indeed, (6.38) is trivial if $\dot\xi(t) = 0$; otherwise we have $|\dot\xi(t)| = A[v,u](t)/|J(v(t),x(t))| = G[v,u](t)/|J(v(t),x(t))|$ with $x(t) \in \partial Z(v(t))$. As a consequence of (6.28) and (A.4.15) we have that $C\,|J(v(t),x(t)))| \ge M^*(v(t),J(v(t),x(t))) = M(v(t),x(t)) = 1$, and (6.38) follows from (6.37).

Lemma 6.6 *Let Hypothesis 6.3 (i) hold, let $(v_i,u_i) \in W^{1,1}(0,T\,;V)\times W^{1,1}(0,T\,;X)$ and $x_i^0 \in Z(v_i(0))$ be given, let $\xi_i \in W^{1,1}(0,T\,;X)$ be the respective solutions to Problem $(\mathcal{P})$, and set $x_i = u_i - \xi_i$ for $i = 1,2$. Then for a. e. $t \in]0,T[$ we have*

$$\begin{aligned}|A[v_1,u_1](t) - A[v_2,u_2](t)| \;&+\; \frac{d}{dt}|B[v_1,u_1](t) - B[v_2,u_2](t)| \\ &\le\; |G[v_1,u_1](t) - G[v_2,u_2](t)|\,,\end{aligned} \tag{6.39}$$

$$\begin{aligned}|\dot\xi_1(t) - \dot\xi_2(t)| \;&\le\; C\,|A[v_1,u_1](t) - A[v_2,u_2](t)| \\ &\quad + C\,(|\dot u_1(t)| + CK_0\|\dot v_1(t)\|)|J(v_1(t),x_1(t)) - J(v_2(t),x_2(t))|\,.\end{aligned} \tag{6.40}$$

Proof. The assertion follows directly from Lemma 6.5 if $\dot\xi_1(t) = \dot\xi_2(t) = 0$. Assume now

- $\dot\xi_1(t) \neq 0$, $\dot\xi_2(t) \neq 0$.

Then (6.39) is again an immediate consequence of Lemma 6.5. To prove (6.40), we use (6.35) and the elementary vector identity

$$\left|\frac{z}{|z|^2} - \frac{z'}{|z'|^2}\right| \;=\; \frac{1}{|z||z'|}\,|z - z'| \quad \text{for} \quad z, z' \in X \setminus \{0\}\,,$$

to obtain

$$\begin{aligned}|\dot\xi_1(t)-\dot\xi_2(t)| &\le |A[v_1,u_1](t)|\left|\frac{J(v_1(t),x_1(t))}{|J(v_1(t),x_1(t))|^2}-\frac{J(v_2(t),x_2(t))}{|J(v_2(t),x_2(t))|^2}\right|\\ &\quad+\frac{1}{|J(v_2(t),x_2(t))|}\,|A[v_1,u_1](t)-A[v_2,u_2](t)|\\ &= \frac{1}{|J(v_1(t),x_1(t))|\,|J(v_2(t),x_2(t))|}\,|G[v_1,u_1](t)|\,|J(v_1(t),x_1(t))-J(v_2(t),x_2(t))|\\ &\quad+\frac{1}{|J(v_2(t),x_2(t))|}\,|A[v_1,u_1](t)-A[v_2,u_2](t)|\,.\end{aligned}$$

By (A.4.15) we have $|J(v_i(t),x_i(t))|\ge 1/C$ for $i=1,2$, and combining the above inequalities with (6.37) we obtain the assertion.

Let us consider now the case

- $\dot\xi_1(t)\neq 0\,,\ \dot\xi_2(t)=0\,.$

Then $|A[v_1,u_1](t)-A[v_2,u_2](t)|=A[v_1,u_1](t)\,,\ B[v_1,u_1](t)-B[v_2,u_2](t)=1/2-B[v_2,u_2](t)\ge 0$, hence

$$\begin{aligned}|A[v_1,u_1](t)-A[v_2,u_2](t)|+\frac{d}{dt}|B[v_1,u_1](t)-B[v_2,u_2](t)| &= A[v_1,u_1](t)-\frac{d}{dt}B[v_2,u_2](t)\\ &= G[v_1,u_1](t)-G[v_2,u_2](t)\,,\end{aligned}$$

hence (6.39) is fulfilled. We further have similarly as above that

$$|\dot\xi_1(t)-\dot\xi_2(t)| = |\dot\xi_1(t)| \le C\,A[v_1,u_1](t) = C\,|A[v_1,u_1](t)-A[v_2,u_2](t)|\,,$$

hence (6.40) holds. The remaining case

- $\dot\xi_1(t)=0\,,\ \dot\xi_2(t)\neq 0$

is analogous, and Lemma 6.6 is proved. ■

We are now ready to prove the following crucial estimate.

Proposition 6.7 *Let Hypothesis 6.3 hold, let* $(v_i,u_i)\in W^{1,1}(0,T\,;\,V)\times W^{1,1}(0,T\,;\,X)$ *and* $x_i^0\in Z(v_i(0))$ *be given, let* $\xi_i\in W^{1,1}(0,T\,;\,X)$ *be the respective solutions to Problem* $(\boldsymbol{\mathcal{P}})$*, and set* $x_i=u_i-\xi_i$ *for* $i=1,2$*. Then for a. e.* $t\in\,]0,T[$ *we have*

$$\begin{aligned}|\dot\xi_1-\dot\xi_2|(t) \ +\ C\frac{d}{dt}|B[v_1,u_1]-B[v_2,u_2]|(t) &\le \frac{C}{c}|\dot u_1-\dot u_2|(t)+CK_0\,\|\dot v_1-\dot v_2\|(t) \qquad (6.41)\\ +\ C\Big(2C_J|\dot u_1(t)|+(C_K+CC_JK_0)\,\|\dot v_1(t)\|\Big)&\Big(\|v_1-v_2\|(t)+|x_1-x_2|(t)\Big)\,.\end{aligned}$$

Proof. We have $c\,|J(v_1(t),x_1(t))|\le M^*(v_1(t),J(v_1(t),x_1(t)))=M(v_1(t),x_1(t))\le 1$ by virtue of (A.4.15), hence $|J(v_1(t),x_1(t))|\le 1/c$ for every $t\in[0,T]$. By Lemma 6.6, we can estimate the left-hand side of (6.41) by

$$C\Big(|G[v_1,u_1](t)-G[v_2,u_2](t)|+(|\dot u_1(t)|+CK_0\|\dot v_1(t)\|)|J(v_1(t),x_1(t))-J(v_2(t),x_2(t))|\Big)$$

which together with the assumptions (6.31) – (6.33) yields the assertion. ■

6.4 Uniqueness in the smooth implicit problem

We now formulate Problem $(\mathcal{I})$ under the following hypothesis.

Hypothesis 6.8 We are given a mapping $g : [0,T] \times X \times X \to Y$ which is continuous in its domain and $g(t,u,\xi) \in V$ for each $(t,u,\xi) \in [0,T] \times X \times X$. Its partial derivatives $\partial_t g, \partial_u g, \partial_\xi g$ exist and satisfy the inequalities

$$
\begin{aligned}
|\partial_\xi g(t,u,\xi)|_{\mathcal{L}(X,Y)} &\leq \gamma\,, && (6.42)\\
|\partial_u g(t,u,\xi)|_{\mathcal{L}(X,Y)} &\leq \omega\,, && (6.43)\\
\|\partial_t g(t,u,\xi)\| &\leq a(t)\,, && (6.44)\\
|\partial_\xi g(t,u,\xi) - \partial_\xi g(t,u',\xi')|_{\mathcal{L}(X,Y)} &\leq C_g\,(|u-u'| + |\xi-\xi'|)\,, && (6.45)\\
|\partial_u g(t,u,\xi) - \partial_u g(t,u',\xi')|_{\mathcal{L}(X,Y)} &\leq C_u\,(|u-u'| + |\xi-\xi'|)\,, && (6.46)\\
\|\partial_t g(t,u,\xi) - \partial_t g(t,u',\xi')\| &\leq b(t)\,(|u-u'| + |\xi-\xi'|) && (6.47)
\end{aligned}
$$

for every $u,u',\xi,\xi' \in X$ and a. e. $t \in]0,T[$ with given functions $a,b \in L^1(0,T)$ and given constants $\gamma, \omega, C_g, C_u > 0$ such that

$$\delta \;=\; CK_0\,\gamma \;<\; 1\,, \tag{6.48}$$

where C, K_0 are as in Hypothesis 6.3.

For a given function g satisfying Hypothesis 6.8, for a given $u \in W^{1,1}(0,T\,;\,X)$ and an initial condition $x^0 \in Z(g(0,u(0),u(0)-x^0))$ (for instance, any $x^0 \in B_c(0)$ satisfies this inclusion) we look for a solution $\xi \in W^{1,1}(0,T\,;\,X)$ of the implicit problem

$$
\begin{aligned}
(\mathcal{I})\quad &\text{(i)} && u(t) - \xi(t) \in Z(g(t,u(t),\xi(t))) \quad \forall t \in [0,T]\,,\\
&\text{(ii)} && u(0) - \xi(0) = x^0\,,\\
&\text{(iii)} && \left\langle \dot\xi(t), u(t) - \xi(t) - y \right\rangle \;\geq\; 0 \quad \forall y \in Z(g(t,u(t),\xi(t))) \quad \text{for a. e. } t \in]0,T[\,.
\end{aligned}
$$

Let us start our analysis with the following necessary condition.

Lemma 6.9 *Let Hypotheses 6.3, 6.8 hold, and let $\xi \in W^{1,1}(0,T\,;\,X)$ be a solution to Problem $(\mathcal{I})$ with some $u \in W^{1,1}(0,T\,;\,X)$ and $x^0 \in Z(g(0,u(0),u(0)-x^0))$. Then we have*

$$|\dot\xi(t)| \leq \frac{1}{1-\delta}\,((1 + CK_0\omega)|\dot u(t)| + CK_0\,a(t)) \quad \textit{a. e.} \tag{6.49}$$

Proof. Inequality (6.49) is an easy consequence of (6.38) with $v(t) = g(t,u(t),\xi(t))$. Indeed, using (6.42), (6.43) we obtain $\|\dot v(t)\| \leq a(t) + \omega|\dot u(t)| + \gamma|\dot\xi(t)|$ and (6.49) follows. ∎

We now prove the converse as the main result of this subsection.

Theorem 6.10 *Let Hypotheses 6.3, 6.8 hold. Then for every* $u \in W^{1,1}(0,T\,;\,X)$ *and every* $x^0 \in Z(g(0,u(0),u(0)-x^0))$ *there exists a unique solution* $\xi \in W^{1,1}(0,T\,;\,X)$ *to Problem* $\boldsymbol{(\mathcal{I})}$ *in the set*

$$\Omega \;=\; \left\{\eta \in W^{1,1}(0,T\,;\,X)\,; \begin{array}{rcl} |\dot\eta(t)| & \le & \frac{1}{1-\delta}\left((1+CK_0\omega)|\dot u(t)| + CK_0\, a(t)\right)\ a.\,e. \\ \eta(0) & = & u(0)-x^0 \end{array}\right\}.$$

Proof. Let $S:\Omega \to W^{1,1}(0,T\,;\,X)$ be the mapping which with each $\eta \in \Omega$ associates the solution ξ to Problem $\boldsymbol{(\mathcal{P})}$ with $v(t) = g(t,u(t),\eta(t))$. By (6.38) we have

$$\begin{aligned} |\dot\xi(t)| &\le |\dot u(t)| + CK_0\|\dot v(t)\| \;\le\; (1+CK_0\omega)|\dot u(t)| + CK_0\, a(t) + \delta|\dot\eta(t)| \\ &\le \frac{1}{1-\delta}\left((1+CK_0\omega)|\dot u(t)| + CK_0\, a(t)\right), \end{aligned} \tag{6.50}$$

hence $S(\Omega)\subset\Omega$. The set Ω is convex and closed in $W^{1,1}(0,T\,;\,X)$. We now check that $S:\Omega\to\Omega$ is a contraction with respect to a suitable norm in $W^{1,1}(0,T\,;\,X)$.

Let $\eta_1,\eta_2 \in \Omega$ be given. By Proposition 6.7, the functions $\xi_i = S(\eta_i)$ for $i=1,2$ satisfy almost everywhere the inequality

$$\begin{aligned} |\dot\xi_1(t)-\dot\xi_2(t)| + \dot\beta(t) &\le \delta|\dot\eta_1(t)-\dot\eta_2(t)| \\ &\quad + C_\delta(|\dot u(t)| + a(t) + b(t))(|\eta_1(t)-\eta_2(t)| + |\xi_1(t)-\xi_2(t)|) \end{aligned} \tag{6.51}$$

with $\beta(t) = C|B[g(\cdot,u,\eta_1),u](t) - B[g(\cdot,u,\eta_2),u](t)| \ge 0$, $\beta(0)=0$, and with a constant $C_\delta > 0$ independent of η_1,η_2.

Let now $\varepsilon > 0$ be chosen so small that

$$\frac{\delta + \varepsilon C_\delta}{1-\varepsilon C_\delta} \;=\; \delta^* \;<\; 1\,, \tag{6.52}$$

and let us define an auxiliary function

$$w(t) \;=\; e^{-\frac{1}{\varepsilon}\int_0^t (|\dot u(\tau)| + a(\tau) + b(\tau))\,d\tau} \qquad \text{for} \quad t\in[0,T]\,. \tag{6.53}$$

We have $w(t) > 0$ for every $t\in[0,T]$ and $\dot w(t) \le 0$ a.e. We test the inequality (6.51) by $w(t)$ and integrate over $[0,T]$. Taking into account the relations

$$\int_0^T \dot\beta(t)\,w(t)\,dt \;=\; [\beta(t)\,w(t)]_0^T - \int_0^T \beta(t)\,\dot w(t)\,dt \;\ge\; 0\,,$$

$$\begin{aligned} &\int_0^T w(t)\,(|\dot u(t)| + a(t) + b(t))(|\eta_1(t)-\eta_2(t)| + |\xi_1(t)-\xi_2(t)|)\,dt \\ &\qquad \le -\varepsilon\int_0^T \dot w(t)\int_0^t (|\dot\eta_1(\tau)-\dot\eta_2(\tau)| + |\dot\xi_1(\tau)-\dot\xi_2(\tau)|)\,d\tau\,dt \\ &\qquad = -\varepsilon\left[w(t)\int_0^t (|\dot\eta_1(\tau)-\dot\eta_2(\tau)| + |\dot\xi_1(\tau)-\dot\xi_2(\tau)|)\,d\tau\right]_0^T \\ &\qquad\quad + \varepsilon\int_0^T w(t)\,(|\dot\eta_1(t)-\dot\eta_2(t)| + |\dot\xi_1(t)-\dot\xi_2(t)|)\,dt \\ &\qquad \le \varepsilon\int_0^T w(t)\,(|\dot\eta_1(t)-\dot\eta_2(t)| + |\dot\xi_1(t)-\dot\xi_2(t)|)\,dt\,, \end{aligned}$$

we obtain from (6.51) that

$$\int_0^T w(t)\,|\dot\xi_1(t)-\dot\xi_2(t)|\,dt \;\le\; (\delta+\varepsilon C_\delta)\int_0^T w(t)\,|\dot\eta_1(t)-\dot\eta_2(t)|\,dt+\varepsilon C_\delta\int_0^T w(t)\,|\dot\xi_1(t)-\dot\xi_2(t)|\,dt$$

hence

$$\int_0^T w(t)\,|\dot\xi_1(t)-\dot\xi_2(t)|\,dt \;\le\; \delta^*\int_0^T w(t)\,|\dot\eta_1(t)-\dot\eta_2(t)|\,dt\,. \tag{6.54}$$

We thus checked that S is a contraction on Ω with respect to the weighted norm

$$\|\eta\|_{1,1,w} \;=\; |\eta(0)|+\int_0^T w(t)\,|\dot\eta(t)|\,dt\,,$$

hence S admits a unique fixed point $\xi\in\Omega$ which is a solution of $(\boldsymbol{\mathcal{I}})$. ■

6.5 Local Lipschitz continuity of the input-output mapping

We now prove even more, namely that the solution mappings for both Problems $(\boldsymbol{\mathcal{P}})$ and $(\boldsymbol{\mathcal{I}})$ satisfy the local Lipschitz condition in their domains of definition.

Theorem 6.11 *Let the assumptions of Proposition 6.7 be fulfilled. Then there exist positive constants C_0, C_1 such that for every $R>0$, every $(v_1,u_1),(v_2,u_2)\in W^{1,1}(0,T\,;V)\times W^{1,1}(0,T\,;X)$ with $\int_0^T(|\dot u_i|+\|\dot v_i\|)\,dt\le R$ and every $x_i^0\in Z(v_i(0))$ for $i=1,2$, the respective solutions $\xi_i\in W^{1,1}(0,T\,;X)$ of problem $(\boldsymbol{\mathcal{P}})$ satisfy the inequality*

$$\int_0^T |\dot\xi_1-\dot\xi_2|\,dt \;\le\; C_1 e^{C_0R}\left(|x_1^0-x_2^0|+\|v_1(0)-v_2(0)\|+\int_0^T(|\dot u_1-\dot u_2|+\|\dot v_1-\dot v_2\|)\,dtdt\right). \tag{6.55}$$

Proof. By Proposition 6.7, there exists a constant $C_0>0$ such that

$$\begin{aligned}|\dot x_1(t)-\dot x_2(t)|+\dot\beta(t) \;\le\;& C_0\Big(|\dot u_1(t)-\dot u_2(t)|+\|\dot v_1(t)-\dot v_2(t)\|\\ &+(|\dot u_1(t)|+\|\dot v_1(t)\|)(|x_1(t)-x_2(t)|+\|v_1(t)-v_2(t)\|)\Big)\end{aligned} \tag{6.56}$$

with $\beta(t)=C\,|B[v_1,u_1](t)-B[v_2,u_2](t)|$. We argue similarly as in the proof of Theorem 6.10 and test (6.56) by the function $w_1(t)\;=\;\exp(-C_0\int_0^t(|\dot u_1|+\|\dot v_1\|)\,d\tau)$. This yields

$$\begin{aligned}\frac{d}{dt}\left(w_1(t)\int_0^t|\dot x_1-\dot x_2|\,d\tau\right)+w_1(t)\,\dot\beta(t) \;\le\;& C_0\,w_1(t)\Big(|\dot u_1(t)-\dot u_2(t)|+\|\dot v_1(t)-\dot v_2(t)\|\Big)\\ &-\dot w_1(t)\left(|x_1^0-x_2^0|+\|v_1(0)-v_2(0)\|+\int_0^t\|\dot v_1-\dot v_2\|\,d\tau\right).\end{aligned} \tag{6.57}$$

Note that

$$\begin{aligned}\int_0^T w_1(t)\,\dot\beta(t)\,dt \;=\;& [w_1(t)\,\beta(t)]_0^T-\int_0^T\dot w_1(t)\,\beta(t)\,dt\;\ge\;-w_1(0)\,\beta(0)\\ \ge\;& -\frac{C}{2}|M^2(v_1(0),x_1^0)-M^2(v_2(0),x_2^0)|\;\ge\;-\left(CK_0\|v_1(0)-v_2(0)\|+\frac{C}{c}|x_1^0-x_2^0|\right).\end{aligned} \tag{6.58}$$

On the other hand, integrating (6.57) from 0 to T and using the fact that for every $t \in [0,T]$ we have $1 \geq w_1(t) \geq w_1(T) \geq e^{-C_0 R}$ we obtain

$$e^{-C_0 R} \int_0^T |\dot{x}_1 - \dot{x}_2|\, dt \;\leq\; -\int_0^T w_1(t)\, \dot{\beta}(t)\, dt + |x_1^0 - x_2^0| + \|v_1(0) - v_2(0)\| + (C_0+1) \int_0^T (|\dot{u}_1 - \dot{u}_2| + \|\dot{v}_1 - \dot{v}_2\|)\, dt\,, \tag{6.59}$$

and the assertion follows from (6.58), (6.59). ∎

Theorem 6.12 *Let the assumptions of Theorem 6.10 be fulfilled. Then there exist positive constants* C_2, C_3 *such that for every* $R > 0$, *every* $u_i \in W^{1,1}(0,T\,;\,X)$ *with* $\int_0^T |\dot{u}_i|\, dt \leq R$ *and every* $x_i^0 \in Z(g(0,u(0),u(0)-x_i^0)$ *for* $i = 1,2$, *the respective solutions* $\xi_1, \xi_2 \in W^{1,1}(0,T\,;\,X)$ *of problem* $(\boldsymbol{\mathcal{I}})$ *satisfy the inequality*

$$\int_0^T |\dot{\xi}_1 - \dot{\xi}_2|\, dt \;\leq\; C_3 e^{C_2 R} \left(|x_1^0 - x_2^0| + |u_1(0) - u_2(0)| + \int_0^T |\dot{u}_1 - \dot{u}_2|\, dt \right). \tag{6.60}$$

Proof. We use Lemma 6.9 and Proposition 6.7 with $v_i(t) = g(t, u_i(t), \xi_i(t))$ for $i = 1,2$, and find a constant $C^* > 0$ such that

$$(1-\delta)|\dot{\xi}_1(t) - \dot{\xi}_2(t)| + \dot{\beta}(t) \;\leq\; C^* \Big(|\dot{u}_1(t) - \dot{u}_2(t)| + (|\dot{u}_1(t)| + a(t) + b(t))(|u_1(t) - u_2(t)| + |\xi_1(t) - \xi_2(t)|) \Big) \tag{6.61}$$

with $\beta(t) = C\, |B[g(\cdot, u_1, \xi_1), u_1](t) - B[g(\cdot, u_2, \xi_2), u_2](t)|$. Repeating the procedure from the proof of Theorem 6.11 with

$$C_2 \;=\; \frac{C^*}{1-\delta}\,, \quad w_2(t) = e^{-C_2 \int_0^t (|\dot{u}_1(\tau)| + a(\tau) + b(\tau))\, d\tau} \tag{6.62}$$

we easily obtain the assertion. ∎

A Appendix: Convex sets

The aim of this section is to recall some basic elements of convex analysis in Hilbert spaces. Most of the results are well-known. We present them in order to fix the notation and keep the presentation consistent (for more information we refer the reader to the monographs [1] and [30]). Throughout the section, X denotes a real separable Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and norm $|x| := \langle x,x\rangle^{1/2}$. For $x_0 \in X$ and $r > 0$ we will denote by

$$B_r(x_0) \;=\; \{x \in X\,;\, |x - x_0| \le r\} \tag{A.0.1}$$

the closed ball in X centered at x_0 with radius r. For $x \in X$ and a set $A \subset X$ we define

$$\mathrm{dist}\,(x, A) \;=\; \inf\{|x - a|\,;\, a \in A\}\,. \tag{A.0.2}$$

We start with a simple lemma.

Lemma A.1 *Let $Z \subset X$ be a non-empty convex closed set. Then for each $x \in X$ there exists a unique $z \in Z$ such that $|x - z| = \mathrm{dist}\,(x, Z) = \min\{|x - y|\,; y \in Z\}$.*

Proof. Let $x \in X$ be given. Put $p = \inf\{|x - y|\,; y \in Z\}$ and let $\{y_n\}$ be a sequence in Z such that $|x - y_n| \to p$. From the identity

$$|u - v|^2 + |u + v|^2 = 2(|u|^2 + |v|^2) \tag{A.0.3}$$

for $u = x - y_n$, $v = x - y_k$, it follows

$$\frac{1}{2}|y_n - y_k|^2 = |x - y_n|^2 + |x - y_k|^2 - 2\left|x - \frac{y_n + y_k}{2}\right|^2 \le |x - y_n|^2 + |x - y_k|^2 - 2p^2,$$

hence $\{y_n\}$ is a Cauchy sequence and it suffices to put $z := \lim_{n\to\infty} y_n$. Uniqueness is obtained in a similar way. ■

Using Lemma A.1 we can define the projection $Q_Z : X \to Z$ onto Z and its complement $P_Z = I - Q_Z$ (I is the identity) by the formula

$$Q_Z x \in Z\,,\ |P_Z x| = \mathrm{dist}\,(x, Z) \quad \text{for } x \in X\,. \tag{A.0.4}$$

In the sequel, we call (P_Z, Q_Z) the *projection pair* associated with Z. We make extensive use of the following lemma.

Lemma A.2 *For every $x, y \in X$ we have*

(i) $\langle P_Z x, Q_Z x - z\rangle \;\ge\; 0 \quad \forall z \in Z\,,$

(ii) $\langle P_Z x - P_Z y, Q_Z x - Q_Z y\rangle \;\ge\; 0\,,$

(iii) $Q_Z(x + \alpha P_Z x) \;=\; Q_Z x \quad \forall \alpha \ge -1\,.$

(iv) $(x \in Z\,,\ \langle y, x - z\rangle \;\ge\; 0\ \ \forall z \in Z) \iff (x = Q_Z(x + y)\,,\ y = P_Z(x + y))\,.$

Proof. (i) For $z \in Z$, $z \neq Q_Z x$ and $\gamma \in]0,1[$ we have $|x - \gamma z - (1-\gamma)Q_Z x|^2 > |P_Z x|^2$, hence $2\langle P_Z x, Q_Z x - z\rangle + \gamma |Q_Z x - z|^2 > 0$ and the assertion follows easily. Statement (ii) is an obvious consequence of (i). To prove (iii) we notice that for all $z \in Z$ we have $|x + \alpha P_Z x - z|^2 = |Q_Z x - z|^2 + (1+\alpha)^2 |P_Z x|^2 + 2(1+\alpha)\langle P_Z x, Q_Z x - z\rangle$, hence the minimum of $|x + \alpha P_Z x - z|$ is attained for $z = Q_Z x$. The implication "$\Leftarrow$" in (iv) is an immediate consequence of (i). Let now the left-hand side of (iv) be fulfilled for some $x \in Z$ and $y \in X$, and put $u = Q_Z(x+y)$, $v = P_Z(x+y)$. By (i) we have $\langle v, u - x\rangle \geq 0$, which together with the hypothesis $\langle y, x - u\rangle \geq 0$ yields that $0 \leq \langle v - y, u - x\rangle = -|v - y|^2$, hence $v = y$, $u = x$. ∎

A.1 Recession cone

At each point $z_0 \in Z$ we define the *recession cone* $C_Z(z_0)$ by the formula

$$C_Z(z_0) \;=\; \{u \in X\,;\; z_0 + tu \in Z \quad \forall t \geq 0\}\,. \tag{A.1.1}$$

Then $C_Z(z_0)$ is a convex closed set with the following property.

Lemma A.3 *For all $z_0, z_1 \in Z$ we have $C_Z(z_0) = C_Z(z_1)$.*

Proof. By symmetry, it suffices to prove the inclusion $C_Z(z_0) \subset C_Z(z_1)$. Let $u \in C_Z(z_0)$ and $t \geq 0$ be arbitrary. For each $\alpha \in]0,1[$ we have

$$z_\alpha := z_1 + tu + \alpha(z_0 - z_1) \;=\; \alpha\left(z_0 + \frac{t}{\alpha}u\right) + (1-\alpha)z_1 \;\in\; Z\,,$$

and letting α tend to 0 we obtain that $z_1 + tu \in Z$, hence $u \in C_Z(z_1)$. ∎

According to Lemma A.3 it is meaningful to put

$$C_Z \;=\; \{u \in X\,;\; \exists z_0 \in Z \;:\; z_0 + tu \in Z \quad \forall t \geq 0\}\,, \tag{A.1.2}$$

and we have $C_Z = C_Z(z_0)$ for all $z_0 \in Z$.

Lemma A.4 *Let $Z \subsetneqq X$ be such that $C_Z \cup (-C_Z) = X$. Then there exist $z_0 \in \partial Z$ and $n \in X$, $|n| = 1$, such that Z is the half-space*

$$Z \;=\; \{z \in X\,;\; \langle n, z_0 - z\rangle \;\geq\; 0\}\,.$$

Proof. For an arbitrary $x_0 \in X \setminus Z$ put $z_0 = Q_Z(x_0)$, $n = P_Z(x_0)/|P_Z(x_0)|$. For all $z \in Z$ we have by Lemma A.2 that $\langle n, z_0 - z\rangle \geq 0$, hence

$$Z \;\subset\; \{z \in X\,;\; \langle n, z_0 - z\rangle \;\geq\; 0\}\,.$$

To obtain the opposite inclusion, we notice that we have $\langle n, u\rangle = \langle n, (z_0 + u) - z_0\rangle \leq 0$ for every $u \in C_Z$, $\langle n, u\rangle \geq 0$ for every $u \in -C_Z$, hence

$$C_Z \;=\; \{u \in X\,;\; \langle n, u\rangle \leq 0\}\,.$$

Assuming that $\langle n, z_0 - z\rangle \geq 0$ we thus obtain that $z - z_0 \in C_Z$, which in turn implies that $z \in Z$, and the proof is complete. ∎

Lemma A.5 *Let $Z \subset X$ be such that $C_Z \cup (-C_Z) \neq X$. Then for every $z \in Z$ there exist $z_1, z_2 \in \partial Z$ and $\alpha \in [0,1]$ such that $z = \alpha z_1 + (1-\alpha) z_2$.*

Proof. The case $z \in \partial Z$ is obvious. Assume that $z \in \mathrm{Int}\ Z$ and fix some $u \in X \setminus (C_Z \cup (-C_Z))$. The numbers $t_1 = \max\{t \geq 0\,;\, z + tu \in Z\}$, $t_2 = \max\{t \geq 0\,;\, z - tu \in Z\}$ are both positive and both $z_1 = z + t_1 u$, $z_2 = z - t_2 u$ belong to ∂Z, it suffices therefore to put $\alpha = t_2/(t_1 + t_2)$ to obtain the assertion. ■

A.2 Tangent and normal cones

A natural generalization of normal vectors and tangent hyperplanes which in general are not uniquely determined, is the concept of *normal cone* $N_Z(x)$ and *tangent cone* $T_Z(x)$ to a convex closed set $Z \subset X$ at a point $x \in Z$. They are defined by the formula

$$\left\{ \begin{array}{l} N_Z(x) := \{y \in X;\ \langle y, x - z\rangle \geq 0 \quad \forall z \in Z\}, \\ T_Z(x) := \{w \in X;\ \langle w, y\rangle \leq 0 \quad \forall y \in N_Z(x)\}. \end{array} \right. \tag{A.2.1}$$

Every element $u \in X$ admits a unique orthogonal decomposition into the sum $u = v + w$ of the normal component $v \in N_Z(x)$ and the tangential component $w \in T_Z(x)$, namely $v = Q_N(u)$, $w = P_N(u)$, where (P_N, Q_N) is the projection pair associated with $N_Z(x)$. Indeed, by Lemma A.2 (i) we have $\langle w, (1-\alpha)\, v\rangle \geq 0$ for all $\alpha \geq 0$, hence $\langle w, v\rangle = 0$ and $\langle w, y\rangle \leq 0$ for every $y \in N_Z(x)$. Uniqueness is easy: assume $v_1 + w_1 = v_2 + w_2$ for some $v_i \in N_Z(x)$, $w_i \in T_Z(x)$, $\langle w_i, v_i\rangle = 0$, $i = 1,2$. Then $0 \leq \langle w_1 - w_2, v_1 - v_2\rangle \leq -|w_1 - w_2|^2$, hence $w_1 = w_2$, $v_1 = v_2$.

For $x \in \mathrm{Int}\ Z$ we obviously have $N_Z(x) = \{0\}$, $T_Z(x) = X$. One might expect that for $x \in \partial Z$ the normal cone should contain nonzero elements. The example $Z := \{x \in X; |\,\langle x, e_k\rangle\,| \leq 1/k\ \forall k \in \mathbb{N}\}$, where $\{e_k\}$ is an orthonormal basis, shows that this conjecture is false, since $0 \in \partial Z$ and $N_Z(0) = \{0\}$. The statement below shows that this cannot happen in 'regular' cases.

Proposition A.6 *If* $\mathrm{Int}\ Z \neq \emptyset$, *then we have $N_Z(x) \setminus \{0\} \neq \emptyset$ for every $x \in \partial Z$.*

Proof. Let $\{z_n; n \in \mathbb{N}\}$ be a sequence in $X \setminus Z$ such that $\lim_{n\to\infty} |z_n - x| = 0$. Put $\varepsilon_n = |P_Z z_n| > 0$, $y_n := z_n + 1/\varepsilon_n\, P_Z z_n$. We have $\varepsilon_n \leq |z_n - x|$ and Lemma A.2 (iii) yields $Q_Z y_n = Q_Z z_n$, $P_Z y_n = (1 + 1/\varepsilon_n)\, P_Z z_n$. By Lemma A.2 (i) we further have $|Q_Z y_n - x|^2 = |Q_Z z_n - x|^2 = |z_n - x|^2 - |P_Z z_n|^2 - 2\,\langle P_Z z_n, Q_Z z_n - x\rangle \leq |z_n - x|^2$ and

$$\langle P_Z y_n, Q_Z y_n - z\rangle \geq 0 \quad \forall z \in Z, \quad \forall n \in \mathbb{N}. \tag{A.2.2}$$

Passing to subsequences we can assume that $\{P_Z y_n\}$ converges weakly to an element ξ which belongs to $N_Z(x)$ by (A.2.2). It remains to verify that $\xi \neq 0$. We fix an arbitrary ball $B_\delta(x_0) \subset \mathrm{Int}\ Z$. Putting $z := x_0 + \delta/(1+\varepsilon_n)\, P_Z y_n$ in (A.2.2) we obtain $\delta \leq \langle \xi, x - x_0\rangle$, hence $\xi \neq 0$. ■

A.3 The Minkowski functional

For a given set $A \subset X$ we define its *polar* A^* by the formula

$$A^* := \{y \in X;\ \langle y, x\rangle \le 1 \quad \forall\, x \in A\}\,. \tag{A.3.1}$$

We immediately see that A^* is convex and closed, $0 \in A^*$. The following duality statement holds.

Lemma A.7 *Let $A \subset X$ be given, and let A^{**} be the polar of A^*. Then A^{**} is the closed convex hull $\overline{\text{conv}}\,(A \cup \{0\})$ of $A \cup \{0\}$, that is, the minimal convex closed set in X containing $A \cup \{0\}$.*

Proof. Put $\hat{A} = \overline{\text{conv}}\,(A \cup \{0\})$. We have by definition

$$A^{**} = \{z \in X;\ \langle y, z\rangle \le 1 \quad \forall\, y \in A^*\}\,, \tag{A.3.2}$$

hence $0 \in A^{**}$ and $A \subset A^{**}$. Since A^{**} is convex and closed, we necessarily have $\hat{A} \subset A^{**}$. To prove the inclusion $A^{**} \subset \hat{A}$, we fix an arbitrary $z \in A^{**}$ and apply Lemma A.2 with the projection pair $(P_{\hat{A}}, Q_{\hat{A}})$ associated with $\hat{A}$. This yields

$$\langle P_{\hat{A}} z\,,\, z - P_{\hat{A}} z - x\rangle \ge 0 \quad \forall\, x \in \hat{A}\,. \tag{A.3.3}$$

For every $k > 0$ we have in particular

$$\langle k\, P_{\hat{A}} z\,,\, z\rangle \ge k\, |P_{\hat{A}} z|^2 + \sup\{\langle k\, P_{\hat{A}} z\,,\, x\rangle\,;\, x \in A\}\,. \tag{A.3.4}$$

Put

$$\kappa := \inf\{k > 0\,;\, k\, P_{\hat{A}} z \notin A^*\}\,. \tag{A.3.5}$$

From inequality (A.3.4) it follows $\kappa > 0$, and we distinguish two cases.

(i) $\kappa = +\infty$: Putting $x = 0$ in inequality (A.3.3), we obtain

$$k\, |P_{\hat{A}} z|^2 \le \langle k\, P_{\hat{A}} z\,,\, z\rangle \le 1 \qquad \forall\, k > 0\,. \tag{A.3.6}$$

(ii) $\kappa < +\infty$: Then $\kappa\, P_{\hat{A}} z \in \partial A^*$, $\sup\{\langle \kappa\, P_{\hat{A}} z, x\rangle\,;\, x \in A\} = 1$, and inequality (A.3.4) yields

$$1 + \kappa\, |P_{\hat{A}} z|^2 \le \langle \kappa\, P_{\hat{A}} z\,,\, z\rangle \le 1\,. \tag{A.3.7}$$

In both cases (A.3.6) and (A.3.7), we conclude $P_{\hat{A}} z = 0$, hence $z \in \hat{A}$. Lemma A.7 is proved. ■

Lemma A.8 *Let $A \subset X$ be a set with polar A^*, and let $C > 0$ be given. Then*

$$A \subset B_C(0) \iff B_{1/C}(0) \subset A^*\,. \tag{A.3.8}$$

Proof. Assume $A \subset B_C(0)$ and fix $y \in B_{1/C}(0)$. Then for $x \in A$ we have $\langle y, x\rangle \le |y|\,|x| \le 1$, hence $y \in A^*$. Conversely, let $B_{1/C}(0) \subset A^*$ and fix $x \in A$. Then $|x| = \sup\{\langle x, w\rangle\,;\, w \in B_1(0)\} = C\, \sup\{\langle x, y\rangle\,;\, y \in B_{1/C}(0)\} \le C$. ■

Definition A.9 *Let $Z \subset X$ be a convex closed set, $0 \in Z$. The functional $M_Z : X \to \mathbb{R}^+ \cup \{+\infty\}$ defined by the formula*

$$M_Z(x) := \inf\left\{s > 0 \,;\, \frac{1}{s}x \in Z\right\} \quad \textit{for } x \in X\,. \tag{A.3.9}$$

is called the Minkowski functional *of Z.*

The functional M_Z is sometimes called *gauge*, cf. [30] (not to be confused with the gauge in Section 1). We list without proof some of its basic properties.

Proposition A.10 *In the situation of Definition A.9, we have*

(i) $Z = \{x \in X\,;\, M_Z(x) \le 1\}$,

(ii) $C_Z = \{x \in X\,;\, M_Z(x) = 0\}$,

(iii) $M_Z(tx) = t\,M_Z(x) \quad \forall x \in X, \quad \forall t \ge 0$,

(iv) $M_Z(x+y) \le M_Z(x) + M_Z(y) \quad \forall x, y \in X$.

As an immediate consequence of the above considerations, we have the following

Proposition A.11 *Let $Z \subset X$ be a convex closed set and let $C > c > 0$ be given numbers such that*

$$B_c(0) \subset Z \subset B_C(0) \quad . \tag{A.3.10}$$

Then

$$B_{1/C}(0) \subset \quad Z^* \quad \subset B_{1/c}(0)\,, \tag{A.3.11}$$

$$\frac{1}{C}\,|x| \le \quad M_Z(x) \quad \le \frac{1}{c}\,|x| \qquad \forall x \in X\,, \tag{A.3.12}$$

$$c\,|x| \le \quad M_{Z^*}(x) \quad \le C\,|x| \qquad \forall x \in X\,, \tag{A.3.13}$$

where Z^* is the polar of Z.

By virtue of Proposition A.10 and inequality (A.3.12), the Minkowski functional of a convex set Z satisfying the hypotheses of Proposition A.11 is convex and Lipschitz continuous. Its subdifferential has the following properties.

Lemma A.12 *Let Z satisfy the hypotheses of Proposition A.11, and let ∂M_Z be the subdifferential of M_Z. Then*

(i) $\partial M_Z(x) \ne \emptyset \quad \forall\, x \in X$,

(ii) $\partial M_Z(tx) = \partial M_Z(x) \quad \forall\, x \in X, \forall\, t > 0$,

(iii) $\langle w, x\rangle = M_Z(x)\,, \quad \langle w, y\rangle \le M_Z(y) \quad \forall\, x, y \in X, \quad \forall\, w \in \partial M_Z(x)$.

(iv) $M_{Z^*}(w) = 1 \quad \forall\, w \in \partial M_Z(x)\,,\ \forall\, x \neq 0$.

Proof.

(i) We have for all $x \in X$

$$w \in \partial M_Z(x) \iff \langle w, x - y\rangle \geq M_Z(x) - M_Z(y) \quad \forall\, y \in X\,, \tag{A.3.14}$$

hence $\partial M_Z(0) = Z^*$. For $x \neq 0$, we choose a sequence $0 < t_n \nearrow M_Z(x)$, $n = 1, 2, \ldots$, and put $x_n := x/t_n$, $x_0 := x/M_Z(x)$. Then $x_n \notin Z$ for $n \geq 1$, hence $P_Z x_n \neq 0$ and

$$\langle P_Z x_n, Q_Z x_n - z\rangle \geq 0 \quad \forall\, z \in Z\,. \tag{A.3.15}$$

On the other hand, we have $Q_Z x_0 = x_0$, and $|Q_Z x_n - x_0| \leq |x_n - x_0| \to 0$ as $n \to \infty$. Selecting a subsequence, if necessary, we may assume that $P_Z x_n/|P_Z x_n|$ converge weakly to some $w_0 \in B_1(0)$. Then (A.3.15) yields

$$\langle w_0, x_0 - z\rangle \geq 0 \quad \forall\, z \in Z\,. \tag{A.3.16}$$

Putting $z := c\, P_Z x_n/|P_Z x_n|$ in (A.3.15) and passing to the limit as $n \to \infty$, we obtain

$$\langle w_0, x_0\rangle \geq c > 0\,. \tag{A.3.17}$$

Inequality (A.3.16) implies

$$\left\langle w_0, \frac{x}{M_Z(x)} - \frac{y}{M_Z(y)}\right\rangle \geq 0 \quad \forall y \in X \setminus \{0\}\,, \tag{A.3.18}$$

or equivalently,

$$\langle w_0, x - y\rangle \geq (M_Z(x) - M_Z(y))\,\langle w_0, x_0\rangle \quad \forall\, y \in X\,. \tag{A.3.19}$$

By virtue of (A.3.14) and (A.3.17), we have $w := w_0/\langle w_0, x_0\rangle \in \partial M_Z(x)$ and (i) is proved. Using Proposition A.10 (iii) we obtain (ii) trivially from (A.3.14), part (iii) follows from (A.3.14) by putting successively $y := 0$ and $y := 2x$ and part (iv) follows from (iii). ■

Remark A.13 Lemma A.12 does not hold for general convex closed sets Z. To see this, we first notice that by (A.3.14), for every x with $M_Z(x) > 0$ and every $w \in \partial M_Z(x)$ we have

$$w \neq 0\,, \tag{A.3.20}$$

$$\left\langle w, \frac{x}{M_Z(x)} - y\right\rangle \geq 0 \quad \forall\, y \in Z\,. \tag{A.3.21}$$

As an example, we choose $X := L^2(0,1)$, $Z := \{z \in X;\ -1 \leq z(t) \leq 1 \text{ a.e. }\}$, $x(t) := t$ for $t \in [0,1]$. Then Z is convex and closed, $0 \in Z$, $M_Z(x) = 1$. Assume that $\partial M_Z(x)$ is nonempty and let $w \in \partial M_Z(x)$ be arbitrary. By (A.3.21), we have

$$\int_0^1 w(t)\, t\, dt \geq \sup\left\{\int_0^1 w(t)\, y(t)\, dt\,;\ y \in X\,,\, -1 \leq y(t) \leq 1 \text{ a.e.}\right\} = \int_0^1 |w(t)|\, dt$$

hence $w = 0$, which contradicts (A.3.20).

The main result of this subsection reads as follows.

Theorem A.14 *Let* Z *satisfy the hypotheses of Proposition A.11, and let* Z^* *be the polar of* Z. *For* $x \in X$ *put* $J_Z(x) := M_Z(x)\,\partial M_Z(x)$, $J_{Z^*}(x) := M_{Z^*}(x)\,\partial M_{Z^*}(x)$. *Then*

(i) $\langle w - z, x - y\rangle \ge (M_Z(x) - M_Z(y))^2 \quad \forall\, x, y \in X\,, \quad w \in J_Z(x)\,, \quad z \in J_Z(y)\,,$

(ii) $\langle w^* - z^*, x - y\rangle \ge (M_{Z^*}(x) - M_{Z^*}(y))^2 \quad \forall\, x, y \in X\,, \quad w^* \in J_{Z^*}(x)\,, \quad z^* \in J_{Z^*}(y)\,,$

(iii) $y \in J_Z(x) \iff x \in J_{Z^*}(y) \quad \forall\, x, y \in X\,,$

(iv) $Z^* = J_Z(Z)\,, \quad Z = J_{Z^*}(Z^*)\,,$ *where* $J_Z(Z) := \bigcup_{x\in Z} J_Z(x)\,,\ J_{Z^*}(Z^*) := \bigcup_{y \in Z^*} J_{Z^*}(y)\,.$

The proof Theorem A.14 uses the following Lemma.

Lemma A.15 *Let the hypotheses of Theorem A.14 hold. Then for all* $x, y \in X \setminus \{0\}$ *we have*

$$\langle y, x\rangle \le M_Z(x)\, M_{Z^*}(y)\,, \tag{A.3.22}$$

$$\langle y, x\rangle = M_{Z^*}(y)\, M_Z(x) \iff \frac{x}{M_Z(x)} \in \partial M_{Z^*}(y) \iff \frac{y}{M_{Z^*}(y)} \in \partial M_Z(x)\,. \tag{A.3.23}$$

Proof of Lemma A.15. Inequality (A.3.22) follows immediately from the definition of Z^* and Lemma A.12 (iii) yields the implications

$$\frac{x}{M_Z(x)} \in \partial M_{Z^*}(y) \ \Rightarrow\ \langle y, x\rangle = M_{Z^*}(y)\, M_Z(x)\,,$$
$$\frac{y}{M_{Z^*}(y)} \in \partial M_Z(x) \ \Rightarrow\ \langle y, x\rangle = M_{Z^*}(y)\, M_Z(x)\,.$$

Assume now

$$\langle x, y\rangle = M_Z(x)\, M_{Z^*}(y) \quad \text{for some } x, y \in X \setminus \{0\}\,. \tag{A.3.24}$$

Then, by (A.3.22) we have

$$\left\langle \frac{x}{M_Z(x)}, y - z \right\rangle \ge M_{Z^*}(y) - M_{Z^*}(z) \quad \forall\, z \in X\,,$$
$$\left\langle \frac{y}{M_{Z^*}(y)}, x - z \right\rangle \ge M_Z(x) - M_Z(z) \quad \forall\, z \in X$$

and the assertion follows. ■

Proof of Theorem A.14. Inequalities (i), (ii) follow from (A.3.14) (and the corresponding inequality for M_{Z^*}). To prove (iii), it suffices to fix $x \in X$ and $y \in J_Z(x)$ and prove that $x \in J_{Z^*}(y)$. The other implication then follows from the duality $Z = Z^{**}$ and $J_Z = J_{Z^{**}}$. The definition of J_Z immediately entails $J_Z(0) = \{0\}$, $J_{Z^*}(0) = \{0\}$, hence it suffices to assume $x \ne 0$. By Lemma A.12 (iii), (iv) we have

$$\langle y, x\rangle = M_Z^2(x)\,, \quad M_{Z^*}(y) = M_Z(x)\,. \tag{A.3.25}$$

and Lemma A.15 yields the assertion. To prove (iv), it suffices to use (iii) and (A.3.25). ■

We call J_Z the *duality mapping* induced by Z. It can be interpreted geometrically by means of the normal cone $N_Z(x)$ in the following way.

Proposition A.16 *Let the hypotheses of Theorem A.14 hold. Then for every* $x \in \partial Z$ *, we have* $J_Z(x) \subset N_Z(x)$ *. Conversely, for each* $y \in N_Z(x)$ *,* $y \neq 0$ *, we have* $\langle y, x\rangle = M_{Z^*}(y)$ *and* $y/\langle y, x\rangle \in J_Z(x)$ *.*

Proof. The inclusion $J_Z(x) \subset N_Z(x)$ follows immediately from the definition. Let now $y \in N_Z(x)$, $y \neq 0$ be given. Then $\langle y, x\rangle \geq \langle y, z\rangle$ for all $z \in Z$, hence $y/\langle y, x\rangle \in Z^*$. We have in particular $M_{Z^*}(y) \leq \langle y, x\rangle$ and from (A.3.22) (note that $M_Z(x) = 1$) we obtain $\langle y, x\rangle = M_{Z^*}(y)$. Lemma A.15 then completes the proof. ■

Remark A.17 It is easy to see that $M^2_{Z^*}/2$ is the *conjugate function* to $M^2_Z/2$ in the sense of [1], that is,

$$\frac{1}{2}M^2_{Z^*}(y) \;=\; \sup\left\{\langle y, x\rangle - \frac{1}{2}M^2_Z(x)\,;\; x \in X\right\} \quad \text{for every} \;\; y \in X\,. \tag{A.3.26}$$

Let us also mention the case of "regular" convex domains $Z \subset X$ such that $N_Z(x)$ reduces to a half-line for each $x \in \partial Z$. By Proposition A.16, this is equivalent to saying that J_Z is a single-valued mapping. They allow for the following dual characterization.

Theorem A.18 *Let* Z *satisfy (A.3.10) and let* Z^* *be its polar. Then the following conditions are equivalent.*

(i) J_Z *is single-valued,*

(ii) Z^* *is strictly convex, that is,* $(y_0 + y_1)/2 \in \mathrm{Int}\, Z^*$ *for all* $y_0, y_1 \in Z^*$ *,* $y_0 \neq y_1$ *.*

Proof.

(ii) $\Rightarrow$ (i): Let $x \in X$ and y_0, $y_1 \in J_Z(x)$ be given. For $x = 0$ we have $y_0 = y_1 = 0$, otherwise we put $y := (y_0 + y_1)/2$. Then $y \in J_Z(x)$ and $M_{Z^*}(y) = M_{Z^*}(y_0) = M_{Z^*}(y_1) = M_Z(x)$. Consequently, all $y_0/M_Z(x)$, $y_1/M_Z(x)$, $y/M_Z(x)$ belong to ∂Z^*, hence $y_0 = y_1$.

non (ii) $\Rightarrow$ non (i): Assume that there exist $y_0 \neq y_1 \in Z^*$ such that $y := (y_0 + y_1)/2 \in \partial Z^*$. Let $x \in J_{Z^*}(y)$ be arbitrarily chosen. Then $M_Z(x) = M_{Z^*}(y) = 1$ and

$$1 \;=\; \langle x, y\rangle = \frac{1}{2}\left(\langle x, y_0\rangle \;+\; \langle x, y_1\rangle\right) \;\leq\; 1\,.$$

This yields $\langle x, y_0\rangle = \langle x, y_1\rangle = 1 = M_{Z^*}(y_0) = M_{Z^*}(y_1)$ and from Lemma A.15 (ii), we conclude $y_0, y_1 \in J_Z(x)$ and Theorem A.18 is proved. ■

Example A.19 If $Z = \{x \in X\,;\, \langle x, n_i\rangle \leq \beta_i\,,\, i = 1, \ldots, p\}$ is a polyhedron with a system $\{n_i\,;\, i = 1, \ldots, p\}$ of unit vectors and with $\beta_i > 0$, then Z^* is the polyhedron $Z^* = \overline{\mathrm{conv}}\,(\{0,\, n_1/\beta_1,\, \ldots,\, n_p/\beta_p\})$.

A.4 Smooth convex sets

The aim of this paragraph is to give a characterization of what we will call a "smooth convex set" in the sequel.

Theorem A.20 *Let* $Z \subsetneqq X$ *be a convex closed set and let* $c > 0$ *be given. Then the following two conditions are equivalent.*

(i) *For every* $x \in \partial Z$ *there exists a unique unit outward normal* $n(x)$ *to* Z *at the point* x*, and for every* $x, y \in \partial Z$ *we have*

$$|n(x) - n(y)| \;\le\; \frac{1}{c}|x - y|\,. \tag{A.4.1}$$

(ii) *There exists a convex closed set* $\tilde{Z} \subsetneqq X$ *such that* $Z \;=\; \tilde{Z} + B_c(0)$*.*

The proof of Theorem A.20 is based on the following Lemma.

Lemma A.21 *Let condition* (i) *in Theorem A.20 hold. Then for every* $x \in \partial Z$ *we have* $B_c(x - c\,n(x)) \;\subset\; Z$*.*

Proof of Lemma A.21. For $x \in \partial Z$ set $x_0 = x - c\,n(x)$ and assume that $B_c(x_0) \not\subset Z$. We distinguish three cases a), b), c) as follows.

a) $x_0 \notin Z$.

Put $y = Q_Z(x_0)$. We then have $0 < |P_Z(x_0)| =: c' \le c$, $n(y) = (1/c')\,P_Z(x_0)$. Furthermore, $n(x) = (1/c)(x - x_0) = (1/c)(x - y) - (c'/c)n(y)$, hence

$$n(x) - n(y) \;=\; \frac{1}{c}(x - y) - \left(1 + \frac{c'}{c}\right) n(y)\,. \tag{A.4.2}$$

From (A.4.1), (A.4.2) we obtain that

$$\begin{aligned} \frac{1}{c}|x - y|^2 \;&\ge\; |n(x) - n(y)|\,|x - y| \;\ge\; \langle n(x) - n(y), x - y\rangle \qquad \text{(A.4.3)} \\ &=\; \frac{1}{c}|x - y|^2 + \left(1 + \frac{c'}{c}\right) \langle n(y), y - x\rangle \;\ge\; \frac{1}{c}|x - y|^2\,, \end{aligned}$$

hence $n(x) - n(y) \;=\; (1/c)\,(x - y) \;=\; n(x) + (c'/c)n(y)$. This yields that $n(y) = 0$ which is a contradiction.

b) $x_0 \in \partial Z$.

We have $\langle n(x_0), x_0 - x\rangle \ge 0$, hence, by (A.4.1),

$$\begin{aligned} \frac{1}{c}|x - x_0|^2 \;&\ge\; |n(x) - n(x_0)|\,|x - x_0| \;\ge\; \langle n(x) - n(x_0), x - x_0\rangle \qquad \text{(A.4.4)} \\ &=\; \langle n(x), x - x_0\rangle + \langle n(x_0), x_0 - x\rangle \;\ge\; \frac{1}{c}|x - y|^2\,, \end{aligned}$$

and arguing similarly as in a) we obtain $n(x_0) = 0$ which is again a contradiction.

c) $x_0 \in \text{Int } Z$, $\text{dist}\,(x_0, \partial Z) = c' < c$.

We fix some $\varepsilon \in]0, c-c'[$ sufficiently small and find $y \in \partial Z$ such that $c' \le |y - x_0| < c' + \varepsilon$. We have $\langle n(y), y - (x_0 + c'n(y))\rangle \ge 0$, hence

$$\langle n(y), y - x_0\rangle \ \ge\ c'. \tag{A.4.5}$$

The rest of the proof is an exercise on the triangle inequality. Put $w = n(y) - (1/c')(y - x_0)$. Then

$$|w|^2 \ \le\ 1 + \left(\frac{c'+\varepsilon}{c'}\right)^2 - \frac{2}{c'}\langle n(y), y - x_0\rangle \ \le\ \left(\frac{c'+\varepsilon}{c'}\right)^2 - 1 \ =:\ \gamma^2(\varepsilon),$$

hence

$$n(x) - n(y) \ =\ \frac{1}{c}(x - y) \ -\ \left(1 - \frac{c'}{c}\right) n(y) - \frac{c'}{c} w. \tag{A.4.6}$$

This yields that

$$\langle n(x) - n(y), x - y\rangle \ \ge\ \frac{1}{c}|x-y|^2 + \frac{c'}{c}\langle w, y - x\rangle \ \ge\ \frac{1}{c}|x-y|^2 - 2c'\gamma(\varepsilon). \tag{A.4.7}$$

From (A.4.1), (A.4.7) it follows that

$$\frac{c}{2}\left|(n(x) - n(y)) - \frac{1}{c}(x - y)\right|^2 \ \le\ 2c'\gamma(\varepsilon), \tag{A.4.8}$$

hence

$$\left|(n(x) - n(y)) - \frac{1}{c}(x - y)\right| \ \le\ 2\sqrt{\gamma(\varepsilon)}. \tag{A.4.9}$$

Combining (A.4.6) with (A.4.9) we obtain that

$$\left|\left(1 - \frac{c'}{c}\right) n(y) + \frac{c'}{c} w\right| \ \le\ 2\sqrt{\gamma(\varepsilon)}, \tag{A.4.10}$$

hence

$$1 - \frac{c'}{c} \ \le\ \gamma(\varepsilon) + 2\sqrt{\gamma(\varepsilon)}$$

which is a contradiction for small ε.

The above cases a), b), c) exhaust all possibilities, and Lemma A.21 is proved. ■

Proof of Theorem A.20. The assertion is a trivial consequence of Lemma A.4 if $C_Z \cup (-C_Z) = X$. Assume that this is not the case and that (i) holds. Putting

$$A \ =\ \{x - c\,n(x)\,;\, x \in \partial Z\}, \quad \tilde{Z} \ =\ \overline{\text{conv}}\, A. \tag{A.4.11}$$

By Lemma A.21 we have $A + B_c(0) \subset Z$, hence $\tilde{Z} + B_c(0) \subset Z$. Conversely, by (A.4.11) we have $\partial Z \subset A + B_c(0) \subset \tilde{Z} + B_c(0)$. From Lemma A.5 it follows that $Z \subset \tilde{Z} + B_c(0)$, hence (ii) is verified.

Let now (ii) be fulfilled. We claim that

$$n(x) \;=\; \frac{1}{c} P_{\tilde{Z}} x \qquad \forall x \in \partial Z\,. \tag{A.4.12}$$

To see that (A.4.12) holds, we first notice that we have $|P_{\tilde{Z}} z| = \operatorname{dist}(z, \tilde{Z}) \le c$ for all $z \in Z$, $|P_{\tilde{Z}} x| = c$ for all $x \in \partial Z$. This yields that for all $x \in \partial Z$ and $z \in Z$ we have

$$\langle P_{\tilde{Z}} x, x - z\rangle \;=\; \langle P_{\tilde{Z}} x, Q_{\tilde{Z}} x - Q_{\tilde{Z}} z\rangle + \langle P_{\tilde{Z}} x, P_{\tilde{Z}} x - P_{\tilde{Z}} z\rangle \;\ge\; c^2 - |P_{\tilde{Z}} x|\,|P_{\tilde{Z}} z| \;\ge\; 0\,,$$

hence $P_{\tilde{Z}} x \in N_Z(x)$ for all $x \in \partial Z$. Let now $x \in \partial Z$ and $n \in N_Z(x)$ be arbitrary, $|n| = 1$. Then we have

$$0 \;\le\; \langle n, x - (Q_{\tilde{Z}} x + c\,n)\rangle = \langle n, P_{\tilde{Z}} x\rangle - c \le |n|\,|P_{\tilde{Z}} x| - c \;=\; 0\,,$$

hence $n = P_{\tilde{Z}} x$. We thus proved that (A.4.12) holds. It follows from Lemma A.2 (ii) that

$$|P_{\tilde{Z}} x - P_{\tilde{Z}} y| \;\le\; |x - y|\,. \tag{A.4.13}$$

This yields for all $x, y \in \partial Z$ that

$$|n(x) - n(y)| \;=\; \frac{1}{c}\,|P_{\tilde{Z}} x - P_{\tilde{Z}} y| \;\le\; \frac{1}{c}\,|x - y|\,,$$

and the proof is complete. ∎

Using Proposition A.16 we now show that the Lipschitz continuity condition (A.4.1) can be equivalently written in terms of the duality mapping J_Z.

Proposition A.22 *Let condition (A.3.10) hold. Then for every $x \in \partial Z$ there exists a unique unit outward normal $n(x)$ to Z at x if and only if $J_Z(x)$ is single-valued for every $x \in X$, and in this case we have*

$$\frac{1}{C}\,|n(x) - n(y)| \;\le\; |J_Z(x) - J_Z(y)| \;\le\; \frac{1}{c}\left(1 + \frac{C}{c}\right)|n(x) - n(y)| \quad \forall x, y \in \partial Z\,, \tag{A.4.14}$$

where we use (by a slight abuse of notation) the same symbol $J_Z(x)$ to denote the unique element of $J_Z(x)$.

Proof. By Proposition A.16 and (A.3.25) we have

$$J_Z(x) \;=\; |J_Z(x)|\,n(x)\,, \quad M_{Z^*}(J_Z(x)) \;=\; \langle J_Z(x), x\rangle = 1 \qquad \forall x \in \partial Z\,, \tag{A.4.15}$$

hence $\langle n(x), x\rangle = M_{Z^*}(n(x)) = 1/|J_Z(x)|$, where we have by virtue of Proposition A.11 that $1/C \le |J_Z(x)| \le 1/c$. For $x, y \in \partial Z$ we thus have

$$\begin{aligned}
|J_Z(x) - J_Z(y)| \;&=\; \left|\frac{n(x)}{M_{Z^*}(n(x))} - \frac{n(y)}{M_{Z^*}(n(y))}\right| \\
&\le\; \frac{1}{M_{Z^*}(n(x))}\left(|n(x) - n(y)| + \frac{1}{M_{Z^*}(n(y))}\,M_{Z^*}\left(n(x) - n(y)\right)\right) \\
&\le\; \frac{1}{c}\left(1 + \frac{C}{c}\right)|n(x) - n(y)|\,.
\end{aligned}$$

To prove the second inequality we refer to the general vector formula

$$\left|\frac{u}{|u|} - \frac{v}{|v|}\right|^2 \;\le\; \frac{1}{|u||v|}\,|u-v|^2$$

with $u = J_Z(x)$, $v = J_Z(y)$, and use the fact that $|u|, |v| \ge 1/C$. ■

Corollary A.23 *Let $C > c > 0$ be given, and let $0 \in \tilde{Z} \subset B_{C-c}(0)$ be a non-empty closed convex set, $Z = \tilde{Z} + B_c(0)$. Let J_Z be the duality mapping associated with Z. Then for every $x, y \in X$ we have*

$$|J_Z(x) - J_Z(y)| \;\le\; \frac{1}{c^2}\left(1 + \left(1 + \frac{C}{c}\right)^2\right)|x-y|\,.$$

Proof. The assertion is obvious if $x = 0$ or $y = 0$. For arbitrary $x, y \in X \setminus \{0\}$ we have by Theorem A.20 and Proposition A.22 that

$$\begin{aligned}
|J_Z(x) - J_Z(y)| \;&=\; \left|M_Z(x)\,J_Z\left(\frac{x}{M_Z(x)}\right) - M_Z(y)\,J_Z\left(\frac{y}{M_Z(y)}\right)\right| \\
&\le\; M_Z(x-y)\left|J_Z\left(\frac{x}{M_Z(x)}\right)\right| + M_Z(y)\left|J_Z\left(\frac{x}{M_Z(x)}\right) - J_Z\left(\frac{y}{M_Z(y)}\right)\right| \\
&\le\; \frac{1}{c^2}|x-y| + \frac{1}{c^2}\left(1+\frac{C}{c}\right)M_Z(y)\left|\frac{x}{M_Z(x)} - \frac{y}{M_Z(y)}\right| \\
&\le\; \frac{1}{c^2}\left(2+\frac{C}{c}\right)|x-y| + \frac{1}{c^2}\left(1+\frac{C}{c}\right)\frac{|x|}{M_Z(x)}\,M_Z(x-y) \\
&\le\; \frac{1}{c^2}\left(2+\frac{C}{c} + \left(1+\frac{C}{c}\right)\frac{C}{c}\right)|x-y|\,,
\end{aligned}$$

which we wanted to prove. ■

A.5 Distance of convex sets

We can measure the distance of two sets Z_1, Z_2 in X either as the *Hausdorff distance*

$$d_H(Z_1, Z_2) \;=\; \max\{\sup_{z_1\in Z_1}\operatorname{dist}(z_1, Z_2),\; \sup_{z_2\in Z_2}\operatorname{dist}(z_2, Z_1)\}\,, \tag{A.5.1}$$

or, if both Z_1 and Z_2 are convex and contain the origin, the *Minkowski distance*

$$d_M(Z_1, Z_2) \;=\; \sup_{|x|=1}|M_{Z_1}(x) - M_{Z_2}(x)|\,. \tag{A.5.2}$$

We first show that these concepts are equivalent in the class of sets satisfying condition (A.3.10).

Lemma A.24 *Let Z_1, Z_2 be convex closed sets such that (A.3.10) holds for both $Z = Z_i$, $i = 1, 2$. Then we have*

$$c^2 d_M(Z_1, Z_2) \;\le\; d_H(Z_1, Z_2) \;\le\; C^2 d_M(Z_1, Z_2)\,. \tag{A.5.3}$$

Proof. Assume first that there exists $x \in Z_1 \setminus Z_2$. Using (A.3.12) we obtain

$$\begin{aligned} \operatorname{dist}(x, Z_2) &\le \left| x - \frac{x}{M_{Z_2}(x)} \right| \le \frac{|x|^2}{M_{Z_1}(x) M_{Z_2}(x)} \left(M_{Z_2}\left(\frac{x}{|x|}\right) - M_{Z_1}\left(\frac{x}{|x|}\right)\right) \\ &\le C^2 d_M(Z_1, Z_2), \end{aligned}$$

and reversing the roles of Z_1 and Z_2 we obtain the right inequality in (A.5.3). To prove the left estimate in (A.5.3), we divide the unit sphere $\partial B_1(0)$ into the sets

$$\begin{aligned} A_0 &= \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) = M_{Z_2}(x)\}, \\ A_1 &= \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) > M_{Z_2}(x)\}, \\ A_2 &= \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) < M_{Z_2}(x)\}. \end{aligned}$$

For $x \in A_2$ set $\bar{x} = x/M_{Z_1}(x)$. We have $M_{Z_2}(\bar{x}) > M_{Z_1}(\bar{x}) = 1$, hence $\bar{x} \notin Z_2$ and $d := |P_{Z_2}\bar{x}| > 0$. Put $m = 1 + d/c$. Then the vector

$$\frac{1}{m}\,\bar{x} = \frac{c}{c+d}\,Q_{Z_2}\bar{x} + \frac{d}{c+d}\,\frac{cP_{Z_2}\bar{x}}{d}$$

is a convex combination of elements of Z_2, hence $M_{Z_2}(\bar{x}) \le m$. This yields

$$M_{Z_2}(\bar{x}) - M_{Z_1}(\bar{x}) \le m - 1 \le \frac{1}{c}\operatorname{dist}(\bar{x}, Z_2) \le \frac{1}{c} d_H(Z_1, Z_2).$$

Using (A.3.12) we conclude that

$$M_{Z_2}(x) - M_{Z_1}(x) \le \frac{1}{c^2} d_H(Z_1, Z_2),$$

and arguing similarly for $x \in A_0 \cup A_1$ we complete the proof. ∎

In Section 6 we solve the uniqueness problem for quasivariational inequalities using a distance criterion involving the mapping J_Z introduced in Theorem A.14. We now prove that it is stronger than the Minkowski distance. The reader will easily construct smoothened versions of Example 6.2 with $\psi(v) = \sqrt{v}$ showing that the square root on the right-hand side of (A.5.4) cannot be removed in general.

Lemma A.25 *Let $C > c > 0$ be given, and let $0 \in \tilde{Z}_i \subset B_{C-c}(0) \subset X$ for $i = 1, 2$ be convex closed sets, $Z_i = \tilde{Z}_i + B_c(0)$ for $Z = Z_i$, $i = 1, 2$. Let L_J be the Lipschitz constant in Corollary A.23. Then for all $x \in \partial B_1(0)$ we have*

$$\frac{2}{C}\,|M_{Z_1}(x) - M_{Z_2}(x)| \le |J_{Z_1}(x) - J_{Z_2}(x)| \le \frac{2\sqrt{2}}{c}\Big(d_M(Z_1, Z_2)\,(cL_J + d_M(Z_1, Z_2))\Big)^{1/2}. \tag{A.5.4}$$

Proof. The left inequality is an easy consequence of Proposition A.11 and Lemma A.12 (iii) which for every $x \in X$ yield that

$$\frac{2|x|}{C}\,|M_{Z_1}(x) - M_{Z_2}(x)| \le |M_{Z_1}^2(x) - M_{Z_2}^2(x)| = |\langle J_{Z_1}(x) - J_{Z_2}(x), x\rangle|.$$

To prove the other estimate, we fix $x \in X$ with $|x| = 1$ such that $J_{Z_1}(x) \neq J_{Z_2}(x)$, and define

$$x_s = x + s\frac{J_{Z_2}(x) - J_{Z_1}(x)}{|J_{Z_2}(x) - J_{Z_1}(x)|} \quad \text{for} \quad s \geq 0\,. \tag{A.5.5}$$

We may assume that $\langle x_s - x, x\rangle \leq 0$, otherwise we interchange Z_1 and Z_2. The functions $\lambda_i(s) := \frac{1}{2}M_{Z_i}^2(x_s)$ are convex and satisfy

$$\lambda_i(0) + s\lambda_i'(0) \ \leq\ \lambda_i(s) \ \leq\ \lambda_i(0) + s\lambda_i'(s) \quad \text{for} \quad s \geq 0\,. \tag{A.5.6}$$

Thus,

$$\begin{aligned} \lambda_2(s) - \lambda_1(s) \ &\geq\ \lambda_2(0) - \lambda_1(0) \ +\ s\,(\lambda_2'(0) - \lambda_1'(s)) \\ &=\ \lambda_2(0) - \lambda_1(0) \ +\ s\,(\lambda_2'(0) - \lambda_1'(0)) \ +\ s\,(\lambda_1'(0) - \lambda_1'(s))\,. \end{aligned} \tag{A.5.7}$$

Note that

$$\lambda_i'(s) = \left\langle J_{Z_i}(x_s), \frac{J_{Z_2}(x) - J_{Z_1}(x)}{|J_{Z_2}(x) - J_{Z_1}(x)|}\right\rangle \quad \text{for} \quad s \geq 0\,,$$

hence

$$\lambda_2'(0) - \lambda_1'(0) \ =\ |J_{Z_2}(x) - J_{Z_1}(x)|\,, \tag{A.5.8}$$

$$|\lambda_1'(s) - \lambda_1'(0)| \ \leq\ |J_{Z_1}(x_s) - J_{Z_1}(x)| \ \leq\ sL_J\,. \tag{A.5.9}$$

We further have by (A.3.12) for all $s \geq 0$ that

$$|\lambda_2(s) - \lambda_1(s)| \ \leq\ \frac{|x_s|}{c}\,|M_{Z_2}(x_s) - M_{Z_1}(x_s)| \ \leq\ \frac{|x_s|^2}{c}\,d_M(Z_1, Z_2) \ \leq\ \frac{1+s^2}{c}\,d_M(Z_1, Z_2)\,. \tag{A.5.10}$$

Combining (A.5.7)–(A.5.10) we obtain for all $s > 0$ that

$$|J_{Z_2}(x) - J_{Z_1}(x)| \ \leq\ \frac{2+s^2}{sc}\,d_M(Z_1, Z_2) \ +\ sL_J\,. \tag{A.5.11}$$

The right-hand side attains its minimum for $s = \sqrt{2\,d_M(Z_1, Z_2)/(cL_J + d_M(Z_1, Z_2))}$, and the assertion follows. ∎

A.6 Parameter-dependent convex sets

To conclude the section, we will consider families of convex sets $Z(v) \subset X$ parametrized by elements v of a closed subset V of a Banach space Y endowed with norm $\|\cdot\|$. We will consecutively make the following hypotheses.

Hypothesis A.26

(i) $Z(v)$ is a non-empty convex closed subset of X for every $v \in V$;

(ii) The mapping $\Delta : V \times V \to \mathbb{R}_+ : (v, w) \mapsto d_H(Z(v), Z(w))$ is continuous.

Hypothesis A.27 For every $v \in V$ there exists $x(v) \in X$ and $\varrho(v) > 0$ such that

$$B_{\varrho(v)}(x(v)) \ \subset \ Z(v)\,. \tag{A.6.1}$$

For simplicity, we denote by (P_v, Q_v) instead of $(P_{Z(v)}, Q_{Z(v)})$ the projection pair associated with $Z(v)$ for $v \in V$. As an easy consequence of the definition, we have the implication

$$v, w \in V\,,\ x \in Z(v) \implies |P_w x| \ \leq\ \Delta(v,w)\,. \tag{A.6.2}$$

Let us consider now arbitrary sequences $\{x_n\}$ in X, $\{v_n\}$ in V such that $x_n \in Z(v_n)$ for all n, $|x_n - x| \to 0$, $\|v_n - v\| \to 0$ as $n \to \infty$. From (A.6.2) it follows that

$$\operatorname{dist}(x, Z(v)) \ \leq\ |x - Q_v x_n| \ \leq\ |x - x_n| + |P_v x_n| \ \leq\ |x - x_n| + \Delta(v, v_n)\,.$$

Under Hypothesis A.26, the right-hand side of this inequality tends to 0 an $n \to \infty$. This enables us to conclude that

$$\Big(x_n \to x\,,\ v_n \to v\,,\ x_n \in Z(v_n)\ \ \forall n \in \mathbb{N}\Big) \quad \Longrightarrow \quad x \in Z(v)\,. \tag{A.6.3}$$

We now derive some further consequences of the definition.

Lemma A.28 *Let Hypothesis A.26 hold, and let $x, y \in X$, $v, w \in V$ be given. Then we have*

$$|P_v x - P_w y|^2 \ \leq\ |x - y|^2 + \Delta^2(v,w) + 4\Delta(v,w)\,|P_v x|\,. \tag{A.6.4}$$

Proof. By (A.6.2) and Lemma A.2 we have

$$\begin{aligned} \langle P_v x, Q_v x - Q_w y\rangle &= \langle P_v x, Q_v x - Q_v Q_w y\rangle + \langle P_v x, P_v Q_w y\rangle \ \geq\ -|P_v x|\,\Delta(v,w)\,,\\ \langle P_w y, Q_w y - Q_v x\rangle &= \langle P_w y, Q_w y - Q_w Q_v x\rangle + \langle P_w x, P_w Q_v y\rangle \ \geq\ -|P_w y|\,\Delta(v,w)\,. \end{aligned}$$

Summing up the above inequalities we obtain

$$\begin{aligned} |P_v x - P_w y|^2 &\leq \langle P_v x - P_w y, x - y\rangle + (|P_v x| + |P_w y|)\,\Delta(v,w)\\ &\leq |P_v x - P_w y|\,(|x-y| + \Delta(v,w)) + 2\Delta(v,w)\,|P_v x| \end{aligned}$$

and (A.6.4) follows easily. ■

Lemma A.29 *Let Hypothesis A.26 hold, and let $K \subset V$ be a compact set. Then there exists a non-decreasing function $\mu_K : \mathbb{R}_+ \to \mathbb{R}_+$, $\mu_K(0) = \mu_K(0+) = 0$, such that*

$$\Delta(v,w) \ \leq\ \mu_K(\|v - w\|) \quad \forall v, w \in K\,. \tag{A.6.5}$$

Proof. For $h \geq 0$ it suffices to set

$$\mu_K(h) \ =\ \max\{\Delta(v,w)\,;\, v, w \in K\,,\ \|v - w\| \leq h\}\,. \tag{A.6.6}$$

The μ_K is non-decreasing and (A.6.5) holds. From the compactness of K and continuity of Δ we easily obtain that $\mu_K(0+) = 0$. ■

Lemma A.30 *Let Hypotheses A.26, A.27 hold, let* $K \subset V$ *be a compact set, and let* $v \in K$ *be given. Let* $x \in X$ *,* $\varrho > 0$ *and* $h > 0$ *be such that*

$$\mu_K(h) < \varrho\,, \quad B_\varrho(x) \subset Z(v)\,. \tag{A.6.7}$$

Then for every $w \in K$ *,* $\|v - w\| \le h$ *we have*

$$B_{\varrho-\mu_K(h)}(x) \ \subset \ Z(w)\,. \tag{A.6.8}$$

Proof. Let $y \in B_{\varrho-\mu_K(h)}(x)$ be arbitrarily given, and assume that $y \notin Z(w)$, that is,

$$|x - y| \ \le \ \varrho - \mu_K(h)\,, \ \ |P_w(y)| \ > \ 0\,. \tag{A.6.9}$$

Put $\alpha = 1 + \mu_K(h)/|P_w y| > 1$ and

$$\tilde{y} \ = \ Q_w y + \alpha P_w y \ = \ y + (\alpha - 1)P_w y\,. \tag{A.6.10}$$

From Lemma A.2 (iii) it follows that $Q_w\tilde{y} = Q_w y$, hence

$$P_w\tilde{y} \ = \ \alpha P_w y\,. \tag{A.6.11}$$

On the other hand, we have $|\tilde{y} - x| \le |x - y| + (\alpha - 1)|P_w(y)| \le \varrho$, hence $\tilde{y} \in B_\varrho(x) \subset Z(v)$. From (A.6.2), (A.6.6), and (A.6.11) we thus obtain that

$$\Delta(v, w) \ \ge \ |P_w\tilde{y}| \ = \ \alpha|P_w y| \ = \ |P_w y| + \mu_K(h) \ > \ \Delta(v, w)$$

which is a contradiction. ∎

Proposition A.31 *Let Hypotheses A.26, A.27 hold, and let* $K \subset V$ *be a compact set. Then there exists* $\tilde{\varrho} > 0$ *and* $x_1, \dots, x_n \in X$ *such that for every* $v \in K$ *there exists* $i \in \{1, \dots, n\}$ *satisfying* $B_{\tilde{\varrho}}(x_i) \ \subset \ Z(v)$*.*

Proof. For every $v \in K$ we find $h(v) > 0$ such that

$$\mu_K(h(v)) \ \le \ \frac{1}{2}\varrho(v) \tag{A.6.12}$$

with $\varrho(v)$ from Hypothesis A.27. From the covering $K \ \subset \ \bigcup_{v\in K} \{w \in K\,;\, \|v - w\| < h(v)\}$ we select a finite subcovering

$$K \ \subset \ \bigcup_{i=1}^{n} \{w \in K\,;\, \|v_i - w\| < h(v_i)\} \tag{A.6.13}$$

with some $v_1, \dots, v_n \in K$. Set $x_i = x(v_i)$ for $i = 1, \dots, n$. From Hypothesis A.27, Lemma A.30 and formula (A.6.12) we obtain the implication

$$v \in K\,, \ \|v - v_i\| < h(v_i) \quad \Longrightarrow \quad B_{\varrho(v_i)/2}(x_i) \ \subset \ Z(v) \tag{A.6.14}$$

for all $i = 1, \dots, n$. Combining (A.6.13) with (A.6.14) we obtain the assertion by putting $\tilde{\varrho} = \min_{i=1,\dots,n} \varrho(v_i)/2$. ∎

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