Characterization of interpolation between Grand, small or classical Lebesgue spaces

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A R T I C L E   I N F O

A B S T R A C T

In this paper, we show that the interpolation spaces between Grand, small or classical Lebesgue are so called Lorentz–Zygmund spaces or more generally $G\Gamma$-spaces. As a direct consequence of our results any Lorentz–Zygmund space $L^{a,r}(\mathrm{Log}\,L)^{\beta}$, is an interpolation space in the sense of Peetre between either two Grand Lebesgue spaces or between two small spaces provided that $1 < a < \infty, \beta \neq 0$. The method consists in computing the so called K-functional of the interpolation space and in identifying the associated norm.

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1. Introduction. Main results

Let $(X_0, \|\cdot\|_0)$, $(X_1, \|\cdot\|_1)$ be two Banach spaces contained continuously in a Hausdorff topological vector space (that is (X_0, X_1) is a compatible couple).

For $g \in X_0 + X_1$, $t > 0$ one defines the so called K functional $K(g, t; X_0, X_1) \doteq K(g, t)$ by setting

$$K(g, t) = \inf_{g = g_0 + g_1} \left(\|g_0\|_0 + t\|g_1\|_1 \right). \tag{1}$$

For $0 \leqslant \theta \leqslant 1$, $1 \leqslant p \leqslant +\infty$, $\alpha \in \mathbb{R}$ we shall consider

$$(X_0, X_1)_{\theta, p; \alpha} = \left\{ g \in X_0 + X_1, \ \|g\|_{\theta, p; \alpha} = \|t^{-\theta - \frac{1}{p}} \left(1 - \mathrm{Log}\, t\right)^{\alpha} K(g, t)\|_{L^p(0,1)} \text{ is finite} \right\}.$$

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Here $\| \cdot \|_V$ denotes the norm in a Banach space V. The weighted Lebesgue space $L^p(0,1;\omega)$, $0 < p \leqslant +\infty$ is endowed with the usual norm or quasinorm, where ω is a weight function on $(0,1)$.

Our definition of the interpolation space is different from the usual one (see [5,21]) since we restrict the norms on the interval $(0,1)$.

If we consider ordered couple, i.e. $X_1 \hookrightarrow X_0$ and $\alpha = 0$, $(X_0, X_1)_{\theta,p;0} = (X_0, X_1)_{\theta,p}$ is the interpolation space as it is defined by J. Peetre (see [5,6,21]).

A particular attention will be brought to the so-called Grand Lebesgue space $L^{p),\alpha}(\Omega)$, with Ω a bounded (open) set of $\mathbb{R}^n$ whose measure is 1, $1 < p < +\infty$, $\alpha > 0$ defined as

$$L^{p),\alpha}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable } \|f\|_{p),\alpha} = \sup_{0<t<1} (1 - \mathrm{Log}\, t)^{-\frac{\alpha}{p}} \left(\int_t^1 f_*^p(\sigma) d\sigma \right)^{\frac{1}{p}} < +\infty \right\}$$

and its associated spaces $L^{(p',\alpha}(\Omega)$, $\frac{1}{p'} + \frac{1}{p} = 1$ defined as (see [11])

$$L^{(p',\alpha}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable}, \|f\|_{(p',\alpha} = \int_0^1 (1 - \mathrm{Log}\, t)^{-\frac{\alpha}{p'}+\alpha-1} \left(\int_0^t f_*^{p'}(\sigma) d\sigma \right)^{\frac{1}{p'}} \frac{dt}{t} < +\infty \right\}.$$

Here, f_* is the decreasing rearrangement of $|f|$, say it is the generalized inverse of the distribution function

$$D_f(t) = measure\{x \in \Omega, |f(x)| > t\}, t \in \mathbb{R}_+.$$

Many works related to Grand and small Lebesgue spaces have been recently done (see for instance [2,10,11,14,15]).

We will show in particular, the

Theorem 1.1. *Let* $1 < p < q$, $\alpha > 0$. *Then*

$$L^{q),\alpha} = (L^{p),\alpha}, L^q)_{1,\infty;-\frac{\alpha}{q}}.$$

An explicit equivalent of $K(f, t; L^{p),\alpha}, L^q)$ is given in Theorem 3.2.

For convenience, we will sometimes drop the dependence with respect to the domain Ω or $(0,1)$ and we shall write $L^p(\Omega) = L^p \dots$ etc.

More, we will write sometimes $\int_E f_*^m(t) dt = \int_E f_*^m$.

Definition 1.1 (*Lorentz–Zygmund Space*). For $1 \leqslant p$, $q \leqslant \infty$, $-\infty < \alpha < +\infty$, the Lorentz–Zygmund space $L^{p,q}(\mathrm{Log}\ L)^\alpha$ consists of all functions f measurable such that

$$\|f\|_{p,q;\alpha} = \begin{cases} \left(\int_0^1 \left[t^{\frac{1}{p}-\frac{1}{q}} (1 - \mathrm{Log}\, t)^\alpha f_*(t) \right]^q dt \right)^{\frac{1}{q}} & \text{if } 1 \leqslant q < +\infty, \\ \sup_{0<t<1} t^{\frac{1}{p}} (1 - \mathrm{Log}\, t)^\alpha f_*(t) & \text{if } q = +\infty \end{cases} \quad \text{is finite.}$$

One of the major theorems of the first section will be

Theorem 1.2. *Let* $0 < \theta < 1$, $1 \leqslant r < +\infty$, $\alpha > 0$, $1 < p < q$. *Then*

$$\left(L^{p),\alpha}, L^{q),\alpha} \right)_{\theta,r} = L^{p_\theta, r} (\mathrm{Log}\ L)^{-\frac{\alpha}{p_\theta}} \qquad \text{where } \frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

We will also use the following extension of Generalized Gamma space (see [17]).

Definition 1.2 (*of a Generalized Gamma Space with Double Weights*)**.** Let w_1, w_2 be two weights on $(0,1)$, $m \in [1, +\infty]$, $1 \leqslant p < +\infty$. We assume the following conditions:

(c1) There exists $K_{12} > 0$ such that $w_2(2t) \leqslant K_{12}w_2(t) \ \forall \, t \in (0, 1/2)$. The space $L^p(0, 1; w_2)$ is continuously embedded in $L^1(0, 1)$.

(c2) The function $\int_0^t w_2(\sigma)d\sigma$ belongs to $L^{\frac{m}{p}}(0, 1; w_1)$.

A generalized Gamma space with double weights is the set:

$$G\Gamma(p, m; w_1, w_2) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable } \int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \text{ is in } L^{\frac{m}{p}}(0, 1; w_1) \right\}.$$

Property 1.1. *Let $G\Gamma(p, m; w_1, w_2)$ be a Generalized Gamma space with double weights and let us define for $v \in G\Gamma(p, m; w_1, w_2)$*

$$\rho(v) = \left[\int_0^1 w_1(t) \left(\int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}$$

with the obvious change for $m = +\infty$.

Then,

1. *ρ is a quasinorm.*
2. *$G\Gamma(p, m; w_1, w_2)$ endowed with ρ is a quasi-Banach function space.*
3. *If $w_2 = 1$*

$$G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1).$$

Proof.

1. Due to the property of the monotone rearrangement we have $\rho(v) = 0$ if and only if $v = 0$ and for $\lambda \in \mathbb{R} \ \rho(\lambda v) = |\lambda|\rho(v)$. Let us show that

$$\rho(v_1 + v_2) \leqslant (2K_{12})^{\frac{1}{p}} \Big(\rho(v_1) + \rho(v_2) \Big) \text{ for } v_1 \text{ and } v_2 \text{ in } G\Gamma(p, m; w_1, w_2).$$

We have for all $\sigma \in (0, 1)$, $v = v_1 + v_2$

$$(v_1 + v_2)_*(\sigma) \leqslant v_{1*}\left(\frac{\sigma}{2} \right) + v_{2*}\left(\frac{\sigma}{2} \right).$$

Therefore, we have (using the triangle inequality)

$$\left(\int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \right)^{\frac{1}{p}} \leqslant \left(\int_0^t v_{1*}^p\left(\frac{\sigma}{2} \right) w_2(\sigma)d\sigma \right)^{\frac{1}{p}} + \left(\int_0^t v_{2*}^p\left(\frac{\sigma}{2} \right) w_2(\sigma)d\sigma \right)^{\frac{1}{p}} \tag{2}$$

$$\leqslant 2^{\frac{1}{p}} \left(\int_0^{\frac{t}{2}} v_{1*}^p(s)w_2(2s)ds \right)^{\frac{1}{p}} + 2^{\frac{1}{p}} \left(\int_0^{\frac{t}{2}} v_{2*}^p(s)w_2(2s)ds \right)^{\frac{1}{p}} \tag{3}$$

Using condition (c1), we have

$$\left(\int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \right)^{\frac{1}{p}} \leqslant (2K_{12})^{\frac{1}{p}} \left[\left(\int_0^t v_{1*}^p(s)w_2(s)ds \right)^{\frac{1}{p}} + \left(\int_0^t v_{2*}^p(s)w_2(s)ds \right)^{\frac{1}{p}} \right].$$

Using again the triangle inequality with the space $L^m(0, 1; w_1)$ we derive that:

$$\rho(v_1 + v_2) \leqslant (2K_{12})^{\frac{1}{p}} \Big[\rho(v_1) + \rho(v_2) \Big].$$

2. The function ρ satisfies:

If $0 \leqslant v_1 \leqslant v_2$ then $v_{1*} \leqslant v_{2*}$ everywhere on $(0,1)$ so that

$$\rho(v_1) \leqslant \rho(v_2).$$

If $0 \leqslant v_{1k} \nearrow v$ almost-everywhere then by the Beppo–Levi's theorem we have for all $t \in [0,1]$

$$\lim_{k \to +\infty} \left(\int_0^t v_{1k*}^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} = \left(\int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}},$$

since $v_{1k*} \nearrow v_*$ everywhere on $[0,1]$.

Because $L^m(0,1;w_1)$ is a Banach function space, we deduce that

$$\lim_{k \to +\infty} \rho(v_{1k}) = \rho(v).$$

The condition (c2) implies that

$$\rho(\chi_E) \leqslant \rho(1) < +\infty$$

for any $E \subset \Omega$, χ_E denoting its characteristic function.

To conclude that it is a complete space, we shall prove the inequality in Proposition 1.1, which has been already given in the frame of $G\Gamma(p,m;w_1)$ (with one weight (see [14,16,17])). $\Diamond$

Proposition 1.1 (*Inequality for* $G\Gamma$). *Let* $v \in G\Gamma(p,m;w_1,w_2)$, $E \subset \Omega$, $|E| > 0$. *Then*

$$\frac{\rho(v) \left[\int_0^{|E|} w_2(\sigma) d\sigma \right]^{\frac{1}{p}}}{\left[\int_0^{|E|} w_1(t) \left(\int_0^t w_2(s) ds \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}} \geqslant \left[\int_0^{|E|} v_*^p(\sigma) w_2(\sigma) d\sigma \right]^{\frac{1}{p}}.$$

Proof. Let us set

$$V(t) = \frac{\int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma}{\int_0^t w_2(\sigma) d\sigma}.$$

Then V is decreasing since v_*^p is decreasing. Therefore, we have for $1 \leqslant m < +\infty$

$$\rho(v) = \left[\int_0^1 V^{\frac{m}{p}}(t) w_1(t) \left(\int_0^t w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}} \geqslant V^{\frac{1}{p}}(|E|) \left[\int_0^{|E|} w_1(t) \left(\int_0^t w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}.$$

This gives the inequality $m < +\infty$.

For $m = +\infty$, the argument is the same, since

$$\sup_{t \leqslant 1} w_1(t) \left(\int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \geqslant \sup_{t \leqslant |E|} V^{\frac{1}{p}}(t) w_1(t) \left(\int_0^t w_2(\sigma) d\sigma \right)^{\frac{1}{p}}$$

$$\geqslant V^{\frac{1}{p}}(|E|) \sup_{t \leqslant |E|} w_1(t) \left(\int_0^t w_2(\sigma) d\sigma \right)^{\frac{1}{p}}. \quad \Diamond$$

One of our results which motivates the introduction of Generalized Gamma space with double weights is

Theorem 1.3. *Let* $1 < p < +\infty$, $0 < \theta < 1$, $1 \leqslant r < +\infty$. *Then*

$$(L^p), L^{(p)})_{\theta,r} = G\Gamma(p,r;w_1,w_2) \text{ with } w_1(t) = t^{-1}(1 - \text{Log } t)^{\theta r - 1}, \ w_2(t) = (1 - \text{Log } t)^{-1}, \ t \in (0,1).$$

The proof is given in the last theorem of this paper.

2. Notations. Preliminary Lemmas

For two positive quantities A and B depending on some parameters like functions (real number as a function on $(0,1)$) we shall write $A \lesssim B$ if there exists a constant $c > 0$ independent of the parameters such that $A \leqslant cB$, and $A \approx B$ if $B \lesssim A$ and $A \lesssim B$.

Proposition 2.1. *Let* $a > 0$, $b \in \mathbb{R}$, $\Phi(t) = t^a (1 - \text{Log}\, t)^b$, $t \in (0,1)$. *Then, there exists an invertible function* φ *from* $[0,1]$ *into* $[0,1]$ *such that*

1. $t \approx \Phi(\varphi(t)) = \varphi(t)^a (1 - \text{Log}\, \varphi(t))^b$
2. $1 + |\text{Log}\, \varphi(t)| \approx 1 + |\text{Log}\, t|$ *for* $t \in (0,1)$.

Proof. Let us set $t_0 = \begin{cases} e^{\frac{a-b}{a}} & \text{if } a < b, \\ 1 & \text{if } a \geqslant b. \end{cases}$ Then Φ is strictly increasing on $[0, t_0]$.

Define $g(t) = \frac{\Phi(t\, t_0)}{\Phi(t_0)}$ for $t \in [0,1]$. Then g is continuous and strictly increasing from $[0,1]$ into itself, and $g(t) \approx t^a (1 - \text{Log}\, t)^b$ and then $1 - \text{Log}\, g(t) \approx 1 - \text{Log}\, t$. Setting $\varphi(t) = g^{-1}(t)$ the inverse of g we have the result. $\Diamond$

Proposition 2.2. *Let* $\beta \in \mathbb{R}$, $-\infty < \alpha < 1$. *Then, there exists* $c_{\alpha\beta} > 0$:

$$\int_0^a t^{-\alpha}(1 - \text{Log}\, t)^\beta dt \leqslant c_{\alpha\beta} a^{1-\alpha}(1 - \text{Log}\, a)^\beta \qquad \forall\, a \in [0,1].$$

Moreover, if $\beta \geqslant 0$ *then*

$$\int_0^a t^{-\alpha}(1 - \text{Log}\, t)^\beta dt \geqslant \frac{1}{1-\alpha} a^{1-\alpha}(1 - \text{Log}\, a)^\beta.$$

Proof. We start with the case $\beta > 0$.

Let $k \in \mathbb{N}$ such $k - 1 \leqslant \beta < k$. Then by integration by parts, we have

$$\int_0^a t^{-\alpha}(1 - \text{Log}\, t)^\beta dt = c_{\alpha\beta k} \int_0^a t^{-\alpha}(1 - \text{Log}\, t)^{\beta-k} dt + a^{1-\alpha} \sum_{j=0}^{k-1} c_{\alpha\beta j}(1 - \text{Log}\, a)^{\beta-j}$$

$$\text{(using the fact that the function } t \to (1 - \text{Log}\, t)^{\beta-k} \text{ is increasing).}$$

$$\leqslant c_{\alpha\beta k}(1 - \text{Log}\, a)^{\beta-k} \int_0^a t^{-\alpha} dt + c_{\alpha\beta k} a^{1-\alpha}(1 - \text{Log}\, a)^\beta$$

$$\leqslant c_{\alpha\beta} a^{1-\alpha}(1 - \text{Log}\, a)^\beta.$$

If $\beta < 0$ the inequality is still true since the function $t \to (1 - \text{Log}\, t)^\beta$ is increasing, we argue as in the last line of the above proof. If $\beta \geqslant 0$ then $t \to (1 - \text{Log}\, t)^\beta$ is decreasing, then the last result follows directly. $\Diamond$

Using an argument of [4], we have:

Proposition 2.3. *For any* $\alpha < -1$, $\beta \in \mathbb{R}$, *we have*

$$\int_a^1 t^\alpha (1 - \text{Log}\, t)^\beta dt \lesssim a^{\alpha+1}(1 - \text{Log}\, a)^\beta.$$

Proof. Let $\varepsilon > 0$ such that $\alpha + \varepsilon < -1$. For $N = 1 + \frac{|\beta|}{\varepsilon}$, $t^{-\varepsilon}(N - \text{Log}\, t)^\beta$ is decreasing. So $\int_a^1 t^\alpha(1 - \text{Log}\, t)^\beta dt \lesssim a^{-\varepsilon}(N - \text{Log}\, a)^\beta \int_a^1 t^{\alpha+\varepsilon} dt \lesssim (1 - \text{Log}\, a)^\beta a^{\alpha+1}$. $\Diamond$

For convenience, we recall some of those Hardy type inequalities (see [4] Theorem 6.4, [18] Lemma 2.7, Corollary 2.9) that we will use.

Theorem 2.1. *Suppose* $\lambda > 0$, $1 \leqslant b \leqslant +\infty$, *and* $-\infty < \beta < +\infty$, Φ *a nonnegative measurable function on* $(0, 1)$. *Then*

$$\int_0^1 \left[t^{-\lambda}(1 - \mathrm{Log}\, t)^\beta \int_0^t \Phi(s)ds \right]^b \frac{dt}{t} \leqslant c \int_0^1 \left[t^{1-\lambda}(1 - \mathrm{Log}\, t)^\beta\, \Phi(t) \right]^b \frac{dt}{t},$$

and

$$\int_0^1 \left[t^\lambda(1 - \mathrm{Log}\, t)^\beta \int_t^1 \Phi(s)ds \right]^b \frac{dt}{t} \leqslant c \int_0^1 \left[t^{1+\lambda}(1 - \mathrm{Log}\, t)^\beta\, \Phi(t) \right]^b \frac{dt}{t}.$$

The constant c is independent of Φ.

If $\Phi(t) = t^{\mu-1}\Phi_1(t)$, $\mu > 0$, Φ_1 decreasing, then the above inequalities hold true for $0 < b < 1$.

In the above formula when $b = +\infty$, the integral is replaced by the supremum. In the case where $\lambda = 0$ we use the following Hardy inequalities given in Bennett–Rudnick ([4] Theorem 6.5).

Theorem 2.2. *Suppose* $1 \leqslant a \leqslant \infty$, $\alpha \in \mathbb{R}$ *and* $\alpha + \frac{1}{a} \neq 0$. *Let ψ be a nonnegative measurable function on* $(0, 1)$. *Then, if* $\alpha + \frac{1}{a} > 0$,

$$\left(\int_0^1 \left[(1 - \mathrm{Log}\, t)^\alpha \int_0^t \psi(s)ds \right]^a \frac{dt}{t} \right)^{\frac{1}{a}} \leqslant c \left(\int_0^1 \left[t(1 - \mathrm{Log}\, t)^{1+\alpha}\psi(t) \right]^a \frac{dt}{t} \right)^{\frac{1}{a}} \tag{4}$$

and $\alpha + \frac{1}{a} < 0$,

$$\left(\int_0^1 \left[(1 - \mathrm{Log}\, t)^\alpha \int_t^1 \psi(s)ds \right]^a \frac{dt}{t} \right)^{\frac{1}{a}} \leqslant c \left(\int_0^1 \left[t(1 - \mathrm{Log}\, t)^{1+\alpha}\psi(t) \right]^a \frac{dt}{t} \right)^{\frac{1}{a}}. \tag{5}$$

The constant c is independent of ψ.

Let $(\mathcal{R}, \mu)$ be a measure space and $\mathcal{M}(\mathcal{R}, \mu)$ be the set of all μ measurable functions over $\mathcal{R}$. A Banach space $X = X(\mathcal{R}, \mu)$ of μ-measurable complex-valued functions in $\mathcal{M}(\mathcal{R}, \mu)$ (set of all μ measurable functions over $\mathcal{R}$), equipped with the norm $\| \cdot \|_X$, is said to be a *rearrangement-invariant Banach function space* (shortly *r.i. space*) over $(\mathcal{R}, \mu)$ (or over $\mathcal{R}$ with respect to μ) if the following five axioms hold:

(P1) $0 \leq g \leq f$ μ-a.e. implies $\|g\|_X \leq \|f\|_X$;

(P2) $0 \leq f_n \nearrow f$ μ-a.e. implies $\|f_n\|_X \nearrow \|f\|_X$;

(P3) $\|\chi_E\|_X < \infty$ for every $E \subset \mathcal{R}$ of finite measure;

(P4) for every $E \subset \mathcal{R}$ with $\mu(E) < \infty$ there exists a constant C_E such that $\int_E |f(x)| \, d\mu(x) \leq C_E \|f\|_X$ for every $f \in X$;

(P5) $\|f\|_X = \|g\|_X$ whenever $f_* = g_*$.

Given an r.i. space X on $(\mathcal{R}, \mu)$, the set

$$X' = \left\{ f \in \mathcal{M}(\mathcal{R}, \mu) : \int_{\mathcal{R}} |f(x)g(x)| \, d\mu(x) < \infty \text{ for every } g \in X \right\},$$

equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_{\mathcal{R}} |f(x)g(x)| \, d\mu(x) = \sup_{\|g\|_X \leq 1} \int_0^{\mu(\mathcal{R})} f_*(t)g_*(t) \, dt,$$

is called the *associate space* of X. It turns out that X' is again an r.i. space over $\mathcal{R}$ with respect to μ and that $X'' = X$.

For every r.i. space X over $(\mathcal{R}, \mu)$, there exists a unique r.i. space $\overline{X}$ over $(0, \infty)$ with respect to the one-dimensional Lebesgue measure, satisfying

$$\|f\|_X = \|f_*\|_{\overline{X}}$$

for every $f \in X$. This space, equipped with the norm

$$\|f\|_{\overline{X}} = \sup_{\|g\|_{X'} \leq 1} \int_0^{\mu(\mathcal{R})} f_*(t) g_*(t)\, dt,$$

is called the *representation space* of X.

The fundamental function of a r.i. Banach space, X, is defined by

$$\Phi_X(t) = \|\chi_{[0,t]}\|_X, \quad t \in (0, \mu(\mathcal{R})).$$

There is no loss of generality if we assume Φ_X to be positive, nondecreasing, absolutely continuous far from the origin, concave and to satisfy

$$\Phi_X(t)\,\Phi_{X'}(t) = t \quad \text{for all} \quad t \in (0, \mu(\mathcal{R})).$$

Details and further material on r.i. spaces can be found in [5, Chapter 2].

Note that (see [8]) grand Lebesgue spaces $L^{p),\alpha}(\Omega)$ and small Lebesgue spaces $L^{(p,\alpha}(\Omega)$ are r.i. spaces over Ω and

$$\Phi_{L^{p),\alpha}}(t) \approx t^{\frac{1}{p}}(1 - \operatorname{Log} t)^{\frac{-\alpha}{p}} \quad \text{and} \quad \Phi_{L^{(p,\alpha}}(t) \approx t^{\frac{1}{p}}(1 - \operatorname{Log} t)^{\frac{\alpha}{p'}}$$

$(L^{p),\alpha})' = L^{(p',\alpha}$.

Let A_0 and A_1 be two r.i. Banach spaces over $[0,1]$ and let Φ_0, Φ_1 be respectively their fundamental functions. We suppose that the following conditions are satisfied:

(i) There exists a constant C such that, for $i = 0; 1$ and for all $t > 0$

$$\int_0^t \frac{ds}{\Phi_i(s)} \leq \frac{Ct}{\Phi_i(t)}, \tag{C.0}$$

(ii) There exists a constant C such that, for all $t > 0$

$$\frac{\Phi_1(t)}{\Phi_0(t)} \left\| \frac{\chi_{[0,t]}}{\Phi_1} \right\|_{A_0} \leq C, \tag{C.1}$$

$$\frac{\Phi_0(t)}{\Phi_1(t)} \left\| \frac{\chi_{[t,1]}}{\Phi_0} \right\|_{A_1} \leq C. \tag{C.2}$$

Lemma 2.1 ([3, Lemma 2.2]). *Let A_0 and A_1 be two r.i. Banach function spaces satisfied* (C.0)–(C.2). *Then*

$$K\left(f, \frac{\Phi_0(t)}{\Phi_1(t)}; A_0, A_1\right) \approx \left\|f_*\chi_{[0,t]}\right\|_{A_0} + \frac{\Phi_0(t)}{\Phi_1(t)} \left\|f_*\chi_{[t,1]}\right\|_{A_1}, \quad 0 < t < 1.$$

Remarks on the choice of the method

We have chosen a direct method for the proof of our results by computing the K-functionals. In some part of the manuscript (for instance Theorem 5.1) we can adopt an alternative proof as using limiting reiteration theorems [1,9,12,13]. Although, we observe that it is not possible to get the result without computations. The feature of our method is that, as a byproduct, we make explicit the behavior of the K-functional.

3. Computation of some K-functionals and characterization of the interpolation spaces $(L^{p),\alpha}, L^{q),\alpha})_{\theta,r}$

We shall need few lemmas before reaching the proof of Theorems 1.1 and 1.2.

Lemma 3.1. *Let $\sigma = \frac{p}{p-1}$, $1 < p < +\infty$, $\alpha > 0$. Then*

$$\sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \int_0^{t^\sigma} f_*(s)ds \leqslant c_{\alpha p} \sup_{0<t<1} (1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \left(\int_t^1 f_*^p(s)ds \right)^{\frac{1}{p}}$$

$0 < t < 1$, *for some constant $c_{\alpha p} > 0$.*

Proof. We have

$$
\begin{aligned}
N_1 &\equiv \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \int_0^{t^\sigma} f_*(s)ds \\
&= \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \int_0^{t^\sigma} \left(\frac{2}{s} \right)^{\frac{1}{p}} \left(\int_{\frac{s}{2}}^s d\tau \right)^{\frac{1}{p}} f_*(s)ds \\
&\lesssim \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \int_0^{t^\sigma} s^{-\frac{1}{p}} \left(\int_{s/2}^s f_*^p(\tau)d\tau \right)^{\frac{1}{p}} ds \\
&\lesssim \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \left(\int_0^{t^\sigma} s^{-\frac{1}{p}} \left(1 - \text{Log}\, \frac{s}{2} \right)^{\frac{\alpha}{p}} ds \right) \\
&\quad \times \sup_{0<s<t} \left(1 - \text{Log}\, \frac{s}{2} \right)^{-\frac{\alpha}{p}} \left(\int_{s/2}^1 f_*^p(\tau)d\tau \right)^{\frac{1}{p}}.
\end{aligned}
$$

$$(6)$$

Applying Proposition 2.2, we have

$$\int_0^{t^\sigma} s^{-\frac{1}{p}} \left(1 - \text{Log}\, \frac{s}{2} \right)^{\frac{\alpha}{p}} ds \leqslant c_{\alpha p} t^{\sigma(1-\frac{1}{p})}(1 - \text{Log}\, t)^{\frac{\alpha}{p}}. \tag{7}$$

From the two last inequalities, we have:

$$N_1 \leqslant c_{\alpha p} \sup_{0<t<1} (1 - \text{Log}\, t)^{-\frac{\alpha}{p}} \left(\int_t^1 f_*^p \right)^{\frac{1}{p}} \approx \|f\|_{p),\alpha}. \quad \Diamond$$

The next lemma has been already proved (see Theorem 4.1 [15]) for a characterization of grand Lebesgue spaces as interpolation spaces between Lebesgue spaces, obtained using the Holmstedt's formula as well.

Lemma 3.2. *For $1 < p < +\infty$, $\alpha > 0$ we have*

$$L^{p),\alpha} = (L^1, L^p)_{1,\infty;-\frac{\alpha}{p}}.$$

Proof. According to the Holmstedt's formula (see [5]), we have for all $f \in L^1 + L^p$, and for all $t \in (0,1)$

$$K(f,t;L^1,L^p) \approx \int_0^{t^\sigma} f_*(s)ds + t \left(\int_{t^\sigma}^1 f_*^p(s)ds \right)^{\frac{1}{p}} \doteq K_{1p}(f,t) \quad \text{with } \sigma = \frac{p}{p-1}.$$

Making use of the definition of the norm in $(L^1, L^p)_{1,\infty;-\frac{\alpha}{p}}$

$$\|f\|_{1,\infty;-\frac{\alpha}{p}} = \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}} K_{1p}(f,t)$$

$$\approx \sup_{0<t<1} t^{-1}(1 - \operatorname{Log} t)^{-\frac{\alpha}{p}} \int_0^{t^\sigma} f_*(s)ds + \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\alpha}{p}} \left(\int_{t^\sigma}^1 f_*^p(s)ds \right)^{\frac{1}{p}}.$$

From Lemma 3.1, we deduce that

$$\|f\|_{1,\infty;-\frac{\alpha}{p}} \approx \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\alpha}{p}} \left(\int_t^1 f_*^p(s)ds \right)^{\frac{1}{p}} = \|f\|_{p),\alpha}.$$

Noticing $1 - \operatorname{Log} t \approx 1 - \operatorname{Log} t^{\frac{1}{\sigma}}$.   ◊

Lemma 3.3. *Let* $1 < p < q$, $\alpha > 0$. *Then*

$$\sup_{0<t<1} t^{\frac{1}{q}-\frac{1}{p}}(1 - \operatorname{Log} t)^{-\frac{\alpha}{q}} \left(\int_0^t f_*^p(s)ds \right)^{\frac{1}{p}} \leqslant c_{\alpha pq} \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\alpha}{q}} \left(\int_t^1 f_*^q(s)ds \right)^{\frac{1}{q}} \qquad (8)$$

for some constant $c_{\alpha pq} > 0$.

Proof. The relation (8) is equivalent to

$$\sup_{0<t<1} t^{-1}(1 - \operatorname{Log} t)^{-\frac{\alpha p}{q}} \left(\int_0^{t^\sigma} f_*(s)ds \right) \lesssim \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\alpha p}{q}} \left(\int_t^1 f_*(s)^{\frac{q}{p}} ds \right)^{\frac{p}{q}}$$

with $\sigma = \frac{r}{r-1}$, $r = \frac{q}{p} > 1$.

So we deduce Lemma 3.3 from Lemma 3.1, replacing p by r.   ◊

Theorem 3.1. *Let* $1 \leqslant p < q$, $\alpha > 0$. *Then*

$$L^{q),\alpha}(\Omega) = \left(L^p(\Omega), L^q(\Omega) \right)_{1,\infty;-\frac{\alpha}{q}}.$$

Proof of Theorem 3.1. According to Holmstedt's formula [5] we have for all $f \in L^p + L^q$, all $t \in (0,1)$

$$K(f,t;L^p,L^q) \approx \left(\int_0^{t^\sigma} f_*^p(s)ds \right)^{\frac{1}{p}} + t \left(\int_{t^\sigma}^1 f_*^q(s)ds \right)^{\frac{1}{q}} \doteq K_{pq}(f,t) \qquad (9)$$

with $\sigma = \frac{pq}{q-p}$.

According to the norm of f in $(L^p, L^q)_{1,\infty;-\frac{\alpha}{q}}$ we have:

$$\|f\|_{1,\infty;-\frac{\alpha}{q}} \approx \sup_{0<t<1} t^{-1}(1 - \operatorname{Log} t)^{-\frac{\alpha}{q}} K_{pq}(f,t)$$

$$\approx \sup_{0<t<1} t^{-1}(1 - \operatorname{Log} t)^{-\frac{\alpha}{q}} \left(\int_0^{t^\sigma} f_*^p(s)ds \right)^{\frac{1}{p}} + \sup_{0<t<1} (1 - \operatorname{Log} t)^{-\frac{\alpha}{q}} \left(\int_{t^\sigma}^1 f_*^q(s)ds \right)^{\frac{1}{q}}$$

$$= I_1 + I_2. \qquad (10)$$

The first term I_1 of relation (10) is equivalent to

$$\sup_{0<\tau<1} \tau^{\frac{1}{q}-\frac{1}{p}}(1 - \operatorname{Log} \tau)^{-\frac{\alpha}{q}} \left(\int_0^\tau f_*^p(s)ds \right)^{\frac{1}{p}},$$

(making use of the change of variables $\tau = t^\sigma$ and knowing as for I_2, $1 - \operatorname{Log} \tau^{\frac{1}{\sigma}} \approx 1 - \operatorname{Log} \tau$).

According Lemma 3.3, this last term is dominated by I_2.

Therefore we have

$$\|f\|_{1,\infty;-\frac{\alpha}{q}} \approx I_1 + I_2 \approx I_2 = \|f\|_{q),\alpha}.   ◊$$

Theorem 3.2 (*Computation of the K-functional for the Couple* $(L^{p),\alpha}, L^q)$). *Let* $1 < p < q$, $\alpha > 0$. *Then*

$$K(f,t; L^{p),\alpha}, L^q) \approx \sup_{0<s<\varphi(t)} (1 - \text{Log}\, s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(x)dx \right)^{\frac{1}{p}} + t \left(\int_{\varphi(t)}^1 f_*^q(s)ds \right)^{\frac{1}{q}}$$

$$= \overline{K}_{pq}(f,t)$$

where φ *is the inverse of the increasing function* $\psi(t) = t^{\frac{1}{p} - \frac{1}{q}}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}}$, $t \in (0,1)$. *Thus*

$$t = \varphi(t)^{\frac{1}{p} - \frac{1}{q}}(1 - \text{Log}\, \varphi(t))^{-\frac{\alpha}{p}}.$$

Proof of Theorem 3.2. To apply Lemma 2.1, we need to check that the conditions (C.0)–(C.2) are satisfied. Let $\Phi_0(t) = \left\|\chi_{[0,t]}\right\|_{L^{p),\alpha}}$ and $\Phi_1(t) = \left\|\chi_{[0,t]}\right\|_{L^q}$. As we have

$$\Phi_0(t) \approx t^{\frac{1}{p}}(1 - \text{Log}\, t)^{-\frac{\alpha}{p}}, \quad \Phi_1(t) \approx t^{\frac{1}{q}},$$

we have that $\psi(t) = \frac{\Phi_0(t)}{\Phi_1(t)}$. The conditions (C.0)–(C.2) easily follow by using Propositions 2.2 and 2.3. and using Lemma 2.1 we get desired result. $\diamondsuit$

By same way we can obtain following theorem.

Theorem 3.3 (*Computation of the K-functional for the Couple* $(L^{p),\alpha}, L^{q),\alpha})$). *Let* $1 < p < q$, $\alpha > 0$. *Then*

$$K(f,t; L^{p),\alpha}, L^{q),\alpha}) \approx \sup_{0<s<\varphi(t)} (1 - \text{Log}\, s)^{-\frac{\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(x)dx \right)^{\frac{1}{p}}$$

$$+ t \sup_{\varphi(t)<s<1} (1 - \text{Log}\, s)^{-\frac{\alpha}{q}} \left(\int_s^1 f_*^q(x)dx \right)^{\frac{1}{q}},$$

where φ *is the inverse of the increasing function* $\psi(t) = t^{\frac{1}{p} - \frac{1}{q}}(1 - \text{Log}\, t)^{-\frac{\alpha}{p} + \frac{\alpha}{q}}$, $t \in (0,1)$. *Thus*

$$t = \varphi(t)^{\frac{1}{p} - \frac{1}{q}}(1 - \text{Log}\, \varphi(t))^{-\frac{\alpha}{p} + \frac{\alpha}{q}}.$$

Proof of Theorem 1.1. Let f be in $(L^{p),\alpha}, L^q)_{1,\infty;-\frac{\alpha}{q}}$. Then, its norm is

$$\|f\|_{1,\infty;-\frac{\alpha}{q}} = \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{\frac{-\alpha}{q}} K(f,t; L^{p),\alpha}, L^q).$$

Following Theorem 3.2, this expression gives:

$$\|f\|_{1,\infty;-\frac{\alpha}{q}} \approx \sup_{0<t<1} t^{-1}(1 - \text{Log}\, t)^{\frac{-\alpha}{q}} \sup_{0<s<\varphi(t)} (1 - \text{Log}\, s)^{\frac{-\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(\tau)d\tau \right)^{\frac{1}{p}}$$

$$+ \sup_{0<t<1} (1 - \text{Log}\, t)^{\frac{-\alpha}{q}} \left(\int_{\varphi(t)}^1 f_*^q(\tau)d\tau \right)^{\frac{1}{q}} = J_1 + J_2. \tag{11}$$

But we have by the definition of φ,

$$t^{-1} = \varphi(t)^{\frac{1}{q} - \frac{1}{p}}(1 + |\text{Log}\, \varphi(t)|)^{\frac{\alpha}{p}}, \quad 1 + |\text{Log}\, \varphi| \approx 1 + |\text{Log}\, t|.$$

Thus the first term J_1 of (11), after a change of variables, gives

$$J_1 \approx \sup_{0<t<1} t^{\frac{p-q}{qp}}(1 - \text{Log}\, t)^{\frac{\alpha(q-p)}{pq}} \sup_{0<s<t} (1 - \text{Log}\, s)^{\frac{-\alpha}{p}} \left(\int_s^t f_*^p(\tau)d\tau \right)^{\frac{1}{p}}. \tag{12}$$

But we can bound the last term of (12) as:

$$\sup_{0<s<t} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{p}} \left(\int_s^t f_*^p(\tau)d\tau \right)^{\frac{1}{p}} \leqslant t^{\frac{q-p}{pq}} \sup_{0<s<t}(1-\mathrm{Log}\,s)^{\frac{\alpha(p-q)}{qp}} \sup_{0<s<t}(1-\mathrm{Log}\,s)^{\frac{-\alpha}{q}} \left(\int_s^t f_*^q(\tau)d\tau \right)^{\frac{1}{q}}$$

(using Hölder inequality and introducing new factor).

Therefore, we can estimate J_1 (after simplifying):

$$J_1 \lesssim \sup_{0<s<1} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{q}} \left(\int_s^1 f_*^q(\tau)d\tau \right)^{\frac{1}{q}} = \|f\|_{q),\alpha}. \tag{13}$$

While for the second term J_2, we have:

$$J_2 \approx \sup_{0<t<1} (1 - \mathrm{Log}\, \varphi(t))^{\frac{-\alpha}{q}} \left(\int_{\varphi(t)}^1 f_*^q(\tau)d\tau \right)^{\frac{1}{q}} \tag{14}$$

which implies

$$J_2 \approx \sup_{0<s<1} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{q}} \left(\int_s^1 f_*^q(\tau)d\tau \right)^{\frac{1}{q}} = \|f\|_{q),\alpha}. \tag{15}$$

We then have from relations (11) to (15)

$$J_2 \lesssim \|f\|_{1,\infty;-\frac{\alpha}{q}} \approx J_1 + J_2. \tag{16}$$

This shows the result. ◊

Next, we want to prove Theorem 1.2.
For this, we shall need the:

Lemma 3.4. *Let* $1 < p < q,\ \alpha > 0,\ 0 < \theta < 1,\ 1 \leqslant r < +\infty.$ *Then*
$$\left(L^{p),\alpha}, L^{q),\alpha} \right)_{\theta,r} = \left(L^{p),\alpha}, L^q \right)_{\theta,r;-\frac{\alpha\theta}{q}}.$$

Proof. From Theorem 1.1, we know that
$$\left(L^{p),\alpha}, L^{q),\alpha} \right)_{\theta,r} = \left(L^{p),\alpha}, \left(L^{p),\alpha}, L^q \right)_{1,\infty;-\frac{\alpha}{q}} \right)_{\theta,r}.$$

In this case, we can apply the Holmstedt formula (see Theorem 3.1 in [18]) valid also for the extreme case $\theta_1 = 1$.

$$K\left(f, t; L^{p),\alpha}, (L^{p),\alpha}, L^q)_{1,\infty;-\frac{\alpha}{q}} \right) \doteq K_\alpha(t) \approx t \sup_{\psi(t)<s<1} s^{-1}(1-\mathrm{Log}\,s)^{\frac{-\alpha}{q}} K(f,s;L^{p),\alpha}, L^q) \tag{17}$$

where ψ is an invertible function from $[0,1]$ into itself according to Proposition 2.1 so that

$$t \approx \psi(t)(1 - \mathrm{Log}\, \psi(t))^{\frac{\alpha}{q}} \qquad \text{for } t \in [0,1].$$

Considering the norm $\|f\|_{\theta,r}$ of f in $\left(L^{p),\alpha}, L^{q),\alpha} \right)_{\theta,r}$ we have:

$$\begin{aligned}
\|f\|_{\theta,r} &\approx \left\| t^{-\theta} K_\alpha(t) \right\|_{L^r(0,1;\frac{dt}{t})} \\
&\approx \left\| t^{1-\theta} \sup_{\psi(t)<s<1} s^{-1}(1-\mathrm{Log}\,s)^{\frac{-\alpha}{q}} K(f,s;L^{p),\alpha}, L^q) \right\|_{L^r(0,1;\frac{dt}{t})} \\
&\approx \left\| \psi(t)^{1-\theta}(1-\mathrm{Log}\,\psi(t))^{\frac{\alpha(1-\theta)}{q}} \sup_{\psi(t)<s<1} s^{-1}(1-\mathrm{Log}\,s)^{\frac{-\alpha}{q}} K(f,s;L^{p),\alpha}, L^q) \right\|_{L^r(\frac{dt}{t})} \\
&\approx \left\| t^{1-\theta}(1-\mathrm{Log}\,t)^{\frac{\alpha(1-\theta)}{q}} \sup_{t<s<1} s^{-1}(1-\mathrm{Log}\,s)^{\frac{-\alpha}{q}} K(f,s;L^{p),\alpha}, L^q) \right\|_{L^r(\frac{dt}{t})}.
\end{aligned}$$

The last line is obtained after the change of variables $t \to \psi(t)$. To control this last term, we need

Proposition 3.1. *Let K_d be a decreasing nonnegative function on $(0,1)$ and p, q, α, θ, r are as in [Lemma](#) 3.4. Then*

$$I_r \dot{=} \int_0^1 \left[t^{1-\theta}(1 - \text{Log}\, t)^{\frac{\alpha(1-\theta)}{q}} \sup_{t<s<1} (1 - \text{Log}\, s)^{\frac{-\alpha}{q}} K_d(s) \right]^r \frac{dt}{t}$$

$$\approx \int_0^1 \left[t^{1-\theta}(1 - \text{Log}\, t)^{\frac{-\alpha\theta}{q}} K_d(t) \right]^r \frac{dt}{t} \dot{=} I_d.$$

Proof. Since $\sup_{t<s<1}(1 - \text{Log}\, s)^{\frac{-\alpha}{q}} K_d(s) \geqslant (1 - \text{Log}\, t)^{\frac{-\alpha}{q}} K_d(t)$, we deduce

$$I_r \geqslant \int_0^1 \left[t^{1-\theta}(1 - \text{Log}\, t)^{\frac{-\alpha\theta}{q}} K_d(t) \right]^r \frac{dt}{t} = I_d.$$

For the reverse inequality, since K_d is decreasing and $s \to (1 - \text{Log}\, s)^{\frac{-\alpha r}{q}}$ increasing then,

$$(1 - \text{Log}\, s)^{\frac{-\alpha r}{q}} K_d^r(s) \lesssim \int_{s/2}^s (1 - \text{Log}\, x)^{\frac{-\alpha r}{q}} K_d^r(x) \frac{dx}{x} \lesssim (1 - \text{Log}\, \frac{s}{2})^{\frac{-\alpha r}{q}} K_d^r \left(\frac{s}{2} \right). \tag{18}$$

Setting

$$I_{1r} = \int_0^1 t^{(1-\theta)r}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} \sup_{t<s<1} \left(\int_{s/2}^s (1 - \text{Log}\, x)^{\frac{-\alpha r}{q}} K_d^r(x) \frac{dx}{x} \right) \frac{dt}{t},$$

thus, we can bound, the integral I_r by I_{1r}:

$$I_r = \int_0^1 t^{(1-\theta)r}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} \sup_{t<s<1} (1 - \text{Log}\, s)^{\frac{-\alpha r}{q}} K_d^r(s) \frac{dt}{t} \lesssim I_{1r} \tag{19}$$

and

$$I_{1r} \lesssim \int_0^1 t^{(1-\theta)r}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} \sup_{t<s<1} (1 - \text{Log}\, \frac{s}{2})^{\frac{-\alpha r}{q}} K_d^r \left(\frac{s}{2} \right) \frac{dt}{t} = I_{2r}.$$

By a change of variables in the term I_{2r}, we have

$$I_{2r} \lesssim I_r. \tag{20}$$

Therefore, relations (19) and (20) imply:

$$I_r \approx I_{1r}.$$

We may bound I_{1r} as:

$$I_{1r} \leqslant \int_0^1 t^{(1-\theta)r}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} \left(\int_{t/2}^1 (1 - \text{Log}\, x)^{\frac{-\alpha r}{q}} K_d^r(x) \frac{dx}{x} \right) \frac{dt}{t}.$$

By Fubini's Theorem, the upperbound in this inequality gives:

$$I_{1r} \lesssim \int_0^1 (1 - \text{Log}\, x)^{\frac{-\alpha r}{q}} K_d^r(x) \left(\int_0^{2x} t^{(1-\theta)r}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} \frac{dt}{t} \right) \frac{dx}{x}.$$

From [Proposition](#) 2.2, we have

$$\int_0^{2x} t^{(1-\theta)r-1}(1 - \text{Log}\, t)^{\frac{\alpha r(1-\theta)}{q}} dt \leqslant c_{0q\alpha} x^{(1-\theta)r}(1 - \text{Log}\, x)^{\frac{\alpha r(1-\theta)}{q}}. \tag{21}$$

From relation (21), the last estimate I_{1r} becomes

$$I_{1r} \lesssim \int_0^1 x^{(1-\theta)r}(1 - \mathrm{Log}\, x)^{\frac{-\alpha r\theta}{q}} K_d^r(x)\frac{dx}{x} = I_d.$$

With relation (19) we get:

$$I_r \lesssim I_{1r} \lesssim I_d \leqslant I_r. \quad \Diamond$$

End of proof of Lemma 3.4

We apply Proposition 3.1 with

$$K_d(t) = t^{-1}K(f, t; L^{p),\alpha}, L^q), \ t \in (0, 1)$$

to derive the:

$$\|f\|_{\theta,r} \approx \left\| t^{1-\theta}(1 - \mathrm{Log}\, t)^{\frac{\alpha(1-\theta)}{q}} \sup_{t<s<1} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{q}} K_d(s) \right\|_{L^r(\frac{dt}{t})}$$

$$\approx I_d^{\frac{1}{r}} = \|f\|_V \text{ with } V = \left(L^{p),\alpha}, L^q\right)_{\theta,r;-\frac{\alpha\theta}{q}}. \quad \Diamond$$

Proof of Theorem 1.2. Let

$$V = \left(L^{p),\alpha}, L^{q),\alpha}\right)_{\theta,r} \quad \text{and} \quad L^{p_\theta,r}(\mathrm{Log}\, L)^{-\frac{\alpha}{p_\theta}} = V_\theta$$

with $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}$.

We recall that, for $f \in V$

$$\|f\|_V^r = \int_0^1 \left[t^{-\theta}(1 - \mathrm{Log}\, t)^{\frac{-\alpha\theta}{q}} K(f, t; L^{p),\alpha}, L^{q),\alpha}) \right]^r \frac{dt}{t}. \tag{22}$$

Applying Theorem 3.3, we have

$$\|f\|_V^r \approx \int_0^1 \left[t^{-\theta} \sup_{0<s<\varphi(t)} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{p}} \left(\int_s^{\varphi(t)} f_*^p(\tau)d\tau \right)^{\frac{1}{p}} \right]^r \frac{dt}{t}$$

$$+ \int_0^1 \left[t^{1-\theta} \sup_{\varphi(t)<s<1} (1 - \mathrm{Log}\, s)^{\frac{-\alpha}{q}} \left(\int_s^1 f_*^q(\tau)d\tau \right)^{\frac{1}{q}} \right]^r \frac{dt}{t}$$

$$= II_1 + II_2.$$

From the first term II_1, we know that

$$t = \varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1 - \mathrm{Log}\,\varphi(t))^{\frac{-\alpha}{p}+\frac{\alpha}{q}}$$

and

$$1 - \mathrm{Log}\, t \approx 1 - \mathrm{Log}\,\varphi(t).$$

Therefore, we have

$$t^{-\theta} \approx \varphi(t)^{\theta(\frac{1}{q}-\frac{1}{p})}(1 - \mathrm{Log}\,\varphi(t))^{\frac{-\alpha\theta(p-q)}{qp}}. \tag{23}$$

Replacing the integrand in the integral by the II_1 of (23) and making the change of variables $x = \varphi(t)$ or equivalently $t = \psi(x)$, we have

$$II_1 \approx \int_0^1 \left[x^{\theta(\frac{1}{q}-\frac{1}{p})}(1-\text{Log}\,x)^{\frac{-\alpha\theta(p-q)}{qp}} \sup_{0<s<x} (1-\text{Log}\,s)^{\frac{-\alpha}{p}} \left(\int_s^x f_*^p(\tau)d\tau \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}.$$

Thus the monotony of f_* leads to:

$$II_1 \gtrsim \int_0^1 \left[x^{\theta(\frac{1}{q}-\frac{1}{p})}(1-\text{Log}\,x)^{\frac{-\alpha\theta(p-q)}{qp}} \left(1-\text{Log}\,\frac{x}{2}\right)^{\frac{-\alpha}{p}} x^{\frac{1}{p}} f_*(x) \right]^r \frac{dx}{x}$$

$$II_1 \gtrsim \int_0^1 \left[t^{\frac{1-\theta}{p}+\frac{\theta}{q}}(1-\text{Log}\,t)^{\frac{-\alpha(q(1-\theta)+\theta p)}{pq}} f_*(t) \right]^r \frac{dt}{t} = \|f\|_{V_\theta}^r. \tag{24}$$

In particular, we have shown that

$$\|f\|_V \gtrsim \|f\|_{V_\theta}. \tag{25}$$

For the reverse of relation (25), let us consider $\varepsilon > 0$ small enough so that

$$\theta\left(\frac{1}{q}-\frac{1}{p}\right) + \frac{\varepsilon}{p} < 0.$$

$$II_1 \lesssim \int_0^1 \left[x^{\theta(\frac{1}{q}-\frac{1}{p})}(1-\text{Log}\,x)^{\frac{-\alpha\theta(p-q)}{qp}} \sup_{0<s<x} (1-\text{Log}\,s)^{\frac{-\alpha}{p}} \left(\int_s^x t^{\varepsilon-1}t^{1-\varepsilon} f_*^p(t)dt \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}.$$

Then

$$II_1 \lesssim \int_0^1 \left[x^{\theta(\frac{1}{q}-\frac{1}{p})+\frac{\varepsilon}{p}}(1-\text{Log}\,x)^{\frac{-\alpha}{p\theta}} \sup_{0<t<x} t^{\frac{1-\varepsilon}{p}} f_*(t) \right]^r \frac{dx}{x}.$$

We have the following

Proposition 3.2. *For all $x > 0$*

$$\sup_{0<t<x} t^{\frac{1-\varepsilon}{p}} f_*(t) \leqslant 2(\text{Log}\,2)^{\frac{1}{r'}} \left(\int_0^x s^{r\frac{1-\varepsilon}{p}-1} f_*^r(s)ds \right)^{\frac{1}{r}},$$

with $\frac{1}{r'} + \frac{1}{r} = 1$.

Proof. For any $s \in \left[\frac{t}{2}, t\right]$, we have

$$t^{\frac{1-\varepsilon}{p}-1} f_*(t) \leqslant s^{\frac{1-\varepsilon}{p}-1} f_*(s),$$

since $\frac{1}{p} - 1 - \frac{\varepsilon}{p} < 0$.

Integrating between $\frac{t}{2}$ and t, and using Hölder inequality, we have

$$t^{\frac{1-\varepsilon}{p}} f_*(t) \leqslant 2 \left(\int_{t/2}^t s^{\frac{1-\varepsilon}{p}r-1} f_*^r(s)ds \right)^{\frac{1}{r}} (\text{Log}\,2)^{\frac{1}{r'}}$$

$$t^{\frac{1-\varepsilon}{p}} f_*(t) \leqslant 2 \left(\int_0^t s^{\frac{1-\varepsilon}{p}r-1} f_*^r(s)ds \right)^{\frac{1}{r}} (\text{Log}\,2)^{\frac{1}{r'}}.$$

From which, we derive the result. $\diamond$

Using this Proposition 3.2, setting $m = \theta \left(\frac{1}{q} - \frac{1}{p} \right) + \frac{\varepsilon}{p} < 0$,

$$II_1 \lesssim \int_0^1 x^{mr-1} (1 - \text{Log } x)^{\frac{-\alpha r}{p_\theta}} \left(\int_0^x s^{\frac{1-\varepsilon}{p} r - 1} f_*^r(s) ds \right) dx. \tag{26}$$

Applying Hardy type inequality (see Theorem 2.1 and [4], Theorem 6.4), relation (26) becomes

$$II_1 \lesssim \int_0^1 \left[s^{\frac{1}{p_\theta}} (1 - \text{Log } s)^{\frac{-\alpha}{p_\theta}} f_*(s) \right]^r \frac{ds}{s} = \|f\|^r_{L^{p_\theta, r} (\text{Log } L)^{-\frac{\alpha}{p_\theta}}}. \tag{27}$$

It remains to show that the second term satisfies:

$$II_2 = \int_0^1 \left[t^{1-\theta} \sup \varphi(t) < s < 1 (1 - \text{Log } s)^{\frac{-\alpha \theta}{q}} \left(\int_s^1 f_*^q(\tau) d\tau \right)^{\frac{1}{q}} \right]^r \frac{dt}{t} \lesssim \|f\|^r_{V_\theta}.$$

For this, we recall that φ is the inverse of the function $\psi(x) = x^{\frac{1}{p} - \frac{1}{q}} (1 - \text{Log } x)^{\frac{-\alpha}{p} + \frac{\alpha}{q}}$, $x \in [0, 1]$ so we have the following relations:

$$t = \varphi(t)^{\frac{1}{p} - \frac{1}{q}} (1 - \text{Log } \varphi(t))^{\frac{-\alpha}{p} + \frac{\alpha}{q}},$$

$$1 - \text{Log } \varphi(t) \approx 1 - \text{Log } t,$$

and

$$\frac{\psi'(x)}{\psi(x)} \approx \frac{1}{x}.$$

Therefore, we rewrite II_2 as:

$$II_2 \approx \int_0^1 \left[\varphi(t)^{(1-\theta)(\frac{1}{p} - \frac{1}{q})} (1 - \text{Log } \varphi(t))^{\frac{-\alpha(1-\theta)}{p} + \frac{\alpha(1-\theta)}{q}} \sup_{\varphi(t) < s < 1} (1 - \text{Log } s)^{\frac{-\alpha}{q}} \left(\int_s^1 f_*^q(\tau) d\tau \right)^{\frac{1}{q}} \right]^r \frac{dt}{t}.$$

Making the change of variables, $t = \psi(x) \iff x = \varphi(t)$

$$II_2 \approx \int_0^1 \left[x^{(1-\theta)(\frac{1}{p} - \frac{1}{q})} (1 - \text{Log } x)^{\frac{-\alpha(1-\theta)}{p} + \frac{\alpha(1-\theta)}{q}} \sup_{x < t < 1} (1 - \text{Log } t)^{\frac{-\alpha}{q}} \left(\int_t^1 f_*^q(s) ds \right)^{\frac{1}{q}} \right]^r \frac{dx}{x}. \tag{28}$$

Applying Proposition 3.1, we have:

$$\int_0^1 \left[x^{(1-\theta)(\frac{1}{p} - \frac{1}{q})} (1 - \text{Log } x)^{\frac{-\alpha(1-\theta)}{p} + \frac{\alpha(1-\theta)}{q}} \sup_{x < t < 1} (1 - \text{Log } t)^{\frac{-\alpha}{q}} \left(\int_t^1 f_*^q(s) ds \right)^{\frac{1}{q}} \right]^r \frac{dx}{x}$$

$$\leqslant c \int_0^1 \left[x^{\frac{1}{p_\theta}} (1 - \text{Log } x)^{\frac{-\alpha}{p_\theta}} f_*(x) \right]^r \frac{dx}{x} \equiv c \|f\|^r_{V_\theta}. \quad \diamond$$

From the preceding results, we can characterize the interpolation spaces for small Lebesgue spaces.

Theorem 3.4. *Let $0 < \theta < 1$, $1 < r < +\infty$, $1 < p < q$, $\alpha > 0$. Then*

$$\left(L^{(p, \alpha)}, L^{(q, \alpha)} \right)_{\theta, r} = L^{p_\theta, r} (\text{Log } L)^{\frac{\alpha}{p'_\theta}}$$

with $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}$, $\frac{1}{p_\theta} + \frac{1}{p'_\theta} = 1$.

Proof. By the duality result on interpolation spaces, (see [7,21]), we have,

$$\left[\left(L^{(p,\alpha)}, L^{(q,\alpha)}\right)_{\theta,r}\right]' = \left(L^{p'),\alpha}, L^{q'),\alpha}\right)_{\theta,r'} = \left(L^{q'),\alpha}, L^{p'),\alpha}\right)_{1-\theta,r'} \tag{29}$$

and noticing $\left(L^{p'),\alpha}\right)' = L^{(p,\alpha}$, $L^\infty(\Omega)$ is in $L^{(p,\alpha} \cap L^{(q,\alpha}$ and dense in each of these spaces.

From Theorem 1.2, since $q' < p'$, we have

$$\left(L^{q'),\alpha}, L^{p'),\alpha}\right)_{1-\theta,r'} = L^{q'_{1-\theta},r'}(\mathrm{Log}\, L)^{-\frac{\alpha}{q'_{1-\theta}}}, \tag{30}$$

with $\frac{1}{q'_{1-\theta}} = \frac{\theta}{q'} + \frac{1-\theta}{p'} = 1 - \frac{1}{p_\theta} = \frac{1}{p'_\theta}$.

Thus taking the associate space in Eq. (30) gives, taking into account (29),

$$\left(L^{(p,\alpha)}, L^{(q,\alpha)}\right)_{\theta,r} = \left[(L^{(p,\alpha)}, L^{(q,\alpha)})_{\theta,r}\right]'' = L^{p_\theta,r}(\mathrm{Log}\, L)^{\frac{\alpha}{p'_\theta}}$$

with $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}$, $\frac{1}{p_\theta} + \frac{1}{p'_\theta} = 1$. $\Diamond$

A main consequence of Theorems 1.2 and 3.4 is the

Theorem 3.5. *Let $1 < a < +\infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$, $1 < r < +\infty$. Then the Lorentz–Zygmund space $L^{a,r}(\mathrm{Log}\, L)^\beta$ is an interpolation space in the sense of Peetre of two Grand Lebesgue spaces if $\beta < 0$ and of two small Lebesgue spaces if $\beta > 0$.*

Remark 3.1. If $\beta = 0$, it is already known that the Lorentz space $L^{a,r}(\Omega)$ is an interpolation space of two classical Lebesgue spaces.

4. Small Lebesgue space as interpolation of usual Lebesgue spaces

The next proposition has been already proved on [9], we drop its proofs:

Proposition 4.1 ([9]). *Let $1 < p < +\infty$, $\alpha > 0$. Then*

$$L^{(p,\alpha} = \left(L^p, L^\infty\right)_{0,1;-\frac{\alpha}{p}+\alpha-1}.$$

Proposition 4.2. *Let $1 < p < q$, $\alpha > 0$. Then*

$$L^{(p,\alpha} = \left(L^p, L^q\right)_{0,1;-1+\alpha-\frac{\alpha}{p}}.$$

Proof. It is similar to Proposition 4.1. Indeed, let us set W_{pq} the space on the RHS, then for $f \in W_{pq}$

$$\|f\|_{W_{pq}} = \int_0^1 (1 - \mathrm{Log}\, t)^{-1-\frac{\alpha}{p}+\alpha} K(f,t; L^p, L^q) \frac{dt}{t}.$$

By the Holmstedt formula, we have

$$K(f,t; L^p, L^q) \approx \left(\int_0^{t^\sigma} f_*^p(s)ds\right)^{\frac{1}{p}} + t\left(\int_{t^\sigma}^1 f_*^q(s)ds\right)^{\frac{1}{q}} \text{ with } \frac{1}{\sigma} = \frac{q-p}{pq}.$$

Then, we deduce the equivalent expression of the norm:

$$\|f\|_{W_{pq}} \approx \|f\|_{L^{(p,\alpha}} + \int_0^1 (1 - \mathrm{Log}\, t)^{-1 - \frac{\alpha}{p} + \alpha} t^{\frac{1}{\sigma}} \left(\int_t^1 f_*^q(s) ds \right)^{\frac{1}{q}} \frac{dt}{t}. \tag{31}$$

Let us show that the last term in relation (31) is less or equal to the norm of f in $L^{(p,\alpha}$. Let us temporarily set $\beta = -1 + \alpha - \frac{\alpha}{p}$, let $1 < \varepsilon < \frac{q}{\sigma} + 1$, then

$$O_1 = \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta q} t^{\frac{q}{\sigma} - \varepsilon + 1} t^{\varepsilon - 1} \int_t^1 f_*^q(x) dx \right]^{\frac{1}{q}} \frac{dt}{t}$$

$$\leqslant \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta q} t^{\frac{q}{\sigma} - \varepsilon + 1} \int_t^1 x^{\varepsilon - 1} f_*^q(x) dx \right]^{\frac{1}{q}} \frac{dt}{t}. \tag{32}$$

Applying Hardy inequality (see Theorem 2.1 and [4] Theorem 6.4), we obtain from the relation (32)

$$O_1 \lesssim \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta} t^{\frac{1}{\sigma} + \frac{1}{q}} f_*(t) \right] \frac{dt}{t}. \tag{33}$$

But $\frac{1}{\sigma} + \frac{1}{q} = \frac{1}{p}$ then $t^{\frac{1}{\sigma} + \frac{1}{q}} f_*(t) \leqslant \left(\int_0^t f_*^p(x) dx \right)^{\frac{1}{p}}$. So we derive

$$O_1 \lesssim \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\beta} \left(\int_0^t f_*^p(x) dx \right)^{\frac{1}{p}} \right] \frac{dt}{t} = \|f\|_{L^{(p,\alpha}}. \tag{34}$$

Thus, we have from relations (31) to (34):

$$\|f\|_{L^{(p,\alpha}} \lesssim \|f\|_{W_{pq}} \lesssim \|f\|_{L^{(p,\alpha}}. \quad \lozenge$$

5. Interpolation between small, Grand Lebesgue spaces and the associated K-functional

In this section, we want to determine the interpolation space $\left(L^{(p,\alpha}, L^{q),\beta} \right)_{\theta,r}$. Due to the technical aspect of the proof, we shall only consider the case where $\alpha = \beta = 1$, the argument remains the same in the general case.

We want to show the following theorem:

Theorem 5.1. *Let $0 < \theta < 1$, $1 \leqslant r < +\infty$, $p < q$. Then*

$$\left(L^{(p}, L^{q)} \right)_{\theta,r} = L^{p_\theta, r} \left(\mathrm{Log}\, L \right)^{\alpha_\theta}$$

where $\frac{1}{p_\theta} = \frac{1 - \theta}{p} + \frac{\theta}{q}$, $\alpha_\theta = 1 - \theta - \frac{1}{p_\theta}$.

We shall need the following K-functional.

Theorem 5.2. *Let $1 < p < q < +\infty$. Then for all $t > 0$, $f \in L^{(p} + L^{q)}$*

$$K(f, t; L^{(p}, L^{q)}) \approx \int_0^{\varphi(t)} (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_0^s f_*^p(\tau) d\tau \right)^{\frac{1}{p}} \frac{ds}{s} + (1 - \mathrm{Log}\, t)^{\frac{p-1}{p}} \left(\int_0^{\varphi(t)} f_*^p(\tau) d\tau \right)^{\frac{1}{p}}$$

$$+ t \sup_{\varphi(t) < s < 1} (1 - \mathrm{Log}\, s)^{\frac{-1}{q}} \left(\int_s^1 f_*^q(\tau) d\tau \right)^{\frac{1}{q}}$$

$$\doteq \sum_{i=1}^3 K_i(t)$$

where φ is an invertible function from $[0,1]$ into itself satisfying the equivalence

$$\varphi(t)^{\frac{1}{p}-\frac{1}{q}}(1 - \mathrm{Log}\,\varphi(t))^{\frac{p-q+pq}{pq}} \approx t.$$

Proof. We can apply Lemma 2.1 for $A_0 = L^{(p}$ and $A_1 = L^{q)}$. Let $\Phi_0(t) = \big\|\chi_{[0,t]}\big\|_{L^{(p}}$ and $\Phi_1(t) = \big\|\chi_{[0,t]}\big\|_{L^{q)}}$. As we have

$$\Phi_0(t) \approx t^{\frac{1}{p}}(1 - \mathrm{Log}\,t)^{\frac{1}{p'}}, \quad \Phi_1(t) \approx t^{\frac{1}{q}}(1 - \mathrm{Log}\,t)^{-\frac{1}{q}},$$

we have that $\psi(t) = \frac{\Phi_0(t)}{\Phi_1(t)}$. As $p < q$ the conditions (C.0)–(C.2) easily follow by using Propositions 2.2 and 2.3. and from Lemma 2.1 we obtain the result. $\Diamond$

Next, we want to prove Theorem 5.1. We need to show that the norm of f in $W \doteq (L^{(p}, L^{q)})_{\theta,r}$ is equivalent to its norm in $V = L^{p_\theta, r}(\mathrm{Log}\,L)^{\alpha_\theta}$.

For commodity, we set

$$\lambda = \theta\left(\frac{1}{p} - \frac{1}{q}\right), \quad \alpha_\theta = 1 - \theta - \frac{1}{p_\theta}, \quad a = \lambda - \theta, \quad \lambda_1 = \frac{1}{p_\theta} - \frac{1}{q} = (1-\theta)\left(\frac{1}{p} - \frac{1}{q}\right).$$

According to Theorem 5.2, the expression of the norm of f in W is composed with 3 terms:

$$\|f\|_W^r = \int_0^1 \left(t^{-\theta}K(f,t;L^{(p}, L^{q)})\right)^r \frac{dt}{t} \approx N_1 + N_2 + N_3$$

with N_i corresponding to the function K_i and which reads as follows after a change of variables

$$N_1 = \int_0^1 \left[t^{-\lambda}(1 - \mathrm{Log}\,t)^a \int_0^t (1 - \mathrm{Log}\,s)^{\frac{-1}{p}}\left(\int_0^s f_*^p(x)dx\right)^{\frac{1}{p}}\frac{ds}{s}\right]^r \frac{dt}{t},$$

$$N_2 = \int_0^1 \left[t^{-\lambda}(1 - \mathrm{Log}\,t)^{\alpha_\theta}\left(\int_0^t f_*^p(x)dx\right)^{\frac{1}{p}}\right]^r \frac{dt}{t},$$

$$N_3 = \int_0^1 \left[t^{\lambda_1}(1 - \mathrm{Log}\,t)^{\alpha_\theta + \frac{1}{q}} \sup_{t \leqslant s < 1}(1 - \mathrm{Log}\,s)^{\frac{-1}{q}}\left(\int_s^1 f_*^q(x)dx\right)^{\frac{1}{q}}\right]^r \frac{dt}{t}.$$

We start with a lower bound for $\|f\|_W$.

Lemma 5.1. *One has*

$$\|f\|_W^r \gtrsim N_3 \gtrsim \|f\|_V^r = \int_0^1 \left[t^{\frac{1}{p_\theta}}(1 - \mathrm{Log}\,t)^{\alpha_\theta} f_*(t)\right]^r \frac{dt}{t}.$$

Proof. One has

$$N_3 \geqslant \int_0^1 \left[t^{\lambda_1}(1 - \mathrm{Log}\,t)^{\alpha_\theta}\left(\int_t^1 f_*^q(x)dx\right)^{\frac{1}{q}}\right]^r \frac{dt}{t}$$

$$\geqslant \int_0^{\frac{1}{2}} \left[t^{\lambda_1}(1 - \mathrm{Log}\,t)^{\alpha_\theta}\left(\int_t^{2t} f_*^q(x)dx\right)^{\frac{1}{q}}\right]^r \frac{dt}{t}$$

$$\gtrsim \int_0^{\frac{1}{2}} \left[t^{\lambda_1 + \frac{1}{q}}(1 - \mathrm{Log}\,t)^{\alpha_\theta} f_*(2t)\right]^r \frac{dt}{t}.$$

Thus, we have, after a change of variables

$$\int_0^1 \left[t^{\frac{1}{p_\theta}}(1 - \mathrm{Log}\,t)^{\alpha_\theta} f_*(t)\right]^r \frac{dt}{t} \lesssim N_3 \lesssim \|f\|_W^r. \quad \Diamond$$

For the upper bound, we start with the estimate of N_2.

Lemma 5.2. *One has*

$$N_2 \lesssim \|f\|_V^r.$$

Proof. We set $b = \frac{r}{p} \in]0, +\infty[$, $\beta = p\alpha_\theta$, we note that $0 < \lambda p < 1$ we can write

$$N_2 = \int_0^1 \left[t^{-\lambda p}(1 - \mathrm{Log}\, t)^\beta \int_0^t f_*^p(x)dx \right]^b \frac{dt}{t}.$$

We may apply the Hardy's inequalities of Theorem 2.1 to derive

$$N_2 \lesssim \int_0^1 \left[t^{1-\lambda p}(1 - \mathrm{Log}\, t)^\beta f_*^p(t) \right]^b \frac{dt}{t}. \tag{35}$$

The right hand side (RHS) of this inequality is equal to

$$\int_0^1 \left[t^{\frac{1}{p}-\lambda}(1 - \mathrm{Log}\, t)^{\alpha_\theta} f_*(t) \right]^r \frac{dt}{t} \tag{36}$$

since $\frac{1}{p} - \lambda = \frac{1-\theta}{p} + \frac{\theta}{q} = \frac{1}{p_\theta}$, we get the result. ◊

Next, we want to estimate N_1.

Lemma 5.3. *One has:*

$$N_1 \lesssim \int_0^1 \left[t^{\frac{1}{p_\theta}}(1 - \mathrm{Log}\, t)^{\alpha_\theta - 1} f_*(t) \right]^r \frac{dt}{t} \leqslant \|f\|_V^r.$$

Proof. Let us set

$$H(s) = s^{-1}(1 - \mathrm{Log}\, s)^{-\frac{1}{p}} \left(\int_0^s f_*^p dx \right)^{\frac{1}{p}}.$$

Then, from the Hardy's inequality given in Theorem 2.1 and the expression of N_1:

$$N_1 \lesssim \int_0^1 \left[t^{1-\lambda}(1 - \mathrm{Log}\, t)^a H(t) \right]^r \frac{dt}{t}. \tag{37}$$

Replacing H, noticing that $a - \frac{1}{p} = -\theta - \frac{1}{p_\theta} = \alpha_\theta - 1$, we then have from relation (37)

$$N_1 \lesssim \int_0^1 \left[t^{-\lambda}(1 - \mathrm{Log}\, t)^{\alpha_\theta - 1} \left(\int_0^t f_*^p dx \right)^{\frac{1}{p}} \right]^r \frac{dt}{t}. \tag{38}$$

Applying again the Hardy inequality as in Lemma 5.2 we have

$$N_1 \lesssim \int_0^1 \left[t^{\frac{1}{p}-\lambda}(1 - \mathrm{Log}\, t)^{\alpha_\theta - 1} f_*(t) \right]^r \frac{dt}{t}. \tag{39}$$

This gives the result. ◊

It remains to estimate the term N_3. We have

Lemma 5.4. *One has*

$$N_3 \lesssim \|f\|_V^r.$$

The key lemma to estimate N_3 is the analogous of Proposition 3.1.

Proposition 5.1. *Let K_d be a decreasing nonnegative function on $(0,1)$, $\nu > 0$ and β two real numbers. Then*

$$\int_0^1 \left[t^\nu (1 - \mathrm{Log}\, t)^\beta \sup_{t \leqslant s < 1} (1 - \mathrm{Log}\, s)^{-\frac{1}{q}} K_d(s) \right]^r \frac{dt}{t} \approx \int_0^1 \left[t^\nu (1 - \mathrm{Log}\, t)^{\beta - \frac{1}{q}} K_d(t) \right]^r \frac{dt}{t}.$$

The proof follows the same argument as for Proposition 3.1.

Applying Proposition 5.1, we deduce

$$N_3 \lesssim \int_0^1 \left[t^{\lambda_1} (1 - \mathrm{Log}\, t)^{\alpha_\theta} \left(\int_t^1 f_*^q(x) dx \right)^{\frac{1}{q}} \right]^r \frac{dt}{t}. \tag{40}$$

(The function $K_d(t) = \left(\int_t^1 f_*^q(x) dx \right)^{\frac{1}{q}}$ is decreasing.)

Again applying Hardy inequalities (see Theorem 2.1) we have

$$N_3 \lesssim \int_0^1 \left[t^{\frac{1}{q} + \lambda_1} (1 - \mathrm{Log}\, t)^{\alpha_\theta} f_*(t) \right]^r \frac{dt}{t}. \tag{41}$$

But we have $\frac{1}{q} + \lambda_1 = \frac{1}{p_\theta}$ so the RHS of (41) is the norm of f in V at the power r. $\diamond$

We have shown

$$\|f\|_W^r \lesssim N_1 + N_2 + N_3 \lesssim \|f\|_V^r$$

and

$$\|f\|_W^r \gtrsim N_3 \gtrsim \|f\|_V^r.$$

This proves Theorem 5.1. $\diamond$

6. The critical case $p = q$. The interpolation space $(L^{p)}, L^{(p})$ and its K-functional

The preceding study can be extended to the case where $p = q$. In this case, the inverse function of $\psi_1(x) = (1 - \mathrm{Log}\, x)^{-1}$, say $\varphi_1(t) = e^{1 - \frac{1}{t}}$ will play the fundamental role to express the K-functional. Note that in this case we cannot use Lemma 2.1. we should do it by direct calculation.

Theorem 6.1 (*K-functional for $(L^{p)}, L^{(p})$*). *For $1 < p < +\infty$, $0 < t < 1$, $f \in L^{p)} + L^{(p}$, one has*

$$K(f, t; L^{p)}, L^{(p}) \approx \sup_{0 < s < e^{1 - \frac{1}{t}}} (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_s^{e^{1 - \frac{1}{t}}} f_*^p(x) dx \right)^{\frac{1}{p}} + t \int_{e^{1 - \frac{1}{t}}}^1 (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_{e^{1 - \frac{1}{t}}}^s f_*^p(x) dx \right)^{\frac{1}{p}} \frac{ds}{s}.$$

Proof. We only sketch it since the arguments are similar to the proof of Theorem 3.2.

We consider $f = g + h \in L^{p)} + L^{(p}$. Then for $t > 0$, $f_*(t) \leqslant g_* \left(\frac{t}{2} \right) + h_* \left(\frac{t}{2} \right)$. We derive

$$K^0(t) = \sup_{0 < s < e^{1 - \frac{1}{t}}} (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_s^{e^{1 - \frac{1}{t}}} f_*^p(x) dx \right)^{\frac{1}{p}} \leqslant I_5 + I_6.$$

with

$$I_5 = \sup_{0 < s < \varphi_1(t)} (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_s^{\varphi_1(t)} g_*^p \left(\frac{x}{2} \right) dx \right)^{\frac{1}{p}}.$$

With a change of variables, we have

$$I_5 \lesssim \sup_{0<s<\varphi_1(t)} \left(1 - \text{Log}\,\frac{s}{2}\right)^{\frac{-1}{p}} \left(\int_{\frac{s}{2}}^{\varphi_1(t)/2} g_*^p(x)dx\right)^{\frac{1}{p}} \lesssim \|g\|_{L^p)}. \tag{42}$$

While for the term

$$I_6 = \sup_{0<s<\varphi_1(t)} \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \left(\int_s^{\varphi_1(t)} h_*^p\left(\frac{x}{2}\right)dx\right)^{\frac{1}{p}},$$

since $1 - \text{Log}\,\varphi_1(t) = \frac{1}{t}$ and $s \to (1 - \text{Log}\,s)^{\frac{-1}{p}}$ is increasing, we have

$$I_6 \leqslant t^{\frac{1}{p}} \left(\int_0^{\varphi_1(t)} h_*^p(x)dx\right)^{\frac{1}{p}} = t \times t^{\frac{1}{p}-1} \left(\int_0^{\varphi_1(t)} h_*^p(x)dx\right)^{\frac{1}{p}}. \tag{43}$$

We replace the quantity $t^{\frac{1}{p}-1}$ by making an integration of the following integral:

$$\int_{\varphi_1(t)}^1 \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \frac{ds}{s} = \frac{1}{1-\frac{1}{p}} \left[t^{\frac{1}{p}-1} - 1\right]. \tag{44}$$

Therefore, we have from (43) to (44)

$$I_6 \lesssim t \int_{\varphi_1(t)}^1 \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \left(\int_0^s h_*^p(x)dx\right)^{\frac{1}{p}} \frac{ds}{s} + t\|h\|_{L^p}, \tag{45}$$

$$I_6 \lesssim t\left(\|h\|_{L^{(p}} + \|h\|_{L^p}\right) \lesssim t\|h\|_{L^{(p}}. \tag{46}$$

Thus, we have

$$K^0(t) \lesssim \|g\|_{L^p)} + t\|h\|_{L^{(p}}. \tag{47}$$

While for the second term in the expression of K, say

$$K^1(t) = t \int_{\varphi_1(t)}^1 \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \left(\int_{\varphi_1(t)}^s f_*^p(x)dx\right)^{\frac{1}{p}} \frac{ds}{s}.$$

One has

$$K^1(t) \leqslant I_7 + I_8,$$

with

$$I_7 = t \int_{\varphi_1(t)}^1 \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \left(\int_{\varphi_1(t)}^s g_*^p\left(\frac{x}{2}\right)dx\right)^{\frac{1}{p}} \frac{ds}{s},$$

and

$$I_8 = t \int_{e^{1-\frac{1}{t}}}^1 \left(1 - \text{Log}\,s\right)^{\frac{-1}{p}} \left(\int_{\varphi_1(t)}^s h_*^p\left(\frac{x}{2}\right)dx\right)^{\frac{1}{p}} \frac{ds}{s}.$$

To estimate I_7, we use the relation $\frac{1}{t} = 1 - \text{Log}\,\varphi_1(t)$ to derive

$$I_7 \lesssim t\left(\int_{\varphi_1(t)/2}^1 g_*^p(x)dx\right)^{\frac{1}{p}} t^{\frac{1}{p}-1} \lesssim t(1 - \text{Log}\,\frac{\varphi_1(t)}{2})^{\frac{p-1}{p}}\left(\int_{\varphi_1(t)/2}^1 g_*^p(x)dx\right)^{\frac{1}{p}} \lesssim \|g\|_{L^{p)}}. \tag{48}$$

While for the second term I_8, we have after a change of variables

$$I_8 \lesssim t\int_0^1 (1 - \text{Log}\,x)^{\frac{-1}{p}}\left(\int_0^x h_*^p(\tau)d\tau\right)^{\frac{1}{p}}\frac{dx}{x} = t\|h\|_{L^{(p}}. \tag{49}$$

Then, we deduce from (48) to (49)

$$K^1(t) \lesssim \|g\|_{L^{p)}} + t\|h\|_{L^{(p}}. \tag{50}$$

Therefore, we have for all $f = g + h$

$$K^0(t) + K^1(t) \lesssim \|g\|_{L^{p)}} + t\|h\|_{L^{(p}}.$$

That is

$$K^0(t) + K^1(t) \lesssim K(f, t; L^{p)}, L^{(p}). \tag{51}$$

For the reverse, we use the same decomposition as before $f \in L^{p)} + L^{(p}$, say

$$g = \left(|f| - f_*(\varphi_1(t))\right)_+ = \left(|f| - f_*(\varphi_1(t))\right)\chi_{\{|f|>f_*(\varphi_1(t))\}}, \quad h = f - g,$$

so that $h_* + g_* = f_*$ and

$$g_* = \left(f_* - f_*(\varphi_1(t))\right)_+, \quad h_* = f_*(\varphi_1(t))\chi_{(0,\varphi_1(t))} + f_*(s)\chi_{(\varphi_1(t),1)}.$$

With those expressions, we derive

$$\|g\|_{L^{p)}} \leqslant \sup_{0<s<\varphi_1(t)} (1 - \text{Log}\,s)^{\frac{1}{p}}\left(\int_s^{\varphi_1(t)} f_*^p(x)dx\right)^{\frac{1}{p}} = K^0(t). \tag{52}$$

While for the term in h,

$$t\|h\|_{L^{(p}} \leqslant t\left(\int_0^{\varphi_1(t)} (1 - \text{Log}\,s)^{\frac{-1}{p}} s^{\frac{1}{p}}\frac{ds}{s}\right)f_*(\varphi_1(t))$$

$$+ t\int_{\varphi_1(t)}^1 (1 - \text{Log}\,s)^{\frac{-1}{p}}\frac{ds}{s}\varphi_1(t)^{\frac{1}{p}}f_*(\varphi_1(t))$$

$$+ t\int_{\varphi_1(t)}^1 (1 - \text{Log}\,s)^{\frac{-1}{p}}\left(\int_{\varphi_1(t)}^s f_*^p(x)dx\right)^{\frac{1}{p}}\frac{ds}{s}. \tag{53}$$

For the first integral in (53), one can use Proposition 2.2 to derive

$$\int_0^{\varphi_1(t)} s^{\frac{1}{p}-1}(1 - \text{Log}\,s)^{\frac{-1}{p}} ds \lesssim \varphi_1(t)^{\frac{1}{p}}(1 - \text{Log}\,\varphi_1(t))^{\frac{-1}{p}}. \tag{54}$$

So we obtain

$$t\int_0^{\varphi_1(t)} (1 - \text{Log}\,s)^{\frac{-1}{p}} s^{\frac{1}{p}}\frac{ds}{s}f_*(\varphi_1(t)) \lesssim t(1 - \text{Log}\,\varphi_1(t))^{\frac{-1}{p}}\left(\int_{\varphi_1(t)/2}^{\varphi_1(t)} f_*^p(x)dx\right)^{\frac{1}{p}}$$

$$\lesssim t\sup_{0<s\leqslant\frac{\varphi_1(t)}{2}} (1 - \text{Log}\,s)^{\frac{-1}{p}}\left(\int_s^{\varphi_1(t)} f_*^p(x)dx\right)^{\frac{1}{p}}$$

$$\lesssim tK^0(t) \leqslant K^0(t). \tag{55}$$

For the second integral, we use relation (44) to derive

$$B = t \int_{\varphi_1(t)}^1 (1 - \operatorname{Log} s)^{\frac{-1}{p}} \frac{ds}{s} \varphi_1(t)^{\frac{1}{p}} f_*(\varphi_1(t)) \lesssim t^{\frac{1}{p}} \varphi_1(t)^{\frac{1}{p}} f_*(\varphi_1(t)). \tag{56}$$

Since $\int_{e^{-\frac{1}{t}}}^{\varphi_1(t)} dx = \frac{e-1}{e} \varphi_1(t)$, we then have

$$\varphi_1(t)^{\frac{1}{p}} f_*(\varphi_1(t)) \lesssim \left(\int_{e^{-\frac{1}{t}}}^{\varphi_1(t)} f_*^p(x) dx \right)^{\frac{1}{p}}. \tag{57}$$

So that relations (56) and (57) imply

$$B = t \int_{\varphi_1(t)}^1 (1 - \operatorname{Log} s)^{\frac{-1}{p}} \frac{ds}{s} \varphi_1(t)^{\frac{1}{p}} f_*(\varphi_1(t)) \lesssim (1 - \operatorname{Log} e^{1-\frac{1}{t}})^{-\frac{1}{p}} \left(\int_{e^{-\frac{1}{t}}}^{\varphi_1(t)} f_*^p(x) dx \right)^{\frac{1}{p}}. \tag{58}$$

Therefore, we obtain

$$B \lesssim \sup_{0 < s < \varphi_1(t)} (1 - \operatorname{Log} s)^{\frac{-1}{p}} \left(\int_s^{\varphi_1(t)} f_*^p(x) dx \right)^{\frac{1}{p}} = K^0(t). \tag{59}$$

The last term in relation (53) is equal to $K^1(t)$.

Combining these last relations (52) to (59), we come to

$$\|g\|_{L^{p)}} + t \|h\|_{L^{(p}} \lesssim K^0(t) + K^1(t),$$

which implies that

$$K(f, t; L^{p)}, L^{(p)}) \lesssim K^0(t) + K^1(t) \lesssim K(f, t; L^{p)}, L^{(p)}). \quad \lozenge$$

We can now identify the interpolation space $(L^{p)}, L^{(p)})_{\theta,r}$ as we state in

Theorem 6.2. *Let $1 < p < +\infty$, $0 < \theta < 1$, $1 \leqslant r < +\infty$. Then $Z_{\theta,r} \doteq (L^{p)}, L^{(p)})_{\theta,r}$ has the following equivalent norm:*

For $f \in Z_{\theta,r}$, $\beta_\theta = \theta - \frac{1}{p} - \frac{1}{r}$

- *if $\theta < \frac{1}{p}$ then*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1 - \operatorname{Log} t)^{\beta_\theta} \left(\int_t^1 f_*^p(s) ds \right)^{\frac{1}{p}} \right]^r \frac{dt}{t} \right]^{\frac{1}{r}}.$$

- *if $\theta > \frac{1}{p}$ then $Z_{\theta,r} = G\Gamma(p, r, w), w(t) = t^{-1}(1 - \operatorname{Log} t)^{\beta_\theta r}$ and*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 \left[(1 - \operatorname{Log} t)^{\beta_\theta} \left(\int_0^t f_*^p(s) ds \right)^{\frac{1}{p}} \right]^r \frac{dt}{t} \right]^{\frac{1}{r}}.$$

- *if $\theta = \frac{1}{p}$*

$$\|f\|_{Z_{\theta,r}} \approx \left[\sum_{k=0}^{+\infty} \left(\int_{2^{1-2^{k+1}}}^{2^{1-2^k}} f_*^p(s) ds \right)^{\frac{r}{p}} \right]^{\frac{1}{r}}.$$

In particular

$$(L^{p)}, L^{(p)})_{\frac{1}{p},p} = L^p \qquad and \quad (L^{p)}, L^{(p)})_{\theta,p} = L^{p,p}(\operatorname{Log} L)^{\theta - \frac{1}{p}}.$$

Proof. According to Theorem 6.1, we have for $f \in Z_{\theta,r}$,

$$\|f\|^r_{Z_{\theta,r}} \approx \int_0^1 \left[t^{-\theta} \sup_{0<s<\varphi_1(t)} (1 - \text{Log } s)^{\frac{-1}{p}} \left(\int_s^{\varphi_1(t)} f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dt}{t}$$

$$+ \int_0^1 \left[t^{1-\theta} \int_{\varphi_1(t)}^1 (1 - \text{Log } s)^{\frac{-1}{p}} \left(\int_{\varphi_1(t)}^s f_*^p(y)dy \right)^{\frac{1}{p}} \frac{ds}{s} \right]^r \frac{dt}{t} \doteq F_1 + F_2.$$

Using the change of variables $x = \varphi_1(t)$ so that $t = (1 - \text{Log } x)^{-1}$, we have

$$F_1 = \int_0^1 \left[(1 - \text{Log } x)^{\theta} \sup_{0<s<x} (1 - \text{Log } s)^{\frac{-1}{p}} \left(\int_s^x f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dx}{(1 - \text{Log } x)x},$$

$$F_2 = \int_0^1 \left[(1 - \text{Log } x)^{\theta-1} \int_x^1 (1 - \text{Log } s)^{\frac{-1}{p}} \left(\int_x^s f_*^p(y)dy \right)^{\frac{1}{p}} \frac{ds}{s} \right]^r \frac{dx}{(1 - \text{Log } x)x}.$$

Let us start with the upper bound of this norm.

Proposition 6.1. *One has*

$$F_1 + F_2 \lesssim \int_0^1 \left[(1 - \text{Log } x)^{\theta-\frac{1}{r}-\frac{1}{p}} \left(\int_0^x f_*^p(s)ds \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}, \quad \text{if } \theta < 1.$$

Proof. For the term F_1, we have

$$\left(\int_s^x f_*^p(t)dt \right)^{\frac{1}{p}} \leqslant \left(\int_0^x f_*^p(t)dt \right)^{\frac{1}{p}} \text{ for } s < x,$$

so, one deduces

$$F_1 \leqslant \int_0^1 \left[(1 - \text{Log } x)^{\theta-\frac{1}{r}} \left(\int_0^x f_*^p(s)ds \right)^{\frac{1}{p}} \sup_{0<s<x} (1 - \text{Log } s)^{\frac{-1}{p}} \right]^r \frac{dx}{x}, \tag{60}$$

which gives the upper estimate since $\sup_{0<s<x}(1 - \text{Log } s)^{\frac{-1}{p}} = (1 - \text{Log } x)^{\frac{-1}{p}}$.

For the term F_2, we have

$$F_2 \leqslant \int_0^1 \left[(1 - \text{Log } x)^{\theta-1-\frac{1}{r}} \int_x^1 (1 - \text{Log } s)^{\frac{-1}{p}} \left(\int_0^s f_*^p(y)dy \right)^{\frac{1}{p}} \frac{ds}{s} \right]^r \frac{dx}{x}. \tag{61}$$

Since $\theta < 1$, one may apply the Hardy inequality Theorem 2.2 (see [4] Theorem 6.5) to derive

$$F_2 \leqslant \int_0^1 \left[(1 - \text{Log } x)^{\theta-\frac{1}{r}-\frac{1}{p}} \left(\int_0^x f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}. \quad \diamond \tag{62}$$

Proposition 6.2. *For $0 < \theta < 1$, we have*

$$F_1 + F_2 \lesssim \int_0^1 \left[(1 - \text{Log } x)^{\theta-\frac{1}{r}-\frac{1}{p}} \left(\int_x^1 f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}.$$

Proof. We start with F_2. Since

$$\int_x^s f_*^p(y)dy \leqslant \int_x^1 f_*^p(y)dy,$$

the expression of F_2 can be estimated as:

$$F_2 \leqslant \int_0^1 \left[(1 - \mathrm{Log}\, x)^{\theta - 1 - \frac{1}{r}} \left(\int_x^1 f_*^p(y)dy \right)^{\frac{1}{p}} \left(\int_x^1 (1 - \mathrm{Log}\, s)^{-\frac{1}{p}} \frac{ds}{s} \right) \right]^r \frac{dx}{x}. \tag{63}$$

By integration as for relation (44), one has

$$\int_x^1 (1 - \mathrm{Log}\, s)^{-\frac{1}{p}} \frac{ds}{s} = \frac{p}{p-1} \left[(1 - \mathrm{Log}\, x)^{1 - \frac{1}{p}} - 1 \right]. \tag{64}$$

Therefore, we deduce from (63) and (64), the

$$F_2 \lesssim \int_0^1 \left[(1 - \mathrm{Log}\, x)^{\theta - \frac{1}{p} - \frac{1}{r}} \left(\int_x^1 f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}. \tag{65}$$

While for the second term F_1, we deduce from Bennett–Rudnick's Lemma ([4] Lemma 6.1)

$$\sup_{0<s<x} (1 - \mathrm{Log}\, s)^{-\frac{1}{p}} \left(\int_s^1 f_*^p(y)dy \right)^{\frac{1}{p}} \leqslant c \int_0^x \left[(1 - \mathrm{Log}\, s)^{-\frac{1}{p} - 1} \left(\int_s^1 f_*^p(y)dy \right)^{\frac{1}{p}} \right] \frac{ds}{s}.$$

Setting temporarily $\psi(s) = (1 - \mathrm{Log}\, s)^{-\frac{1}{p} - 1} \left(\int_s^1 f_*^p(y)dy \right)^{\frac{1}{p}}$, we then deduce

$$F_1 \lesssim \int_0^1 \left[(1 - \mathrm{Log}\, x)^{\theta - \frac{1}{r}} \left(\int_0^x \psi(s) \frac{ds}{s} \right) \right]^r \frac{dx}{x}.$$

By the Hardy inequality   Theorem 2.2 (see [4] Theorem 6.5).

$$F_1 \lesssim \int_0^1 \left[(1 - \mathrm{Log}\, x)^{\theta - \frac{1}{p} - \frac{1}{r}} \left(\int_x^1 f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r \frac{dx}{x}. \quad \diamond$$

For the lower bound, we need few lemmas.

The first lemma is a consequence of a general lemma given in Goldman Heinig and Stepanov ([20], see also [19, Lemma 3.1]).

Lemma 6.1. *Let $t_k = 2^{1 - 2^k}$, $\forall\, k \in \mathbb{N}$, H a nonnegative locally integrable function on $(0, 1)$, $(q, \lambda) \in]0, +\infty[^2$. Then*

1. $\sum_{k \in \mathbb{N}} \left(\int_0^{t_k} H(x)dx \right)^q 2^{\lambda k q} \approx \sum_{m \in \mathbb{N}} \left(2^{\lambda m} \int_{t_{m+1}}^{t_m} H(x)dx \right)^q,$
2. $\sum_{k \in \mathbb{N}} \left(\int_{t_{k+1}}^1 H(x)dx \right)^q 2^{-\lambda k q} \approx \sum_{m \in \mathbb{N}} \left(2^{-\lambda m} \int_{t_{m+1}}^{t_m} H(x)dx \right)^q.$

Here $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers.

The next lemma can be obtained by straightforward computation.

Lemma 6.2. *Let* $t_k = 2^{1-2^k}$, $k \in \mathbb{N}$, $\lambda \neq 0$. *Then, one has*

1. *For any* $s \in [t_{k+1}, t_k]$

$$2^k \approx 1 - \operatorname{Log} s,$$

2.

$$\int_{t_{k+1}}^{t_k} (1 - \operatorname{Log} t)^{\lambda - 1} \frac{dt}{t} \approx 2^{k\lambda},$$

for all $k \in \mathbb{N}$.

As a corollary of the above lemmas we have:

Lemma 6.3. *Let* $t_k = 2^{1-2^k}$, $k \in \mathbb{N}$, $\lambda \neq 0$, $q > 0$ *and* H *a nonnegative locally integrable function on* $(0, 1)$. *Then*

• *If* $\lambda > 0$ *then*

$$\sum_{k \in \mathbb{N}} \left(\int_0^{t_k} H(x) dx \right)^q 2^{\lambda k q} \gtrsim \int_0^1 \left[(1 - \operatorname{Log} t)^\lambda \int_0^t H(x) dx \right]^q \frac{dt}{(1 - \operatorname{Log} t)t}$$

and one has the equivalence if $\int_0^1 H(x) dx \lesssim \int_0^{\frac{1}{2}} H(x) dx$ *(for instance* H *decreasing).*
• *If* $\lambda < 0$

$$\sum_{k \in \mathbb{N}} \left(\int_{t_{k+1}}^1 H(x) dx \right)^q 2^{\lambda k q} \approx \int_0^1 \left[(1 - \operatorname{Log} t)^\lambda \int_t^1 H(x) dx \right]^q \frac{dt}{(1 - \operatorname{Log} t)t}.$$

Proof. If $\lambda > 0$, we use statement (2.) of Lemma 6.2

$$2^{\lambda k q} \approx \int_{t_{k+1}}^{t_k} (1 - \operatorname{Log} t)^{\lambda q - 1} \frac{dt}{t}. \tag{66}$$

Thus, we have

$$\sum_{k \in \mathbb{N}} \left(\int_0^{t_k} H(x) dx \right)^q 2^{\lambda k q} \approx \sum_{k \in \mathbb{N}} \left(\int_0^{t_k} H(x) dx \right)^q \int_{t_{k+1}}^{t_k} (1 - \operatorname{Log} t)^{\lambda q - 1} \frac{dt}{t}. \tag{67}$$

Since for $t < t_k$, $\int_0^t H(x) dx \leqslant \int_0^{t_k} H(x) dx$, then we derive from relation (67)

$$\sum_{k \in \mathbb{N}} \left(\int_0^{t_k} H(x) dx \right)^q 2^{\lambda k q} \gtrsim \sum_{k \in \mathbb{N}} \int_{t_{k+1}}^{t_k} \left(\int_0^t H(x) dx \right)^q (1 - \operatorname{Log} t)^{\lambda q - 1} \frac{dt}{t}, \tag{68}$$

and we have for any G nonnegative

$$\sum_{k \in \mathbb{N}} \int_{t_{k+1}}^{t_k} G(t) dt = \int_0^1 G(t) dt. \tag{69}$$

This ends the proof of the lower bound of the first statement.

For the upper bound, we change the index of summation and use relation (66)

$$\sum_{k=1}^{+\infty} \left(\int_0^{t_k} H(x)dx \right)^q 2^{\lambda k q} \approx \sum_{j\in\mathbb{N}} \left(\int_0^{t_{j+1}} H(x)dx \right)^q 2^{\lambda j q}$$

$$\lesssim \sum_{j\in\mathbb{N}} \int_{t_{j+1}}^{t_j} \left(\int_0^t H(x)dx \right)^q (1 - \text{Log}\, t)^{\lambda q - 1} \frac{dt}{t}$$

$$\lesssim \int_0^1 (1 - \text{Log}\, t)^{\lambda q - 1} \left(\int_0^t H(x)dx \right)^q \frac{dt}{t}. \tag{70}$$

Moreover, if $\int_0^1 H(x)dx \lesssim \int_0^{\frac{1}{2}} H(x)dx$ and since

$$\left(\int_0^{\frac{1}{2}} H(x)dx \right)^q \lesssim \int_{\frac{1}{2}}^1 \left(\int_0^t H(x)dx \right)^q (1 - \text{Log}\, t)^{\lambda q - 1} t^{-1} dt$$

$$\lesssim \int_0^1 \left[(1 - \text{Log}\, t)^\lambda \int_0^t H(x)dx \right]^q \frac{dt}{(1 - \text{Log}\, t)t}, \tag{71}$$

then from the two relations (70) and (71), we obtain the upper bound for $\lambda > 0$.

Same argument holds for $\lambda < 0$ for having the second statement.

Lemma 6.3 holds for $\lambda = 0$, noticing that $\int_{t_{k+1}}^{t_k} \frac{dt}{(1 - \text{Log}\, t)t} \approx 1$. Then under the same conditions as before, we have

$$\sum_{k\in\mathbb{N}} \left(\int_0^{t_k} H(x)dx \right)^q \approx \int_0^1 \left[\int_0^t H(x)dx \right]^q \frac{dt}{(1 - \text{Log}\, t)t}$$

and

$$\sum_{k\in\mathbb{N}} \left(\int_{t_{k+1}}^1 H(x)dx \right)^q \approx \int_0^1 \left[\int_t^1 H(x)dx \right]^q \frac{dt}{(1 - \text{Log}\, t)t}. \qquad \diamond$$

To obtain the lower bound for F_1, we first show the

Lemma 6.4. *One has*

$$F_1 \gtrsim \sum_{k\in\mathbb{N}} 2^{k(\theta - \frac{1}{p})r} \left(\int_{t_{k+1}}^{t_k} f_*^p(s)ds \right)^{\frac{r}{p}} \text{with} t_k = 2^{1-2^k}, \ k \in \mathbb{N}.$$

Proof. Let us set $G(x) = \sup_{0<s<x}(1 - \text{Log}\, s)^{-\frac{r}{p}} \left(\int_s^x f_*^p(y)dy \right)^{\frac{r}{p}}$, $q = \frac{r}{p}$ and write

$$F_1 = \sum_{k\in\mathbb{N}} \int_{t_{k+1}}^{t_k} G(x)(1 - \text{Log}\, x)^{\theta r - 1} \frac{dx}{x}.$$

Then, we derive from Lemma 6.2 and the definition of G

$$F_1 \gtrsim \sum_{k\in\mathbb{N}} G(t_k) \int_{t_{k+1}}^{t_k} (1 - \text{Log}\, x)^{\theta r - 1} \frac{dx}{x} \approx \sum_{k\in\mathbb{N}} G(t_k) 2^{k\theta r}. \tag{72}$$

But $G(t_k) = \sup_{i\geqslant k}\sup_{t_{i+1}<s<t_i}(1 - \text{Log}\, s)^{-q} \left(\int_s^{t_k} f_*^p(y)dy \right)^q$ and Lemma 6.2 implies that for $t_{i+1} < s < t_i$, $(1 - \text{Log}\, s)^{-q} \approx 2^{-iq}$, so that

$$G(t_k) \approx \sup_{i\geqslant k} 2^{-iq} \sup_{t_{i+1}<s<t_i} \left(\int_s^{t_k} f_*^p(y)dy \right)^q,$$

that is

$$G(t_k) \approx \sup_{i \geqslant k} 2^{-iq} \left(\int_{t_{i+1}}^{t_k} f_*^p(y)dy \right)^q \tag{73}$$

and then

$$G(t_k) \approx \sup_{i \geqslant k} 2^{-iq} \left(\int_{t_{i+1}}^{t_k} f_*^p(y)dy \right)^q \geqslant 2^{-kq} \left(\int_{t_{k+1}}^{t_k} f_*^p(y)dy \right)^q. \tag{74}$$

Combining relation (74) and (72), we deduce

$$F_1 \gtrsim \sum_{k \in \mathbb{N}} 2^{k\theta r - k\frac{r}{p}} \left(\int_{t_{k+1}}^{t_k} f_*^p(y)dy \right)^{\frac{r}{p}}.$$

This ends the proof.   ◊

As a corollary of the above lemmas, one has

Theorem 6.3. *On has for $f \in Z_{\theta,r}$*

1. *If $\theta > \frac{1}{p}$ then*

$$\|f\|_{Z_{\theta,r}}^r \gtrsim \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\theta - \frac{1}{p}} \left(\int_0^t f_*^p(x)dx \right)^{\frac{1}{p}} \right]^r \frac{dt}{(1 - \mathrm{Log}\, t)t}.$$

2. *If $\theta < \frac{1}{p}$ then*

$$\|f\|_{Z_{\theta,r}}^r \gtrsim \int_0^1 \left[(1 - \mathrm{Log}\, t)^{\theta - \frac{1}{p}} \left(\int_t^1 f_*^p(x)dx \right)^{\frac{1}{p}} \right]^r \frac{dt}{(1 - \mathrm{Log}\, t)t}.$$

Proof. Let us set $q = \frac{r}{p}$, $\lambda = \left| \theta - \frac{1}{p} \right| p$.
  If $\theta - \frac{1}{p} > 0$, we may apply statement (1) of Lemma 6.1 with $H(x) = f_*^p(x)$ to derive from Lemma 6.4

$$F_1 \gtrsim \sum_{k \in \mathbb{N}} \left(2^{\lambda k} \int_{t_{k+1}}^{t_k} f_*^p(y)dy \right)^q \approx \sum_{k \in \mathbb{N}} \left[\int_0^{t_k} f_*^p(x)dx \right]^q 2^{\lambda kq}. \tag{75}$$

So that relation (75) gives with the help of Lemma 6.3

$$F_1 \gtrsim \sum_{k \in \mathbb{N}} \left(\int_0^{t_k} f_*^p(x)dx \right)^q 2^{\lambda kq} \gtrsim \int_0^1 \left[(1 - \mathrm{Log}\, t)^\lambda \int_0^t f_*^p(x)dx \right]^q \frac{dt}{(1 - \mathrm{Log}\, t)t}.$$

This last inequality implies the first statement of Theorem 6.3, noticing that

$$\|f\|_{Z_{\theta,r}}^r \gtrsim F_1.$$

While for the second statement, we apply the statement (2) of Lemma 6.1 to derive from Lemma 6.4:

$$F_1 \gtrsim \sum_{k \in \mathbb{N}} \left(2^{-\lambda k} \int_{t_{k+1}}^{t_k} f_*^p(y)dy \right)^q \approx \sum_{k \in \mathbb{N}} \left[\int_{t_{k+1}}^1 f_*^p(x)dx \right]^q 2^{-\lambda kq}.$$

So that

$$F_1 \gtrsim \sum_{k \in \mathbb{N}} \left(\int_{t_{k+1}}^{1} f_*^p(x) dx \right)^q 2^{-\lambda kq} \gtrsim \int_0^1 \left[(1 - \operatorname{Log} t)^{-\lambda} \int_t^1 f_*^p(x) dx \right]^q \frac{dt}{(1 - \operatorname{Log} t)t}.$$

The last inequality comes from the second statement of Lemma 6.3 and this ends the proof of Theorem 6.3 since $\|f\|_{Z_{\theta,r}}^r \gtrsim F_1$.   ◇

It remains to investigate in the particular case $\theta = \frac{1}{p}$. The lower bound for $\|f\|_{Z_{\theta,r}}$ comes from Lemma 6.4. It is sufficient to show the

Lemma 6.5. *For any $\theta \in]0, 1[$, we have*

$$\max(F_1; F_2) \leqslant \sum_{k \in \mathbb{N}} 2^{kr(\theta - \frac{1}{p})} \left(\int_{t_{k+1}}^{t_k} f_*^p(s) ds \right)^{\frac{r}{p}}.$$

Proof. The estimates for F_1 and F_2 follow the same argument, nevertheless we detail both estimates for clarity reason.

Following the notation in the proof of Lemma 6.4, we have

$$F_1 = \sum_{k \in \mathbb{N}} \int_{t_{k+1}}^{t_k} G(x)(1 - \operatorname{Log} x)^{\theta r - 1} \frac{dx}{x}.$$

Following the above argument as in Lemma 6.4, we have

$$F_1 \lesssim \sum_{k \in \mathbb{N}} G(t_k) 2^{k\theta r} \lesssim \sum_{k \in \mathbb{N}} 2^{k\theta r} \sup_{i \geqslant k} 2^{-iq} \left(\int_{t_{i+1}}^{t_k} f_*^p(y) dy \right)^q. \tag{76}$$

Writing

$$\int_{t_{i+1}}^{t_k} f_*^p(y) dy = \sum_{j=k}^{i} 2^j 2^{-j} \int_{t_{j+1}}^{t_j} f_*^p(y) dy,$$

one deduces

$$\sup_{i \geqslant k} 2^{-iq} \left(\int_{t_{i+1}}^{t_k} f_*^p(y) dy \right)^q \leqslant \sup_{i \geqslant k} 2^{-iq} \left(\sum_{j=k}^{i} 2^j \right)^q \sup_{j \geqslant k} 2^{-jq} \left(\int_{t_{j+1}}^{t_j} f_*^p(x) dx \right)^q.$$

Therefore, we have

$$F_1 \lesssim \sum_{k \in \mathbb{N}} 2^{k\theta r} \sup_{j \geqslant k} 2^{-jq} \left(\int_{t_{j+1}}^{t_j} f_*^p(x) dx \right)^q.$$

So, setting $A_j \doteq \left(\int_{t_{j+1}}^{t_j} f_*^p(y) dy \right)^q$ estimating the supremum term by a sum, one has

$$F_1 \lesssim \sum_{k \in \mathbb{N}} 2^{k\theta r} \sum_{j=k}^{+\infty} 2^{-jq} A_j = \sum_{j \in \mathbb{N}} 2^{-jq} A_j \sum_{k=0}^{j} 2^{k\theta r}.$$

$$F_1 \lesssim \sum_{j \in \mathbb{N}} 2^{j\theta r - jq} A_j = \sum_{j \in \mathbb{N}} 2^{j(\theta - \frac{1}{p})r} \left(\int_{t_{j+1}}^{t_j} f_*^p(y) dy \right)^q.$$

For estimating F_2, we set $G_0(x) = \left[\int_x^1 (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_x^s f_*^p(y)dy \right)^{\frac{1}{p}} \frac{ds}{s} \right]^r$ and write

$$F_2 = \sum_{k \in \mathbb{N}} \int_{t_{k+1}}^{t_k} (1 - \mathrm{Log}\, x)^{(\theta-1)r-1} G_0(x) \frac{dx}{x}.$$

Then, using Lemma 6.2 and the fact that $G_0(x) \leqslant G_0(t_{k+1})$ for $t_{k+1} < x < t_k$, one has

$$F_2 \lesssim \sum_{k \in \mathbb{N}} G_0(t_{k+1}) 2^{kr(\theta-1)}. \tag{77}$$

Writing $G_0(t_{k+1})$ as

$$G_0(t_{k+1}) = \left[\sum_{i=0}^k \int_{t_{i+1}}^{t_i} (1 - \mathrm{Log}\, s)^{\frac{-1}{p}} \left(\int_{t_{k+1}}^s f_*^p(y)dy \right)^{\frac{1}{p}} \frac{ds}{s} \right]^r,$$

and using Lemma 6.2, we then have

$$G_0(t_{k+1}) \lesssim \left[\sum_{i=0}^k \left(\int_{t_{k+1}}^{t_i} f_*^p(y)dy \right)^{\frac{1}{p}} 2^{(1-\frac{1}{p})i} \right]^r. \tag{78}$$

Using Lemma 6.1 with an adequate step function H, the RHS of relation (78) can be estimated as

$$\sum_{i=0}^k \left(\int_{t_{k+1}}^{t_i} f_*^p(y)dy \right)^{\frac{1}{p}} 2^{(1-\frac{1}{p})i} = \sum_{i=0}^k 2^{(1-\frac{1}{p})i} \left(\sum_{j=i}^k \int_{t_{j+1}}^{t_j} f_*^p(y)dy \right)^{\frac{1}{p}}$$

$$\approx \sum_{i=0}^k 2^{(1-\frac{1}{p})i} \left(\int_{t_{i+1}}^{t_i} f_*^p(y)dy \right)^{\frac{1}{p}}.$$

Summarizing the above relations, we come to

$$F_2 \lesssim \sum_{k \in \mathbb{N}} 2^{kr(\theta-1)} \left[\sum_{i=0}^k 2^{(1-\frac{1}{p})i} \left(\int_{t_{i+1}}^{t_i} f_*^p(y)dy \right)^{\frac{1}{p}} \right]^r,$$

using the second statement of Lemma 6.1, by taking H a step function on $[0,1]$ such that $\int_{t_{i+1}}^{t_i} H(x)dx = 2^{(1-\frac{1}{p})i} \left(\int_{t_{i+1}}^{t_i} f_*^p(y)dy \right)^{\frac{1}{p}}$, we deduce from this last inequality

$$F_2 \lesssim \sum_{i \in \mathbb{N}} 2^{i(\theta-\frac{1}{p})r} \left(\int_{t_{i+1}}^{t_i} f_*^p(y)dy \right)^{\frac{r}{p}}.$$

This ends the proof. ◊

Corollary 6.1 (*of Lemmas 6.4 and 6.5*). *One has for any $\theta \in]0,1[$ and $f \in Z_{\theta,r}$ $\|f\|_{Z_{\theta,r}}^r \approx \sum_{k \in \mathbb{N}} 2^{kr(\theta-\frac{1}{p})} \left(\int_{t_{k+1}}^{t_k} f_*^p(y)dy \right)^{\frac{r}{p}}.$*

As a consequence of the above corollary, we have

Theorem 6.4. *One has*

$$\|f\|_{Z_{\theta,r}} \approx \left[\int_0^1 (1 - \mathrm{Log}\, t)^{\theta r} \left(\int_0^t (1 - \mathrm{Log}\, x)^{-1} f_*^p(x)dx \right)^{\frac{r}{p}} \frac{dt}{(1 - \mathrm{Log}\, t)t} \right]^{\frac{1}{r}}.$$

Proof. From Lemma 6.2

$$\int_{t_{k+1}}^{t_k} 2^{-k} f_*^p(y) dy \approx \int_{t_{k+1}}^{t_k} (1 - \mathrm{Log}\, y)^{-1} f_*^p(y) dy,$$

and from Corollary 6.1

$$\begin{aligned}
\|f\|_{Z_{\theta,r}}^r &\approx \sum_{k \in \mathbb{N}} 2^{kr\theta} \left(\int_{t_{k+1}}^{t_k} (1 - \mathrm{Log}\, x)^{-1} f_*^p(x) dx \right)^{\frac{r}{p}} \\
&\approx \sum_{k \in \mathbb{N}} 2^{kr\theta} \left(\int_0^{t_k} (1 - \mathrm{Log}\, x)^{-1} f_*^p(x) dx \right)^{\frac{r}{p}} \text{ (using Lemma 6.1)} \\
&\approx \int_0^1 (1 - \mathrm{Log}\, x)^{r\theta} \left(\int_0^t (1 - \mathrm{Log}\, t)^{-1} f_*^p(x) dx \right)^{\frac{r}{p}} \frac{dt}{(1 - \mathrm{Log}\, t)t} \text{ (using Lemma 6.3).} \quad \diamond
\end{aligned}$$

Acknowledgments

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