INCOMPRESSIBLE VISCOUS STATIONARY FLOW THROUGH A CASCADE OF PROFILES - EXISTENCE OF A WEAK SOLUTION, TRANSFERABILITY OF RESULTS FROM ONE PERIOD BACK TO ORIGINAL DOMAIN

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Introduction

We study the steady flow through a simplified plane cascade of profiles. The model of cascade of profiles describes e.g. the flow through a turbine or thorough a general blade machine. If we consider the intersection of the real 3D region filled by the moving fluid with a surface defined along the streamlines of the flow, and expand the surface in the x_1, x_2–plane we will naturally arrive at a 2D domain. The obtained two dimensional domain (denoted by D) is unbounded, however periodic in the x_2–direction. Its complement in R^2 consists of the infinite number of profiles, numbered from $-\infty$ to $+\infty$. From the definition of the domain it is reasonable to assume that the flow through the cascade is periodic in the x_2–direction with the period τ. Consequently, the problem then can be formulated in a bounded domain (Ω) of the form of one space period and completed by the Dirichlet boundary condition on the inlet (Γ_i) and the profile (Γ_w), a suitable natural boundary condition on the outlet (Γ_o) and periodic boundary conditions on artificial cuts (Γ_+, Γ_-). In this paper we study the possibility to periodically extend the weak solution obtained on one period Ω to get the solution on the whole cascade. This is necessary to legitimate the idea, to solve the problem analytically or numerically on just a part of the infinite but periodical domain.

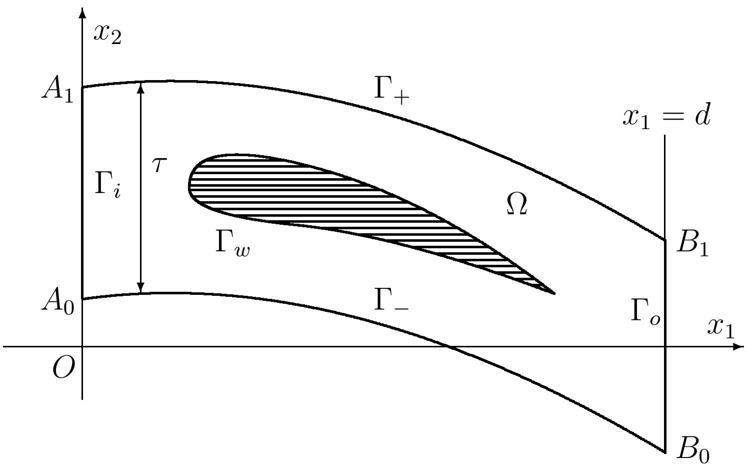

Figure 1: One spatial period of the profile cascade

Used Equations and Boundary conditions

We assume that the fluid is viscous, stationary, incompressible and newtonian. For simplicity we suppose that the unit system is chosen in such a way that the constant density of the fluid is one. The conservation of momentum is described by the Navier-Stokes equations in the form

$$(\mathbf{u}\cdot\nabla)\mathbf{u} \,=\, \mathbf{f} \,-\, \nabla p \,+\, \nu\,\Delta\mathbf{u}, \tag{1}$$

where $\mathbf{u}$ ($= (u_1, u_2)$) is the velocity of the fluid and p the pressure in the fluid, $\mathbf{f}$ ($= (f_1, f_2)$) is the density of the volume force and constant $\nu > 0$ is the kinematic viscosity. The conservation of mass is described by the equation of continuity

$$\mathrm{div}\;\mathbf{u} \,=\, 0. \tag{2}$$

We prescribe the inhomogeneous Dirichlet boundary condition on the inlet and the no slip Dirichlet boundary condition on the profile:

$$\mathbf{u}\,|_{\Gamma_i} \,=\, \mathbf{g}, \qquad\qquad \mathbf{u}\,|_{\Gamma_w} \,=\, \mathbf{0}. \tag{3}$$

According to the definition of the model we suppose that the following conditions of periodicity are fulfilled on the artificial boundaries Γ_+ and Γ_-:

$$\begin{aligned}
\mathbf{u}(x_1, x_2+\tau) &= \mathbf{u}(x_1, x_2) && \text{for } (x_1, x_2) \in \Gamma_-, && (4)\\
\frac{\partial\mathbf{u}}{\partial\mathbf{n}}(x_1, x_2+\tau) &= -\frac{\partial\mathbf{u}}{\partial\mathbf{n}}(x_1, x_2). && \text{for } (x_1, x_2) \in \Gamma_-, && (5)\\
p(x_1, x_2+\tau) &= p(x_1, x_2) && \text{for } (x_1, x_2) \in \Gamma_-. && (6)
\end{aligned}$$

We use the the nonlinear form of the do-nothing type of boundary condition proposed Bruneau and Fabri.

$$-\nu\,\frac{\partial\mathbf{u}}{\partial\mathbf{n}} \,+\, p\cdot\mathbf{n} \,-\, \frac{1}{2}\,(\mathbf{u}\cdot\mathbf{n})^-\,\mathbf{u} \,=\, \mathbf{h} \qquad \text{on } \Gamma_o \tag{7}$$

where $\mathbf{n}$ is the outer normal vector and $\mathbf{h}$ is a given function. For $a \in R$ we set $a^+ = (|a|+a)/2$ and $a^- = (|a| - a)/2$.

Weak formulation

We denote by $H^1(\Omega)$ the usual Sobolev space of functions defined a.e. in Ω. The space of vector–functions (with values in R^2), whose each component belongs to $H^1(\Omega)$, is denoted by $H^1(\Omega)^2$. Furthermore, V denotes the space of vector–functions $\mathbf{v} = (v_1, v_2) \in H^1(\Omega)^2$ such that $\mathrm{div}\,\mathbf{v} = 0$ a.e. in Ω, $\mathbf{v} = \mathbf{0}$ a.e. in $\Gamma_i \cup \Gamma_w$ and $\mathbf{v}(x_1, x_2+\tau) = \mathbf{v}(x_1, x_2)$ for a.a. $(x_1, x_2) \in \Gamma_-$. (The conditions on the curves Γ_i, Γ_w and Γ_- are interpreted in the sense of traces.) We equip the linear space V by the norm $|||\mathbf{v}||| \,:=\, (\int_\Omega \sum_{i,j=1}^2 (\frac{\partial v_i}{\partial x_j})^2 \;\mathrm{d}\mathbf{x})^{1/2}$ which is equivalent to the norm of the space $H^1(\Omega)^2$. In order to derive formally the weak formulation of the problem, we multiply equation (1) by an arbitrary test function $\mathbf{v} = (v_1, v_2) \in V$, integrate in Ω, apply the Green's theorem and use all the boundary conditions (3)–(7). We obtain

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \,=\, \nu\int_\Omega \sum_{i,j=1}^2 \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\mathbf{x} \;+\; \int_\Omega \sum_{i,j=1}^2 u_j\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\mathbf{x} + \int_{\Gamma_o} \frac{1}{2}\,(\mathbf{u}\cdot\mathbf{n})^-\,\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}S$$
$$+ \int_{\Gamma_o} \mathbf{h}\cdot\mathbf{v}\,\mathrm{d}S, \quad \mathbf{v} \in V. \tag{8}$$

In order to simplify its form, we introduce the following notation: for $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2) \in H^1(\Omega)^2$, we put

$$\begin{aligned} a_1(\mathbf{u}, \mathbf{v}) &:= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\mathbf{x}, \qquad a_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial v_i}{\partial x_j} w_i \, \mathrm{d}\mathbf{x}, \\ a_3(\mathbf{u}, \mathbf{v}, \mathbf{w}) &:= \int_{\Gamma_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \, \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}S, \\ a(\mathbf{u}, \mathbf{v}) &:= a_1(\mathbf{u}, \mathbf{v}) + a_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) + a_3(\mathbf{u}, \mathbf{u}, \mathbf{v}), \\ (\mathbf{f}, \mathbf{v}) &:= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}, \qquad b(\mathbf{h}, \mathbf{v}) := -\int_{\Gamma_o} \mathbf{h} \cdot \mathbf{v} \, \mathrm{d}S. \end{aligned}$$

Obviously, all these forms are well defined for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(\Omega)^2$, $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{h} \in L^2(\Gamma_o)^2$. Now the identity (8) can shortly be written as

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + b(\mathbf{h}, \mathbf{v}), \qquad \mathbf{v} \in V. \tag{9}$$

Let the function $\mathbf{g}$, appearing in the boundary condition (3), belongs to $H^s(\Gamma_i)^2$ for $s \in (\frac{1}{2}, 1\rangle$ and $\mathbf{g}(A_1) = \mathbf{g}(A_0)$. (Let us recall that A_0 and A_1 are the end–points of Γ_i.) Let $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{h} \in L^2(\Gamma_o)^2$ be given functions. We seek a vector function $\mathbf{u} \in H^1(\Omega)^2$ which satisfies the equation of continuity (2) a.e. in Ω, the boundary conditions (3) in the sense of traces on Γ_i (respectively on Γ_w), the condition of periodicity (4) a.e. on Γ_- and such that identity (9) holds for all test functions $\mathbf{v} \in V$. The solution $\mathbf{u}$ of this problem is called *a weak solution* in the domain Ω. Now let us suppose that we have a weak solution $\mathbf{u} \in V$ to the problem (1)–(7). The existence of such a weak solution is proved for e.g. in [2] or in [3] for nonstationary problem.

The extension of the weak solution from Ω to D

The next theorem shows that the weak solution in the unbounded domain D can be obtained by means of an appropriate extension of the weak solution in the domain Ω.

Theorem 1: Let $\mathbf{u}$ be a weak solution of problem in the domain Ω. Then the periodic extension (with period τ) in the variable x_2 of the function $\mathbf{u}$ onto the domain D is a weak solution of problem in D.

Proof. We can naturally extend all functions appearing in the weak formulation of the problem in the domain Ω as τ–periodic functions in the variable x_2. Because of simplicity the extended functions will be denoted by the same symbols. Thus, the function $\mathbf{f}$ is now defined a.e. in D, the function $\mathbf{g}$ is defined a.e. in G_i and the function $\mathbf{h}$ is defined a.e. in Γ_o. Clearly, the extended functions $\mathbf{f}$ and $\mathbf{h}$ fulfill the assumptions. It can be easily shown that the extended function $\mathbf{g}$ also fulfills the assumptions.

A function from $H^1(\Omega)$, extended τ–periodically in variable x_2, need not generally belong to $H^1(D')$ for an arbitrary bounded sub–domain D' of D. In the sequel we shall show that the extended function $\mathbf{u}$, however, has this property. Obviously, due to the periodicity of $\mathbf{u}$ in the x_2–direction, it is sufficient to verify that $\mathbf{u}$ (precisely, its restriction on $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$) belongs to $H^1(\Omega_0 \cup \Gamma_- \cup \Omega_{-1})$.

The function $\mathbf{u}$ has generalized first order derivatives $\partial\mathbf{u}/\partial x_i$ $(i = 1, 2)$ in domains Ω_0 and Ω_{-1} and these derivatives are square integrable both in Ω_0 and Ω_{-1}. To verify that $\mathbf{u} \in H^1(\Omega_0 \cup \Gamma_- \cup \Omega_{-1})$, we need to show that the function $D_i\mathbf{u}$, defined by

$$D_i\mathbf{u}(x) = \begin{cases} \dfrac{\partial\mathbf{u}}{\partial x_i}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_0, \\[2mm] \dfrac{\partial\mathbf{u}}{\partial x_i}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_{-1}, \end{cases}$$

is a generalized derivative of $\mathbf{u}$ in the union $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$. Thus, suppose that $\boldsymbol{\varphi} \in C_0^\infty(\Omega_0 \cup \Gamma_- \cup \Gamma_{-1})^2$. Then, denoting by $n_{0,i}$ the i–th component of the outer normal vector to the boundary of Ω_0 on the curve Γ_- and by $n_{-1,i}$ the i–th component of the outer normal vector to the boundary of Ω_{-1} on the curve Γ_-, we have

$$\begin{aligned} \int_{\Omega_0\cup\Gamma_-\cup\Omega_{-1}} D_i\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} &= \int_{\Omega_0} \frac{\partial\mathbf{u}}{\partial x_i}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} + \int_{\Omega_{-1}} \frac{\partial\mathbf{u}}{\partial x_i}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \\ = \int_{\Gamma_-} T_0\mathbf{u}\cdot\boldsymbol{\varphi}\, n_{0,i}\,\mathrm{d}S &- \int_{\Omega_1} \mathbf{u}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_-} T_{-1}\mathbf{u}\cdot\boldsymbol{\varphi}\, n_{-1,i}\,\mathrm{d}S \\ - \int_{\Omega_{-1}} \mathbf{u}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\mathbf{x} &= -\int_{\Omega_0\cup\Gamma_-\cup\Omega_{-1}} \mathbf{u}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{10}$$

(We have denoted by $T_0\mathbf{u}$ the trace of $\mathbf{u}$ on Γ_- as the trace of a function from $H^1(\Omega_0)^2$ and by $T_{-1}\mathbf{u}$ the trace of $\mathbf{u}$ on Γ_- as the trace of a function from $H^1(\Omega_{-1})^2$.) Due to the periodicity condition (4), both the traces on Γ_- coincide, i.e. $T_0\mathbf{u} = T_{-1}\mathbf{u}$ on Γ_-. Moreover, $n_{0,i} = -n_{-1,i}$ on Γ_-. Hence,

$$\int_{\Gamma_-} T_0\mathbf{u}\cdot\varphi\, n_{0,i}\,\mathrm{d}S + \int_{\Gamma_-} T_{-1}\mathbf{u}\cdot\varphi\, n_{-1,i}\,\mathrm{d}S = 0.$$

If we use this equality in (10), we can observe that $D_i\mathbf{u}$ is a generalized derivative of function $\mathbf{u}$ with respect to x_i in $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$. The square integrability of function $D_i\mathbf{u}$ in $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$ now follows from the definition of $D_i\mathbf{u}$ and from the square integrability of $\partial\mathbf{u}/\partial x_i$ in Ω_0 and in Ω_{-1}.

To prove that $\mathbf{u}$ is the solution of problem in domain D, we must show that function $\mathbf{u}$, moreover, fulfills the integral identity (8) rewritten for domain D for an arbitrary acceptable test function $\mathbf{w}$ in D (i.e. a test function $\mathbf{w}$ that is an element of $H^1(D')^2$, where $D' \subset D$ is compact, $\mathbf{w}$ has zero traces on G_i and G_w, its divergence is equal to zero a.e. in D and it has a compact support in $D \cup G_o$). Due to the periodicity of $\mathbf{u}$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ in the x_2–direction, it is sufficient to consider only such test functions $\mathbf{w} \in C^\infty(\overline{D})^2$ that have a compact support in $D \cup G_o$. It means that we can work only with test functions $\mathbf{w}$ such that $\mathbf{w}(x_1, x_2) = \mathbf{0}$, if $|x_2| > K(\mathbf{w})$, where $K(\mathbf{w})$ is a positive constant depending on $\mathbf{w}$. The validity of (8) rewritten for domain D for $\mathbf{w} \in H^1(D)^2$ having a compact support in $D \cup G_o$ can afterwards be proven by means of an appropriate limit procedure. (We can use the density of the space of all infinitely differentiable test functions $\mathbf{w}$ with the described properties in the space of test functions $\mathbf{w} \in H^1(D)^2$.)

First let us show the validity of (8) rewritten for domain D for a smooth admissible test function $\mathbf{w}$ that equals zero outside $\overline{\Omega_0} \cup \overline{\Omega_{-1}}$.

Lemma 2: Suppose that $\mathbf{u}$ is a function which satisfies the assumptions of Theorem 1. Then $\mathbf{u}$ fulfills the integral identity (8) rewritten for domain D for each test function $\mathbf{w} \in C^\infty(\overline{D})^2$ such that $\mathbf{w} = \mathbf{0}$ on $G_i \cup G_w$, $\operatorname{div} \mathbf{w} = 0$ in D and $\mathbf{w} = \mathbf{0}$ in $D - (\overline{\Omega}_0 \cup \overline{\Omega}_{-1})$.
Proof. Function $\mathbf{u}$ is a weak solution in the domain $\Omega \equiv \Omega_0$. Hence,

$$\int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = \nu \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}S + \int_{\Gamma_o} \mathbf{h} \cdot \mathbf{v} \, \mathrm{d}S. \tag{11}$$

for each test function $\mathbf{v} \in V$. We shall prove the validity of (8) rewritten for domain D with a test function $\mathbf{w}$, satisfying the assumptions of Lemma 2. Since $\mathbf{w}$ can differ from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_{-1}$, we can integrate only over $\Omega_0 \cup \Omega_{-1}$ instead of D. Thus, (8) rewritten for domain D takes the form

$$\int_{\Omega_0 \cup \Omega_{-1}} \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = \nu \int_{\Omega_0 \cup \Omega_{-1}} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, \mathrm{d}\mathbf{x} + \int_{\Omega_0 \cup \Omega_{-1}} \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} w_i \, \mathrm{d}\mathbf{x} + \int_{B_{-1}B_1} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \mathbf{u} \cdot \mathbf{w} \, \mathrm{d}S + \int_{B_{-1}B_1} \mathbf{h} \cdot \mathbf{w} \, \mathrm{d}S. \tag{12}$$

The integrals on $\Omega_0 \cup \Omega_{-1}$ are equal to the sum of two integrals on Ω_0 and on Ω_{-1}. Similarly, the integrals on the line segment $B_{-1}B_1$ are equal to the sum of two integrals on line segments $B_{-1}B_0$ and B_0B_1. (The line segment B_0B_1 coincides with Γ_o.) The integrals over Ω_{-1} (respectively along $B_{-1}B_0$) can easily be transformed (just shifting the system of coordinates) to the integrals over Ω_0 (respectively along B_0B_1). We can show it, for example, in the case of the first integral on the right-hand side of (12):

$$\int_{\Omega_{-1}} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1,x_2) \frac{\partial w_i}{\partial x_j}(x_1,x_2) \, \mathrm{d}\mathbf{x} = \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1,x_2-\tau) \frac{\partial w_i}{\partial x_j}(x_1,x_2-\tau) \, \mathrm{d}\mathbf{x}$$
$$= \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1,x_2) \frac{\partial w_i}{\partial x_j}(x_1,x_2-\tau) \, \mathrm{d}\mathbf{x}.$$

(We have used the τ–periodicity of function $\mathbf{u}$ in variable x_2.) If we apply the same procedure to all integrals over Ω_{-1} or along $B_{-1}B_0$ in (12) and then sum the integrals over Ω_0 and along B_0B_1, we obtain

$$\int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = \nu \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\mathbf{x} + \int_{\Omega_0} \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}S + \int_{\Gamma_o} \mathbf{h} \cdot \mathbf{v} \, \mathrm{d}S, \tag{13}$$

where $\mathbf{v}(x_1,x_2) = \mathbf{w}(x_1,x_2) + \mathbf{w}(x_1,x_2-\tau)$ for $(x_1,x_2) \in \Omega_0$. This identity is of the same form as (8). In order to verify that (13) holds, we need to show that function $\mathbf{v}$ used in (13) has all the properties required in (8), i.e. that $\mathbf{v} \in V$ and then the validity of (13) will immediately follow from (8). Due to the assumption that $\mathbf{w} = \mathbf{0}$ outside $(\overline{\Omega}_0 \cup \overline{\Omega}_{-1})$ and the continuity of $\mathbf{w}$, we have $\mathbf{w}(x_1, x_2 \pm \tau) = \mathbf{0}$ for $(x_1, x_2) \in \Gamma_-$. This implies that $\mathbf{v}$ satisfies, for $(x_1, x_2) \in \Gamma_-$, the condition of periodicity in the x_2–direction:

$$\begin{aligned} \mathbf{v}(x_1, x_2 + \tau) &= \mathbf{w}(x_1, x_2 + \tau) + \mathbf{w}(x_1, x_2) \\ &= \mathbf{w}(x_1, x_2) + \mathbf{w}(x_1, x_2 - \tau) \;=\; \mathbf{v}(x_1, x_2). \end{aligned}$$

Thus, $\mathbf{v} \in V$, (13) is satisfied and consequently, the identities (12) and (8) (rewritten for D) are also satisfied. $\square$

Suppose further that $\mathbf{w}$ is an infinitely differentiable divergence–free vector function in $\overline{D}$, equal to zero on G_i and on G_w, and such that $\mathbf{w}(x_1, x_2) = \mathbf{0}$ if $|x_2| > K(\mathbf{w})$. In order to complete the proof of Theorem 1, we need one more lemma.

Lemma 3: Let a function $\mathbf{w}$ satisfy the above assumptions. Then there exists a function $\psi \in C^\infty(\overline{D})$ (the so-called "stream function") such that

a) $\psi(x_1, x_2) = 0$ for $|x_2| > K(\mathbf{w})$,

b) $w_1 = \dfrac{\partial \psi}{\partial x_2}, \quad w_2 = -\dfrac{\partial \psi}{\partial x_1} \quad$ in D.

Proof. We can choose an integer N so large that $\mathbf{w} = \mathbf{0}$ in $\overline{\Omega}_i$ for all $i \in Z$ such that $|i| \geq N$. Let us denote

$$D_N \;=\; \Omega_{-N} \cup \bigcup_{k=N-1}^{N} \Big(\Omega_k \cup \Gamma^k \Big).$$

D_N is a bounded domain. Its boundary ∂D_N has the following components:

a) the line segment $A_{-N}A_N$ (which lies on the straight line G_i),

b) the curves Γ^{-N} and Γ^{N+1},

c) the line segment $B_{-N}B_N$ (which lies on the straight line G_o)

d) and the curves C_i for $i = -N, \dots, N$.

Theorem 3.1 in [5], page 37, provides the existence of a function $\psi \in C^\infty(\overline{D_N})$ such that

$$w_1 \;=\; \frac{\partial \psi}{\partial x_2}, \qquad w_2 \;=\; -\frac{\partial \psi}{\partial x_1} \qquad \text{in } D_N. \tag{14}$$

(An analogous theorem can be found in [4].) Due to the smoothness of the function ψ in $\overline{D_N}$, that first formula in (14) also holds on the open line segment $A_{-N}A_N$. Since the function w_1 equals zero on this line segment, the derivative of ψ with respect to x_2 also equals zero and consequently, the function ψ is constant on the line segment $A_{-N}A_N$. The constant can be chosen to be zero because the function ψ is given uniquely up to an additive constant. Using the identity $\mathbf{w} = \mathbf{0}$ and the second formula in (14) in Ω_N and in Ω_{-N}, we can derive that $\psi = 0$ in both domains Ω_N and Ω_{-N}. (Since $w_2 = 0 = -\frac{\partial \psi}{\partial x_1}$ and $\psi = 0$ on $A_{-N}A_N$).

If we extend function ψ from domain D_N onto the whole domain D by zero, we obtain a function with all the properties stated in Lemma 3. $\square$

Continuation of the proof of Theorem 1. Let us denote by η an infinitely differentiable function of one variable defined in the interval $(-\infty, +\infty)$ such that its support is contained in $(-\tau, \tau)$, its range is $[0, 1]$ and

$$\eta(x_2) + \eta(x_2 + \tau) = 1 \qquad \text{for } x_2 \in [-\tau, 0]. \tag{15}$$

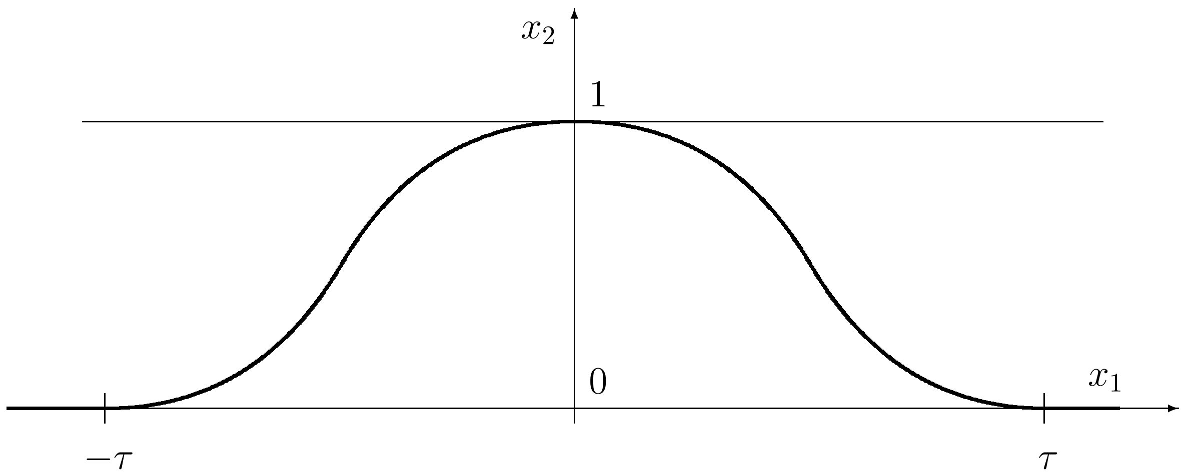

Fig. 3 (the example of function η)

If N is an integer then

$$\sum_{k=-N}^{+N} \eta(x_2 + k\tau) \quad \begin{cases} = 0 & \text{for } x_2 \in \big(-\infty, -(N+1)\tau\big] \cup \big[(N+1)\tau, +\infty\big), \\ \in [0, 1] & \text{for } x_2 \in \big[-(N+1)\tau, -N\tau\big] \cup \big[N\tau, (N+1)\tau\big], \\ = 1 & \text{for } x_2 \in \big[-N\tau, N\tau\big]. \end{cases}$$

(In the first case, $x_2 + k\tau$ is outside the interval $(-\tau, \tau)$ for all $k = -N, \dots, N$. In the second case, just one of the points $x_2 + k\tau$ belongs to $(-\tau, \tau)$. In the third case, just two of the points $x_2 + k\tau$ find themselves in the region where $\eta \neq 0$ and the sum of the function values of η at these points equals one due to (15).)

Further, we put $\zeta(x_1, x_2) := \eta(x_2 - \gamma(x_1))$. We can observe that

$$\sum_{k=-N}^{+N} \zeta(x_1, x_2 + k\tau) \quad \begin{cases} = 0 & \text{for } (x_1, x_2) \in \overline{\Omega}_i; \quad |i| \geq N+1, \\ \in [0, 1] & \text{for } (x_1, x_2) \in \overline{\Omega}_N \cup \overline{\Omega}_{-N}, \\ = 1 & \text{for } (x_1, x_2) \in \overline{\Omega}_i; \quad |i| < N. \end{cases} \tag{16}$$

(In the first case, $x_2 + k\tau - \gamma(x_1)$ is outside the interval $(-\tau, \tau)$ for all $k = -N, \dots, N$. In the second case, just one of the points $x_2 + k\tau - \gamma(x_1)$ belongs to $(-\tau, \tau)$. In the third case, just two of the points $x_2 + k\tau - \gamma(x_1)$ are in the region where $\eta \neq 0$.) Obviously, the functions $\zeta(x_1, x_2 + k\tau)$ represent an appropriate partition of unity.

If N is the same number as in the proof of Lemma 3, then $\psi = 0$ in $\overline{\Omega}_i$ for $|i| \geq N$ and

$$\psi(x_1, x_2) = \psi(x_1, x_2) \sum_{k=-N}^{+N} \zeta(x_1, x_2 + k\tau) \qquad \text{for } (x_1, x_2) \in D. \tag{17}$$

For $k \in Z$ we define, a vector function $\mathbf{w}^k \equiv (w_1^k, w_2^k)$ by the formulas

$$\begin{aligned} w_1^k(x_1, x_2) &:= \frac{\partial}{\partial x_2} [\, \psi(x_1, x_2)\, \zeta(x_1, x_2 + k\tau)\,], \\ w_2^k(x_1, x_2) &:= -\frac{\partial}{\partial x_1} [\, \psi(x_1, x_2)\, \zeta(x_1, x_2 + k\tau)\,]. \end{aligned}$$

The function $\mathbf{w}^0$ differs from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_1$. By analogy, the function $\mathbf{w}^k$ (for a general $k \in Z$) differs from zero only in $\overline{\Omega}_k \cup \overline{\Omega}_{k+1}$ and $\mathbf{w}^{-k}$ differs from zero only in $\overline{\Omega}_{-k} \cup \overline{\Omega}_{-k-1}$. From (17) it follows that

$$\mathbf{w}(x_1, x_2) = \sum_{k=-N}^{+N} \mathbf{w}^k(x_1, x_2)$$

for $(x_1, x_2) \in D$.

Now we use this function $\mathbf{w}$ in the integral identity (8) rewritten for domain D. Obviously, if this identity is separately satisfied for each function $\mathbf{w}^k$ ($k = -N, -N+1, \ldots, N$), then it is also satisfied for the test function $\mathbf{w}$.

The identity (8) rewritten for domain D, with the test function $\mathbf{w}^k$, has the form

$$\begin{aligned} \int_D \mathbf{f} \cdot \mathbf{w}^k \, \mathrm{d}\mathbf{x} &= \nu \int_D \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i^k}{\partial x_j} \, \mathrm{d}\mathbf{x} + \int_D \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} w_i^k \, \mathrm{d}\mathbf{x} \\ &\quad + \int_{G_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \, \mathbf{u} \cdot \mathbf{w}^k \, \mathrm{d}S + \int_{G_o} \mathbf{h} \cdot \mathbf{w}^k \, \mathrm{d}S. \end{aligned}$$

In order to simplify the integrals, we use the substitution $x_1 = \bar{x}_1,\ x_2 = \bar{x}_2 + k\tau$. If we denote $\bar{x} = (\bar{x}_1, \bar{x}_2),\ \bar{\mathbf{w}}^k(\bar{x}_1, \bar{x}_2) = \mathbf{w}^k(x_1, x_2 - k\tau)$ and use the equality $\mathbf{u}(\bar{x}_1, \bar{x}_2) = \mathbf{u}(x_1, x_2 - k\tau) = \mathbf{u}(x_1, x_2)$ (following from the periodicity of the function $\mathbf{u}$), we obtain,

$$\begin{aligned} \int_D \mathbf{f} \cdot \bar{\mathbf{w}}^k \, \mathrm{d}\bar{\mathbf{x}} &= \nu \int_D \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial \bar{x}_j} \frac{\partial \bar{w}_i^k}{\partial \bar{x}_j} \, \mathrm{d}\bar{\mathbf{x}} + \int_D \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial \bar{x}_j} \bar{w}_i^k \, \mathrm{d}\bar{\mathbf{x}} \\ &\quad + \int_{G_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \, \mathbf{u} \cdot \bar{\mathbf{w}}^k \, \mathrm{d}S + \int_{G_o} \mathbf{h} \cdot \bar{\mathbf{w}}^k \, \mathrm{d}S. \end{aligned} \tag{18}$$

Function $\bar{\mathbf{w}}^k$ differs from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_{-1}$ and fulfills all the assumptions put on a test function in Lemma 2. Thus, we can apply Lemma 2 and we see that identity (18) holds. This completes the proof. $\square$

Conclusion. The presented result shows that its legal to study analytically or numerically just one period of the cascade of profiles. The obtained result can be extended to get the weak solution to the whole infinite domain. This is very important for numerical calculations. Similar result can be obtained for classical solution.

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