# On the motion of rigid bodies in an incompressible or compressible viscous fluid under the action of gravitational forces

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## 1 Introduction and weak formulation

We consider the motion of several rigid bodies in a non-Newtonian fluid of power-law type (see Chapter 1 in Málek et al. [M] for details), where the viscous stress tensor $\mathbb{S}$ depends on the symmetric part $\mathbb{D}[\mathbf{u}]$, $\mathbb{D}[\mathbf{u}] = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}$ of the gradient of the velocity field $\mathbf{u}$ in the following way: Assumptions (A1): $\mathbb{S} = \mathbb{S}[\ \mathbb{D}[\mathbf{u}]\ ]$, $\mathbb{S} : R^{3\times 3}_{\rm sym} \to R^{3\times 3}_{\rm sym}$ is continuous, $\Big(\mathbb{S}[\mathbb{M}] - \mathbb{S}[\mathbb{N}]\Big) : \Big(\mathbb{M} - \mathbb{N}\Big) > 0$ for all $\mathbb{M} \neq \mathbb{N}$, and $c_1|\mathbb{M}|^p \leqslant \mathbb{S}[\mathbb{M}] : \mathbb{M} \leqslant c_2(1 + |\mathbb{M}|^p)$ for a certain $p \geqslant 4$ and newtonian case for compressible case.

For the description of the initial position of the bodies see [DN]. The mass density $\varrho = \varrho(t, \mathbf{x})$ and the velocity field $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ at a time $t \in (0, T)$ and the spatial position $\mathbf{x} \in \Omega$ satisfy the integral identity $\int_0^T \int_\Omega \Big(\rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla_x \phi\Big) {\rm d}x\ {\rm d}t = -\int_\Omega \rho_0 \phi\ {\rm d}x,\ \ \phi \in C^1([0,T) \times \bar{\Omega})$,

$\int_0^T \int_\Omega \Big(\rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x [\varphi] - \mathbb{S} : \mathbf{D}[\varphi]\Big)\ {\rm d}x\ {\rm d}t$
$= -\int_0^T \int_\Omega \rho G \nabla_x \int_{R^3} \frac{\rho}{|x-y|} dy \cdot \varphi\ {\rm d}x\ {\rm d}t - \int_\Omega \rho_0 \mathbf{u}_0 \cdot \varphi\ {\rm d}x\ {\rm d}t$
$\varphi \in C^1([0,T) \times \bar{\Omega})$, $\varphi(t, \cdot) \in \mathcal{R}(t)$,
$\mathcal{R}(t) = \{\phi \in C^1(\bar{\Omega})\ |\ {\rm div}\ \Phi = 0$ in $\Omega$, $\phi = 0$ on a neighborhood of $\partial\Omega$, $\mathbf{D}[\Phi] = 0$ on a neighborhood of $\cup_{i=1}^n \bar{\mathbf{B}}_i(t)\}$, where $\int_0^T \int_\Omega \rho G \nabla_x \Big(\int_{R^3} \frac{\rho}{|x-y|} dy\Big) \varphi\ dx\ dt =$

$\int_0^T \int_\Omega \rho G \nabla_x F\ dx\ {\rm d}t$, with $F = \Big(\sum_{i \neq j} \int_{R^3} \frac{\rho_j^{B_j}}{|x-y|}\ dy + \int_{R^3} \frac{\rho^f}{|x-y|}\ dy\Big)$. Finally, we require the velocity field $u$ to be compatible with the motion of bodies. As the mappings $\eta_i(t, i)$ are isometries on $R^3$, they can be written in the form $\eta_i(t, \mathbf{x}) = x_i(t) + \mathcal{O}_i(t)\mathbf{x}$. Accordingly, we impose to the velocity field $u$ to be compatible with the family of motions $\{\eta_1, \dots, \eta_n\}$ if $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}^{B_i}(t, \mathbf{x}) = \mathbf{U}_i(t) + \mathcal{Q}_i(t)(\mathbf{x} - x_i(t))$ for a.a. $x \in \bar{\mathbf{B}}_i(t)$, $i = 1, \dots, n$ for a.a. $t \in [0, T)$, where $\frac{\rm d}{{\rm d}t} x_i = \mathbf{U}_i$, $\Big(\frac{\rm d}{{\rm d}t} \mathcal{O}_i\Big) \mathcal{O}_i^T = \mathcal{Q}_i$ a.a. on $(0, T)$.

**Problem P**
*Let the initial distribution of the density and the velocity field be determined through given functions* $\rho_0$, $\mathbf{u}_0$, *respectively. The initial position of the rigid bodies being* $B^i \subset \Omega$, $i = 1, ..., m$. *We say that a family* $\rho$, $\mathbf{u}$, $\eta^i$, $i = 1, ..., m$, *represent a variational solution of* **problem (P)** *on a time interval* $(0, T)$ *if the following conditions are satisfied:*

(1) The density $\rho$ is a non-negative bounded function, the velocity field $\mathbf{u}$ belongs to the space $L^\infty(0,T;L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3))$, and they satisfy energy inequality (EI) for $t_1 = 0$ and a.a. $t_2 \in (0,T)$,

(2) The continuity equation holds on $(0,T) \times R^3$ provided $\rho$ and $\mathbf{u}$ are extended to be zero outside $\Omega$.

(3) Momentum equation (the integral identity) holds for any admissible test function $\mathbf{w} \in \mathcal{R}(t)$ .

(4) The mappings $\eta^i$, $i = 1,...,m$ are affine isometries of $R^3$ compatible with the velocity field $\mathbf{u}$ in the sense of compatibility conditions.

Let us formulate one of our main existence results.

**Theorem 1.1** *Let the initial position of the rigid bodies be given through a family of open sets*

$$\mathbf{B}_i \subset \Omega \subset R^3,\ \mathbf{B}_i \textit{ diffeomorphic to the unit ball for } i = 1,\dots,n,$$

*where both $\partial\mathbf{B}_i$, $i = 1,\dots,n$, and $\partial\Omega$ belong to the regularity class see [DN]. In addition, suppose that*

$$\mathrm{dist}[\overline{\mathbf{B}}_i, \overline{\mathbf{B}}_j] > 0 \textit{ for } i \neq j,\ \mathrm{dist}[\overline{\mathbf{B}}_i, R^3 \setminus \Omega] > 0 \textit{ for any } i = 1,\dots,n$$

*and we assume that boundary of $\Omega$ and $\mathbf{B}_i$ belong to $C^{2,\nu}$, $\nu \in (0,1)$. Furthermore, let the viscous stress tensor $\mathbb{S}$ satisfy hypotheses (A1), with $p \geqslant 4$.*

*Finally, let the initial distribution of the density be given as*

$$\varrho_0 = \begin{cases} \varrho_f = \mathrm{const} > 0 \textit{ in } \Omega \setminus \cup_{i=1}^n \overline{\mathbf{B}}_i, \\ \\ \varrho_{\mathbf{B}_i} \textit{ on } S_i, \textit{ where } \varrho_{\mathbf{B}_i} \in L^\infty(\Omega),\ \mathrm{ess\,inf}_{\mathbf{B}_i} \varrho_{\mathbf{B}_i} > 0,\ i = 1,\dots,n, \end{cases}$$

*while*

$$\mathbf{u}_0 \in L^2(\Omega;R^3),\ \mathrm{div}_x \mathbf{u}_0 = 0 \textit{ in } \mathcal{D}'(\Omega),\ \mathbb{D}[\mathbf{u}_0] = 0 \textit{ in } \mathcal{D}'(\mathbf{B}_i;R^{3\times 3}) \textit{ for } i = 1,\dots,n.$$

*Then there exist a density function $\varrho$,*

$$\varrho \in C([0,T];L^1(\Omega)),\ 0 < \operatorname*{ess\,inf}_\Omega \varrho(t,\cdot) \leqslant \operatorname*{ess\,sup}_\Omega \varrho(t,\cdot) < \infty \textit{ for all } t \in [0,T],$$

*a family of isometries $\{\eta_i(t,\cdot)\}_{i=1}^n$, $\eta_i(0,\cdot) = \mathrm{I}$, and a velocity field $\mathbf{u}$,*

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3)),$$

*compatible with $\{\eta_i\}_{i=1}^n$ in the sense specified in (3.7), (3.8), such that $\varrho$, $\mathbf{u}$ satisfy the integral identity (3.3) for any test function $\phi \in C^1([0,T) \times R^3)$, and the integral identity (3.4) for any $\varphi$ satisfying (3.5), (3.6).*

For compressible case see [DN1].

**References**


[DN] B. Ducomet, Š. Nečasova: On the motion of several rigid bodies in an incompressible viscous fluid under the influence of selfgravitating forces, *"Parabolic Problems: Herbert Amann Festschrift", Birkhäuser*, To appear

[DN1] On the motion of rigid bodies in an incompressible or compressible viscous fluid under the action of gravitational forces, Preprint 2011

[M] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDE's.* Chapman and Hall, London, 1996.