Dispersion

due to

- space discretization
- time discretization
- geometry
- material

It is known that

the spatial and temporal discretizations always accompany the finite element modelling of transient problems,

the topic is of academic interest but has important practical consequences as well.

In the contribution

the spatial dispersive properties of typical Lagrangian and Hermitian elements will be presented,

the dispersion due to time discretization will be briefly mentioned and

examples of spatial and temporal dispersions will be shown.

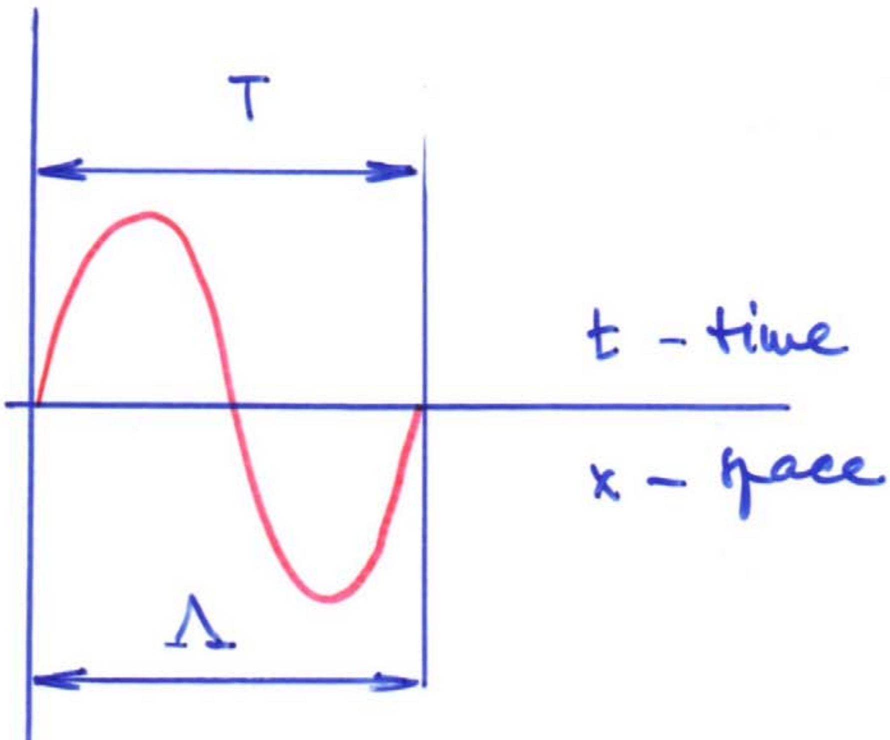
NOTATION and TERMINOLOGY

a periodic (harmonic) monochromatic wave can be described

in space by Λ ... wavelength

in time by $T = 2\pi/\omega$ ... period

ω ... angular freq.



$k = \frac{2\pi}{\Lambda}$ ... angular wavenumber

PHASE VELOCITY

$$v = \frac{\Lambda}{T} = \frac{\omega}{k}$$

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The phase velocity is very often related to the speed of a 1D wave in a thin bar given by

$$c_0 = \sqrt{E/\rho}$$

Assumed solution of a discrete system

$$y = A e^{i(\gamma x \pm \omega t)} = A e^{i\gamma\left(x - \frac{\omega}{\gamma} t\right)}$$

$$= A e^{i\gamma(x - v t)}$$

A DISPERSIVE SYSTEM IS ADMITTING SOLUTIONS IN THE FORM

$$u = A\, e^{i\,(k\,x - \omega\, t)} \;,$$

where

A = A (ω) ... function of frequency ω,

x,t ... spatial and temporal variables,

k = ω/c ..... wavenumber,

c ..... velocity of propagation.

the quantity u could be displacement, voltage, etc.

Other quantities entering into consideration are

$\lambda = 2\pi / k$ ... wavelength,

$T = 2\pi / \omega$ ... period.

If $\omega = \omega(k)$

... a so called dispersion relation ...

is a linear function of k ... the wavenumber,

then

the system is called NONDISPERSIVE,

otherwise it is DISPERSIVE.

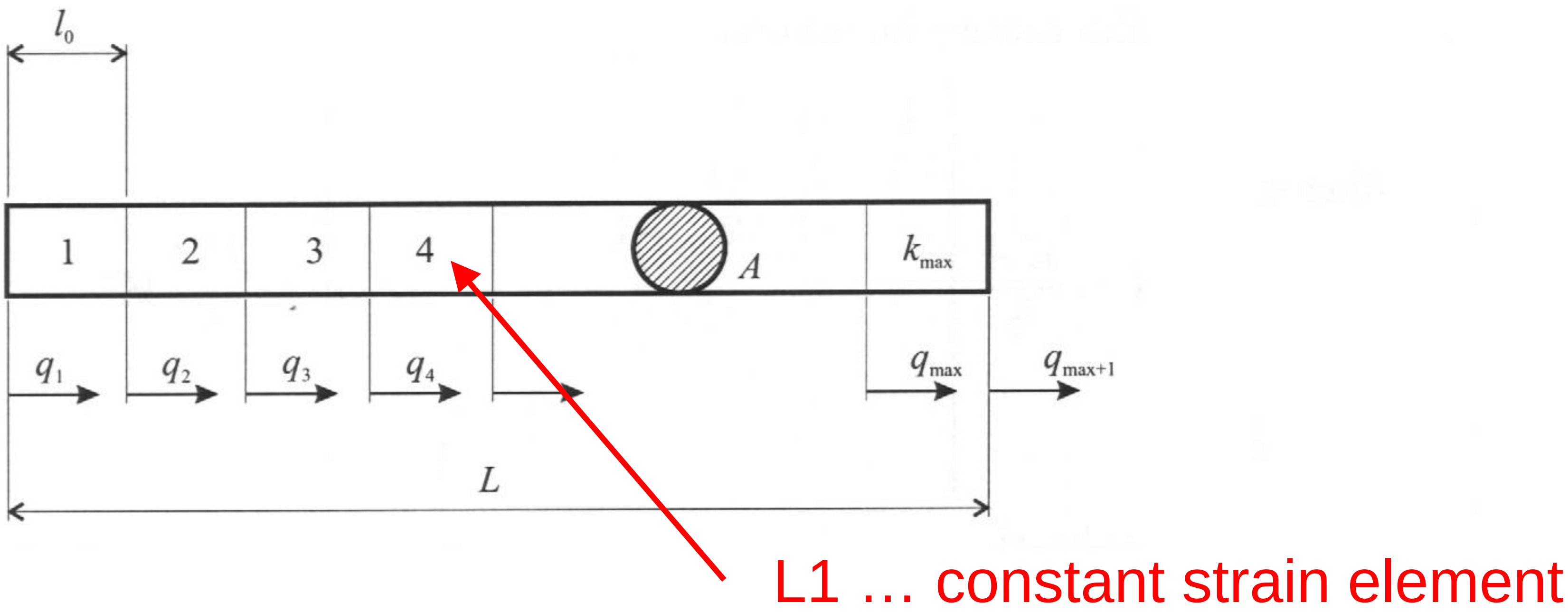
For non-dispersive systems the velocity of propagation is constant – does not depend on frequency.

Analytical approach ... 1D example

$$q = C\, e^{i\,(k\,x - \omega\, t)}$$

Assumed solution (a harmonic wave) written for nodes is substituted into 'characteristic equations' taken out of equations of motion

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\}\,.$$



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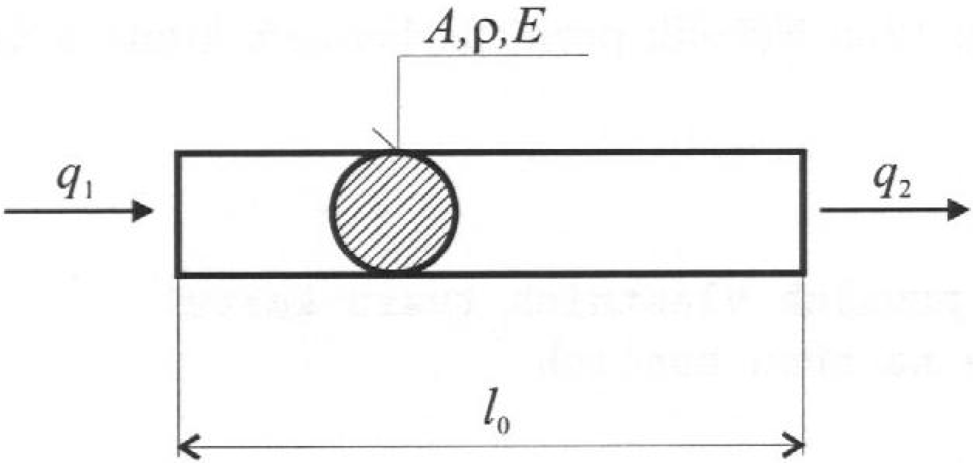

Constant strain 1D element

Mass and stiffness matrices

$$\mathbf{m}_\mathrm{C} = \frac{\rho A l_0}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$ consistent

$$\mathbf{m}_\mathrm{D} = \frac{\rho A l_0}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$ diagonal

$$\mathbf{k} = \frac{EA}{l_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

This gives a system of homogenous algebraic equations in the form

$$[A(\omega, a)]\{c\} = \{0\}, \qquad (*)$$

where

ω ... frequency,
a ... dimensionless wave number kl_0.

Algebra manipulation languages could help a lot in this process

Nontrivial solution of (*) only if $\det [A] = 0$

$\Rightarrow$ dispersive equation(s) in the form

$$F(\omega, a) = 0, \qquad \omega = f(a) \quad .$$

Dispersion for 1D constant strain elements with consistent and diagonal mass matrix formulations

$$\omega^* = \sqrt{\frac{6(1-\cos(kl_0))}{2+\cos(kl_0)}} \quad \text{for consistent mass matrix}$$

$$\omega^* = \sqrt{2(1-\cos(kl_0))} \quad \text{for diagonal mass matrix}$$

$$\text{where } \omega^* = \frac{\omega}{\omega_0} \quad \text{and} \quad \omega_0 = \frac{c_0}{l_0} \quad \text{with} \quad c_0 = \sqrt{\frac{E}{\rho}}$$

Numerical approach L1 elements

$$\frac{\rho A\, l_0}{6}[M]\{\ddot{q}\} + \frac{E\,A}{l_0}[K]\{q\} = \{0\}$$

assumption

$$\{q\} = \{Q\}\, e^{i\omega\, t} \Rightarrow \qquad \{\ddot{q}\} = -\,\omega^2\{Q\}\, e^{i\omega\, t}$$

$$\left[[K] - \underbrace{\omega^2 \frac{l_0^2}{c_0^2}\frac{1}{6}}_{\lambda}\, [M] \right]\{Q\} = \{0\}$$

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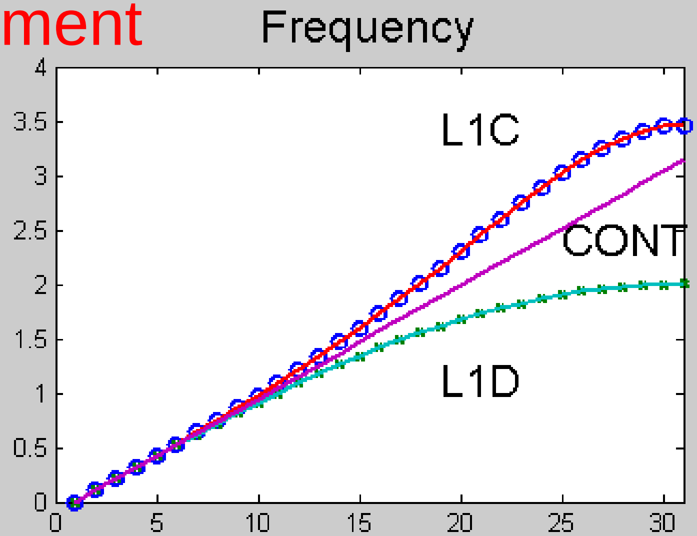
... generalized eigenvalue problem, where

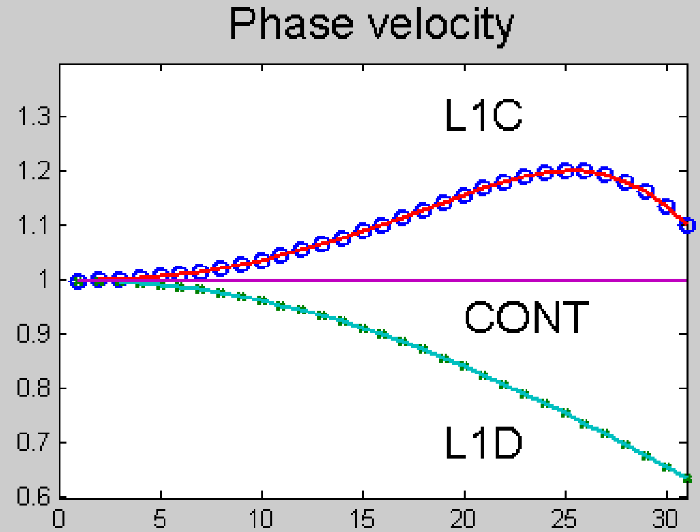
$$\omega_k = \sqrt{\lambda_k}\,\frac{\sqrt{6}\,c_0}{l_0} = \sqrt{6}\,\sqrt{\lambda_k}\,\omega_0$$

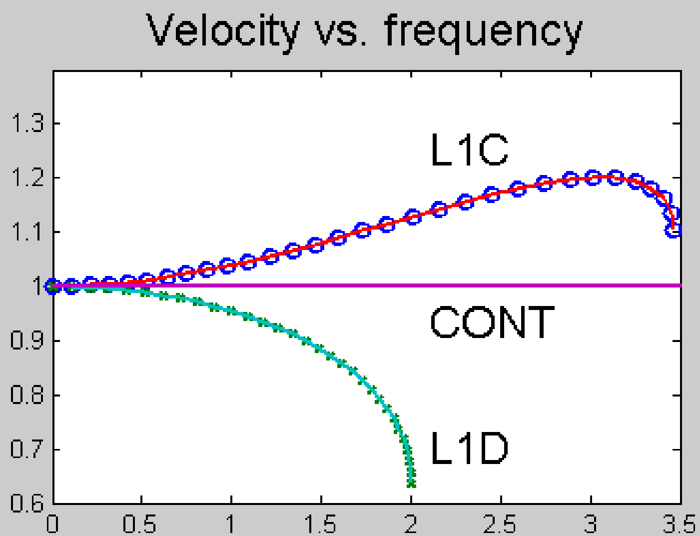
and

$$\omega_0 = \frac{c_0}{l_0}$$

L1 element





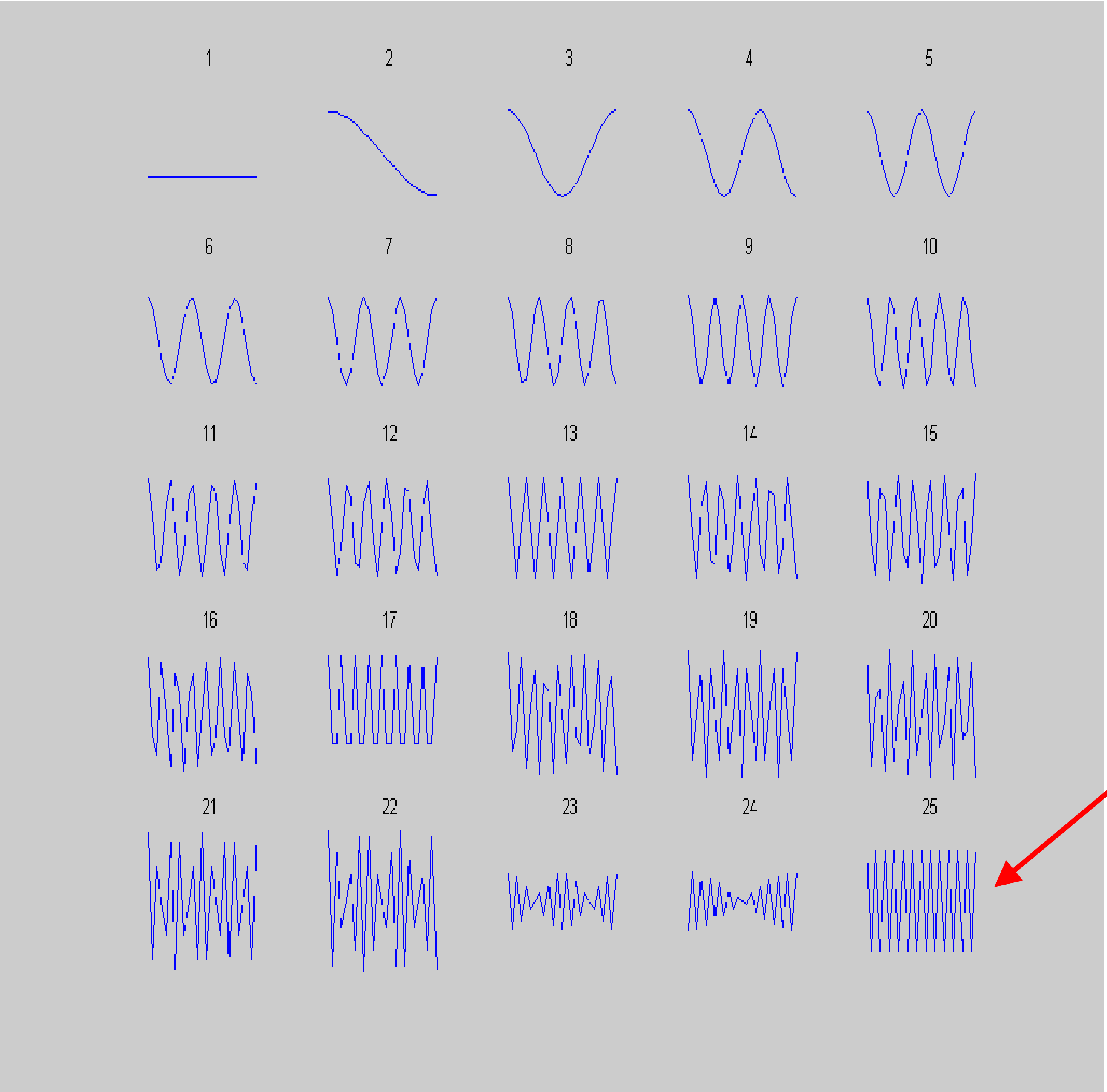


1D continuum is a non-dispersive medium, it has infinite number of frequencies, phase velocity is constant regardless of frequency.

Discretized model is dispersive,
it has finite number of frequencies,
velocity depends on frequency,
spectrum is bounded,
there are cut-offs.

Frequency (velocity) is overestimated / underestimated with consistent / diagonal mass matrix.

Eigenvectors, unconstrained rod, 24 L1C elements



Notice the rigid body mode

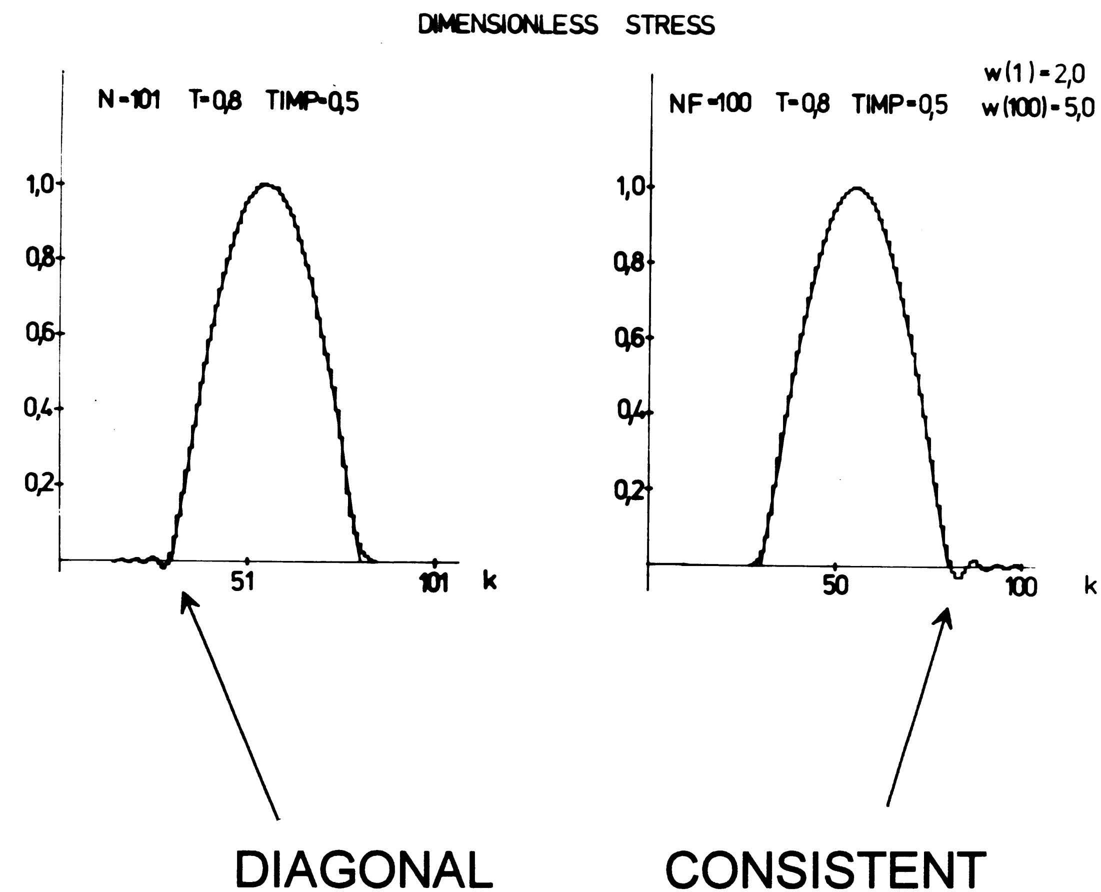
Higher modes have wrong shapes

Not in accordance with assumed harmonic solution

Nodes in opposition

The system cannot be forced to vibrate more violently

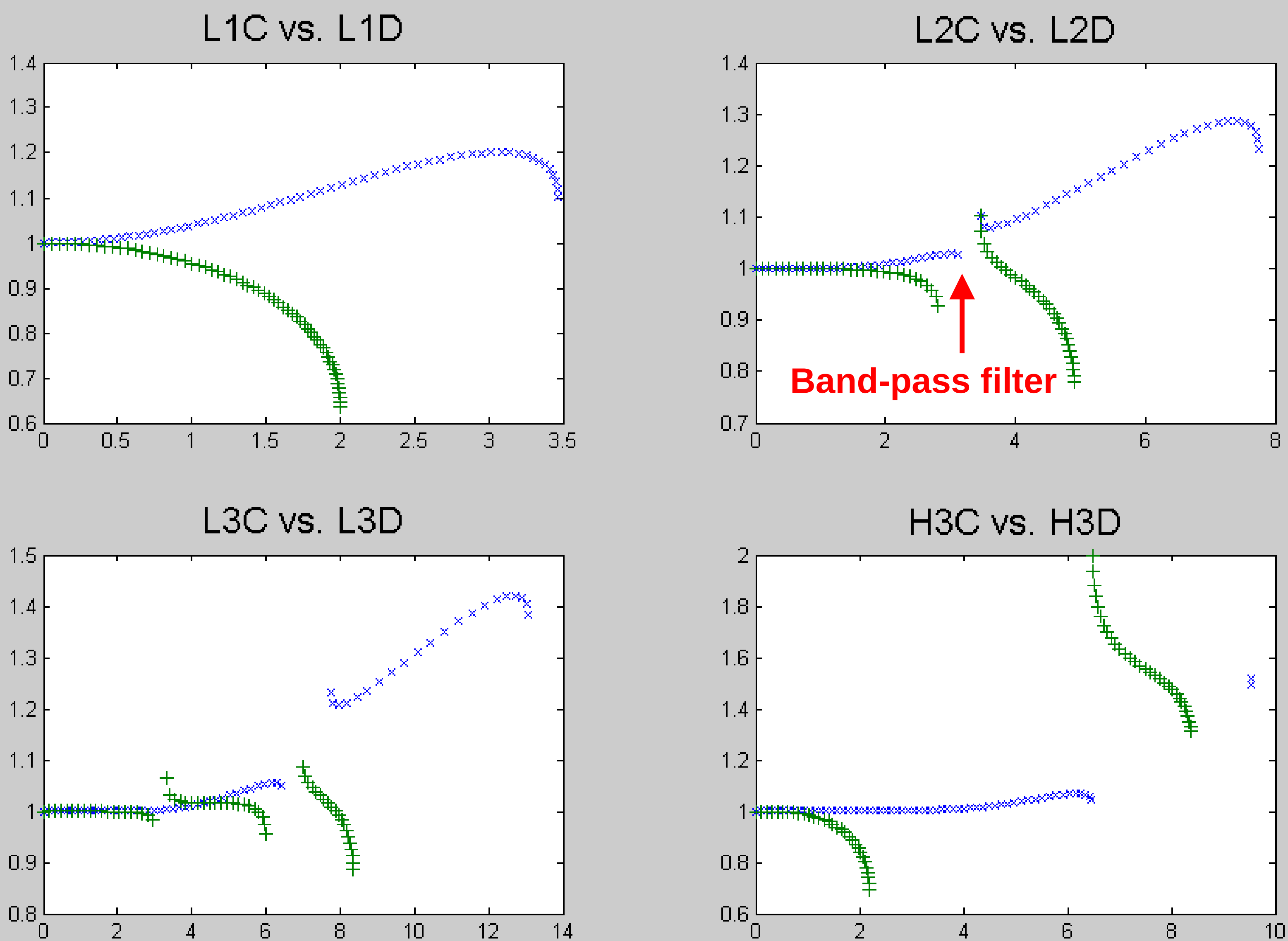
THE INFLUENCE OF MASS MATRIX FORMULATION



- SPURIOUS HIGH FREQUENCY COMPONENTS

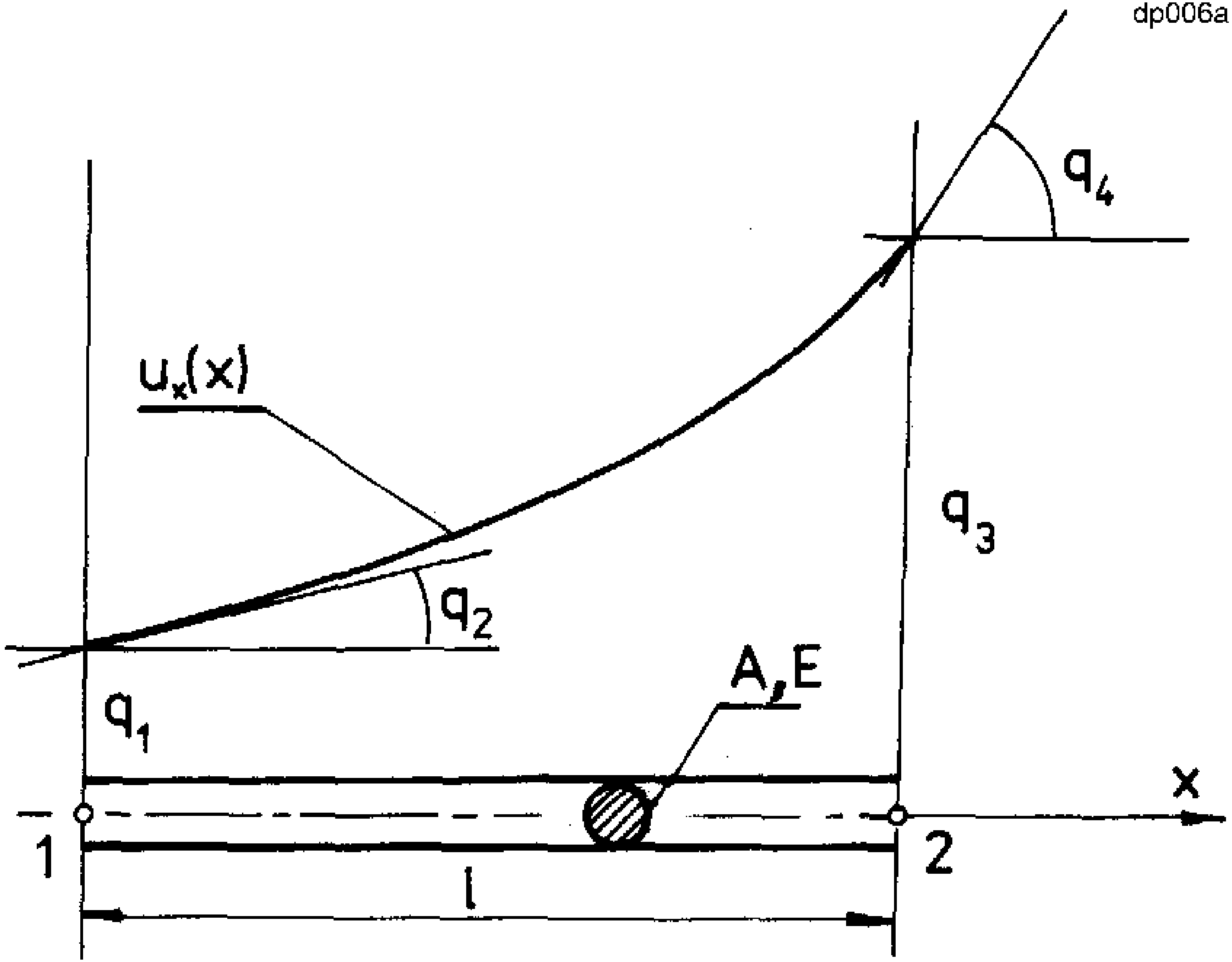
- FALSE ARTIFACTS, PURE EFFECTS OF DISCRETIZATION

1D – velocity vs. frequency for higher order elements: L2, L3, H3

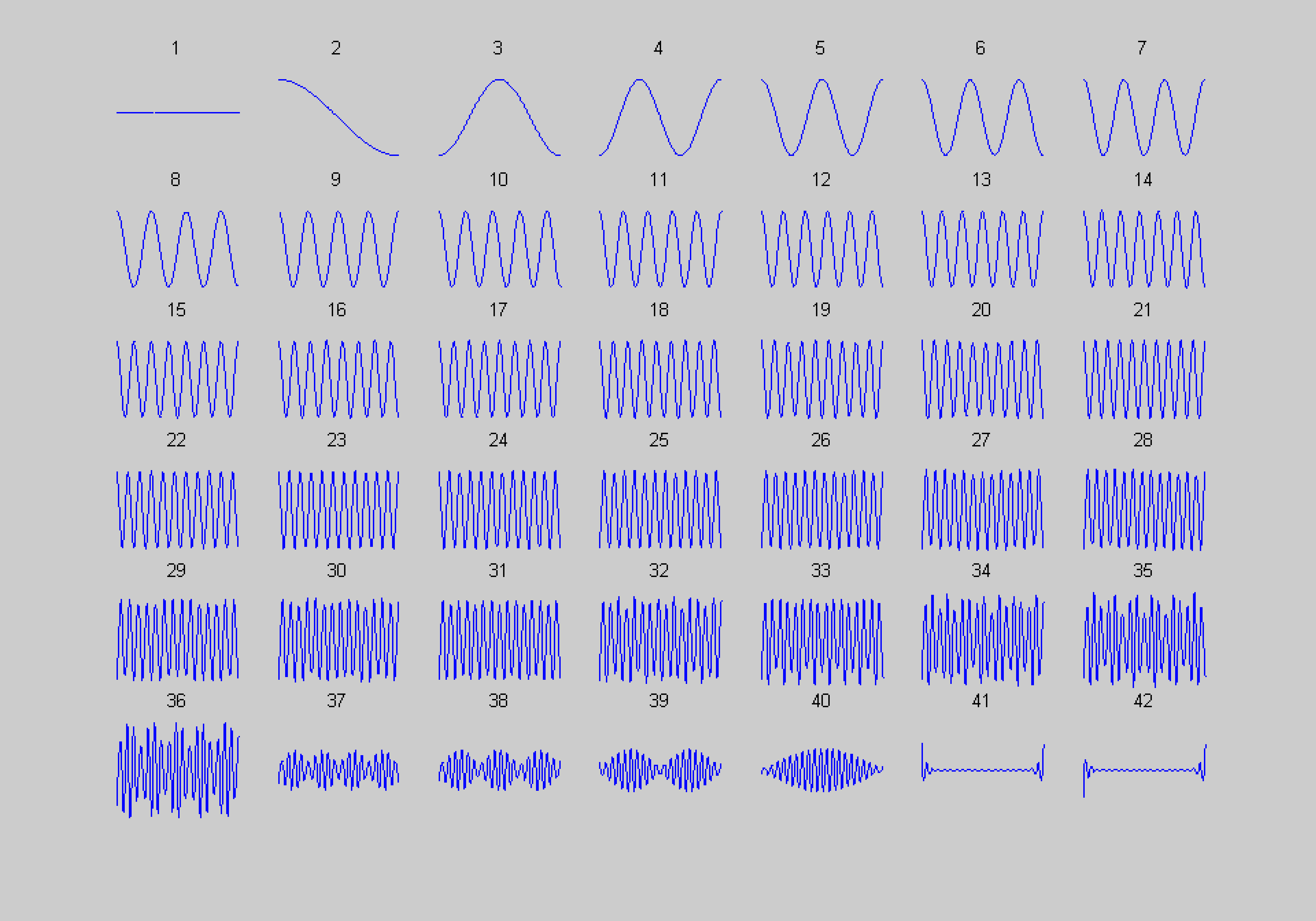


A mixed blessing of a longer spectrum

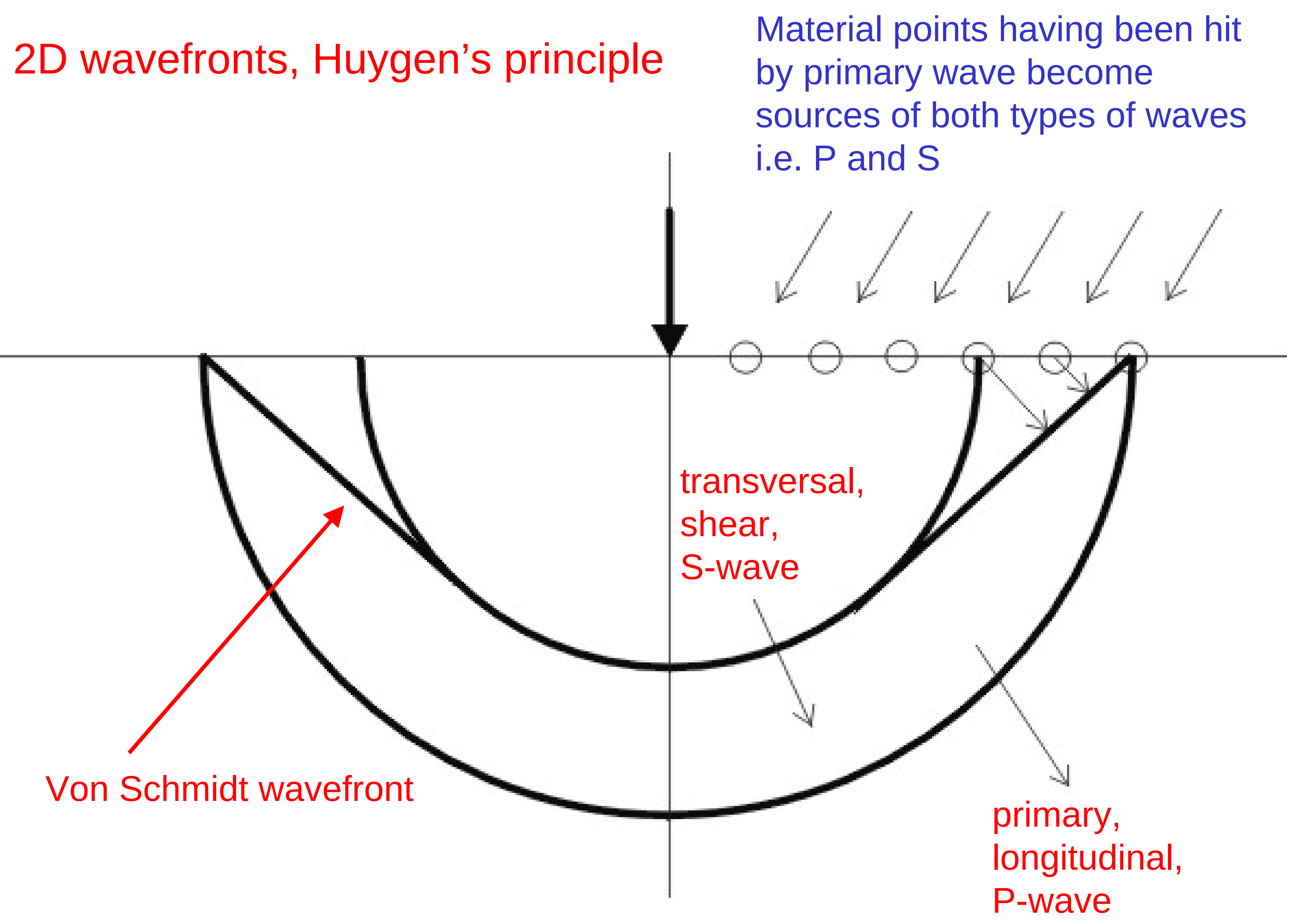
1D Hermitian cubic element, H3



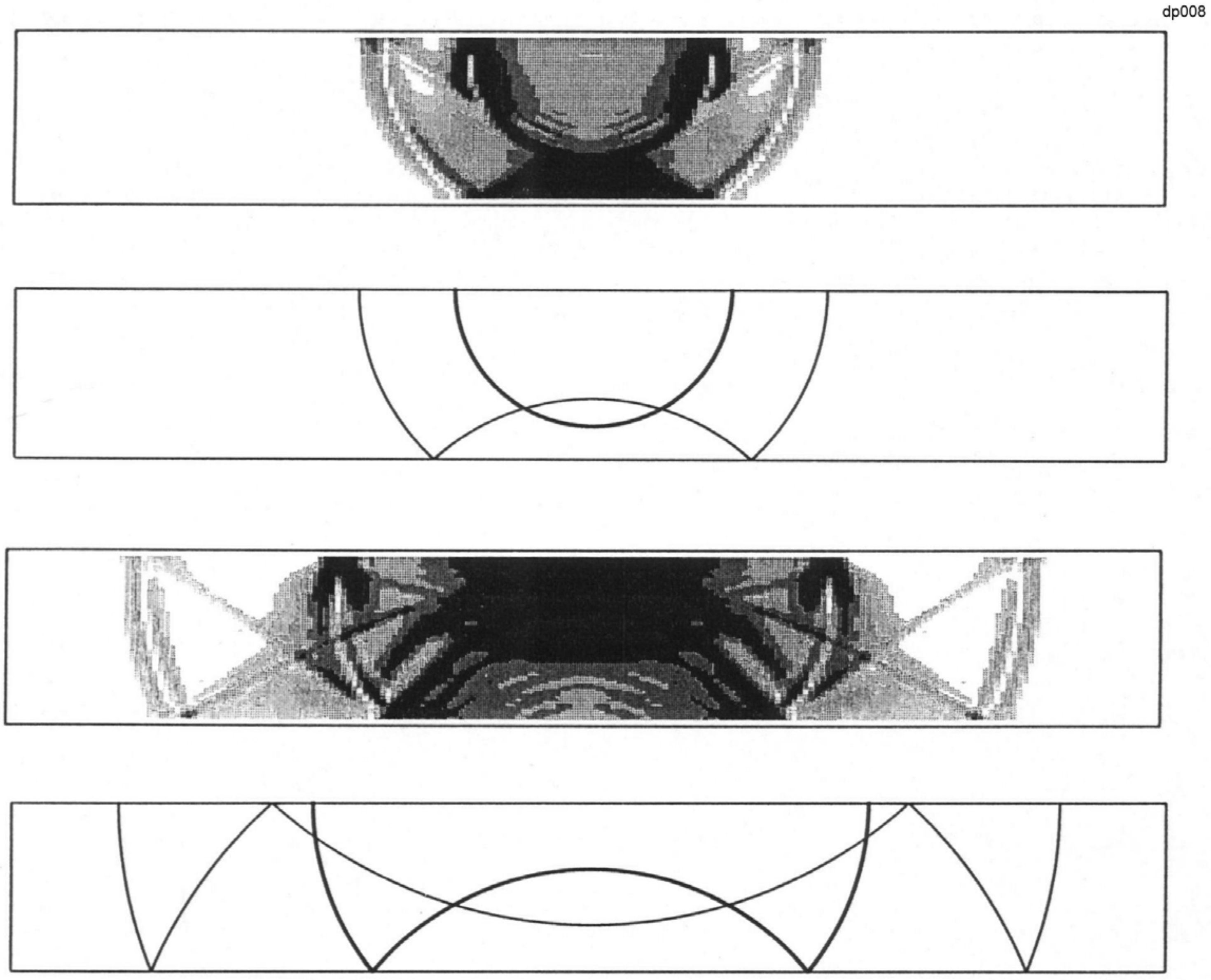
Eigenvectors of a free-free rod modeled by H3C elements



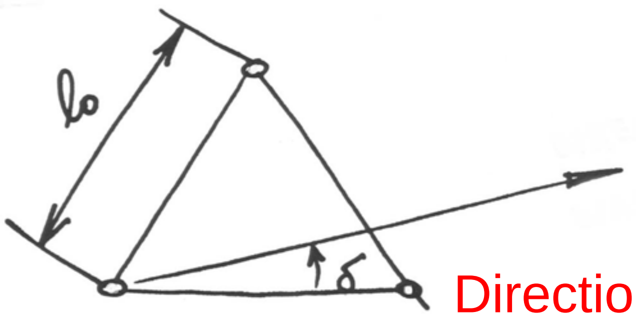
2D wavefronts, Huygen's principle



2D isotropic medium, plane stress, FE simulation



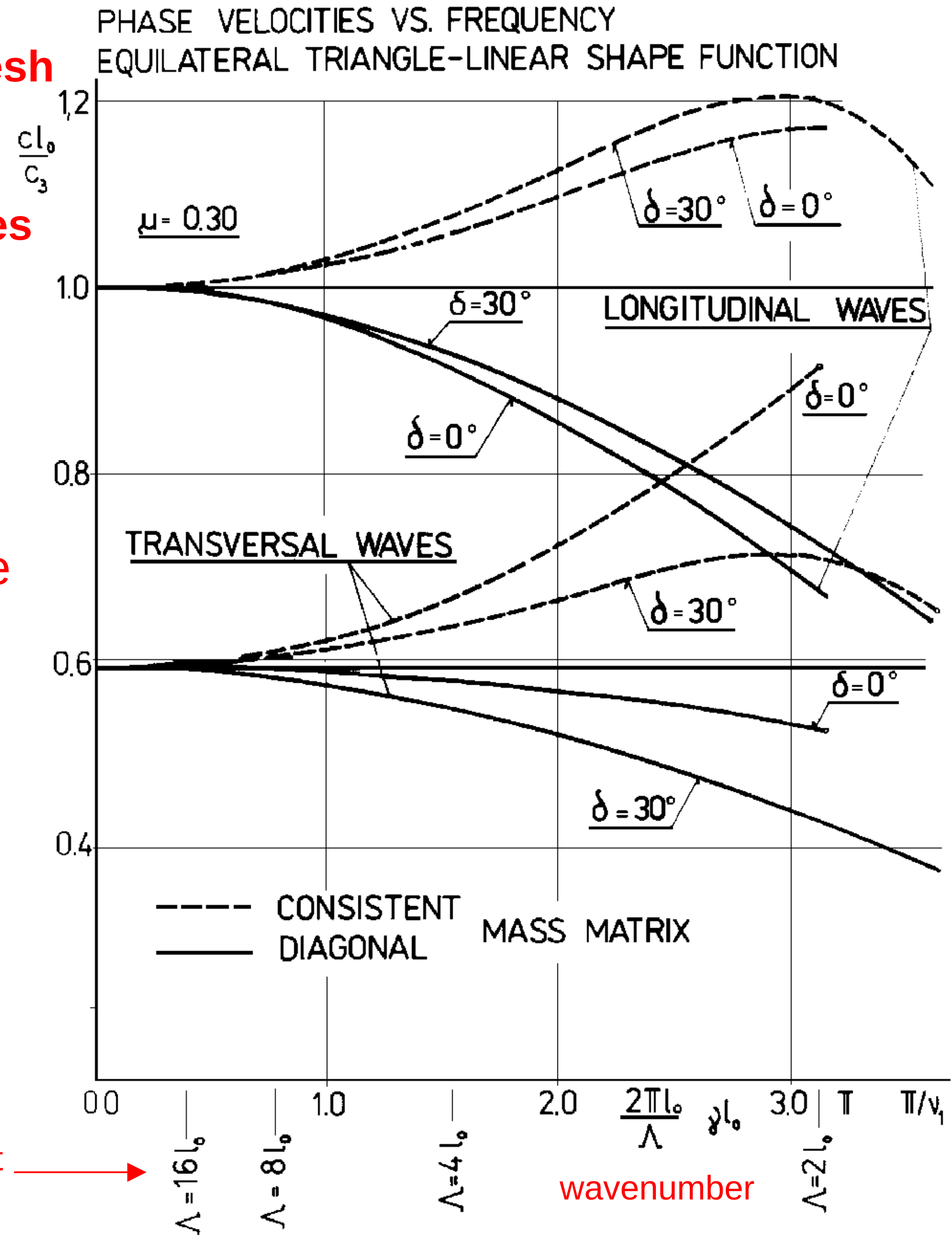
Dispersive properties of a uniform mesh (plane stress) assembled of equilateral elements, full integration consistent and diagonal mass matrices



Direction of wave propagation

Dispersion effects depend also on the direction of wave propagation

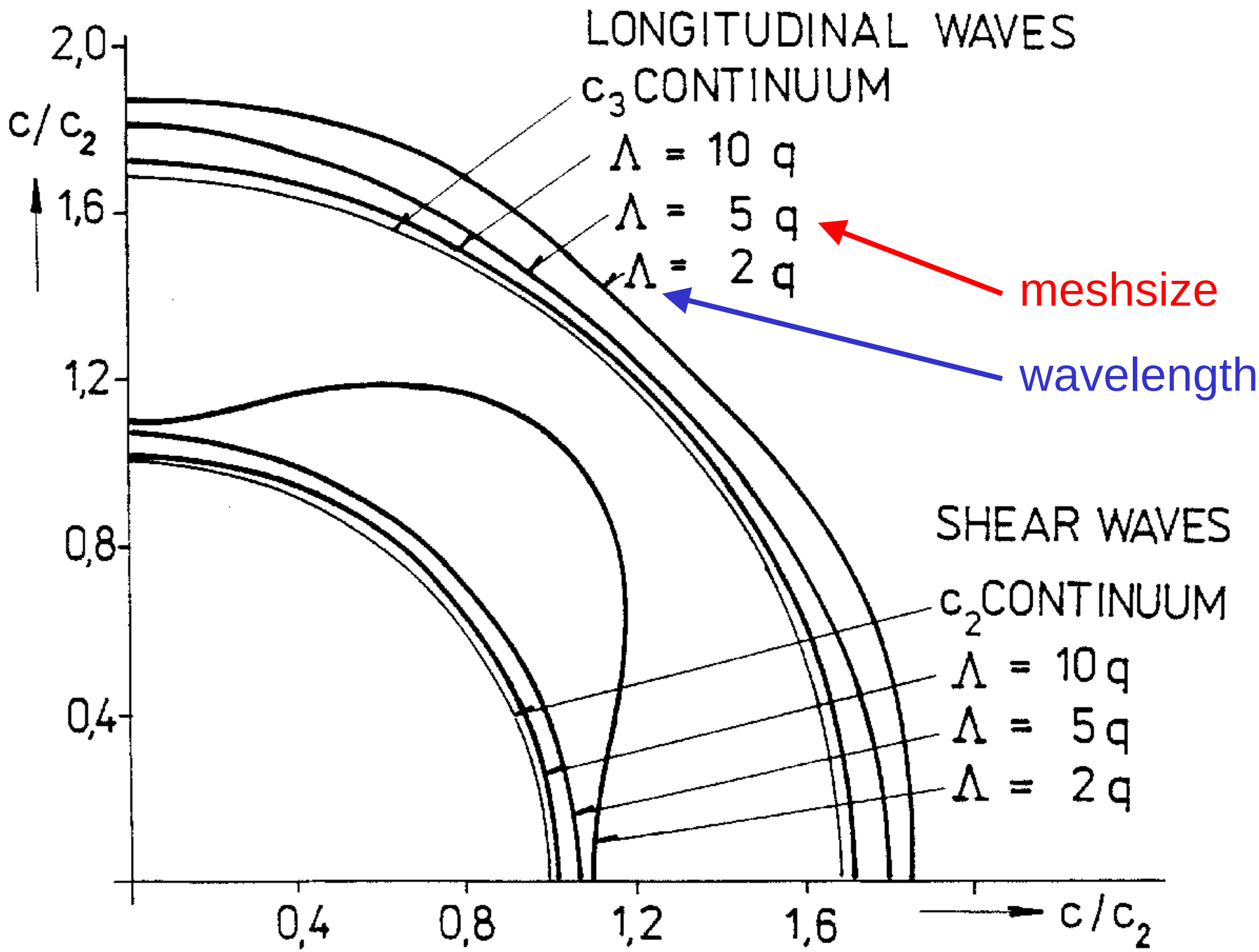
Artificial (false) anisotropy



How many elements to a wavelenght →

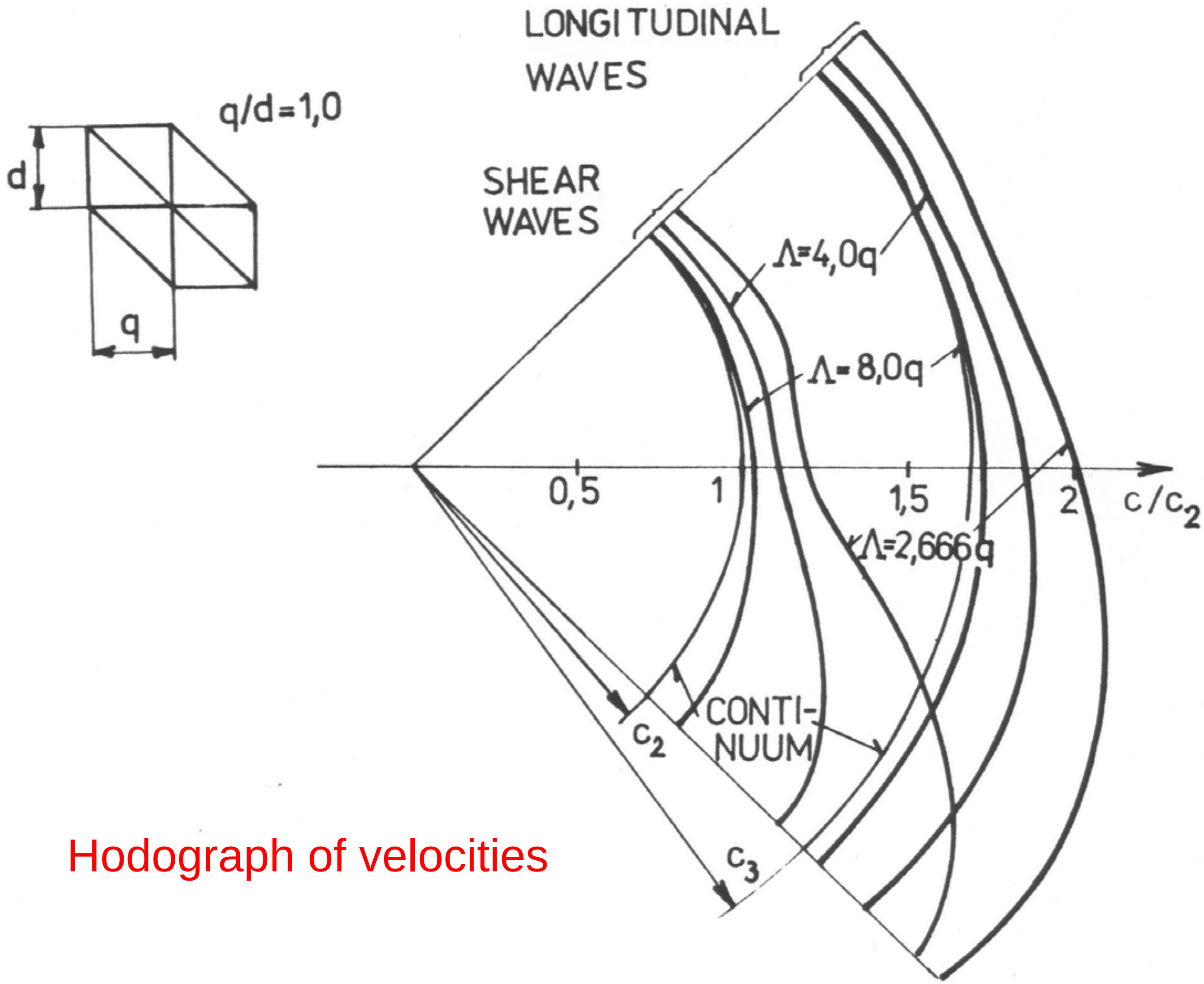
wavenumber

Dispersive properties of a uniform mesh (plane stress) assembled of square isoparametric elements, full integration, consistent mass matrix

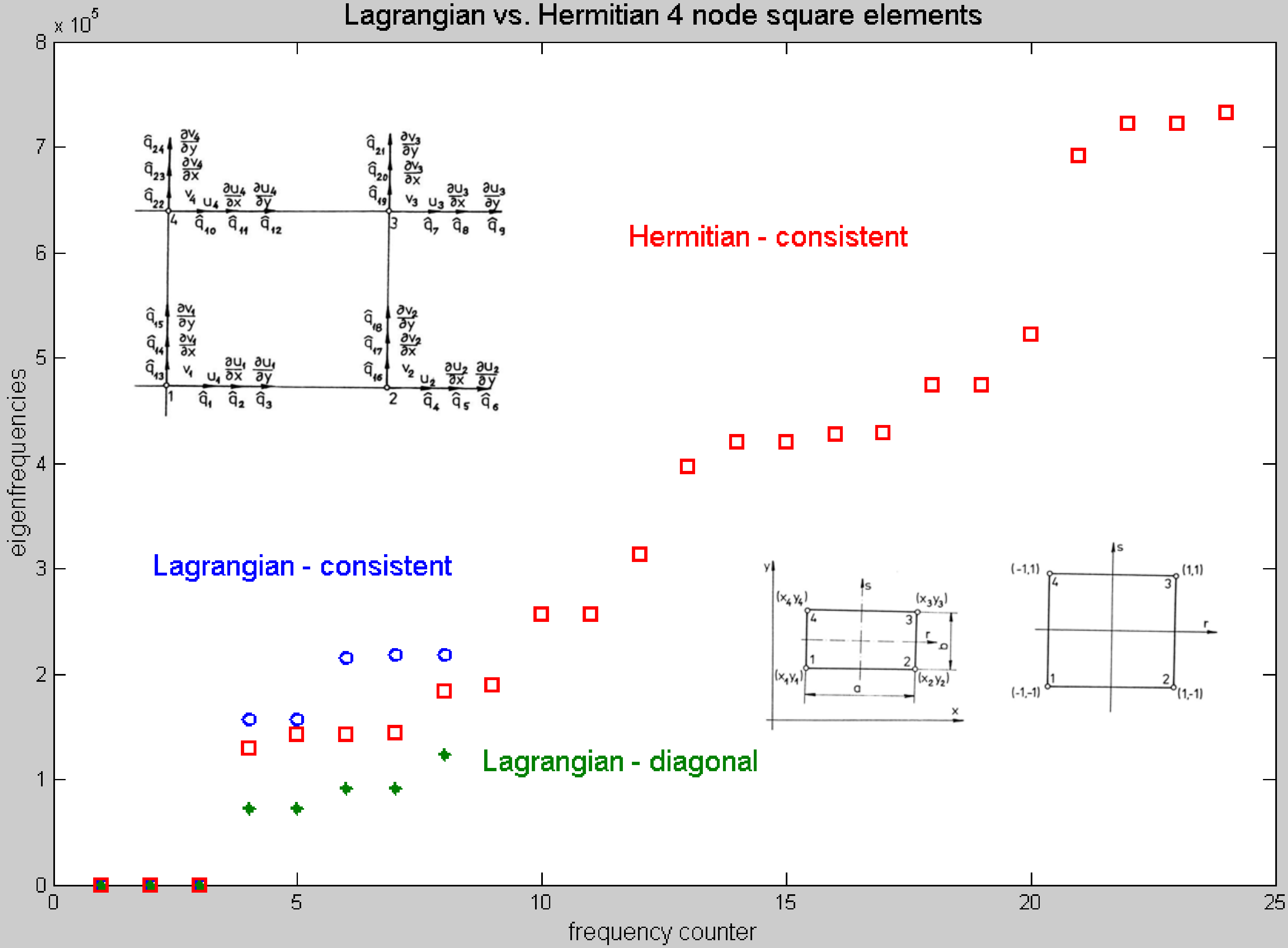


Hodograph of velocities

Dispersive properties of a uniform mesh assembled of right-angle triangular elements, consistent and diagonal mass matrices

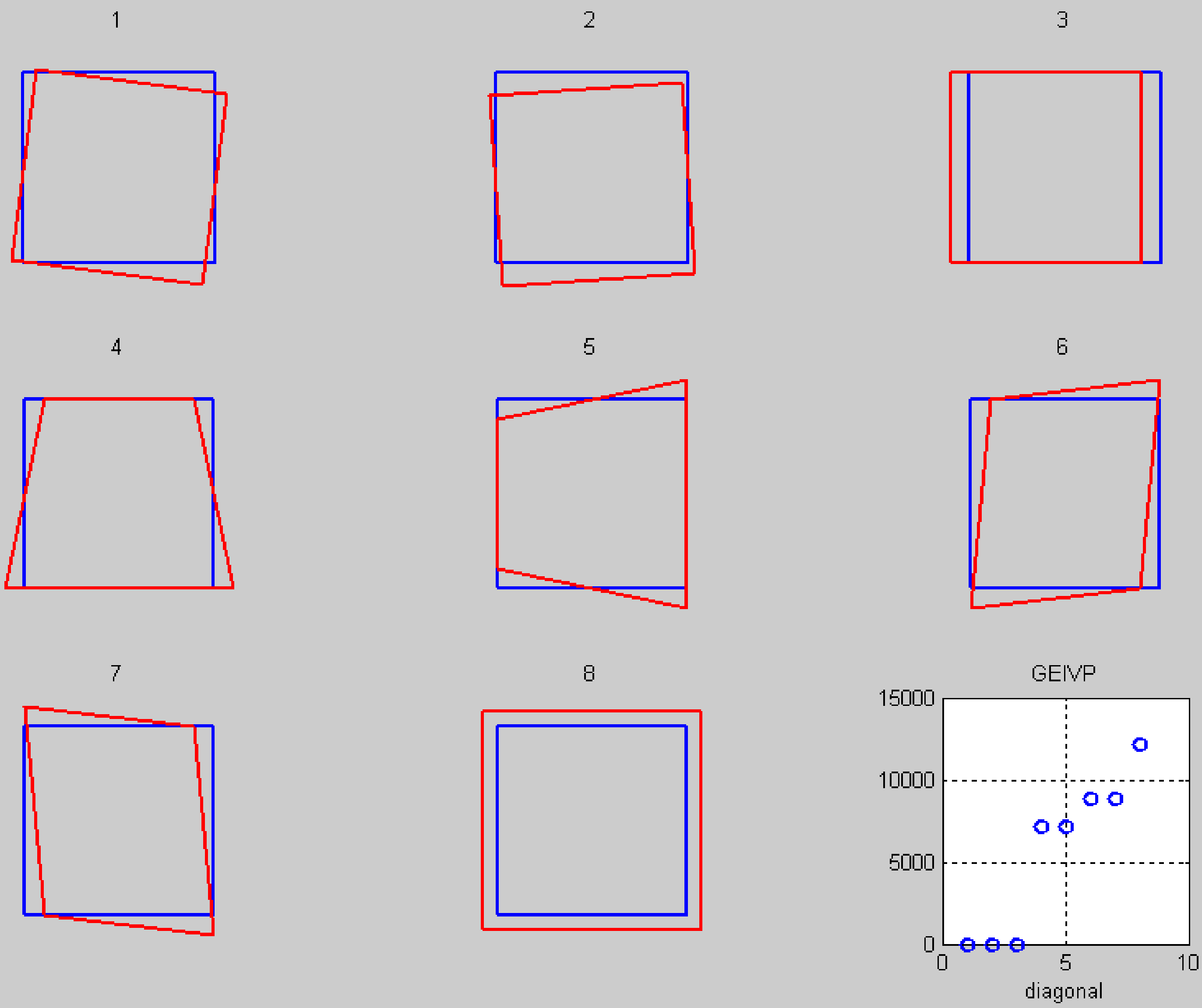


Hodograph of velocities

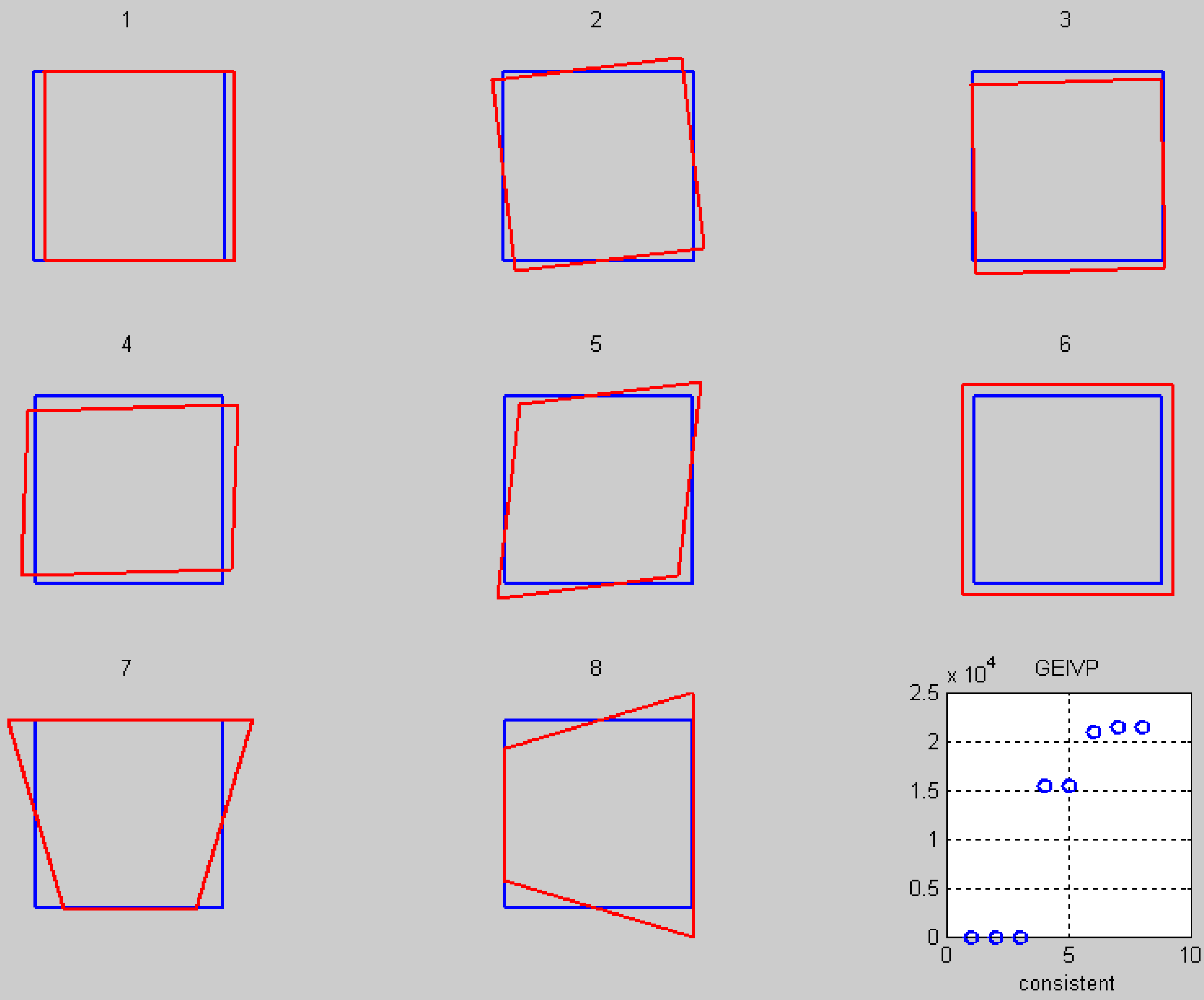


Hermitian elements are disqualified due to their 'long' spectrum

Eigenmodes and eigenvalues of a four-node bilinear element



Eigenmodes and eigenvalues of a four-node bilinear element



WE ARE NOT, HOWEVER, LIVING IN A LINEAR WORLD, SO CONCLUSIONS ARE TWOFOLD

IN LINEAR CASES

HERMITIAN ELEMENTS ARE MORE EFFICIENT,
THEY HAVE LONGER FREQUENCY SPECTRUM,
CONSISTENT MASS MATRIX SHOULD BE USED,
NEWMARK METHOD IS THE BEST CHOICE

IN NON - LINEAR CASES

EXPICIT FORMULATIONS MUST BE USED

- SINCE THEY ALLOW A SIMPLER FORMULATION OF COMPICATED CONSTITUTIVE EQUATIONS.
- ON THE OTHER HAND THEY ARE ONLY CONDITIONALLY STABLE AND REQUIRE A SMALL TIMESTEP
- THE TIMESTEP DEPENDS ON THE HIGHEST FREQUENCY OF THE MOST UNFAVOURABLE ELEMENT OF THE MESH
- $$(\Delta t)_{critical} = \max_{all\ elem.} \left(\frac{2}{\omega_{max}} \right)$$
- IN THIS CASE SIMPLE LAGRANGIAN ELEMENTS ARE PREFERED (THEY ALLOW UNDERINTEGRATION)
- DIAGONAL MASS MATRIX IS THE WINNER, NO ASSEMBLY - NO MATRIX SOLVER NEEDED

TESTING THE TIME MARCHING ALGORITHMS

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{P(t)\}$$

in the past:

Runge-Kutta, Sarafian, Nyström, Hamming, etc

later:

central differences, Wilson, Newmark.

EXPLICIT (central differences) IMPLICIT (Newmark)

↖ classical representatives ↗

Equilibrium is considered

at time t at time $t + \Delta t$

and leads to the repeated solutions of

$$\left[\tilde{M}\right]\{q_{t+\Delta t}\} = \{\tilde{P}\} \qquad \left[\tilde{K}\right]\{q_{t+\Delta t}\} = \{\bar{P}\}$$

Conditionally stable for $\Delta t < \frac{2}{\omega_{max}}$ unconditionally stable

highest eigenvalue of the system

EFFICIENT ONLY if $\left[\tilde{M}\right]$ is diagonal !!

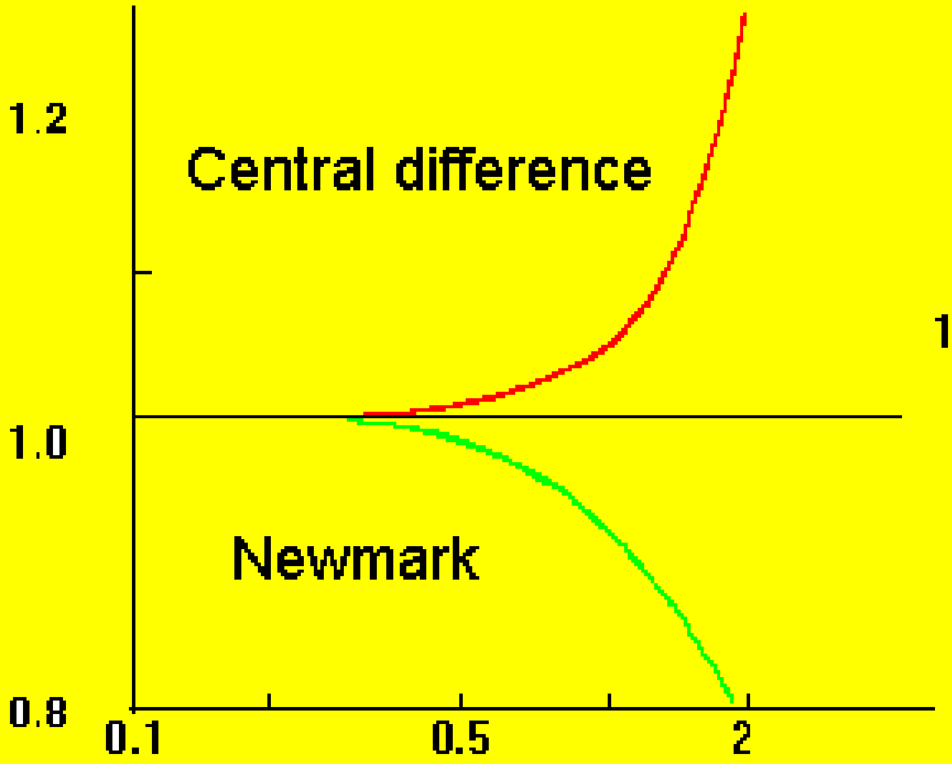
THE CORRECT COMBINATION IS

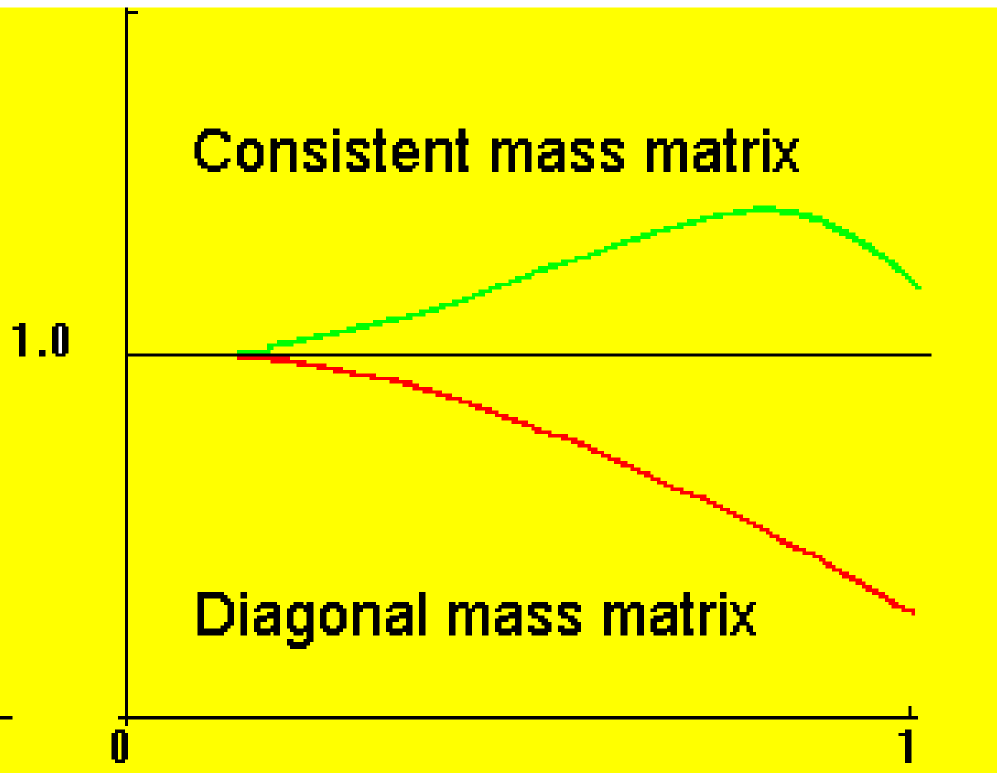
DIAGONAL MASS MATRIX EXPLICIT METHODS	CONSISTENT MASS MATRIX IMPLICIT METHODS

SINCE THE SPACE AND TIME DISCRETISATION ERRORS TEND TO CANCEL OUT

ok 1236 testing.sam1

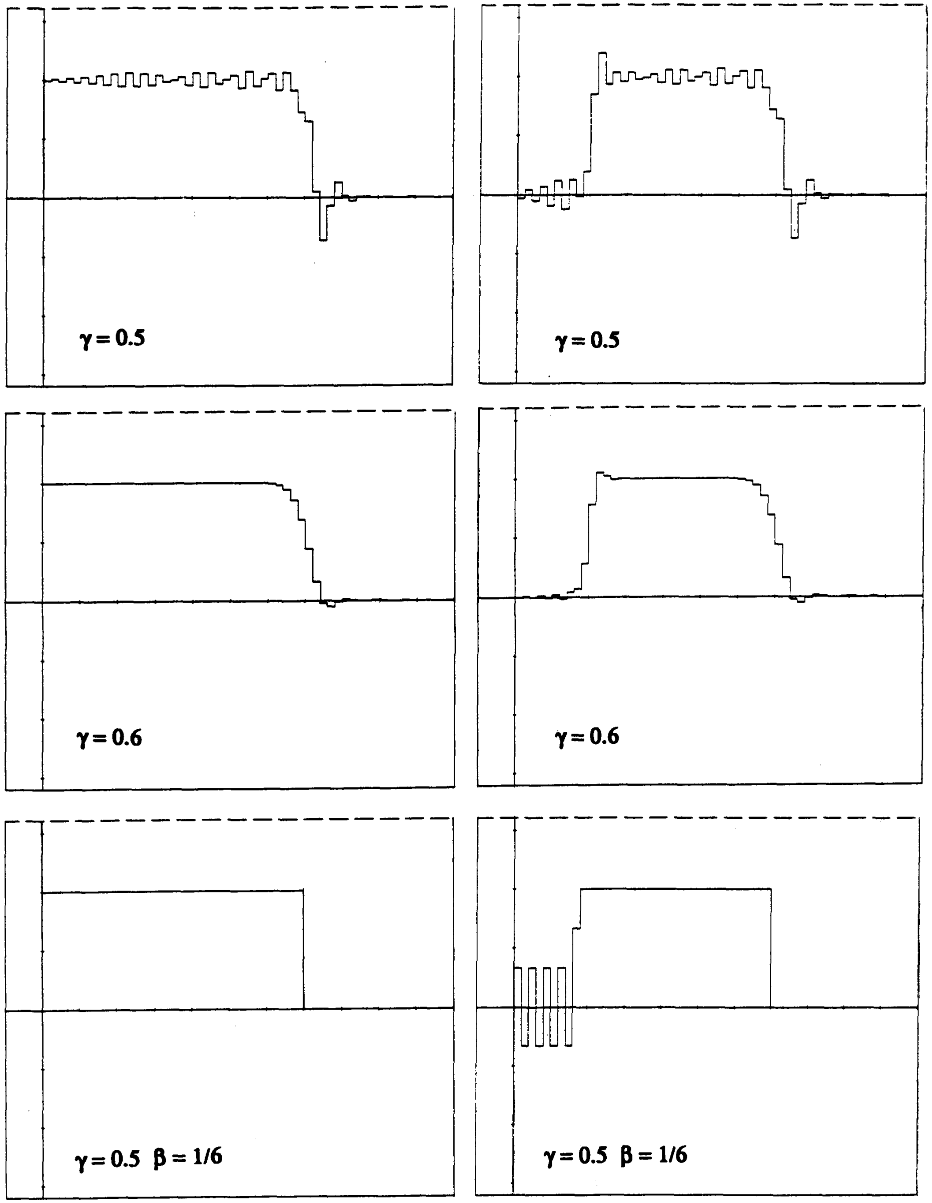
Time and space discretization errors vs. frequency





Mother nature is kind to us

Propagation of Heaviside and rectangular pulses in a bar



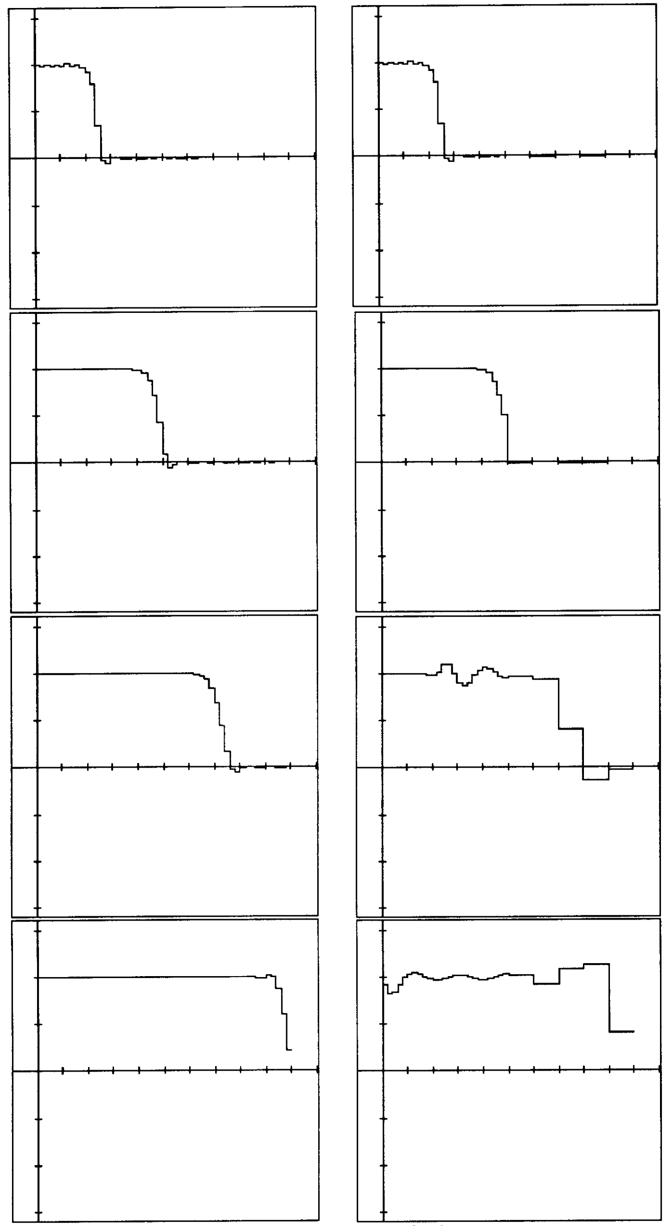
No algorithmic damping

Small algorithmic damping

The effect of space and time discretization cancels out

Newmark, 50 L1C elements

THE INFLUENCE OF NONUNIFORM MESH



25 + 25 L1C elements 25 + 5 L1C elements

Newmark, gamma=0.6, Heaviside, Strain vs. length of bar

DISPERSIVE PROPERTIES

different frequencies propagate with different velocities
⇒ a pulse is distorted

ARTIFICIAL ANISOTROPY

different velocities of propagation in different directions

FILTERING EFFECTS

cut-off and band-pass frequencies
⇒ high frequency components are strongly attenuated

MESH REGULARITY

false reflections at mesh boundaries

FOURIER SPECTRUM OF LOADING

determines what frequency components will be filtered
- out with respect to the current mesh size