MULTIPLICITY RESULTS FOR SIGN-CHANGING SOLUTIONS OF AN OPERATOR EQUATION IN ORDERED BANACH SPACE

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Abstract. In this paper, we prove some multiplicity results for sign-changing solutions of an operator equation in an ordered Banach space. The methods to show the main results of the paper are to associate a fixed point index with a strict upper or lower solution. The results can be applied to a wide variety of boundary value problems to obtain multiplicity results for sign-changing solutions.

Keywords: sign-changing solution, operator equation in ordered Banach space, fixed point index

MSC 2000: 47H07, 47H10

1. Introduction

In recent years, many authors studied the existence of sign-changing solutions for various nonlinear problems, see [4]–[10] and the references therein. For example, by using the fixed point index method, the authors of [12] obtained a result of at least one sign-changing solution for the three-point boundary value problem

$$\begin{cases} y''(t) + f(y) = 0, & 0 \leqslant t \leqslant 1, \\ y(0) = 0, \ \alpha y(\eta) = y(1), \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $\alpha \in [0, 1)$, $\eta \in (0, 1)$.

Recently, the authors of [11] considered the four-point boundary value problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) = 0, & 0 < t < 1, \\ y(0) = \alpha_1 y(\eta_1), \ y(1) = \alpha_2 y(\eta_2), \end{cases} \tag{1.2}$$

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where $0 \leqslant \alpha_1 \leqslant 1$, $0 \leqslant \alpha_2 \leqslant 1$, $0 < \eta_1 < \eta_2 < 1$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$. They obtained in [11] the existence of at least four sign-changing solutions for the four-point boundary value problem (1.2).

By using the Leray-Schauder degree and the fixed point index method, the authors of [10] obtained the existence of at least two sign-changing solutions for some m-point boundary value problems. In [14] the authors obtained by using bifurcation theory some results concerning infinite sign-changing solutions for some m-point boundary value problems.

Let E be a real Banach space which is ordered by a cone P, that is, $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, we write $x < y$. Consider the operator equation in a real Banach space E

$$x = Ax, \tag{1.3}$$

where $A = KF, K\colon E \to E$ is a completely continuous linear operator, $F\colon E \to E$ is a nonlinear continuous and bounded operator.

Let $\bar{x}$ be a non-zero solution of the operator equation (1.3). If $\bar{x} \in (-P)$ or $\bar{x} \in P$ or $\bar{x} \in E \setminus ((-P) \cup P)$, then we say $\bar{x}$ is a negative or positive or sign-changing solution of the equation (1.3), respectively. The purpose of this paper is to prove some multiplicity results for sign-changing solutions of the equation (1.3). The methods to show the results are to associate a fixed point index with a strict upper or lower solution. The results can be applied to a wide variety of boundary value problems to obtain multiplicity results for sign-changing solutions.

2. Main results

Let θ denote the zero element of E. In this section we will always assume that P is a solid normal cone, $e \in P \setminus \{\theta\}$ and $\|e\| \leqslant 1$. Let $Q = \{x \in P;\ x \geqslant \|x\| e\}$. Then Q is also a cone of E.

Definition 2.1 [12]. An operator $T\colon \mathcal{D}(T)(\supset P) \to E$ is called e-positive if for every $u \in P \setminus \{\theta\}$, there are numbers $\alpha = \alpha(u)$, $\beta = \beta(u) > 0$ such that

$$\alpha e \leqslant Tu \leqslant \beta e.$$

Definition 2.2 [10]. An operator $T\colon \mathcal{D}(T) \subset E \to E$ is called e-continuous at $x_0 \in \mathcal{D}(T)$ if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$-\varepsilon e \leqslant Tx - Tx_0 \leqslant \varepsilon e$$

for every $x \in \mathcal{D}(T)$ with $\|x - x_0\| < \delta$. The operator T is called e-continuous on $\mathcal{D}(T)$ if T is e-continuous at every $x \in \mathcal{D}(T)$. From [4, Lemma 5.2] we have the following result.

Lemma 2.1. *Let E be an ordered Banach space with a solid cone P. Let $K\colon E \to E$ be a compact, e-positive, linear operator and let $F\colon E \to E$ be a map such that, for some $u_0 \in E$, $u_0 = KF(u_0)$. Suppose F is Gâteaux differentiable at u_0 with a strictly positive derivative $F'(u_0)$. Denote by $r(T)$ the spectral radius of the operator $T = KF'(u_0)$ and by h_0 a positive eigenfunction of T corresponding to $r(T)$. Then there exists a $\tau_0 > 0$ such that, for all $0 < \tau < \tau_0$,*

$$r(T) > 1 \quad \textit{implies} \quad \begin{cases} KF(u_0 + \tau h_0) > u_0 + \tau h_0 \\ KF(u_0 - \tau h_0) < u_0 - \tau h_0 \end{cases}$$

and

$$r(T) < 1 \quad \textit{implies} \quad \begin{cases} KF(u_0 + \tau h_0) < u_0 + \tau h_0 \\ KF(u_0 - \tau h_0) > u_0 - \tau h_0. \end{cases}$$

From [14, Theorem 19.2] we have the following Lemma 2.2.

Lemma 2.2 (Krein-Rutman)**.** *Let E be a Banach space, $P \subset E$ a total cone and let $K \in L(E)$ be compact positive with $r(K) > 0$. Then $r(K)$ is an eigenvalue with a positive eigenvector.*

Let us list the following conditions which will be used in this section.

- (H_1) *$K\colon E \to E$ is e-positive, e-continuous and linear completely continuous, $K(P) \subset Q$, $r(K) > 0$.*
- (H_2) *$F\colon E \to E$ is strictly increasing, bounded and continuous, $F(\theta) = \theta$, $F'(\theta) = \beta_0 I$, where $0 < \beta_0 < (r(K))^{-1}$, I is the identical operator of E, $F'(\theta)$ denotes the Fréchet derivative of F at θ.*

We have the following main results.

Theorem 2.1. *Suppose that* (H_1) *and* (H_2) *hold. Moreover, let there exist $u_1, v_1 \in E \setminus ((-P) \cup P)$ and $m_0 > 0$ such that*

$$-m_0 e \leqslant u_1 < v_1 \leqslant m_0 e$$

and $u_1 < Au_1, Av_1 < v_1$. Then (1.3) *has at least three sign-changing solutions x_1, x_2 and x_3. Also,* (1.3) *has at least one positive solution x_4 and one negative solution x_5.*

P r o o f. The proof is completed in four steps.

S t e p 1. Clearly, $A\colon E \to E$ is a strictly increasing operator. From the Krein-Rutman Theorem, the eigenvalue $r(K)$ of the operator K has a corresponding positive eigenvector h_0. Since $K\colon E \to E$ is e-positive, then there are numbers $\alpha_{h_0}, \beta_{h_0} > 0$ such that

$$\alpha_{h_0} e \leqslant h_0 \leqslant \beta_{h_0} e. \tag{2.1}$$

By Lemma 2.1, there exists $\tau_0 > 0$ such that for every $\tau \in (0, \tau_0]$

$$-\tau h_0 < A(-\tau h_0), \quad A(\tau h_0) < \tau h_0. \tag{2.2}$$

We claim that there exist $\tau_1, \tau_2 \in (0, \tau_0]$ such that

$$-\tau_1 h_0 \nleqslant v_1, \quad \tau_2 h_0 \ngeqslant u_1. \tag{2.3}$$

By contradiction, assume that $-\tau h_0 \leqslant v_1$ for all $\tau \in (0, \tau_0]$. Letting $\tau \to 0$, we have $\theta \leqslant v_1$, which contradicts $v_1 \in E \setminus ((-P) \cup P)$. The second relation can be proved analogously. Hence, (2.3) holds. Let $u_2 = -\tau_1 h_0$ and $v_2 = \tau_2 h_0$. From (2.1)–(2.3), we have

$$u_2 < Au_2, \quad Av_2 < v_2, \quad u_2 \nleqslant v_1, \quad v_2 \ngeqslant u_1$$

and

$$-m_1 e \leqslant u_2 < v_2 \leqslant m_1 e, \tag{2.4}$$

where m_1 is a positive number.

S t e p 2. Let $D_1 = Au_1 + Q$. Then D_1 is a closed convex subset of E. For any $x \in D_1$ we have $x \geqslant Au_1 > u_1$. Since F is strictly increasing and $K\colon P \to Q$, we have

$$Ax - Au_1 = K(Fx - Fu_1) \geqslant \|K(Fx - Fu_1)\| e = \|Ax - Au_1\| e,$$

that is

$$Ax \geqslant \|Ax - Au_1\| e + Au_1.$$

This implies that $Ax \in D_1$, and so $A(D_1) \subset D_1$.

Let sets Ω_{10}, Ω_{11} and Ω_{12} be defined by

$$\begin{aligned}
\Omega_{10} &= \{x \in D_1;\ Ax \ngeqslant u_2\},\\
\Omega_{11} &= \{x \in D_1;\ \text{ there exists } \tau > 0 \text{ such that } Ax \leqslant Av_1 - \tau e\},\\
\Omega_{12} &= \{x \in D_1;\ Ax \ngeqslant u_2, Ax \nleqslant v_1\}.
\end{aligned}$$

From (2.4) we have

$$u_2 - u_1 \leqslant (m_1 + m_0)e. \tag{2.5}$$

For any $x \in D_1$, if $\|x\| \geqslant m_1 + m_0 + \|Au_1\|$, then

$$x \geqslant \|x - Au_1\|e + Au_1 \geqslant (\|x\| - \|Au_1\|)e + u_1 \geqslant u_2$$

and so

$$Ax \geqslant Au_2 > u_2.$$

This implies Ω_{10} is a bounded set. Clearly, $\Omega_{11} \subset \Omega_{10}$, $\Omega_{12} \subset \Omega_{10}$ and $\Omega_{11} \cap \Omega_{12} = \emptyset$. Thus, Ω_{11}, Ω_{12} and Ω_{10} are three bounded sets. Since K is e-positive, we have $Au_1 \in \Omega_{11} \subset \Omega_{10}$, and thus Ω_{11} and Ω_{10} are two nonempty bounded sets. We claim that $\Omega_{12} \neq \emptyset$. Indeed, if $\Omega_{12} = \emptyset$, then $D_1 = S_{11} \cup S_{12}$, where $S_{11} = \{x \in D_1;\ Ax \leqslant v_1\}$ and $S_{12} = \{x \in D_1;\ Ax \geqslant u_2\}$. Since $u_2 \nleqslant v_1$, we have $S_{11} \cap S_{12} = \emptyset$. S_{11} is a nonempty closed set since $Au_1 \in S_{11}$. Take $z_0 \in D_1$ with $\|z_0\| \geqslant m_0 + m_1 + \|Au_1\|$. By (2.5) we have

$$z_0 \geqslant \|z_0 - Au_1\|e + Au_1 \geqslant (\|z_0\| - \|Au_1\|)e + Au_1 \geqslant (m_0 + m_1)e + u_1 \geqslant u_2$$

and so $Az_0 \geqslant Au_2 > u_2$, $z_0 \in S_{12}$. S_{12} is a nonempty closed set. Hence, the connected set D_1 can be represented as a union of two disjoint nonempty closed sets S_{11} and S_{12}, which is a contradiction. Therefore, $\Omega_{12} \neq \emptyset$.

It is easy to see that Ω_{10} and Ω_{12} are two open subsets of D_1. For any $x_0 \in \Omega_{11}$ there exists $\tau' > 0$ such that $Ax_0 \leqslant Av_1 - \tau' e$. By ($H_1$) and ($H_2$), $A\colon E \to E$ is e-continuous. Hence, there exists $\delta_0 > 0$ such that

$$-\frac{\tau'}{2}\, e \leqslant Ax - Ax_0 \leqslant \frac{\tau'}{2}\, e$$

for all $x \in D_1$ with $\|x - x_0\| < \delta_0$. Thus, for all $x \in D_1$ with $\|x - x_0\| < \delta_0$

$$Ax \leqslant Ax_0 + \frac{\tau'}{2}\, e \leqslant Av_1 - \frac{\tau'}{2}\, e.$$

This implies that $x \in \Omega_{11}$. Hence, Ω_{11} is a nonempty open subset of D_1.

Step 3. Now we will show that

$$x \neq \lambda Ax + (1 - \lambda)Au_1, \quad x \in \partial_{D_1}\Omega_{11}, \quad \lambda \in [0, 1], \tag{2.6}$$

where $\partial_{D_1}\Omega_{11}$ denotes the boundary of Ω_{11} in D_1. Suppose this is not the case. Then there exist $x_0 \in \partial_{D_1}\Omega_{11}$ and $\lambda_0 \in [0,1]$ such that $x_0 = \lambda_0 Ax_0 + (1-\lambda_0)Au_1$. Since $Au_1 \in \Omega_{11}$, we have $\lambda_0 \in (0,1]$. It is easy to see that $Ax_0 \leqslant Av_1 < v_1$, and so

$$x_0 = \lambda_0 Ax_0 + (1-\lambda_0)Au_1 < \lambda_0 v_1 + (1-\lambda_0)Au_1 \leqslant v_1.$$

Consequently, we have

$$Av_1 - Ax_0 \geqslant \|Av_1 - Ax_0\|e,$$

that is

$$Ax_0 \leqslant Av_1 - \|Av_1 - Ax_0\|e.$$

This implies that $x_0 \in \Omega_{11}$, which contradicts $x_0 \in \partial_{D_1}\Omega_{11}$. Thus, (2.6) holds.

From the homotopy invariance and normalization properties of the fixed point index, we have

$$i(A,\Omega_{11},D_1) = i(Au_1,\Omega_{11},D_1) = 1. \tag{2.7}$$

Then A has at least one fixed point $x_1 \in \Omega_{11}$. Clearly,

$$u_1 < Au_1 \leqslant x_1 = Ax_1 \leqslant Av_1 < v_1. \tag{2.8}$$

From (2.8) and the fact that $u_1, v_1 \in E\setminus((-P)\cup P)$ we see that $x_1 \in E\setminus((-P)\cup P)$.

Next we will show that A has at least one other fixed point $x_2 \in \mathrm{Cl}_{D_1}\Omega_{10}\setminus\Omega_{11}$, where $\mathrm{Cl}_{D_1}\Omega_{10}$ denotes the closure of Ω_{10} in D_1. Assume on the contrary that A has no fixed point on $\mathrm{Cl}_{D_1}\Omega_{10}\setminus\Omega_{11}$. We claim that

$$x - Ax \neq \lambda e,\ x \in \partial_{D_1}\Omega_{10}, \lambda \geqslant 0. \tag{2.9}$$

In fact, assuming contrary, there exist $x_0 \in \partial_{D_1}\Omega_{10}$ and $\lambda_0 \geqslant 0$ such that $x_0 - Ax_0 = \lambda_0 e$. The fact that A has no fixed point on $\partial_{D_1}\Omega_{10} \subset \mathrm{Cl}_{D_1}\Omega_{10}\setminus(\Omega_{11}\cup\Omega_{12})$ implies that $\lambda_0 > 0$. Since $x_0 \in \partial_{D_1}\Omega_{10}(x_0 \notin \Omega_{10})$, we have $Ax_0 \geqslant u_2$, and so $x_0 = Ax_0 + \lambda_0 e \geqslant u_2 + \lambda_0 e$. By (H$_1$) and (H$_2$) we have

$$Ax_0 \geqslant A(u_2+\lambda_0 e) \geqslant Au_2 + \|A(u_2+\lambda_0 e) - Au_2\|e.$$

Let $\gamma_0 = \|A(u_2+\lambda_0 e) - Au_2\| > 0$. Then there exists $\delta_0 > 0$ small enough such that

$$-\frac{\gamma_0}{2}e \leqslant Ax - Ax_0 \leqslant \frac{\gamma_0}{2}e$$

for any $x \in D_1$ with $\|x - x_0\| < \delta_0$. Thus, we have for any $x \in D_1$ with $\|x - x_0\| < \delta_0$

$$Ax \geqslant Ax_0 - \frac{\gamma_0}{2}e \geqslant Au_2 + \frac{\gamma_0}{2}e > Au_2 > u_2.$$

Take $z_0 \in \Omega_{10}$ with $\|z_0 - x_0\| < \delta_0$. Then we have $Az_0 > u_2$, which is a contradiction. Thus, (2.9) holds. For any $x \in D_1$ and $\lambda \geqslant 0$ we have

$$\begin{aligned} Ax + \lambda e &\geqslant \|Ax - Au_1\|e + \lambda e + Au_1 \\ &\geqslant \|Ax + \lambda e - Au_1\|e + Au_1 + \lambda(1 - \|e\|)e \\ &\geqslant \|Ax + \lambda e - Au_1\|e + Au_1. \end{aligned}$$

Thus, $Ax + \lambda e \in D_1$ for all $x \in D_1$. Let $a = \sup_{x \in \mathrm{Cl}_{D_1}\Omega_{10}} \|Ax\|$ and $b = \sup_{x \in \mathrm{Cl}_{D_1}\Omega_{10}} \|x\|$. Take $s_0 > 0$ such that $s_0\|e\| > a+b$. Let an operator A_1 be defined by $A_1 x = Ax + s_0 e$ for all $x \in \mathrm{Cl}_{D_1}\Omega_{10}$. Then we have

$$\|A_1 x\| \geqslant s_0\|e\| - \|Ax\| > b \geqslant \|x\|, \quad x \in \mathrm{Cl}_{D_1}\Omega_{10}.$$

From the solution property of the fixed point index we have

$$i(A_1, \Omega_{10}, D_1) = 0. \tag{2.10}$$

Let $H(t,x) = (1-t)Ax + tA_1x$ for all $(t,x) \in [0,1] \times \mathrm{Cl}_{D_1}\Omega_{10}$. From (2.9) we see that $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial_{D_1}\Omega_{10}$. Then, by the homotopy invariance property of the fixed point index and (2.10), we have

$$i(A, \Omega_{10}, D_1) = i(A_1, \Omega_{10}, D_1) = 0. \tag{2.11}$$

From (2.7) and (2.11) we have

$$i(A, \Omega_{12}, D_1) = i(A, \Omega_{10}, D_1) - i(A, \Omega_{11}, D_1) = -1.$$

Therefore, A has at least one fixed point in $\Omega_{12} \subset \mathrm{Cl}_{D_1}\Omega_{10} \setminus \Omega_{11}$,which is a contradiction. The contradiction obtained proves that A has at least one fixed point $x_2 \in \mathrm{Cl}_{D_1}\Omega_{10} \setminus \Omega_{11}$. Now we show that $u_2 \not< x_2$. Indeed, if $u_2 < x_2$, then we have

$$Ax_2 - Au_2 \geqslant \|Ax_2 - Au_2\|e.$$

Let $\gamma_1 = \|Ax_2 - Au_2\| > 0$. Since A is e-continuous, there exists $\delta_1 > 0$ such that for any $x \in E$ with $\|x - x_2\| < \delta_1$

$$-\frac{\gamma_1}{2}\, e \leqslant Ax - Ax_2 \leqslant \frac{\gamma_1}{2}\, e$$

and so

$$Ax \geqslant Ax_2 - \frac{\gamma_1}{2}\, e \geqslant Au_2 + \frac{\gamma_1}{2}\, e > Au_2 > u_2.$$

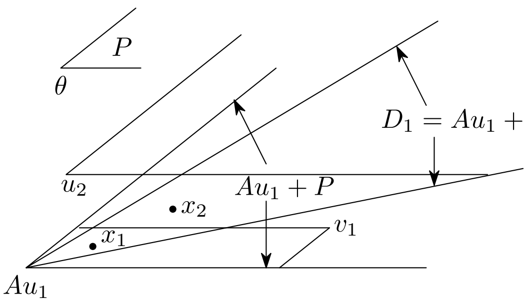
This implies that $B(x_2, \delta_1) \cap \Omega_{10} = \emptyset$, which contradicts $x_2 \in \mathrm{Cl}_{D_1}\Omega_{10}$. Thus, $u_2 \not\leqslant x_2$, and so $x_2 \not\geqslant \theta$. If $x_2 \leqslant \theta$, since $u_1 < Au_1 \leqslant Ax_2 = x_2$, we have $u_1 < \theta$, which contradicts $u_1 \in E \setminus ((-P) \cup P)$. Therefore, $x_2 \in E \setminus ((-P) \cup P)$, and x_2 is a sign-changing solution of (1.3).

Step 4. Let $D_2 = Av_1 - Q$, $\Omega_{20} = \{x \in D_2;\ Ax \not\leqslant v_2\}$ and $\Omega_{21} = \{x \in D_2;$ there exists $\tau > 0$ such that $Ax \geqslant Au_1 + \tau e\}$. Essentially the same argument as in Step 3 shows that A has at least one fixed point $x_3 \in \mathrm{Cl}_{D_2}\Omega_{20} \setminus \Omega_{21}$ and x_3 is a sign-changing solution of (1.3).

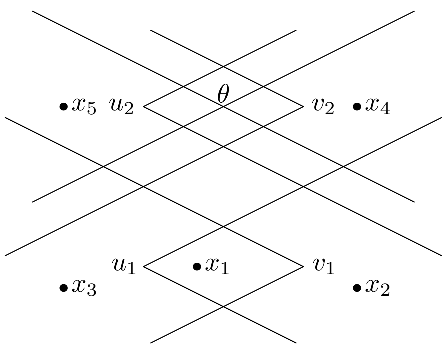
Finally, we shall show the existence of positive solutions and negative solutions. Let $D_3 = Q(= A\theta + Q)$, $\Omega_{30} = \{x \in D_3;\ Ax \not\geqslant u_1\}$ and $\Omega_{31} = \{x \in D_3;$ there exists $\tau > 0$ such that $Ax \leqslant Av_2 - \tau e\}$. Then A has at least one fixed point x_4 such that $x_4 \in \mathrm{Cl}_{D_3}\Omega_{30} \setminus \Omega_{31}$ and x_4 is a positive solution of (1.3).

Let $D_4 = -Q(= A\theta - Q)$, $\Omega_{40} = \{x \in D_4;\ Ax \not\leqslant v_1\}$ and $\Omega_{41} = \{x \in D_4;$ there exists $\tau > 0$ such that $Ax \geqslant Au_2 + \tau e\}$. Then A has at least one fixed point x_5 such that $x_5 \in \mathrm{Cl}_{D_4}\Omega_{40} \setminus \Omega_{41}$ and x_5 is a negative solution of (1.3). This completes the proof. □

Remark 2.1. The position of u_1, u_2, v_1 can be illustrated roughly by the following figure.



Remark 2.2. The position of u_1, u_2, v_1, v_2 and x_1, x_2, x_3, x_4, x_5 in Theorem 2.1 can be illustrated roughly by the following figure.



R e m a r k 2.3. The two pairs of strict lower and upper solutions u_1, v_1 and u_2, v_2 in Theorem 2.1 satisfy $u_1 \nleqslant v_2$ and $u_2 \nleqslant v_1$. We say these two pairs of strict lower and upper solutions are parallel to each other. We should point out that this condition was first put forward in [14]. The above u_1, v_2 and u_2, v_1 are also two pairs of non-well-ordered upper and lower solutions. For other discussions concerning the non-well-ordered upper and lower solutions, the reader is refered to [11, 5.4B].

3. Applications

Consider the two-point boundary value problem

$$\begin{cases} u'' + f(t,u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing in the second argument, $f(\cdot,0) \equiv 0$.

Theorem 3.1. *Suppose that there exist $u_1, v_1 \in C^2[0,1]$ which are sign-changing on $[0,1]$, $m_0 > 0$ such that $u_1 \not\equiv v_1$ on $[0,1]$, and*

$$\begin{cases} u_1''(t) + f(t,u_1(t)) > 0, & 0 < t < 1, \\ u_1(0) \leqslant 0, u_1(1) \leqslant 0, \end{cases} \tag{3.2}$$

$$\begin{cases} v_1''(t) + f(t,v_1(t)) < 0, & 0 < t < 1, \\ v_1(0) \geqslant 0, v_1(1) \geqslant 0, \end{cases} \tag{3.3}$$

$$-m_0 t(1-t) \leqslant u_1(t) \leqslant v_1(t) \leqslant m_0 t(1-t), \quad t \in [0,1]$$

and

$$0 < \beta_0 = \lim_{u\to 0} \frac{f(t,u)}{u} < \pi^2 \quad \textit{uniformly on } [0,1].$$

Then (3.1) has at least three sign-changing solutions. Moreover, (3.1) has at least one non-zero non-negative solution and one non-zero non-positive solution.

P r o o f. Let E be the Banach space $C[0,1]$ with the maximum norm. Let $P = \{x \in E;\ x(t) \geqslant 0,\ t \in [0,1]\}$. Then E is a real Banach space and P is a solid cone of E. Let $e(t) = t(1-t)$ for $t \in [0,1]$ and $Q = \{x \in P;\ x(t) \geqslant \|x\| e(t), t \in [0,1]\}$. Q is also a cone of E. Let operators K, F and A be defined by

$$(Kx)(t) = \int_0^1 G(t,s)x(s)\,\mathrm{d}s,\ t \in [0,1],\ x \in E,$$
$$(Fx)(t) = f(t,x(t)),\ t \in [0,1],\ x \in E$$

and $A = KF$, where

$$G(t,s) = \begin{cases} t(1-s), & t \leqslant s, \\ s(1-t), & t > s. \end{cases}$$

It is easy to see that

$$e(t)G(\tau,s) \leqslant G(t,s) \leqslant e(t), \quad t,s,\tau \in [0,1]. \tag{3.4}$$

For each $x \in P \setminus \{\theta\}$, we have from (3.4)

$$\|x\|e(t) \geqslant (Kx)(t) \geqslant (Kx)(\tau)e(t), \quad t,\tau \in [0,1],\ x \in P,$$

and thus

$$\|x\|e(t) \geqslant (Kx)(t) \geqslant \|Kx\|e(t), \quad t \in [0,1],\ x \in P.$$

This implies that K is e-positive. Thus, we have for each $x, y \in E$

$$-\|x-y\|e(t) \leqslant (K(x-y))(t) \leqslant \|x-y\|e(t), \quad t \in [0,1].$$

This implies that K is e-continuous. The sequence of eigenvalues of K is $\{(n^2\pi^2)^{-1}\}$. Since $\lim\limits_{x\to 0} f(t,x)/x = \beta_0$, F is Fréchet differentiable at θ and $r(A'(\theta)) = \beta_0\pi^{-2} < 1$. From (3.2) and (3.3), it is easy to prove that $u_1 < Au_1$ and $Av_1 < v_1$. Consequently, all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, the conclusion of Theorem 3.1 holds. □

Remark 3.1. Obviously, Theorem 2.1 can be applied to other types of nonlinear boundary value problems to obtain multiplicity results for sign-changing solutions.

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