ON THE LINEAR PROBLEM ARISING FROM MOTION OF A FLUID AROUND A MOVING RIGID BODY

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Abstract. We study a linear system of equations arising from fluid motion around a moving rigid body, where rotation is included. Originally, the coordinate system is attached to the fluid, which means that the domain is changing with respect to time. To get a problem in the fixed domain, the problem is rewritten in the coordinate system attached to the body. The aim of the present paper is the proof of the existence of a strong solution in a weighted Lebesgue space. In particular, we prove the existence of a global pressure gradient in L^2.

Keywords: incompressible fluid; rotating rigid body; strong solution

MSC 2010: 35Q35

1. Mathematical formulation

In the present paper we study the initial-boundary value problem of the motion of a viscous fluid around a moving rigid body. First we will give the mathematical formulation of the problem.

Let $\mathcal{B}$ denote an open, connected and bounded C^2 domain, representing a rigid body in a fluid motion in $\mathcal{D} := \mathbb{R}^3 \setminus \overline{\mathcal{B}}$. Clearly, $\mathcal{D}$ defines an exterior C^2 domain in $\mathbb{R}^3$ with boundary $\Sigma = \partial\mathcal{D} = \partial\mathcal{B}$.

The motion of the fluid and the body will be governed by the following system of equations.

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Equations of fluid in $\mathcal{D} \times (0,T)$,

$$\text{(1.1)} \qquad \begin{cases} \operatorname{div} \boldsymbol{w} = 0, \\ \dfrac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{w} - \boldsymbol{U}) \cdot \nabla \boldsymbol{w} + \boldsymbol{\omega} \times \boldsymbol{w} = \operatorname{div} \mathbb{T}(\boldsymbol{w}, \pi) + \boldsymbol{Q}^\top \cdot \mathcal{F}(x,t) \end{cases}$$

where $\boldsymbol{U} = \boldsymbol{\xi} + \boldsymbol{\omega} \times y$. Here by $\boldsymbol{w}$ we denote the velocity of the fluid and by $\boldsymbol{U}$ the velocity of the body, where $\boldsymbol{\xi}$ stands for its translation and $\boldsymbol{\omega}$ for its rotation. Furthermore,

$$\mathbb{T}(\boldsymbol{w}, p) = 2\nu \boldsymbol{D}\boldsymbol{w} - \boldsymbol{I}p \quad \text{the Cauchy stress tensor,}$$

$$\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t) \quad \text{the symmetrical stress tensor,}$$

where the constant $\nu > 0$ denotes the viscosity, p is the pressure, $\boldsymbol{I}$ denotes the identity matrix (δ_{ij}).

In addition, the term $\boldsymbol{Q}^\top \mathcal{F}$ represents a given external force, while the tensor $Q^\top$ is related to ω in the following way

$$\text{(1.2)} \qquad \frac{\mathrm{d}\boldsymbol{Q}^\top}{\mathrm{d}t} = \Omega(\omega)\boldsymbol{Q}^\top, \quad \boldsymbol{Q}^\top(0) = \boldsymbol{I}, \quad \Omega(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

The above system will be completed by the following boundary and initial conditions

$$\text{(1.3)} \qquad \boldsymbol{w} = \boldsymbol{w}_* + \boldsymbol{U}, \quad \text{on } \partial\mathcal{D} \times (0,T),$$

$$\text{(1.4)} \qquad \lim_{|y| \to \infty} \boldsymbol{w}(y,t) = 0,$$

$$\text{(1.5)} \qquad \boldsymbol{w}(0) = \boldsymbol{w}_0.$$

Equations of motion of the body

$$\text{(1.6)} \qquad m\dot{\boldsymbol{\xi}} + m\boldsymbol{\omega} \times \boldsymbol{\xi} = \boldsymbol{Q} \cdot \boldsymbol{F} - \int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{w}, \pi) \cdot \boldsymbol{n} - \boldsymbol{w}(\boldsymbol{w} - \boldsymbol{U}) \cdot \boldsymbol{n}) \, \mathrm{d}S,$$

$$\text{(1.7)} \quad \boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega} = \boldsymbol{Q}^\top \cdot \boldsymbol{M}_C - \int_{\partial\mathcal{D}} (y \times (\mathbb{T}(\boldsymbol{w}, \pi) \cdot \boldsymbol{n} - \boldsymbol{w}(\boldsymbol{w} - \boldsymbol{U}) \cdot \boldsymbol{n})) \, \mathrm{d}S,$$

where $\boldsymbol{F}$ is the external force acting on the body, w_* is a velocity distribution, which takes into account that the body may generate a nonzero momentum flux through its boundary. $\boldsymbol{M}_C$ is the external torque (by the subscript C we denote the center of mass), while $\boldsymbol{n}$ stands for the outward unit normal on $\partial\mathcal{D}$. Finally, $\boldsymbol{J}$ is the inertial tensor with respect to the center of mass.

For the sake of simplicity we assume $\mathcal{F} = 0$, $\boldsymbol{F} = 0$, $\boldsymbol{M}_C = 0$ and $\boldsymbol{w}_* = 0$. The above system of equations in a fixed exterior domain is obtained by applying the so-called global transformation to the equations of the moving body in a fluid motion in the whole space, which clearly coincides with the classical Navier-Stokes equation in an time dependent exterior domain combined with appropriate boundary condition and asymptotic condition as $|x| \to \infty$. In particular, the conservation of energy is invariant under this transformation. Thus, using the usual energy method, global existence of weak solutions and local existence of strong solutions to the above system can be proved similarly as in the case of the Navier-Stokes equations. There are several results in this direction. The existence of a global weak solution of the Leray-Hopf type has been proved by Borchers [1] (see also [18]). The asymptotic behaviour in time of such solution was investigated by Chen and Miyakawa in [2]. The first result of existence for more regular data is due to Hishida [11]. The generalization of Hishida's results in L^p spaces was done by Hieber, Heck and Geissert in [10]. They proved the existence of a unique local mild solution to the Navier-Stokes problem. The existence of a global strong solution under a smallness assumption on the data with respect to the L^2-norm has been studied by Galdi and Silvestre [8], [9] and by Takahashi and Tucsnak [20] for a rigid body being a disk in the two-dimensional situation. Local in time existence and uniqueness of the strong solution have been proved by Cumsille and Tucsnak [4]. The global time existence and uniqueness were investigated in the work of Cumsille and Takahashi [3]. However, in the three dimensional case the uniqueness is valid only under a smallness assumption on the data. $L^p - L^q$ estimates of the problem were studied in [12].

Alternatively, the problem has been studied in [4], [5], [19], [20] by using the local transformation introduced by Inue and Wakimoto in [13], and in domains depending on time in [14]–[17].

The aim of this paper is the study of the corresponding linear system by neglecting the nonlinear term $(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}$ in the momentum equation of (1.1) and moving the term $-\boldsymbol{U}\cdot\nabla\boldsymbol{w} + \omega\times\boldsymbol{w}$ to the right hand side. Our main result is the existence of global strong solution to this linear problem in a suitable weighted Sobolev space together with estimates of the pressure and the pressure gradient as well. This result will be used for the study of global strong solutions to the full nonlinear problem which will be the subject of a forthcoming paper. In Section 2 we introduce the notion of a weak solution belonging to an appropriate weighted Sobolev space and state our main result (cf. Theorem 2.1). The proof of the main theorem will be divided into two parts. The first part concerns the existence of a strong solution to the linear problem coupled with a motion of the body with a right hand side $\boldsymbol{f}$ in $\boldsymbol{L}^2$. The second part deals with a weighted approach of the heat equation with a right hand side f in weighted L^2 space.

2. The linearized problem

In this section we study the following linear problem which describes the movement of a rigid body inside a fluid, neglecting the nonlinear term $(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}$ and moving the term $\boldsymbol{U}\cdot\nabla\boldsymbol{w}+\boldsymbol{\omega}\times\boldsymbol{w}$ to the right hand side. The equation of the fluid is given by the following Stokes system in $\mathcal{D}\times(0,T)$,

$$\begin{cases} \operatorname{div}\boldsymbol{u}=0, \\ \dfrac{\partial\boldsymbol{u}}{\partial t}-\Delta\boldsymbol{u}=\boldsymbol{f}-\nabla p \end{cases} \tag{2.1}$$

with the boundary and initial conditions

$$\boldsymbol{u}=\boldsymbol{U}\quad\text{on } \partial\mathcal{D}\times(0,T), \tag{2.2}$$

$$\lim_{|x|\to\infty}\boldsymbol{u}(x,t)=0, \tag{2.3}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \tag{2.4}$$

where $\boldsymbol{U}=\boldsymbol{\xi}+\boldsymbol{\omega}\times x$. The equation of the motion of the body is given by

$$m\dot{\boldsymbol{\xi}}=\boldsymbol{\gamma}_1-\int_{\partial\mathcal{D}}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{2.5}$$

$$\boldsymbol{J}\dot{\boldsymbol{\omega}}=\boldsymbol{\gamma}_2-\int_{\partial\mathcal{D}}x\times(\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\,\mathrm{d}S. \tag{2.6}$$

Here $\boldsymbol{f}(x,t)$, $\boldsymbol{u}_0(x)$, $\boldsymbol{\gamma}_1(t)$ and $\boldsymbol{\gamma}_2(t)$ are given data, while $\boldsymbol{u}$, p, $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$ denote the unknown quantities. (Recall that $\boldsymbol{\xi}$ stands for the translation and $\boldsymbol{\omega}$ stands for the rotation of the body.)

Our aim is to study the above system for a right hand side $\boldsymbol{f}=\mathbf{rot}(\mathbf{rot}\,\boldsymbol{a}\times\boldsymbol{b})$, where $\boldsymbol{a}$ denotes a smooth vector field such that $|a(x)|$ behaves like $|x|^2$ ($x\in\mathbb{R}^3$).

Remark 2.1. In order to treat the nonlinear system we may move the term $(\boldsymbol{w}-\boldsymbol{U})\cdot\nabla\boldsymbol{w}+\boldsymbol{\omega}\times\boldsymbol{w}$ of equation (1.1) to the right hand side. Neglecting the convective term $\boldsymbol{w}\cdot\nabla\boldsymbol{w}$ we end up with a linearized system with $\boldsymbol{f}=\boldsymbol{U}\cdot\nabla\boldsymbol{w}-\boldsymbol{\omega}\times\boldsymbol{w}$. Calculating

$$\boldsymbol{\omega}\times\boldsymbol{w}=w_i\omega\times e_i=w_i\frac{\partial}{\partial x_i}(\boldsymbol{\omega}\times x)=\boldsymbol{w}\cdot\nabla\boldsymbol{U},$$

$$\boldsymbol{U}=\boldsymbol{\xi}+\boldsymbol{\omega}\times x=\mathbf{rot}\,\boldsymbol{\psi},\quad\text{where}\quad\boldsymbol{\psi}=\frac{1}{2}(\boldsymbol{\xi}\times x-\boldsymbol{\omega}|x|^2),\quad x\in\mathbb{R}^3,$$

we see that

$$\boldsymbol{U}\cdot\nabla\boldsymbol{w}-\boldsymbol{\omega}\times\boldsymbol{w}=\boldsymbol{U}\cdot\nabla\boldsymbol{w}-\boldsymbol{w}\cdot\nabla\boldsymbol{U}=\mathbf{rot}(\mathbf{rot}\,\boldsymbol{\psi}\times\boldsymbol{w}), \tag{2.7}$$

which has the desired form.

Clearly, for such forces $\boldsymbol{f}$ we can expect the existence of weak solutions in an appropriate Sobolev space rather then in a usual Sobolev space. For the notion of such weak solutions we will introduce the following weight function

$$\eta(x) = (1 + |x|^2)^{-1/2}, \quad x \in \mathbb{R}^3.$$

Then we define the spaces

$$\boldsymbol{L}^2_\eta(\mathcal{D}) = \{\boldsymbol{v} \in \boldsymbol{L}^2_{\mathrm{loc}}(\mathcal{D});\ \eta\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{D})\},$$
$$\boldsymbol{W}^{1,2}_\eta(\mathcal{D}) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{\mathrm{loc}}(\mathcal{D});\ \eta\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\mathcal{D})\}.$$

In addition, by $\mathcal{C}(\mathcal{D})$ we denote the space of all solenoidal smooth vector fields $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^3)$ for which there exist constant vectors Φ_1 and Φ_2 such that

$$\boldsymbol{\varphi} = \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 \times x \quad \text{in a neighbourhood of } \partial\mathcal{D}.$$

Then we define $\mathcal{V}(\mathcal{D})$ and $\mathcal{V}_\eta(\mathcal{D})$ as the closure of $\mathcal{C}(\mathcal{D})$ with respect to the norm in $\boldsymbol{W}^{1,2}(\mathcal{D})$ and $\boldsymbol{W}^{1,2}_\eta(\mathcal{D})$, respectively.

Definition 2.1 (Weak solution)**.** Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathcal{D}$. We assume $\boldsymbol{f} = \mathbf{rot}\,\boldsymbol{g}$, where $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. A triple $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ is called a *weak solution* to (2.1)–(2.6) if

(i) $\boldsymbol{u} \in L^2(0,T;\mathcal{V}_\eta(\mathcal{D})) \cap C_w([0,T);\boldsymbol{L}^2_\eta(\mathcal{D}))$,
(ii) $\boldsymbol{\xi}, \boldsymbol{\omega} \in \boldsymbol{C}([0,T])$,
(iii) for every $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{C}(\mathcal{D}))$ there holds the identity

$$\begin{aligned}
(2.8) \qquad & \int_0^t\!\!\int_{\mathcal{D}} \Big(-\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + \boldsymbol{D}\boldsymbol{u} : \boldsymbol{D}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{\mathcal{D}} \boldsymbol{u}(t) \cdot \boldsymbol{\varphi}(t)\,\mathrm{d}x + \boldsymbol{\xi}(t) \cdot \boldsymbol{\Phi}_1(t) + \boldsymbol{J}\boldsymbol{\omega}(t) \cdot \boldsymbol{\Phi}_2(t) \\
&= \int_0^t (m\boldsymbol{\xi} \cdot \dot{\boldsymbol{\Phi}}_1 - \boldsymbol{\gamma}_1 \cdot \boldsymbol{\Phi}_1 + \boldsymbol{J}\boldsymbol{\omega} \cdot \dot{\boldsymbol{\Phi}}_2 - \boldsymbol{\gamma}_2 \cdot \boldsymbol{\Phi}_2)\,\mathrm{d}s \\
&\quad + \int_{\mathcal{D}} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0)\,\mathrm{d}x + \boldsymbol{\xi}_0 \cdot \boldsymbol{\Phi}_1(0) + \boldsymbol{J}\boldsymbol{\omega}_0 \cdot \boldsymbol{\Phi}_2(0) + \int_0^t\!\!\int_{\mathcal{D}} \boldsymbol{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

for all $0 < t < T$.

Our main result is the following:

Theorem 2.1. *Let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathcal{D}$. Let $\boldsymbol{f} = \mathbf{rot}\, \boldsymbol{g}$, such that $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. Then there exists a weak solution $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ to* (2.1)–(2.6) *according to Definition* 2.1*, such that*

$$\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \in L^2(0,T;\boldsymbol{L}^2_\eta(\mathcal{D})), \quad i,j = 1,2,3 \tag{2.9}$$

and there exists a pressure $p \in L^2(0,T;L^2_{\mathrm{loc}}(\overline{\mathcal{D}}))$ with

$$\nabla p \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D})). \tag{2.10}$$

Furthermore, there holds

$$\begin{aligned} \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} &+ \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{W}^{2,2}_\eta)} + \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;L^2_\eta)} \\ &+ \|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}(0,T)} + \|\boldsymbol{\omega}\|_{\boldsymbol{W}^{1,2}(0,T)} + \|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} \leqslant cK_0, \end{aligned} \tag{2.11}$$

where $K_0 := \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + |\boldsymbol{\omega}_0| + |\boldsymbol{\xi}_0| + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2(0,T)} + \|\boldsymbol{\gamma}_2\|_{\mathbf{L}^2(0,T)}$ and $c = \mathrm{const}$ depending on $\mathcal{D}$ only.

Remark 2.2. Since $\boldsymbol{f} \notin \boldsymbol{L}^2$ we are not allowed to test equation $(2.1)_2$ with the solution $\boldsymbol{u}$. Therefore an estimate based on the usual energy method is not possible. To overcome this difficulty we divide the problem into a Stokes-like problem in the whole space with non-decaying right hand side and a linear problem (2.1)–(2.6) with a right hand side belonging to $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$.

3. Estimates for auxiliary problems

Our first result is related to the a priori estimate of weak solutions of the Stokes-like system in the weighted Sobolev space with solenoidal right hand side. As we will see below, such system coincides with the system of heat equations. Therefore, it will be sufficient to consider the case of the heat equation.

Lemma 3.1. *Let $f \in L^2(0,T;L^2_\eta(\mathbb{R}^3))$. Then there exists a weak solution*

$$z \in L^2(0,T;W^{1,2}_\eta(\mathbb{R}^3)) \cap L^\infty(0,T;L^2_\eta(\mathbb{R}^3))$$

to the heat equation

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } \mathbb{R}^3 \times (0,T), \tag{3.1}$$

$$z(0) = 0. \tag{3.2}$$

In addition, there holds $\partial z/\partial t, \partial^2 z/\partial x_i \partial x_j \in L^2(0,T;L^2_\eta(\mathbb{R}^3))$, $i,j=1,2,3$, *together with the estimate*

$$\Bigl\|\frac{\partial z}{\partial t}\Bigr\|_{L^2(0,T;L^2_\eta)} + \|z\|_{L^2(0,T;W^{2,2}_\eta)} \leqslant c\|f\|_{L^2(0,T;L^2_\eta)}. \tag{3.3}$$

(Here by $W^{2,2}_\eta(\mathbb{R}^3)$ we denote the space of all $v \in W^{2,2}_{\rm loc}(\mathbb{R}^3)$ such that $\eta D^\alpha v \in L^2(\mathbb{R}^3)$ for every multi-index $\alpha \leqslant 2$.)

Proof. We divide the proof into two steps. First, we consider the case $f \in L^2(0,T;L^2(\mathbb{R}^3))$ and prove the a priori estimate (3.3). Second, for general f we get an approximate weak solution z_m for the truncated right hand side f_m and pass to the limit $m \to \infty$ by using a priori estimate (3.3).

1° Let $f \in L^2(0,T;L^2(\mathbb{R}^3))$. Clearly, there exists a weak solution

$$z \in L^2(0,T;W^{1,2}(\mathbb{R}^3)) \cap C([0,T];L^2(\mathbb{R}^3)),$$

such that

$$\frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x_i \partial x_j} \in L^2(0,T;L^2(\mathbb{R}^3)), \quad i,j=1,2,3.$$

Setting $h(x,t) = z(x,t)\eta(x)$ and using the product rule the equation (3.1) turns into

$$\frac{\partial h}{\partial t} - \Delta h = \eta f - 2\nabla z \cdot \nabla \eta - z\Delta\eta \quad \text{in } \mathbb{R}^3 \times (0,T). \tag{3.4}$$

By elementary calculus, the equation (3.4) can be rewritten as

$$\frac{\partial h}{\partial t} - \Delta h = \eta f + 2x \cdot \eta^2 \nabla h + (\eta^4 + 2\eta^2)h \quad \text{in } \mathbb{R}^3 \times (0,T). \tag{3.5}$$

Next, we multiply both sides of (3.5) by h, integrate the obtained equation over $\mathbb{R}^3 \times (0,t)$ $(t \in (0,T))$ and apply integration by parts. This yields

$$\frac{1}{2}\|h(t)\|^2_{L^2} + \int_0^t\!\!\int_{\mathbb{R}^3} |\nabla h|^2 \,\mathrm{d}x\,\mathrm{d}s = \int_0^t\!\!\int_{\mathbb{R}^3} \eta f h \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\mathbb{R}^3} (\eta^2 - \eta^4)|h|^2 \,\mathrm{d}x\,\mathrm{d}s$$

for a.e. $t \in (0,T)$. Using Young's inequality and Gronwall's lemma we obtain the following a priori estimate

$$\|h\|_{L^\infty(0,T;L^2)} + \|\nabla h\|_{L^2(0,T;L^2)} \leqslant c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.6}$$

Recalling the definition of h from (3.6) we immediately obtain

$$\|z\|_{L^\infty(0,T;L^2_\eta)} + \|\nabla z\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} \leqslant c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.7}$$

On the other hand, multiplying equation (3.5) by $\partial h/\partial t$ and by Δh, and applying integration by parts, observing (3.6) and (3.7) we get

$$\Big\|\frac{\partial z}{\partial t}\Big\|_{L^2(0,T;L^2_\eta)} + \|z\|_{L^2(0,T;W^{2,2}_\eta)} \leqslant c\|\eta f\|_{L^2(0,T;L^2)}. \tag{3.8}$$

2° Now, let $f \in L^2(0,T;L^2_\eta(\mathbb{R}^3))$. We define

$$f_\varepsilon(x) = (1+\varepsilon|x|)^{-1} f, \quad x \in \mathbb{R}^3,\ \varepsilon > 0.$$

Clearly, $f_\varepsilon \in L^2(0,T;L^2(\mathbb{R}^3))$ and $\|f_\varepsilon\|_{L^2(0,T;L^2_\eta)} \leqslant \|f\|_{L^2(0,T;L^2_\eta)}$ for all $\varepsilon > 0$. As it has been shown in 1°, for each $\varepsilon > 0$ there exists a weak solution $z_\varepsilon \in L^2(0,T;W^{1,2}(\mathbb{R}^3)) \cap C([0,T];L^2(\mathbb{R}^3))$ to (3.1), (3.2) replacing f by f_ε therein. In addition, we have $\partial z_\varepsilon/\partial t \in L^2(0,T;L^2(\mathbb{R}^3))$ and $\partial^2 z_\varepsilon/\partial x_i \partial x_j \in L^2(0,T;L^2(\mathbb{R}^3))$, $i,j = 1,2,3$. From (3.7) and (3.8) it follows that

$$\Big\|\frac{\partial z_\varepsilon}{\partial t}\Big\|_{L^2(0,T;L^2_\eta)} + \|z_\varepsilon\|_{L^2(0,T;W^{2,2}_\eta)} \leqslant c\|\eta f_\varepsilon\|_{L^2(0,T;L^2)} \leqslant c\|f\|_{L^2(0,T;L^2_\eta)}. \tag{3.9}$$

By means of reflexivity of $L^2(0,T;W^{2,2}_\eta)$ there exists a sequence (ε_k) with $\varepsilon_k \to 0^+$ as $k \to \infty$ and $z \in L^2(0,T;W^{2,2}_\eta)$ with $\partial z/\partial t \in L^2(0,T;L^2_\eta)$ such that

$$z_{\varepsilon_k} \to z \quad \text{weakly in } L^2(0,T;W^{2,2}_\eta) \text{ as } k \to \infty.$$

In the equation for z_{ε_k}, taking the passage to the limit $\varepsilon_k \to 0^+$ on both sides we see that z solves (3.1), (3.2) in weak sense. Finally, by virtue of (3.9), using the lower semicontinuity of the norm we get (3.3). □

Next, let us consider the problem (2.1)–(2.6) with $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. In this case we have the following existence result.

Lemma 3.2. *Let $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and let $\boldsymbol{u}_0 \in \mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0 = \boldsymbol{\xi}_0 + \boldsymbol{\omega}_0 \times x$ on $\partial\mathcal{D}$, where $\boldsymbol{\xi}_0, \boldsymbol{\omega}_0 \in \mathbb{R}$ are given. In addition, let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \boldsymbol{L}^2(0,T)$. Then there exists a weak solution $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ to (2.1)–(2.6), such that*

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2} + \|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^\infty(0,T)} \\ \leqslant c\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0| + |\boldsymbol{\gamma}_1|_{\boldsymbol{L}^2(0,T)} + |\boldsymbol{\gamma}_2|_{\boldsymbol{L}^2(0,T)}. \end{aligned} \tag{3.10}$$

In addition, we have

$$\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j}, \nabla p \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D})) \quad i,j = 1,2,3, \quad \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^2(0,T)$$

and there holds

$$\left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_{\boldsymbol{L}^2} + \|\nabla^2 \boldsymbol{u}\|_{\boldsymbol{L}^2} + \|\nabla p\|_{\boldsymbol{L}^2} + \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^2(0,T)} \leqslant c\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{f}\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0| + \|\dot{\gamma}_1\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\gamma}_2\|_{\boldsymbol{L}^2(0,T)}. \tag{3.11}$$

P r o o f. 1° The existence and uniqueness of a weak solution $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ can be shown easily by applying the linear theory of evolutionary equations in Hilbert spaces (e.g. see in [6]).

2° Assume $\boldsymbol{u}_0 = 0$. Let $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ be a weak solution to (2.1)–(2.6). First, we assume

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T; \boldsymbol{L}^2(\mathcal{D})), \quad \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^2(0,T). \tag{3.12}$$

Next, let $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta \equiv 1$ in a neighborhood of $\mathcal{B}$. Set

$$\Psi(x,t) := \frac{1}{2}\,\mathbf{rot}((\boldsymbol{\xi}(t) \times x - \boldsymbol{\omega}(t)|x|^2)\zeta(x)), \quad (x,t) \in \mathbb{R}^3 \times (0,T).$$

Since

$$\frac{1}{2}\,\mathbf{rot}(\boldsymbol{\xi} \times x - \boldsymbol{\omega}|x|^2) = \boldsymbol{\xi} + \boldsymbol{\omega} \times x, \quad x \in \mathbb{R}^3$$

it follows that $\boldsymbol{u} - \Psi = 0$ on $\partial\mathcal{D}$. Thus, for almost all $t \in (0,T)$ the function $\boldsymbol{v} := \boldsymbol{u}(\cdot,t) - \Psi(\cdot,t)$ is a solution to the Stokes system

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathcal{D}, \\ -\Delta \boldsymbol{v} = \boldsymbol{f}(t) + \Delta\Psi(t) - \dfrac{\partial \boldsymbol{u}}{\partial t}(t) - \nabla p(t) \quad \text{in } \mathcal{D}, \\ \boldsymbol{v}|_{\partial\mathcal{D}} = 0, \qquad \lim\limits_{|x|\to\infty} \boldsymbol{v}(x) = 0. \end{cases}$$

By the well-known theory of the Stokes equation one gets $\nabla p(t) \in \boldsymbol{L}^2(\mathcal{D})$ together with the estimate

$$\|\nabla p(t)\|_{\boldsymbol{L}^2} \leqslant c\Big(\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^2} + \left\|\frac{\partial \boldsymbol{u}}{\partial t}(t)\right\|_{\boldsymbol{L}^2} + |\boldsymbol{\xi}(t)| + |\boldsymbol{\omega}(t)|\Big), \tag{3.13}$$

where $c =$ const independent of $t \in (0,T)$. Hence, from (3.13), the equation $(2.1)_2$ and the assumption $\partial\boldsymbol{u}/\partial t \in L^2(0,T; \boldsymbol{L}^2(\mathcal{D}))$ then $\nabla p, \Delta\boldsymbol{u} \in L^2(0,T; \boldsymbol{L}^2(\mathcal{D}))$. Moreover, there holds

$$\|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2 \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2)} \leqslant c\Big(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{\omega}\|_{\boldsymbol{L}^\infty}\Big). \tag{3.14}$$

Then, multiplying both sides of $(2.1)_2$ by $\partial \boldsymbol{u}/\partial t$, integrating the result over $\mathcal{D} \times (0,t)$ ($t \in (0,T)$) and applying integration by parts we are led to

$$\int_0^t\!\!\int_{\mathcal{D}} \Big|\frac{\partial \boldsymbol{u}}{\partial t}\Big|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2 \,\mathrm{d}x = \int_0^t\!\!\int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{u},p)\cdot \boldsymbol{n})\cdot(\dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\omega}}\times x)\,\mathrm{d}S\,\mathrm{d}s + \int_0^t\!\!\int_{\mathcal{D}} \boldsymbol{f}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.15}$$

(Note that by virtue of (3.13) the trace of $\mathbb{T}(\boldsymbol{u},p)$ upon $\partial\mathcal{D}$ is well defined.)

Next, from (2.5) we induce

$$\int_{\partial\mathcal{D}} (\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\cdot\dot{\boldsymbol{\xi}}(s)\,\mathrm{d}S = \gamma_1(s)\cdot\dot{\boldsymbol{\xi}}(s) - m|\dot{\boldsymbol{\xi}}(s)|^2 \quad \text{for a.e. } s\in(0,T).$$

Moreover, from (2.6) we obtain

$$\begin{aligned}\int_{\partial\mathcal{D}} ((\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\cdot\dot{\boldsymbol{\omega}}(s)\times x)\,\mathrm{d}S &= \int_{\partial\mathcal{D}} x\times(\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{n})\,\mathrm{d}S\cdot\dot{\boldsymbol{\omega}}(s)\,\mathrm{d}S \\ &= (\gamma_2(s) - \boldsymbol{J}\dot{\boldsymbol{\omega}}(s))\cdot\dot{\boldsymbol{\omega}}(s)\end{aligned}$$

for a.e. $s\in(0,T)$. Inserting these identities into (3.15) and applying integration by parts we see that

$$\begin{aligned}\int_0^t\!\!\int_{\mathcal{D}} \Big|\frac{\partial \boldsymbol{u}}{\partial t}\Big|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{D}} |\nabla \boldsymbol{u}(t)|^2\,\mathrm{d}x + \int_0^t (m|\dot{\boldsymbol{\xi}}|^2 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2)\,\mathrm{d}s \\ = \int_0^t (\dot{\boldsymbol{\xi}}\cdot\gamma_1 + \dot{\boldsymbol{\omega}}\cdot\gamma_2)\,\mathrm{d}s + \int_0^t\!\!\int_{\mathcal{D}} \boldsymbol{f}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

for a.e. $t\in(0,T)$. Here $\boldsymbol{R}$ denotes the square root of $\boldsymbol{J}$, i.e. $\boldsymbol{R}^2 = \boldsymbol{J} = \boldsymbol{J}^\top$. By the aid of Cauchy-Schwarz's inequality and Young's inequality one finds

$$\begin{aligned}\Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}\|_{\boldsymbol{L}^2(0,T)} + \|\nabla\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{L}^2)} \\ \leqslant 2(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\gamma_1\|_{L^2(0,T)} + \|\gamma_2\|_{L^2(0,T)}).\end{aligned} \tag{3.16}$$

Finally, combining (3.14) and (3.16) we obtain the a priori estimate

$$\begin{aligned}\|\nabla p\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{L}^2)} \\ \leqslant c(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\gamma_1\|_{L^2(0,T)} + \|\gamma_2\|_{L^2(0,T)}).\end{aligned} \tag{3.17}$$

Second, let us consider the general case, without assuming (3.12). To begin with, we introduce the Steklov mean as follows. Let $f\in L^1(0,T;L^1(\mathcal{D}))$. Define

$$f_\lambda(x,t) = -\frac{1}{\lambda}\int_{\max\{t+\lambda,0\}}^t f(x,s)\,\mathrm{d}s, \quad (x,s)\in\mathcal{D}\times(0,T),\ \lambda<0.$$

Applying the Steklov mean to both sides of the equations $(2.1)_2$, (2.5) and (2.6), recalling that $\boldsymbol{u}(0)=0$ we see that $(\boldsymbol{u}_\lambda,\boldsymbol{\xi}_\lambda,\boldsymbol{\omega}_\lambda)$ is a weak solution to (2.1)–(2.6) with $\boldsymbol{f}_\lambda$, $\gamma_{1,\lambda}$, $\gamma_{2,\lambda}$, $\boldsymbol{\omega}_\lambda(0)$, $\boldsymbol{\xi}_\lambda(0)$ instead of $\boldsymbol{f}$, γ_1, γ_2, $\boldsymbol{\omega}_0$, $\boldsymbol{\xi}_0$. In addition, this weak solution satisfies (3.12). Thus, from (3.16) and (3.17) it follows that

$$
\begin{aligned}
(3.18)\qquad \Big\|\frac{\partial \boldsymbol{u}_\lambda}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} &+ \|\nabla^2\boldsymbol{u}_\lambda\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla\boldsymbol{u}_\lambda\|_{L^\infty(0,T;\boldsymbol{L}^2)}\\
&+\|\nabla p_\lambda\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}_\lambda\|_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}_\lambda\|_{\boldsymbol{L}^2(0,T)}\\
&\leqslant 2(\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\gamma_1\|_{L^2(0,T)} + \|\gamma_2\|_{L^2(0,T)}).
\end{aligned}
$$

Here $c=\text{const}>0$ depending on the geometry $\mathcal{D}$ only. Thus, by means of reflexivity of $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and $\boldsymbol{L}^2(0,T)$ the assertion of the lemma follows from (3.18).

3° Let $\boldsymbol{u}_0\in\mathcal{V}(\mathcal{D})\cap\boldsymbol{W}^{2,2}(\mathcal{D})$. Let $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ be a weak solution to (2.1)–(2.6). Clearly, $\boldsymbol{u}-\boldsymbol{u}_0$ solves the system (2.1)–(2.6) too, with vanishing initial data and right hand side in $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Hence, applying the result of step 2°, wee see that

$$
\frac{\partial\boldsymbol{u}}{\partial t},\Delta\boldsymbol{u},\nabla p\in L^2(0,T;\boldsymbol{L}^2(\mathcal{D})),\quad \dot{\boldsymbol{\xi}},\dot{\boldsymbol{\omega}}\in\boldsymbol{L}^2(0,T).
$$

Now, we are in a position to multiply the equation $(2.1)_2$ by $\partial\boldsymbol{u}/\partial t$ and integrate both sides over $\mathcal{D}\times(0,t)$ ($t\in(0,T)$). Then arguing similarly as in step 2°, by using integration by parts we achieve the identity

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathcal{D}}\Big|\frac{\partial\boldsymbol{u}}{\partial t}\Big|^2\,\mathrm{d}x\,\mathrm{d}s &+ \frac12\int_{\mathcal{D}}|\nabla\boldsymbol{u}(t)|^2\,\mathrm{d}x + \int_0^t(|\dot{\boldsymbol{\xi}}|^2+|\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2)\,\mathrm{d}s\\
&= \frac12\int_{\mathcal{D}}|\nabla\boldsymbol{u}_0|^2\,\mathrm{d}x + \int_0^t\!\!\int_{\mathcal{D}}\boldsymbol{f}\cdot\frac{\partial\boldsymbol{u}}{\partial t}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t(\dot{\boldsymbol{\xi}}\cdot\gamma_1+\dot{\boldsymbol{\omega}}\cdot\gamma_2)\,\mathrm{d}s
\end{aligned}
$$

for a.e. $t\in(0,T)$. As above, from this identity the assertion easily follows.

4° Finally, in case $\boldsymbol{u}_0\in\mathcal{V}(\mathcal{D})$ the proof is completed by a standard density argument using the a priori estimate obtained in 3°. □

4. Proof of main theorem

We divide the proof into two steps. First, we prove the assertion for the case when $\boldsymbol{f}$ belongs to $L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Then, we will complete the proof by applying a standard approximation argument, passing to the limit on the basis of the a priori estimate obtained in the first step.

1° Let $\boldsymbol{f}=\mathbf{rot}\,\boldsymbol{g}$, with $\boldsymbol{g}\in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D}))$, $\boldsymbol{u}_0\in\mathcal{V}(\mathcal{D})$ with $\boldsymbol{u}_0=\boldsymbol{\xi}+\boldsymbol{\omega}\times x$ a.e. on $\partial\mathcal{D}$ and $\gamma_1,\gamma_2\in\boldsymbol{L}^2(0,T)$. According to Lemma 3.2 there exists a weak solution $(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\omega})$ to (2.1)–(2.6) which is strong in the sense that $\partial\boldsymbol{u}/\partial t, \Delta\boldsymbol{u}, \nabla p \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ and $\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\omega}} \in \boldsymbol{L}^2(0,T)$.

Fix $0 < R_0 < \infty$, such that $\mathcal{B} \subset B_{R_0/2}(0)$. Let $\zeta \in C_0^\infty(\mathbb{R}^3)$ denote a cut-off function with $\operatorname{supp}(\zeta) \subset B_{R_0}$, such that $\zeta \equiv 1$ in a neighborhood of $\mathcal{B}$. We write $\boldsymbol{f}$ as the sum $\boldsymbol{f}_1 + \boldsymbol{f}_2$, where $\boldsymbol{f}_1 = \mathbf{rot}(\zeta\boldsymbol{g})$, $\boldsymbol{f}_2 = \mathbf{rot}((1-\zeta)\boldsymbol{g})$ a.e. in $\mathcal{D}\times(0,T)$. Once more, applying Lemma 3.2 there exists a strong solution $(\boldsymbol{u}_1,\boldsymbol{\xi}_1,\boldsymbol{\omega}_1)$ to the system (2.1)–(2.6) with right hand side $\boldsymbol{f}_1$ in place of $\boldsymbol{f}$. In particular, we have the a priori estimate

$$
\begin{aligned}
(4.1)\quad & \Big\|\frac{\partial \boldsymbol{u}_1}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla^2\boldsymbol{u}_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla p_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}_1\|_{\boldsymbol{L}^2} + \|\dot{\boldsymbol{\omega}}_1\|_{\boldsymbol{L}^2}\\
&\leqslant c(\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\mathcal{D}_{R_0}))} + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2} + \|\boldsymbol{\gamma}_2\|_{\boldsymbol{L}^2}).
\end{aligned}
$$

Additionally, by means of (3.10) we get

$$
\begin{aligned}
(4.2)\quad & \|\nabla\boldsymbol{u}_1\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}_1\|_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}_1\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{\omega}_1\|_{\boldsymbol{L}^\infty}\\
&\leqslant c(\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2(\mathcal{D}_{R_0}))} + |\boldsymbol{\xi}_0| + |\boldsymbol{\omega}_0| + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2} + \|\boldsymbol{\gamma}_2\|_{\boldsymbol{L}^2}).
\end{aligned}
$$

Next, let $\boldsymbol{z} \in L^2(0,T;\boldsymbol{W}^{2,2}(\mathbb{R}^3)) \cap W^{1,2}(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$, such that z_j is the strong solution to the heat equation

$$
\begin{aligned}
\frac{\partial z_j}{\partial t} - \Delta z_j &= f_{2,j} \quad \text{in } \mathbb{R}^3\times(0,T),\\
z_j(0) &= 0 \quad \text{in } \mathbb{R}^3
\end{aligned}
$$

$(j=1,2,3)$ (cf. Lemma 3.1). Since $\operatorname{div}\boldsymbol{f}_2 = 0$ one sees that $\operatorname{div}\boldsymbol{z}$ is a weak solution to the heat equation with zero data. By a standard uniqueness argument it follows that $\operatorname{div}\boldsymbol{z} = 0$ a.e. in $\mathbb{R}^3\times(0,T)$. Furthermore, applying Lemma 3.1 one gets the estimate

$$
(4.3)\qquad \Big\|\frac{\partial \boldsymbol{z}}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\boldsymbol{z}\|_{L^2(0,T;\boldsymbol{W}^{2,2}_\eta)} \leqslant c\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2_\eta)}.
$$

Setting $\boldsymbol{u}_2 = \boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{z}$ and $p_2 = p - p_1$ we see that $\boldsymbol{u}_2 \in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D})) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathcal{D}))$ solves the system in $\mathcal{D}\times(0,T)$,

$$
(4.4)\qquad \begin{cases} \operatorname{div}\boldsymbol{u}_2 = 0,\\ \dfrac{\partial \boldsymbol{u}_2}{\partial t} - \Delta\boldsymbol{u}_2 = -\nabla p_2 \end{cases}
$$

fulfilling the following boundary and initial conditions

$$
(4.5)\qquad \boldsymbol{u}_2 = \boldsymbol{\xi} - \boldsymbol{\xi}_1 + (\boldsymbol{\omega} - \boldsymbol{\omega}_1)\times x - \boldsymbol{z} \quad \text{on} \quad \partial\mathcal{D}\times(0,T),
$$

$$
(4.6)\qquad \lim_{|x|\to\infty} \boldsymbol{u}_2(x,t) = 0,
$$

$$
(4.7)\qquad \boldsymbol{u}_2(0) = 0.
$$

We multiply the equation $(4.4)_2$ by $\boldsymbol{u}_2$, integrate both sides over $\mathcal{D}\times(0,t)$ $(t\in(0,T))$ and apply integration by parts. This leads to

$$
\begin{aligned}
(4.8)\qquad \frac{1}{2}\|\boldsymbol{u}_2(t)\|_{2,\mathcal{D}}^2 &- \int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&- \int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u}_2(1-\zeta)\,\mathrm{d}x\,\mathrm{d}s = 0
\end{aligned}
$$

for all $t\in(0,T)$. Recalling the definition of $\mathbb{T}$ we calculate

$$
(4.9)\qquad -\mathbb{T}(\boldsymbol{u}_2,p_2) = -\mathbb{T}(\boldsymbol{u},p)+\mathbb{T}(\boldsymbol{u}_1,p_1)+\boldsymbol{D}\boldsymbol{z}.
$$

Noticing $\boldsymbol{u}_2=\boldsymbol{u}-\boldsymbol{u}_1-\boldsymbol{z}$ and taking into account $\operatorname{div}\mathbb{T}(\boldsymbol{u},p)=\partial\boldsymbol{u}/\partial t-\boldsymbol{f}$ (cf. $(2.1)_2$ for $\boldsymbol{u}$), the first integral on the left of (4.8) satisfies the following identity

$$
\begin{aligned}
(4.10)\quad -\int_0^t\!\!\int_{\mathcal{D}} &\operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{u}\zeta\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\mathcal{D}} \Big(\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{f}\Big)\cdot(\boldsymbol{u}_1+\boldsymbol{z})\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\!\!\int_{\mathcal{D}} (\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1)+\Delta\boldsymbol{z})\cdot\boldsymbol{u}_2\zeta\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Using integration by parts, observing equations (2.5) and (2.6) assigned to $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, we infer

$$
\begin{aligned}
(4.11)\quad -\int_0^t\!\!\int_{\mathcal{D}} &\operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\boldsymbol{u}\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t\!\!\int_{\mathcal{D}} |\nabla\boldsymbol{u}|^2\zeta\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}(|\boldsymbol{\xi}(t)|^2+|\boldsymbol{R}\boldsymbol{\omega}(t)|^2-|\boldsymbol{\xi}_0|^2-|\boldsymbol{R}\boldsymbol{\omega}_0|^2) \\
&\quad + \int_0^t\!\!\int_{\mathcal{D}} \mathbb{T}(\boldsymbol{u},p-p_{\mathcal{D}_{R_0}}):\boldsymbol{u}\otimes\nabla\zeta\,\mathrm{d}x\,\mathrm{d}s - \int_0^t(\boldsymbol{\gamma}_1\cdot\boldsymbol{\xi}+\boldsymbol{\gamma}_2\cdot\boldsymbol{\omega})\,\mathrm{d}s.
\end{aligned}
$$

(Note that for every $\boldsymbol{v}\in\mathcal{V}(\mathcal{D})$ there holds $\int_{\mathcal{D}}\boldsymbol{v}\cdot\nabla\zeta\,\mathrm{d}x=\int_{\partial\mathcal{D}}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S=0$.)

On the other hand, integration by parts gives

$$
\begin{aligned}
(4.12)\quad -\int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\boldsymbol{u}_2(1-\zeta)\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\!\!\int_{\mathcal{D}} |\nabla\boldsymbol{u}_2|^2(1-\zeta)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t\!\!\int_{\mathcal{D}} \mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}}):\boldsymbol{u}_2\otimes\nabla\zeta\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Combining identities (4.10), (4.11), (4.12) and inserting the result into the left hand side of (4.8) we get

$$
\begin{aligned}
(4.13)\quad & \frac{1}{2}(\|\boldsymbol{u}_2(t)\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}(t)|^2 + |\boldsymbol{R}\boldsymbol{\omega}(t)|^2) + \int_0^t\int_{\mathcal{D}} (|\nabla \boldsymbol{u}|^2\zeta + |\nabla \boldsymbol{u}_2|^2(1-\zeta))\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{2}(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0|^2 + |\boldsymbol{R}\boldsymbol{\omega}_0|^2) + \int_0^t (\boldsymbol{\gamma}_1\cdot\boldsymbol{\xi} + \boldsymbol{\gamma}_2\cdot\boldsymbol{\omega})\,\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathcal{D}} ((\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot \boldsymbol{u}_2 - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u})\cdot\nabla\zeta)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t\int_{\mathcal{D}} \Big(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{f}\Big)\cdot(\boldsymbol{u}_1 + \boldsymbol{z})\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathcal{D}} (\operatorname{div}\mathbb{T}(\boldsymbol{u}_1, p_1) + \Delta \boldsymbol{z})\cdot \boldsymbol{u}_2\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{2}(\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + |\boldsymbol{\xi}_0|^2 + |\boldsymbol{R}\boldsymbol{\omega}_0|^2) + I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

for almost all $t \in (0,T)$.

(i) Using Cauchy-Schwarz's inequality and Young's inequality, taking into account (4.1), (4.2) and (4.3) we easily get

$$
I_1 + I_4 \leqslant cK_0^2 + \frac{1}{8}(\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\omega}\|^2_{\boldsymbol{L}^\infty(0,T)} + \|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^\infty(0,T)})
$$

(where $K_0 := \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + |\boldsymbol{\omega}_0| + |\boldsymbol{\xi}_0| + \|\boldsymbol{\gamma}_1\|_{\boldsymbol{L}^2(0,T)} + \|\boldsymbol{\gamma}_2\|_{\mathbf{L}^2(0,T)}$).

(ii) In order to estimate I_2 we first notice the following identity

$$
\begin{aligned}
&\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u}_2 - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u} \\
&\quad = -(\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z})\cdot\boldsymbol{u}_2 - \boldsymbol{D}\boldsymbol{u}_2\cdot(\boldsymbol{u}_1 + \boldsymbol{z}) - (\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z})\cdot(\boldsymbol{u}_1 + \boldsymbol{z}) \\
&\qquad + \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}})\cdot\boldsymbol{u}_2 - \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}})\cdot(\boldsymbol{u}_1 + \boldsymbol{z}) \\
&\qquad - \boldsymbol{I}(p_2 - (p_2)_{\mathcal{D}_{R_0}})\cdot(\boldsymbol{u}_1 + \boldsymbol{z}).
\end{aligned}
$$

Again using Cauchy-Schwarz's and Young's inequality and the Poincaré inequality for the term involving the pressure $p_1 - (p_1)_{\mathcal{D}_{R_0}}$ together with (4.2) and (4.3) we get

$$
I_2 \leqslant cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}K_0 + \frac{1}{8}\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)}.
$$

(iii) For the estimation of I_3 we apply integration by parts. This gives

$$
\begin{aligned}
&-\int_0^t\int_{\mathcal{D}} \frac{\partial \boldsymbol{u}_2}{\partial t}\cdot(\boldsymbol{u}_1 + \boldsymbol{z})\zeta\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad = \int_0^t\int_{\mathcal{D}} \boldsymbol{u}_2\cdot\frac{\partial}{\partial t}(\boldsymbol{u}_1 + \boldsymbol{z})\zeta\,\mathrm{d}x\,\mathrm{d}s - \int_{\mathcal{D}} (\boldsymbol{u}_2(t)\cdot(\boldsymbol{u}_1(t) + \boldsymbol{z}(t))\zeta)\,\mathrm{d}x.
\end{aligned}
$$

By the aid of Cauchy-Schwarz's inequality and Young's inequality, again observing (4.1), (4.2) and (4.3) we get

$$I_3 \leqslant cK_0^2 + \frac{1}{8}\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)}.$$

Now, inserting the estimates of $I_1 - I_4$ into (4.13) we obtain the estimate

$$\begin{aligned}\|\boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\nabla\boldsymbol{u}_2\|^2_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{\xi}\|^2_{\boldsymbol{L}^2(0,T)} + \|\boldsymbol{\omega}\|^2_{\boldsymbol{L}^2(0,T)} \\ \leqslant cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}K_0.\end{aligned} \tag{4.14}$$

Next, we are going to estimate the L^2 norm of $\partial\boldsymbol{u}_2/\partial t$. To begin with, we multiply both sides of the equation $(4.4)_2$ by $\partial\boldsymbol{u}_2/\partial t$ and integrate the obtained result over $\mathcal{D}\times(0,t)$ ($t\in(0,T)$). Then, using integration by parts we are led to

$$\begin{aligned}\Big\|\frac{\partial\boldsymbol{u}_2}{\partial t}\Big\|^2_{L^2(0,t;\boldsymbol{L}^2)} - \int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\frac{\partial\boldsymbol{u}_2}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{D}}|\nabla\boldsymbol{u}_2(t)|^2(1-\zeta)\,\mathrm{d}x \\ = \int_0^t\!\!\int_{\mathcal{D}} \mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial\boldsymbol{u}_2}{\partial t}\otimes\nabla\zeta\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_{\mathcal{D}}|\nabla\boldsymbol{u}_0|^2(1-\zeta)\,\mathrm{d}x\end{aligned} \tag{4.15}$$

for almost all $t\in(0,T)$. By elementary calculus we see that

$$\begin{aligned}-\int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u}_2,p_2)\cdot\frac{\partial\boldsymbol{u}_2}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s = -\int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\frac{\partial\boldsymbol{u}}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\!\!\int_{\mathcal{D}} \Big(\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{f}\Big)\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\mathcal{D}} (\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1)+\Delta\boldsymbol{z})\cdot\frac{\partial\boldsymbol{u}_2}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Observing (2.5) and (2.6), arguing as in the proof of Lemma 3.2 we infer

$$\begin{aligned}-\int_0^t\!\!\int_{\mathcal{D}} \operatorname{div}\mathbb{T}(\boldsymbol{u},p)\cdot\frac{\partial\boldsymbol{u}}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}s = \int_0^t (|\dot{\boldsymbol{\xi}}|^2 - \dot{\boldsymbol{\xi}}\cdot\gamma_1 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2 - \dot{\boldsymbol{\omega}}\cdot\gamma_2)\,\mathrm{d}s \\ + \frac{1}{2}\int_{\mathcal{D}}|\nabla\boldsymbol{u}(t)|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathcal{D}}|\nabla\boldsymbol{u}_0|^2\,\mathrm{d}x + \int_0^t\!\!\int_{\mathcal{D}} \mathbb{T}(\boldsymbol{u},p-p_{\mathcal{D}_{R_0}}) : \frac{\partial\boldsymbol{u}}{\partial t}\otimes\nabla\zeta\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

(Note that $p_{\mathcal{D}_{R_0}}\int_{\mathcal{D}}(\partial\boldsymbol{u}/\partial t)\cdot\nabla\zeta\,\mathrm{d}x = 0$.)

Combining the above identities together with (4.15) we obtain

$$\begin{aligned}\Big\|\frac{\partial\boldsymbol{u}_2}{\partial t}\Big\|^2_{L^2(0,t;\boldsymbol{L}^2)} + \frac{1}{2}\int_{\mathcal{D}}|\nabla\boldsymbol{u}(t)|^2\zeta + |\nabla\boldsymbol{u}_2(t)|^2(1-\zeta)\,\mathrm{d}x + \int_0^t(|\dot{\boldsymbol{\xi}}|^2 + |\boldsymbol{R}\dot{\boldsymbol{\omega}}|^2)\,\mathrm{d}s \\ = \int_0^t\!\!\int_{\mathcal{D}} \Big(\mathbb{T}(\boldsymbol{u}_2,p_2-(p_2)_{\mathcal{D}_{R_0}})\cdot\frac{\partial\boldsymbol{u}_2}{\partial t} - \mathbb{T}(\boldsymbol{u},p-p_{\mathcal{D}_{R_0}})\cdot\frac{\partial\boldsymbol{u}}{\partial t}\Big)\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\!\!\int_{\mathcal{D}} \Big(\Big(\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{f}\Big)\cdot\frac{\partial(\boldsymbol{u}_1+\boldsymbol{z})}{\partial t}\zeta + (\operatorname{div}\mathbb{T}(\boldsymbol{u}_1,p_1)+\Delta\boldsymbol{z})\cdot\frac{\partial\boldsymbol{u}_2}{\partial t}\zeta\Big)\,\mathrm{d}x\,\mathrm{d}s \\ + \frac{1}{2}\|\nabla\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2} + \int_0^t(\dot{\boldsymbol{\xi}}\cdot\gamma_1 + \dot{\boldsymbol{\omega}}\cdot\gamma_2)\,\mathrm{d}s.\end{aligned} \tag{4.16}$$

Clearly, recalling $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{z}$ one calculates

$$\begin{aligned}
&\mathbb{T}(\boldsymbol{u}_2, p_2 - (p_2)_{\mathcal{D}_{R_0}}) \cdot \frac{\partial \boldsymbol{u}_2}{\partial t} - \mathbb{T}(\boldsymbol{u}, p - p_{\mathcal{D}_{R_0}}) \cdot \frac{\partial \boldsymbol{u}}{\partial t} \\
&\quad = -(\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z}) \cdot \frac{\partial \boldsymbol{u}_2}{\partial t} - \boldsymbol{D}\boldsymbol{u}_2 \cdot \frac{\partial(\boldsymbol{u}_1 + \boldsymbol{z})}{\partial t} - (\boldsymbol{D}\boldsymbol{u}_1 + \boldsymbol{D}\boldsymbol{z}) \cdot \frac{\partial(\boldsymbol{u}_1 + \boldsymbol{z})}{\partial t} \\
&\qquad + \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}}) \cdot \frac{\partial \boldsymbol{u}_2}{\partial t} - \boldsymbol{I}(p_1 - (p_1)_{\mathcal{D}_{R_0}}) \cdot \frac{\partial(\boldsymbol{u}_1 + \boldsymbol{z})}{\partial t} \\
&\qquad - \boldsymbol{I}(p_2 - (p_2)_{\mathcal{D}_{R_0}}) \cdot \frac{\partial(\boldsymbol{u}_1 + \boldsymbol{z})}{\partial t}.
\end{aligned}$$

By the aid of this identity, using Young's inequality and taking into account estimates (4.1), (4.2) and (4.3) we obtain

$$\begin{aligned}
\text{(4.17)} \qquad &\Big\| \frac{\partial \boldsymbol{u}_2}{\partial t} \Big\|^2_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \|\dot{\boldsymbol{\xi}}\|^2_{\boldsymbol{L}^2(0,T)} + \|\dot{\boldsymbol{\omega}}\|^2_{\boldsymbol{L}^2(0,T)} \\
&\qquad \leqslant cK_0^2 + c\|p_2 - (p_2)_{\mathcal{D}_{R_0}}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))} K_0.
\end{aligned}$$

For the estimation of the pressure on the right hand side of (4.17) we make use of $(4.4)_2$. That is,

$$-\nabla p_2 = \frac{\partial \boldsymbol{u}_2}{\partial t} - \Delta \boldsymbol{u}_2 \quad \text{a.e. in } \mathcal{D}_{R_0} \times (0,T).$$

Consulting [7] (Theorem III. 3.1, Theorem III. 5.2) we obtain the estimate

$$\|p_2 - (p_2)_{L^2(0,T;L^2(\mathcal{D}_{R_0}))}\|_{L^2(0,T;L^2(\mathcal{D}_{R_0}))} \leqslant c\Big(\Big\| \frac{\partial \boldsymbol{u}_2}{\partial t} \Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla \boldsymbol{u}_2\|_{L^2(0,T;\boldsymbol{L}^2)}\Big).$$

Inserting this estimate into the right hand side of (4.17), using Young's inequality and taking into account (4.14) we get

$$\text{(4.18)} \qquad \Big\| \frac{\partial \boldsymbol{u}_2}{\partial t} \Big\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\boldsymbol{u}_2\|_{L^\infty(0,T;\boldsymbol{W}^{1,2})} + \|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{\omega}\|_{\boldsymbol{W}^{1,2}} \leqslant cK_0.$$

Finally, it remains to estimate ∇p_2 and $\nabla^2 \boldsymbol{u}_2$. Let $t \in (0,T)$ be fixed such that $(\partial \boldsymbol{u}_2/\partial t)(t), (\partial^2 \boldsymbol{z}/\partial x_i \partial x_j)(t) \in \boldsymbol{L}^2$, $i,j = 1,2,3$. We define

$$\boldsymbol{\Psi}_2(x) = \frac{1}{2}\,\mathbf{rot}(\zeta(\boldsymbol{\xi}_2(t) \times x - \boldsymbol{\omega}_2(t)|x|^2)), \quad x \in \mathbb{R}^3,\ t \geqslant 0,$$

where $\boldsymbol{\xi}_2 = \boldsymbol{\xi} - \boldsymbol{\xi}_1$ and $\boldsymbol{\omega}_2 = \boldsymbol{\omega} - \boldsymbol{\omega}_1$. Then $\boldsymbol{v} = \boldsymbol{u}_2(\cdot,t) - \boldsymbol{\Psi}_2(\cdot,t) - \zeta \boldsymbol{z}(\cdot,t)$ solves the steady problem

$$\begin{aligned}
\operatorname{div} \boldsymbol{v} &= -\nabla\zeta \cdot \boldsymbol{z}(t) \quad \text{in} \quad \mathcal{D}, \\
-\Delta \boldsymbol{v} &= \frac{\partial \boldsymbol{u}_2}{\partial t}(t) + \Delta \boldsymbol{\Psi}_2(t) + \Delta(\zeta \boldsymbol{z}(t)) - \nabla p_2(t) \quad \text{in} \quad \mathcal{D}, \\
\boldsymbol{v} &= 0 \quad \text{on} \quad \partial\mathcal{D}.
\end{aligned}$$

Consulting [7] we see that $\partial^2 \boldsymbol{v}/\partial x_i \partial x_j, \nabla p_2 \in \boldsymbol{L}^2(\mathcal{D})$ together with the estimate

$$\begin{aligned}(4.19)\qquad &\|\nabla^2 \boldsymbol{v}\|^2_{\boldsymbol{L}^2} + \|\nabla p_2(t)\|^2_{\boldsymbol{L}^2}\\ &\leqslant c\Big(\Big\|\frac{\partial \boldsymbol{u}_2}{\partial t}(t)\Big\|^2_{\boldsymbol{L}^2} + \|\Delta(\zeta \boldsymbol{z}(t))\|^2_{\boldsymbol{L}^2(\mathcal{D}_{R_0})} + \|\Delta \boldsymbol{\Psi}_2(t)\|^2_{\boldsymbol{L}^2}\Big).\end{aligned}$$

Integrating both sides of (4.19) over $(0,T)$ using (4.18), (4.3) and (4.1) we see that

$$(4.20)\qquad \|\nabla^2 \boldsymbol{u}_2\|_{L^2(0,T;\boldsymbol{L}^2)} + \|\nabla p_2\|_{L^2(0,T;\boldsymbol{L}^2)} \leqslant cK_0.$$

P r o o f. 2° Proof of the theorem for general $\boldsymbol{f} = \mathbf{rot}(\boldsymbol{g})$ for $\boldsymbol{g} \in L^2(0,T;\boldsymbol{W}^{1,2}_\eta(\mathcal{D}))$. For $\varepsilon > 0$ we define

$$\boldsymbol{g}_\varepsilon(x,t) = (1+\varepsilon|x|)^{-1}\boldsymbol{g}(x,t), \quad (x,t) \in \mathcal{D}\times(0,T), \quad \boldsymbol{f}_\varepsilon = \mathbf{rot}(\boldsymbol{g}_\varepsilon).$$

Clearly, $\boldsymbol{g}_\varepsilon \in L^2(0,T;\boldsymbol{W}^{1,2}(\mathcal{D}))$ and $\boldsymbol{f}_\varepsilon \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$ for all $\varepsilon > 0$. In addition, we immediately see that for a.e. $t \in (0,T)$

$$\|\boldsymbol{g}_\varepsilon(t)\|_{\boldsymbol{W}^{1,2}_\eta(\mathcal{D})} \leqslant 2\|\boldsymbol{g}(t)\|_{\boldsymbol{W}^{1,2}_\eta(\mathcal{D})}, \quad \varepsilon > 0.$$

By Lebesgue's theorem of dominated convergence it follows that

$$\boldsymbol{f}_\varepsilon \to \boldsymbol{f} \quad \text{in } L^2(0,T;\boldsymbol{L}^2_\eta(\mathcal{D})) \quad \text{as } \varepsilon \to 0.$$

As it has been proved above in 1° there exists a strong solution $(\boldsymbol{u}_\varepsilon, \boldsymbol{\xi}_\varepsilon, \boldsymbol{\omega}_\varepsilon)$ to the system (2.1)–(2.6) with $\boldsymbol{f}_\varepsilon$ in place of $\boldsymbol{f}$. Furthermore, there exists a pressure $p_\varepsilon \in L^2(0,T;L^2_{\rm loc}(\overline{\mathcal{D}}))$ with $\nabla p_\varepsilon \in L^2(0,T;\boldsymbol{L}^2(\mathcal{D}))$. Using (4.1), (4.2), (4.3), (4.18) and (4.20) we get the a priori bound

$$\begin{aligned}(4.21)\qquad &\Big\|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\Big\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\nabla^2 \boldsymbol{u}_\varepsilon\|_{L^2(0,T;\boldsymbol{L}^2_\eta)} + \|\boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;\boldsymbol{W}^{1,2}_\eta)}\\ &\quad + \|\boldsymbol{\xi}_\varepsilon\|_{\boldsymbol{W}^{1,2}} + \|\boldsymbol{\omega}_\varepsilon\|_{\boldsymbol{W}^{1,2}} + \|\nabla p_\varepsilon\|_{L^2(0,T;\boldsymbol{L}^2)} \leqslant cK_0,\end{aligned}$$

where c denotes a constant depending on $\mathcal{D}$ only. Hence, by means of reflexivity we may choose a sequence of positive numbers $\varepsilon_j \to 0$ as $j \to \infty$, such that

$$\begin{aligned}\boldsymbol{u}_{\varepsilon_j} \to \boldsymbol{u} &\quad \text{weakly in } L^2(0,T;\boldsymbol{W}^{2,2}_\eta(\mathcal{D})),\\ \frac{\partial \boldsymbol{u}_{\varepsilon_j}}{\partial t} \to \frac{\partial \boldsymbol{u}}{\partial t} &\quad \text{weakly in } L^2(0,T;\boldsymbol{L}^2_\eta(\mathcal{D})),\\ \boldsymbol{\xi}_{\varepsilon_j}, \boldsymbol{\omega}_{\varepsilon_j} \to \boldsymbol{\xi}, \boldsymbol{\omega} &\quad \text{weakly in } \boldsymbol{W}^{1,2}(0,T),\\ \nabla p_{\varepsilon_j} \to \nabla p &\quad \text{weakly in } L^2(0,T;\boldsymbol{L}^2(\mathcal{D})) \text{ as } j \to \infty.\end{aligned}$$

As one easily checks, the triple $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{\omega})$ is a strong solution to the system (2.1)–(2.6) with pressure p satisfying (2.9), (2.10). □

References

[1] *W. Borchers*: Zur Stabilität und Faktorisienrungsmethode für die Navier-Stokes Gleichungen inkompressibler viskoser Flüssigkeiten. Habilitationsschrift, University of Paderborn, 1992. (In German.)

[2] *Z.-M. Chen, T. Miyakawa*: Decay properties of weak solutions to a perturbed Navier-Stokes system in $\mathbb{R}^n$. Adv. Math. Sci. Appl. *7* (1997), 741–770. zbl MR

[3] *P. Cumsille, T. Takahashi*: Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid. Czech. Math. J. *58* (2008), 961–992. zbl MR

[4] *P. Cumsille, M. Tucsnak*: Wellposedness for the Navier-Stokes flow in the exterior of a rotating obstacle. Math. Methods Appl. Sci. *29* (2006), 595–623. zbl MR

[5] *E. Dintelmann, M. Geissert, M. Hieber*: Strong L^p-solutions to the Navier-Stokes flow past moving obstacles: The case of several obstacles and time dependent velocity. Trans. Am. Math. Soc. *361* (2009), 653–669. zbl MR

[6] *G. P. Galdi*: On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications. Handbook of Mathematical Fluid Dynamics 1 (S. Friedlander et al., eds.). Elsevier, Amsterdam, 2002, pp. 653–791. zbl MR

[7] *G. P. Galdi*: An Introduction to the Mathematical Theory of the Navier-Stokes Equations I. Linearized Steady Problems. Springer Tracts in Natural Philosophy 38, Springer, New York, 1994. zbl MR

[8] *G. P. Galdi, A. L. Silvestre*: Strong solutions to the Navier-Stokes equations around a rotating obstacle. Arch. Ration. Mech. Anal. *176* (2005), 331–350. zbl MR

[9] *G. P. Galdi, A. L. Silvestre*: Strong solutions to the problem of motion of a rigid body in a Navier-Stokes liquid under the action of prescribed forces and torques. Nonlinear Problems in Mathematical Physics and Related Topics I. (M. S. Birman et al., eds.). Int. Math. Ser. (N. Y.) 1, Kluwer Academic/Plenum Publishers, New York, 2002, pp. 121–144. zbl MR

[10] *M. Geissert, H. Heck, M. Hieber*: L^p-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle. J. Reine Angew. Math. *596* (2006), 45–62. zbl MR

[11] *T. Hishida*: An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle. Arch. Ration. Mech. Anal. *150* (1999), 307–348. zbl MR

[12] *T. Hishida, Y. Shibata*: L_p-L_q estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle. Arch. Ration. Mech. Anal. *193* (2009), 339–421. zbl MR

[13] *A. Inoue, M. Wakimoto*: On existence of solutions of the Navier-Stokes equation in a time dependent domain. J. Fac. Sci., Univ. Tokyo, Sect. I A *24* (1977), 303–319. zbl MR

[14] *O. A. Ladyzhenskaya*: An initial-boundary value problem for the Navier-Stokes equations in domains with boundary changing in time. Semin. Math., V. A. Steklov Math. Inst., Leningrad *11* (1968), 35–46; translation from Zap. Nauchn. Semin. Leningrad. Otdel. Mat. Inst. Steklov. *11* (1968), 97–128. (In Russian.) zbl

[15] *J. Neustupa*: Existence of a weak solution to the Navier-Stokes equation in a general time-varying domain by the Rothe method. Math. Methods Appl. Sci. *32* (2009), 653–683. zbl MR

[16] *J. Neustupa, P. Penel*: A weak solvability of the Navier-Stokes equation with Navier's boundary condition around a ball striking the wall. Advances in Mathematical Fluid Mechanics (R. Rannacher et al., eds.). Springer, Berlin, 2010, pp. 385–407. MR

[17] *J. Neustupa, P. Penel*: A weak solution to the Navier-Stokes system with Navier's boundary condition in a time varying domain. Accepted to "Recent Developments of Mathematical Fluid Mechanics", Series: Advances in Math. Fluid Mech. (G. P. Galdi, J. G. Heywood, R. Rannacher., eds.). Birkhäuser.

[18] *D. Serre*: Free fall of a rigid body in an incompressible viscous fluid. Existence. Japan J. Appl. Math. *4* (1987), 99–110. (In French.) zbl MR
[19] *T. Takahashi*: Existence of strong solutions for the problem of a rigid-fluid system. C. R., Math., Acad. Sci. Paris *336* (2003), 453–458. zbl MR
[20] *T. Takahashi, M. Tucsnak*: Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. J. Math. Fluid Mech. *6* (2004), 53–77. zbl MR

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