GLOBAL DOMINATION AND NEIGHBORHOOD NUMBERS IN BOOLEAN FUNCTION GRAPH OF A GRAPH

T. N. Janakiraman, Tiruchirappalli,
S. Muthammai, M. Bhanumathi, Pudukkottai

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Abstract. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. The Boolean function graph $B(G, L(G), \mathrm{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \mathrm{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G. In this paper, global domination number, total global domination number, global point-set domination number and neighborhood number for this graph are obtained.

Keywords: Boolean function graph, global domination number, neighborhood number

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1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. For a connected graph G, the *eccentricity* $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \max\{d_G(u, v) \colon u \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted $r(G)$ and $\mathrm{diam}(G)$ respectively. A set D of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex in $V - D$ is adjacent to some vertex in D. Further, D is a *global dominating set* if it is a dominating set of both G and its complement $\overline{G}$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *global domination number* γ_{g} of G is defined similarly [11]. A dominating set D is called a *connected dominating set* if the induced subgraph $\langle D \rangle$ is connected. The *connected domination number* $\gamma_{\mathrm{c}}(G)$ of G is the minimum cardinality of a connected dominating set [14]. A *total dominating set* T of G is a dominating set such that the induced subgraph $\langle T \rangle$ has no isolated vertices. The *total domination number* $\gamma_{\rm t}(G)$ of G is the minimum cardinality of a total dominating set. This concept was introduced in Cockayne et al [4]. A total dominating set T of G is a *total global dominating set* (t.g.d. set) if T is also a total dominating set of $\overline{G}$. The *total global dominating number* $\gamma_{\rm tg}(G)$ of G is the minimum cardinality of a t.g.d. set [8]. $\gamma_{\rm t}(G)$ is defined for G with $\delta(G) \geqslant 1$ and $\gamma_{\rm tg}(G)$ is only defined for G with $\delta(G) \geqslant 1$ and $\delta(\overline{G}) \geqslant 1$, where $\delta(G)$ is the minimum degree of G.

For a connected graph $G = (V, E)$, a set D of vertices is a *point-set dominating set* (psd-set) of G, if for each set $S \subseteq V - D$, there exists a vertex $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ induced by $S \cup \{v\}$ is connected. The *point-set domination number* $\gamma_{\rm ps}(G)$ is the minimum cardinality of a psd-set of G [13]. We say that a graph G is co-connected if both G and $\overline{G}$ are connected. For a co-connected graph $G = (V, E)$, a set $D \subseteq V$ is said to be a *global psd-set* if it is a psd-set of both G and $\overline{G}$. The *global point-set domination number* $\gamma_{\rm pg}$ of G is defined as the minimum cardinality of a global psd-set [10]. A γ-set is a minimum dominating set. Similarly, a $\gamma_{\rm g}$-set, $\gamma_{\rm t}$-set, $\gamma_{\rm tg}$-set and $\gamma_{\rm pg}$-set are defined.

For $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set of all vertices adjacent to v in G. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of $V(G)$ is a *neighborhood set* (n-set) of G if $G = \bigcup\limits_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G [12].

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. As for every graph (undirected, uniformly weighted) there exists an adjacency $(0, 1)$ matrix, we call the general operation a Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incidence relationship between vertices and edges and defines new structure from the given graph. This extracts information from the original graph and encodes it into a new structure. If it is possible to decode the given graph from the encoded graph in polynomial time, such operation may be used to analyze various structural properties of original graph in terms of the Boolean graph. If it is not possible to decode the original graph in polynomial time, then that operation can be used in graph coding or coding of certain grouped signals.

Whitney [16] introduced the concept of the line graph $L(G)$ of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The middle graph $M(G)$ of a graph G was introduced by Hamada and Yoshimura [5]. Chikkodimath and Sampathkumar [3] also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad [2] in 1966. Sastry and Raju [15] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations. Using $L(G)$, G, incidence and non-incidence, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) have already been defined and studied, we study all other similar remaining graph operations. This is illustrated below.

$$\boxed{G/\overline{G}/K_p/\overline{K}_p} \text{ --- } \boxed{\text{incident (INC), not incident (NINC)}} \text{ --- } \boxed{L(G)/\overline{L}(G)/K_q/\overline{K}_q}.$$

Here, $\overline{G}$ and $L(G)$ denote the complement and the line graph of G respectively. K_p is the complete graph on p vertices.

The *Boolean function graph* $B(G, L(G), \mathrm{NINC})$ of G is the graph with vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G [7]. For brevity, this graph is denoted by $B_1(G)$. In other words, $V(B_1(G)) = V(G) \cup V(L(G))$; and $E(B_1(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L}(G)))] \cup (E(G) \cup E(L(G)))$, where $\overline{G}$, $L(G)$ and $T(G)$ denote the complement, the line graph and the total graph of G respectively. Note that G and $L(G)$ are induced subgraphs of $B_1(G)$. The vertices of G and $L(G)$ are referred to as point and line vertices respectively.

In this paper, we obtain the bounds for the global, total global and global point-set domination numbers and neighborhood number for this Boolean function graph. The definitions and details not furnished in this paper may be found in [6].

2. Prior results

Theorem 2.1 [8]. *A total dominating set T of G is a total global dominating set (t.g.d. set) if and only if for each vertex $v \in V(G)$ there exists a vertex $u \in T$ such that v is not adjacent to u.*

Theorem 2.2 [8]. *Let G be a graph with* $\operatorname{diam}(G) \geqslant 5$. *Then $T \subseteq V(G)$ is a total dominating set of G if and only if T is a total global dominating set.*

Theorem 2.3 [10]. *Let $G=(V,E)$ be a connected graph. A set $D\subseteq V(G)$ is a point-set dominating set of G if and only if for every independent set W in $V-D$, there exists u in D such that $W\subseteq N(u)\cap(V-D)$ in G.*

Theorem 2.4 [10]. *For a co-connected graph G, a set $D\subseteq V(G)$ is a global point-set dominating set if and only if the following conditions are satisfied:*

(i) *For every independent set W in $V-D$, there exists u in D such that $W\subseteq N(u)\cap(V-D)$ in G; and*

(ii) *For every set $S\subseteq V-D$ such that $\langle S\rangle$ is complete in G, there exists v in D such that $S\cap N(v)=\emptyset$ in G.*

Theorem 2.5 [10]. *For a co-connected graph of order $p\geqslant 5$, $3\leqslant\gamma_{\rm pg}(G)\leqslant p-2$.*

Proposition 2.6 [12]. *For a graph G of order p, the neighborhood number n_0 of G is 1 if and only if G has a vertex of degree $p-1$.*

Theorem 2.7 [7]. *$B_1(G)$ is disconnected if and only if G is one of the following graphs: nK_1, K_2, $2K_2$ and $K_2\cup nK_1$, for $n\geqslant 1$; and contains isolated vertices if and only if G is either K_2 or nK_1, for $n\geqslant 1$.*

Theorem 2.8 [7]. *If G is a connected graph with at least 3 vertices, then $B_1(G)$ is Hamiltonian.*

Theorem 2.9 [7]. *For any graph G with at least one edge, $\gamma_{\rm c}(B_1(G))=2$ if and only if G contains a triangle in which at least one vertex has degree 2.*

3. Main results

In the following, the global domination number $\gamma_{\rm g}$ of $B_1(G)$ is determined. We find the graphs G for which $\gamma_{\rm g}(B_1(G))=2$.

Theorem 3.1. *For any graph G not totally disconnected, $\gamma_{\rm g}(B_1(G))=2$ if and only if one of the following is true.*

(i) *There exists an independent point cover D of G containing exactly two vertices which is also a dominating set of $\overline{G}$.*

(ii) *G contains K_2 or C_3 as one of its components.*

P r o o f. Assume $\gamma_{\rm c}(B_1(G))=2$. Let D be a $\gamma_{\rm g}$-set of $B_1(G)$.

C a s e (i): Both vertices of D are point vertices. Let $D = \{v_1, v_2\} \subseteq V(B_1(G))$, where $v_1, v_2 \in V(G)$. For D to be a $\gamma_{\rm g}$-set of $B_1(G)$, D must be an independent point cover of G and is also a dominating set of $\overline{G}$.

C a s e (ii): D contains one point vertex and one line vertex. Let $D = \{v, e'\}$, where $v \in V(G)$ and e' is the line vertex corresponding to an edge e in G.

S u b c a s e (i): e is incident with v in G.

Since D is a global dominating set of $B_1(G)$, it is a dominating set of $\overline{B}_1(G)$. Hence, G contains K_2 as one of its components.

S u b c a s e (ii): e is not incident with v in G.

Since D is a dominating set of $B_1(G)$, v and e lie on a triangle in G with $\deg_G(v) = 2$, where e is not incident with v in G, by Theorem 2.9. Also since D is a dominating set of $\overline{B}_1(G)$, G contains C_3 as one of its components.

C a s e (iii): D contains two line vertices. Let $D = \{e_1', e_2'\} \subseteq V(B_1(G))$, where e_1' and e_2' are the line vertices in $B_1(G)$ corresponding to the independent edges e_1 and e_2 in G respectively. Since D is a global dominating set of $B_1(G)$, $G \cong 2K_2$.

By Case (i), Case (ii) and Case (iii), it follows that G is one of the graphs given in the Theorem. The converse is obvious. □

Theorem 3.2. *For any graph G, $\gamma_{\rm g}(B_1(G)) \leqslant 3$ if there exists at least one non 3-cyclic edge.*

P r o o f. Let $e = (u, v)$ be an edge in G, where $u, v \in V(G)$, not lying in any C_3 in G and e' be the line vertex in $B_1(G)$ corresponding to the edge e. Then $\{u, v, e'\} \subseteq V(B_1(G))$ is a global dominating set of $B_1(G)$. Hence, $\gamma_{\rm g}(B_1(G)) \leqslant 3$.

This bound is attained if $G \cong C_n$, for $n \geqslant 4$. □

Theorem 3.3. *For any graph G, $\gamma_{\rm g}(B_1(G)) \leqslant 4$ if each edge in G lies on exactly one C_3 in G.*

P r o o f. Let u, v, w be the vertices of a C_3 in G with $e = (u, v) \in E(G)$. Let e' be the line vertex in $B_1(G)$ corresponding to the edge e in G. Then $\{u, v, w, e'\} \subseteq V(B_1(G))$ is a global dominating set of $B_1(G)$. Thus, $\gamma_{\rm g}(B_1(G)) \leqslant 4$.

The next theorem relates $\gamma_{\rm g}$ of $B_1(G)$ with the point covering number of G. □

Theorem 3.4. *If G is any graph other than a star, then $\gamma_{\rm g}(B_1(G)) \leqslant \alpha_0(G) + 1$, where $\alpha_0(G)$ is the point covering number of G.*

P r o o f. Let D be a point cover of G with $|D| = \alpha_0(G)$. Since G is not a star, $|D| \leqslant 2$. If $v \in V(G) - D$, then $D \cup \{v\} \subseteq V(B_1(G))$ is a dominating set of both $B_1(G)$ and $\overline{B}_1(G)$. Hence, $\gamma_{\rm g}(B_1(G)) \leqslant \alpha_0(G) + 1$.

This bound is attained if $G \cong K_4$. □

N o t e 3.1. If $G \cong K_{1,n}$, $n \geqslant 2$, then $\gamma_{\rm g}(B_1(G)) = 3$.

The following theorems relate $\gamma_{\rm g}(B_1(G))$ to the maximum number of independent edges $\beta_1(G)$. We omit the proof, since it is similar to that of Theorem 3.4.

Theorem 3.5. *For any graph G with $\beta_1(G) \geqslant 3$, $\gamma_{\rm g}(B_1(G)) \leqslant \beta_1(G) + 1$, where $\beta_1(G)$ is the line independence number of G. This bound is attained if $G \cong C_{2n+1}$, for $n \geqslant 3$.*

Theorem 3.6. *If $\beta_1(G) = 2$ and there exist two independent edges e_1, e_2 such that no edge in G is adjacent to both e_1 and e_2, then $\gamma_{\rm g}(B_1(G)) \leqslant \beta_1(G) + 1$.*

Theorem 3.7. *If G has a perfect matching having at least three edges, then $\gamma_{\rm g}(B_1(G)) \leqslant \beta_1(G)$.*

In the following theorems, we give upper bounds of $\gamma_{\rm g}(B_1(G))$ in terms of the domination number γ of $\overline{G}$.

Theorem 3.8. *For any graph G not totally disconnected, $\gamma_{\rm g}(B_1(G)) \leqslant \gamma(\overline{G}) + 2$.*

P r o o f. Let D be a dominating set of $\overline{G}$ and let $e = (u, v)$ be an edge in G with $u \in D$, $v \in V(G)$. If e' is the line vertex in $B_1(G)$ corresponding to the edge e, then $D \cup \{v, e'\} \subseteq V(B_1(G))$ is a global dominating set of $B_1(G)$. Hence, $\gamma_{\rm g}(B_1(G)) \leqslant \gamma(\overline{G}) + 2$. □

Theorem 3.9. $\gamma_{\rm g}(B_1(G)) \leqslant \gamma(\overline{G}) + 1$ *if there exists a dominating set D of $\overline{G}$ such that $\langle D\rangle$ is not totally disconnected in G.*

P r o o f. Let D be a γ-set of $\overline{G}$ such that $\langle D\rangle$ is not totally disconnected in G and let e be an edge in $\langle D\rangle$. If e' is the line vertex in $B_1(G)$ corresponding to the edge e, then $D \cup \{e\} \subseteq V(B_1(G))$ is a global dominating set of $B_1(G)$. Hence, $\gamma_{\rm g}(B_1(G)) \leqslant \gamma(\overline{G}) + 1$.

This bound is attained if G is a wheel on six vertices. □

Theorem 3.10. *A global dominating set D of a graph G is also a global dominating set of $B_1(G)$ if and only if either D is a point cover of G containing at least 3 vertices or D is an independent point cover of G containing exactly two vertices.*

P r o o f. Let D be a global dominating set of both G and $B_1(G)$. If D is not a point cover of G, then there exists an edge in G which is not covered by any of the vertices in D. Hence, the corresponding line vertex in $\overline{B}_1(G)$ is not adjacent to any of the vertices in D, which is a contradiction. If $|D| = 2$ and D is not independent, then D is not a dominating set of $B_1(G)$. Conversely, let D be a global dominating set of G. If one of the conditions is true, then D is a global dominating set of $B_1(G)$. $\square$

Example 3.1.

(a) $\gamma_g(B_1(C_n)) = 2$, if $n = 3$; and

$= 3$, if $n \geqslant 4$.

(b) $\gamma_g(B_1(P_n)) = 3$, if $n \geqslant 4$.

(c) $\gamma_g(B_1(K_{1,n})) = 3$, if $n \geqslant 2$.

(d) $\gamma_g(B_1(W_n)) = 4$, if $n \geqslant 4$, where W_n is a wheel on n vertices.

(e) $\gamma_g(B_1(K_n)) = 2$, if $n = 3$.

$= 4$, if $n = 4, 5$; and

$= \{n/2\}$, if $n \geqslant 6$,

where $\{x\}$ is the smallest integer not less than x.

In the following, various bounds for the total global domination number γ_{tg} of $B_1(G)$ are obtained. Here we consider the graphs G for which both $B_1(G)$ and its complement $\overline{B}_1(G)$ are connected. We prove the following theorems by applying Theorem 2.1.

Theorem 3.11. *For any graph G with $\delta(G) = 1$, $\gamma_{tg}(B_1(G)) = 4$.*

Proof. For any graph G, $\gamma_{tg}(B_1(G)) \geqslant 4$. Let v be a pendant vertex in G and u be the vertex adjacent to v such that $e = (u, v) \in E(G)$. Let w be any vertex in $V(G) - \{u, v\}$ and e' be the line vertex in $B_1(G)$ corresponding to e. Then $D = \{u, v, w, e'\} \subseteq V(B_1(G))$ and $\langle D\rangle \cong P_4$ or $2K_2$ in $B_1(G)$ and D is a total dominating set of $B_1(G)$. To prove that D is a total global dominating (t.g.d.) set of $B_1(G)$, by Theorem 2.1., it is enough to prove that for each vertex x' in $B_1(G)$ there exists a vertex, say y, in D such that x' is not adjacent to y. Let $x' \in V(B_1(G))$. If x' is a vertex in D, then since $\langle D\rangle \cong P_4$ or $2K_2$, there exists a vertex in D not adjacent to x'. If x' is a point vertex in $V((B_1(G)) - D$, then x' is not adjacent to v in D. Assume x' is a line vertex in $V((B_1(G)) - D$. Let x be an edge in G corresponding to x'. Then $x \in E(G) - \{e\}$. If x is adjacent to e in G, then $x' \in V(B_1(G) - D$ is not adjacent to u in D and if x is not adjacent to e in G, then $x' \in V(B_1(G)) - D$ is not adjacent to e' in D. Hence, D is a t.g.d. set of $B_1(G)$. Thus, $\gamma_{tg}(B_1(G)) \leqslant 4$, whence $\gamma_{tg}(B_1(G)) = 4$. $\square$

Theorem 3.12. *If $\delta(G) = 0$, then $\gamma_{\rm tg}(B_1(G)) = 4$.*

P r o o f. Let v be an isolated vertex in G. Since $B_1(G)$ is connected, G is none of the graphs $K_2 \cup nK_1$, $2K_2$, nK_1, for $n \geqslant 1$ by Theorem 2.7. Let $e = (u, w)$ be an edge in G, where u, $w \in V(G)$ and let e' be the line vertex in $B_1(G)$ corresponding to e. Then $D = \{v, w, u, e'\} \subseteq V(B_1(G))$ is a total global dominating set of $B_1(G)$. Hence, $\gamma_{\rm tg}(B_1(G)) = 4$. □

Theorem 3.13. *Let G be any graph with $\delta(G) \geqslant 2$ and v a vertex of minimum degree in G. If the radius of the subgraph $\langle N(v)\rangle$ of G induced by $N(v)$ is at least two, then $\gamma_{\rm tg}(B_1(G)) \leqslant \delta(G) + 2$.*

P r o o f. Assume $\deg_G(v) = \delta(G) \geqslant 2$ and $r(\langle N(v)\rangle) \geqslant 2$. Let e be an edge in G incident with v and e' the corresponding line vertex in $B_1(G)$. Then $D = \{N[v], e'\} \subseteq V(B_1(G))$ is a total dominating set of G. Let x be a vertex in $B_1(G)$.

C a s e (i): x is a point vertex in D. If $x = v$, then x is not adjacent to e in D. If $x \neq v$, since $r(\langle N(v)\rangle) \geqslant 2$, there exists at least one vertex in D not adjacent to x.

C a s e (ii): x is a point vertex in $V(B_1(G)) - D$. Then x is not adjacent to v in D.

C a s e (iii): x is a line vertex in $B_1(G)$. If $x = e'$, then x is not adjacent to v. If $x \neq e'$, then $x \in V(B_1(G)) - D$ and there exists a point or a line vertex in D not adjacent to x. Hence, D is a total global dominating set of $B_1(G)$. Thus, $\gamma_{\rm tg}(B_1(G)) \leqslant \delta(G) + 2$.

This bound is attained if $G \cong C_5$. □

Theorem 3.14. *For any connected (p, q) graph G, if $r(G) \geqslant 2$, then $\gamma_{\rm tg}(B_1(G)) \leqslant p$, where $r(G)$ is the radius of G.*

P r o o f. Let $D = V(G)$. Since $r(G) \geqslant 2$, D is a total global dominating set of $B_1(G)$. Hence, $\gamma_{\rm tg}(B_1(G)) \leqslant p$, and this bound is attained if $G \cong C4$.

Next, we obtain a necessary and sufficient condition for a t.g.d. set of G to be also a t.g.d. set of $B_1(G)$. □

Theorem 3.15. *$\gamma_{\rm tg}(B_1(G)) \leqslant \gamma_{\rm tg}(G)$ if and only if there exists a total global dominating set D of G with $V(G) - D$ independent.*

P r o o f. Let D be a total global dominating set of both G and $B_1(G)$. Then $D \subseteq V(G)$ and D contains at least four vertices. If $V(G) - D$ is not independent in G, then there exists an edge e in $\langle V(G) - D\rangle$. If e' is the line vertex in $B_1(G)$ corresponding to e, then e' is adjacent to all the vertices in D, which is a contradiction. Hence, $V(G) - D$ is independent. Conversely, assume D is a t.g.d. set of G such that $V(G) - D$

is independent. Since G is an induced subgraph of $B_1(G)$, to prove D is a t.g.d. set of $B_1(G)$, it is enough to prove that for each line vertex e' in $V(B_1(G))$, there exists a vertex u in D such that e' is not adjacent to u. Since $|D| \geqslant 4$ and $V(G) - D$ is independent, each edge in G is incident with at least one of the vertices in D. Hence, each line vertex in $V(B_1(G))$ is not adjacent to at least one of the vertices in D, implying that D is a t.g.d. set of $B_1(G)$. Thus, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{tg}}(G)$. □

Theorem 3.16. $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma'_{\mathrm{tg}}(G)$, $\gamma'_{\mathrm{tg}}(G) \geqslant 4$, *if and only if there exists a total global edge dominating set D' of G with $|D'| \geqslant 4$ and D' is a line cover for G.*

Proof. Assume $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma'_{\mathrm{tg}}(G)$, where $\gamma'_{\mathrm{tg}}(G) \geqslant 4$. Let D' be a total global edge dominating set of G with $|D'| \geqslant 4$. If D' is not a line cover for G, then there exists a vertex v in G which is not incident with any of the edges in D'. If D'' is the set of line vertices in $B_1(G)$ corresponding to the edges in D', then D'' is a t.g.d. set of $B_1(G)$ by the assumption and $v \in V(B_1(G)) - D''$ is adjacent to all the vertices in D''. This is a contradiction. Hence, D' is a line cover for G. Conversely, let D' be a line cover for G which is also a total global edge dominating set of G with $|D'| \geqslant 4$. Let D'' be the set of line vertices in $B_1(G)$ corresponding to the edges in D'. Then it is obvious that D'' is a total global dominating set of $B_1(G)$. Hence, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma'_{\mathrm{tg}}(G)$. □

In the following theorems, we give upper bounds of $\gamma_{\mathrm{tg}}(B_1(G))$ in terms of total domination number, global domination number and point covering number of G.

Theorem 3.17. $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)$, $\gamma_{\mathrm{t}}(G) \geqslant 4$, *if only if there exists a total dominating set D of G satisfying*

(i) *$r(\langle D\rangle) \geqslant 2$, where $r(\langle D\rangle)$ is the radius of the subgraph $\langle D\rangle$ of G induced by D;*

(ii) *D is a dominating set of $\overline{G}$; and*

(iii) *$V(G) - D$ is independent. That is, D is a point cover for G.*

Proof. Assume $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)$, where $\gamma_{\mathrm{t}}(G) \geqslant 4$. Let D be a total dominating set of G with $|D| \geqslant 4$. Then D is a total global dominating set of $B_1(G)$ and by Theorem 2.1., for each vertex v in $V(B_1(G))$ there exists a vertex u in D such that v is not adjacent to u. If $r(\langle D\rangle) = 1$, then there exists a vertex in D adjacent to all the vertices in D, which is a contradiction. Hence, $r(\langle D\rangle) \geqslant 2$. If D is not a dominating set of $\overline{G}$, then there exists a vertex in $V(G) - D$ adjacent to all the vertices in D, which is a contradiction. Hence, D must be a dominating set of $\overline{G}$. If $V(G) - D$ is not independent, then there exists an edge, say e, in $E(\langle V(G) - D\rangle)$. Then the corresponding line vertex in $B_1(G)$ is adjacent to all the vertices in D, which is contrary to the assumption. Hence, $V(G) - D$ is independent. □

Conversely, let D be a total dominating set of G with $|D| \geqslant 4$ and satisfying (i), (ii) and (iii). Then it is clear that D is a total global dominating set of G. Hence, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)$.

Theorem 3.18. *Let G be any graph having no isolated vertices and* $\operatorname{diam}(G) \geqslant 5$. *Then $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)(\gamma_{\mathrm{t}}(G) \geqslant 4)$ if and only if there exists a total dominating set D of G with $|D| \geqslant 4$ and $V(G) - D$ independent.*

P r o o f. Let D be a total dominating set of G with $|D| \geqslant 4$ and $V(G) - D$ independent. Since $\operatorname{diam}(G) \geqslant 5$, D is a total global dominating set of G by Theorem 2.2. Then D is a total global dominating set of $B_1(G)$ by Theorem 3.15. Thus, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)$. The converse is obvious. □

R e m a r k 3.1. Theorem 3.16 can be rephrased as follows: for any graph G having no isolated vertices and $\operatorname{diam}(G) \geqslant 5$, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G)$ if and only if $\gamma_{\mathrm{t}}(G) = \alpha_0(G)$, where $\alpha_0(G) \geqslant 4$.

Similarly, the following theorem can be proved.

Theorem 3.19. *For any graph G having no isolated vertices and* $\operatorname{diam}(G) \geqslant 5$, *$\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{g}}(G)$ if and only if there exists a global dominating set D of G with $|D| \geqslant 4$, $\delta(\langle D\rangle) \geqslant 1$ and $V(G) - D$ independent.*

Theorem 3.20. *For any graph G with* $\operatorname{diam}(G) = 4$, *$\gamma_{\mathrm{tg}}(B_1(G)) = \gamma_{\mathrm{t}}(G) + 1$ if there exists a total dominating set D of G with $|D| \geqslant 3$ and $V(G) - D$ independent.*

P r o o f. Let D be a total dominating set of G with $|D| \geqslant 3$ and $V(G) - D$ independent. Since $\operatorname{diam}(G) = 4$, there exist vertices u, v in G such that $d(u, v) = 4$. Then either $D \cup \{u\}$, or $D \cup \{v\}$ is a total global dominating set in G; denote it by D'. Since $D \cup \{u\}$ or $D \cup \{v\}$ contains at least four vertices and $V(G) - D$ is independent, D' is a total global dominating set of $B_1(G)$. Hence, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G) + 1$. □

Theorem 3.21. *For any graph G with* $\operatorname{diam}(G) = 3$, *$\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G) + 2$ if there exists a total dominating set D of G with $|D| \geqslant 2$ and $V(G) - D$ independent.*

P r o o f. Let D be a total dominating set of G with $|D| \geqslant 2$ and $V(G) - D$ independent. Let u, v be two vertices in G such that $d(u, v) = 3$. Then $D' = D \cup \{u, v\}$ is a total global dominating set of G having at least four vertices. Since $V(G) - D$ is independent, D' is a total global dominating set of $B_1(G)$. Hence, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{t}}(G) + 2$. □

R e m a r k 3.2. If D contains exactly two vertices, then G is a double star.

Similarly, the following theorems can be proved.

Theorem 3.22. *For any graph G with* $\operatorname{diam}(G) = 4$, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{g}}(G) + 1$ *if there exists a global dominating set D of G with $|D| \geqslant 3$ and $V(G) - D$ independent.*

Theorem 3.23. *For any graph G with* $\operatorname{diam}(G) = 3$, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \gamma_{\mathrm{g}}(G) + 2$ *if there exists a global dominating set D of G such that $V(G) - D$ is independent.*

Theorem 3.24. $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \alpha_0(G) + 1$ *if there exists a point cover D of G having at least three vertices with $\delta(\langle D\rangle) \geqslant 1$ and a vertex $v \in V(G) - D$ such that $r(\langle D \cup \{v\}\rangle) \geqslant 2$ and $\delta(\langle D \cup \{v\}\rangle) \geqslant 1$, where $\alpha_0(G)$ is the point covering number of G.*

P r o o f. Let D be a point cover of G with $|D| = \alpha_0(G) \geqslant 3$ and $\delta(\langle D\rangle) \geqslant 1$ and assume that there exist a vertex $v \in V(G) - D$ such that $r(\langle D \cup \{v\}\rangle) \geqslant 2$ and $\delta(\langle D \cup \{v\}\rangle) \geqslant 1$. Then D is a total dominating set of $B_1(G)$. Let $D' = D \cup \{v\}$, so that $r(\langle D'\rangle) \geqslant 2$ and $\delta(\langle D'\rangle) \geqslant 1$. It remains to prove that D' is a total global dominating set of $B_1(G)$. For any vertex u in D', since radius $(\langle D'\rangle) \geqslant 2$, there exists a vertex u' in D' not adjacent to u. If $u \in V(G) - D'$, then since $V(G) - D$ is independent, u is not adjacent to v. Similarly, for any line vertex e' in $B_1(G)$, there exists a vertex in D' not adjacent to e'. Hence, D' is a total global dominating set of $B_1(G)$. Thus, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant \alpha_1(G) + 1$. □

R e m a r k 3.3. For any graph G with $r(G) \geqslant 2$ having no isolated vertices, $\gamma_{\mathrm{tg}}(B_1(G)) \leqslant 2\alpha_0(G)$.

In the following, some bounds for the global point-set domination number γ_{tg} of $B_1(G)$ are obtained. Here the graphs G for which $B_1(G)$ is co-connected are considered.

Proposition 3.1. *For any (p, q) graph G with $p \geqslant 3$ and $q \geqslant 2$,* $3 \leqslant \gamma_{\mathrm{pg}}(B_1(G)) \leqslant p + q - 2$.

P r o o f. Follows from Theorem 2.5. □

The lower bound is attained if G is a path on 3 vertices, and the upper bound is attained if G is a cycle on 3 vertices.

Theorem 3.25. *If $r(G) = 1$, then $\gamma_{\mathrm{pg}}(B_1(G)) \leqslant p + q - \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.*

P r o o f. Let v be a vertex of maximum degree in G and $\deg_G(v) = \Delta(G)$. Let D' be the set of line vertices in $B_1(G)$ corresponding to the edges incident with v in G. If $D = V(B_1(G)) - D'$, then D is a global point-set dominating set (global psd-set) of $B_1(G)$. Hence, $\gamma_{\mathrm{pg}}(B_1(G)) \leqslant |D| = p + q - \Delta(G)$.

This bound is attained when G is a star. □

Theorem 3.26. *$\gamma_{\rm pg}(B_1(G)) \leqslant q$ if G is triangle-free and for every independent set W in G, $\langle V(G)-W\rangle$ is not totally disconnected.*

Proof. Let D be the set of all line vertices in $B_1(G)$. Since for every independent set W in G, $\langle V(G)-W\rangle$ is not totally disconnected, D is a psd-set of $B_1(G)$ in view of Theorem 2.3. Also since G is triangle-free, D is a global psd-set of $B_1(G)$. Hence, $\gamma_{\rm pg}(B_1(G)) \leqslant q$. □

Theorem 3.27. *For any triangle-free graph G having no perfect matching, $\gamma_{\rm pg}(B_1(G)) \leqslant p$, where $p \geqslant 4$.*

Proof. Let D be the set of all point vertices in $B_1(G)$. Since G has no perfect matching and D has at least four vertices, D is a psd-set of $B_1(G)$. Since G is triangle-free, for every complete subgraph S in $V(B_1(G))-D$ there exists a point vertex v in D such that $S\cap N(v)=\emptyset$ in $B_1(G)$. Hence, D is a global psd-set of $B_1(G)$ by Theorem 2.4. Thus, $\gamma_{\rm pg}(B_1(G)) \leqslant p$.

This bound is attained if $G\cong C_{2n+1}$, for $n\geqslant 2$. □

Remark 3.4. Let G be any graph having no perfect matching. Then $\gamma_{\rm pg}(B_1(G)) \leqslant p+k$, where $p\geqslant 4$ and k is the number of edges to be deleted in G so that G becomes triangle-free.

In the following, the bounds for the neighborhood number n_0 of $B_1(G)$ are obtained. Here we consider the graphs G for which $B_1(G)$ has no isolated vertices. Since there is no vertex of degree $p+q-1$ in $B_1(G)$, $n_0(B_1(G))\geqslant 2$, and $n_0(B_1(G))=2$ if $G\cong 2K_2$.

Theorem 3.28. *If $\gamma_{\rm c}(G)=2$, then $n_0(B_1(G))\leqslant 3$.*

Proof. Let D be a minimal connected dominating set of G such that $|D| = \gamma_{\rm c}(G)=2$ and let $D=\{v_1,v_2\}$, where v_1, $v_2\in V(G)$ and $e_{12}=(v_1,v_2)\in E(G)$. Let e'_{12} be the line vertex in $B_1(G)$ corresponding to the edge e_{12} and $D'=\{v_1,v_2,e'_{12}\}\subseteq V(B_1(G))$. Let xy be an edge in $B_1(G)$.

(i) If either $x\in D$ and $y\in V(B_1(G))$ or both x and y are line vertices, then $xy\in \bigcup\limits_{w\in D'} E(\langle N[w]\rangle)$.

(ii) If x and y are point vertices in $V(B_1(G))-D'$, then $xy\in E(\langle N[e'_{12}]\rangle)$.

(iii) If x is a point vertex in $V(B_1(G))-D'$ and y is a line vertex ($y\neq e'_{12}$) such that the corresponding edge in G has one end in D, then $xy\in E(\langle N[e'_{12}]\rangle)$.

(iv) If x is a point vertex in $V(B_1(G))-D'$ and y is a line vertex such that the corresponding edge in G has both ends in $V(G)-D$, then $xy\in E(\langle N[v_1]\rangle)\cup E(\langle N[v_2]\rangle)$.

From (i), (ii), (iii) and (iv), it follows that $B_1(G) = \bigcup_{w \in D'} E(\langle N[w] \rangle)$. Hence, D' is a neighborhood set for $B_1(G)$. Thus, $n_0(B_1(G)) \leqslant 3$. $\square$

Theorem 3.29. *Let D be an independent neighborhood set (n-set) for G with $|D| = 2$. If there exists exactly one vertex v in $V(G) - D$ such that $|N(v) \cap D| = 1$ and $|N(u) \cap D| = 2$, for all u in $V(G) - (D \cup \{v\})$, then $n_0(B_1(G)) \leqslant 3$.*

Proof. Let D be an independent n-set for G with $|D| = 2$. Assume $D = \{v_1, v_2\}$, where $v_1, v_2 \in V(G)$ and v is the only vertex in $V(G) - D$ such that $|N(v) \cap D| = 1$. Let $D' = \{v_1, v_2, v\}$. Then $D' \subseteq V(B_1(G))$. Let xy be an edge in $B_1(G)$.

(i) If both x and y are point vertices in $V(B_1(G)) - D'$, then since D is an n-set of G, $xy \in E(\langle N[v_1] \rangle)$ or $E(\langle N[v_2] \rangle)$.

(ii) If both x and y are line vertices in $V(B_1(G)) - D'$, then since $|D'| = 3$, $xy \in \bigcup_{w \in D'} E(\langle N[w] \rangle)$.

(iii) If x is a point vertex and y is a line vertex in $V(B_1(G)) - D'$, then by the conditions given in the proposition, $xy \in E(\langle N[v_1] \rangle)$ or $E(\langle N[v_2] \rangle)$. Hence, D' is an n-set for $B_1(G)$ and $n_0(B_1(G)) \leqslant 3$. $\square$

Theorem 3.30. *Let D be an n-set for G with $|D| = 3$ such that $\langle D \rangle$ is not totally disconnected. Then $n_0(B_1(G)) \leqslant 4$ if and only if either*

(a) *$|N(u) \cap D\rangle| \geqslant 2$, for all u in $V(G) - D$; or*

(b) *If $|N(u) \cap D| = 1$, for at least one $u \in V(G) - D$, then the vertex in $N(u) \cap D$ is not adjacent to any of the vertices in $V(G) - D$ other than u.*

Proof. Let D be an n-set for G with $|D| = 3$ such that $\langle D \rangle$ is not totally disconnected and the condition (a) or (b) is satisfied. Let e be an edge in D and e' be the line vertex in $B_1(G)$ corresponding to the edge e. Then $D' = D \cup \{e'\} \subseteq V(B_1(G))$. Let xy be an edge in $B_1(G)$.

(i) If x and y are both point vertices, then since D is an n-set for G, $xy \in \bigcup_{w \in D'} E(\langle N[w] \rangle)$.

(ii) If x and y are both line vertices in $V(B_1(G)) - D'$, then since $|D| = 3$ and $L(G)$ is an induced subgraph of $B_1(G)$, $xy \in \bigcup_{w \in D'} E(\langle N[w] \rangle)$.

(iii) If x is a point vertex and y is a line vertex in $V(B_1(G)) - D'$, then by condition (a) or (b), there exists at least one vertex v in D' such that $N(v)$ contains both x and y, since $L(G)$ is an induced subgraph of $B_1(G)$. Thus, D' is an n-set for $B_1(G)$. Hence, $n_0(B_1(G)) \leqslant n_0(G) + 1$.

Conversely, if conditions (a) and (b) are not true, then there exists a vertex $u \in V(G) - D$ with $N(u) \cap D = \{v\}$, $v \in V(G)$, and a vertex, say w, in $V(G) - D$

adjacent to v, where $w \neq u$. Let $(v, w) = e_1 \in E(G)$ and let e_1' be the line vertex in $B_1(G)$ corresponding to e_1. Then $(u, e_1') \in E(B_1(G))$ and there exists no vertex in D' adjacent to both u and e_1', which is a contradiction. Thus, the theorem follows. $\square$

Theorem 3.31. *Let D be a dominating set of G such that D contains at least four vertices and $|D| = \gamma(G)$. If D is an n-set for G and $|N(u) \cap D| \geqslant 2$, for all $u \in V(G) - D$, then $n_0(B_1(G)) \leqslant \gamma(G)$.*

Proof. Let D be a γ-set of G having at least four vertices. Then $D \subseteq V(B_1(G))$. Let xy be an edge in $B_1(G)$. Since G is an induced subgraph of $B_1(G)$ and D is an n-set with $|D| \geqslant 4$, if x and y are both point or line vertices in $V(B_1(G)) - D$, then $xy \in \bigcup\limits_{w \in D} E(\langle N[w] \rangle)$. Since for all $u \in V(G) - D$, $|N(u) \cap D| \geqslant 2$, if x is a point vertex and y is a line vertex, then $xy \in \bigcup\limits_{w \in D} E(\langle N[w] \rangle)$. Thus, D is an n-set for $B_1(G)$. Hence, $n_0(B_1(G)) \leqslant \gamma(G)$. $\square$

Corollary 3.31.1. *Let D be an n-set with $|D| = n_0(G) \geqslant 4$. Then $n_0(B_1(G)) \leqslant n_0(G)$ if $|N(u) \cap D| \geqslant 2$, for all $u \in V(G) - D$.*

Theorem 3.32. *$n_0(B_1(G)) \leqslant \alpha_0(G)$ ($\alpha_0(G) \geqslant 4$) if and only if there exists a minimal point cover D of G with $|D| \geqslant 4$ and for each pair (v, e), where $e \in E(G)$ is not incident with $v \in V(G) - D$, there exists at least one vertex in $N(v)$ not incident with e in G.*

Proof. Assume that $n_0(B_1(G)) \leqslant \alpha_0(G)$ and D is a point cover of G with $|D| = \alpha_0(G) \geqslant 4$. Let $v \in V(G) - D$ and let $e \in E(G)$ be not incident with v. Assume that all the vertices in $N(v)$ are incident with e in G. That is, the vertices of $N(v)$ are the end vertices of e. Since D is a point cover, $v \in V(G) - D$ is adjacent to at least one vertex, say w, in D. By the assumption, w is incident with e. Let e' be the line vertex in $B_1(G)$ corresponding to the edge e. Then the edge (v, e') in $B_1(G)$ is not covered by $D \subseteq V(B_1(G))$ and hence D is not an n-set for $B_1(G)$, which is a contradiction. Conversely, assume the given condition. Let D be a point cover of G with $|D| = \alpha_0(G) \geqslant 4$. Then D is an n-set for G. Let xy be an edge in $B_1(G)$. If $x, y \in D$, $x \in D$ is a point vertex and y is a line vertex or both x and y are line vertices, then $xy \in \bigcup\limits_{w \in D} E(\langle N[w] \rangle)$. Also by the assumption, if $x \in V(G) - D$ is a point vertex and y is a line vertex, then $xy \in \bigcup\limits_{w \in D} E(\langle N[w] \rangle)$. Hence, D is an n-set for $B_1(G)$. Thus, $n_0(B_1(G)) \leqslant \alpha_0(G)$. $\square$

Remark 3.4. (i) If $|D| = 3$ and $\langle D \rangle \cong P_3$ or C_3, then with the condition given in the theorem, $n_0(B_1(G)) \leqslant \alpha_0(G)$.

(ii) If $|D| = 3$, $\langle D\rangle \cong P_3$ or C_3, for every minimal point cover D of G, then $D \cup \{e'\}$, where e' is the line vertex in $B_1(G)$ corresponding to an edge in P_3 or C_3, is an n-set for $B_1(G)$. Hence, $n_0(B_1(G)) \leqslant 4$.

(iii) If $|D| = 2$, then $n_0(B_1(G)) = \alpha_0(G)$, for $G \cong 2K_2$.

Corollary 3.32.1. *Let D be an independent point cover of G with $|D| = \alpha_0(G) \geqslant 3$. Then $n_0(B_1(G)) \leqslant \alpha_0(G)$ if and only if either $\deg_G(v) \geqslant 2$, for all $v \in V(G) - D$, or $\deg_G(v) = 1$, for at least one $v \in V(G) - D$ with $\delta(\langle N(v)\rangle) = 1$.*

This bound is attained if $G \cong C_{2n}$, for $n \geqslant 3$.

Theorem 3.33. *$n_0(B_1(G)) \leqslant p/2$ if there exists a perfect matching D in G with $|D| \geqslant 3$ such that the set of vertices in $L(G)$ corresponding to the edges in D is an n-set for $L(G)$.*

Proof. Let $D \subseteq E(G)$ be a perfect matching in G with $|D| \geqslant 3$. Let D' be the set of vertices in $L(G)$ corresponding to the edges in D. Assume D' is an n-set for $L(G)$. Then it is clear that, D' is also an n-set for $B_1(G)$. Hence, $n_0(B_1(G)) \leqslant p/2$.

This bound is attained if G is a cycle on 6 vertices or a complete graph on four vertices. □

Remark 3.5. Let $D \subseteq E(G)$ be a perfect matching in G with $|D| \geqslant 2$ and D' be the vertices in $L(G)$ corresponding to the edges in D. Then D' is an n-set for $B_1(G)$ if and only if $G \cong 2K_2$.

Note 3.2. (i) For any graph G, $n_0(G) \leqslant p - \Delta(G)$. Hence, $n_0(B_1(G)) \leqslant p + q - \Delta(B_1(G))$ and $n_0(B_1(G)) \leqslant \max\{p, q + 2 - \Delta_e(G)\}$, where $\Delta_e(G)$ is the maximum degree of $L(G)$.

(ii) If $r(L(G)) = 1$, then $n_0(B_1(G)) \leqslant \max\{p, 3\}$.

(iii) $n_0(B_1(K_n)) = n$, if $n \geqslant 3$.

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Authors' addresses: *T. N. Janakiraman*, National Institute of Technology, Tiruchirappalli 620 015, India, e-mail: `janaki@nitt.edu`; *S. Muthammai*, *M. Bhanumathi*, Government Arts College for Women, Pudukkottai 622 001, India.