REACTION-DIFFUSION SYSTEMS: DESTABILIZING EFFECT OF CONDITIONS GIVEN BY INCLUSIONS

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Abstract. Sufficient conditions for destabilizing effects of certain unilateral boundary conditions and for the existence of bifurcation points for spatial patterns to reaction-diffusion systems of the activator-inhibitor type are proved. The conditions are related with the mollification method employed to overcome difficulties connected with empty interiors of appropriate convex cones.

Keywords: bifurcation, spatial patterns, reaction-diffusion system, mollification, inclusions

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0. Introduction

Systems of reaction-diffusion and the effect of diffusion driven instability, the growth of spatial patterns (stationary but spatially nonconstant solutions) and related eigenvalue and bifurcation problems have been studied for a long time by many authors. The motivation for the study of such problems comes from biology and ecology where the behaviour of two or more species is modeled ([11], [21], [22]); the effect of diffusion driven instability was described for the first time in [27]. Multivalued boundary conditions can describe e.g. a certain control process, a semipermeable or another type of the membrane on a part of the boundary. The system with various types of unilateral boundary conditions was studied by M. Kučera, P. Quittner, M. Bosák, P. Drábek in [2], [3], [4], [6], [12], [15], [16], [19], [26] (the destabilizing effect—the bifurcation for the unilateral problem occurs in a domain of stability of

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the system with classical Dirichlet and/or Neumann boundary conditions) and in [13], [17], [18] (stabilizing effect). For a detailed survey see e.g. [8], [6].

In this paper, the results of [16], i.e. the existence of a bifurcation point for system with multivalued boundary conditions proved for an interval, are generalized to domains with higher dimension and the localization of bifurcation points is specified. In [16] the fact that the Sobolev space $W^{1,2}(\Omega)$ is embedded into the space of continuous functions was used. Therefore, the cone $K := \{v \in W^{1,2}(0,1);\ v(0) = 0, v(1) \geqslant 0\}$ has a nonempty interior. An analogue of this cannot hold for higher dimension. In order to prove the existence of a bifurcation point by a similar process as in [16], we can either define a pseudointerior of K like in [26], [4] or [6] and use a technique similar to [3]—this requires an additional condition for the reaction terms (see Remark 8.1 in Appendix)—or approximate our problem (see Section 3) where the corresponding approximate cone K^{δ}, defined with help of mollification, has a nonempty interior. Similarly to [16] we show the existence of a bifurcation point for the approximate problem and obtain a bifurcation point for the original one by the limiting process for $\delta \to 0$.

1. Problem formulation

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitzian boundary, let Γ_D, Γ_N, Γ_U be open (in $\partial\Omega$) disjoint subsets of $\partial\Omega$. Let $\partial\Gamma_U$ be Lipschitz with respect to $\partial\Omega$, $\operatorname{meas}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N \cup \Gamma_U)) = 0$ and

$$\operatorname{meas}\Gamma_D > 0,\ \operatorname{dist}(\Gamma_D, \Gamma_U) > \delta_0 \text{ with } \delta_0 > 0 \text{ small.} \tag{1.1}$$

Let us consider a reaction-diffusion system

$$\begin{aligned} u_t &= d_1 \Delta u + f(u,v), \\ v_t &= d_2 \Delta v + g(u,v) \end{aligned} \qquad \text{in } [0,+\infty) \times \Omega \tag{RD}$$

with multivalued boundary conditions

$$\begin{aligned} &u = \tilde{u},\ v = \tilde{v} \quad \text{on } [0,+\infty) \times \Gamma_D, \\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} \in -\frac{m(v-\tilde{v})}{d_2} \quad \text{on } [0,+\infty) \times \Gamma_U, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } [0,+\infty) \times \Gamma_N, \end{aligned} \tag{MC}$$

where d_1, d_2 are positive diffusion parameters, $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions such that $f(\tilde{u},\tilde{v}) = g(\tilde{u},\tilde{v}) = 0$, $\tilde{u}, \tilde{v} \in \mathbb{R}$ are constants, $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ is a suitable multivalued function (e.g. $m(\xi) = 0$ for $\xi > 0$, $m(0) = [m^0, 0]$, $m(\xi)$ is singlevalued, negative for $\xi < 0$).

We will prove that there is a bifurcation point $d_I = [d_1^I, d_2^I]$ at which stationary spatially nonconstant solutions ("spatial patterns") for the system (RD) with (MC) bifurcate from a branch of the trivial solution $[\tilde{u}, \tilde{v}]$. Moreover, this bifurcation point can lie in the region of stability of $[\tilde{u}, \tilde{v}]$ as a solution of (RD) with classical boundary conditions

$$\text{(CC)} \qquad \begin{aligned} &u = \tilde{u}, \ v = \tilde{v} \quad \text{on } [0, +\infty) \times \Gamma_D, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } [0, +\infty) \times (\Gamma_N \cup \Gamma_U), \end{aligned}$$

where the bifurcation for (RD), (CC) is excluded.

Set $b_{11} = \frac{\partial f}{\partial u}(\tilde{u}, \tilde{v})$, $b_{12} = \frac{\partial f}{\partial v}(\tilde{u}, \tilde{v})$, $b_{21} = \frac{\partial g}{\partial u}(\tilde{u}, \tilde{v})$, $b_{22} = \frac{\partial g}{\partial v}(\tilde{u}, \tilde{v})$. It is known that under the assumption

$$\text{(SIGN)} \qquad \begin{aligned} &b_{11} > 0, \qquad b_{12} < 0, \qquad b_{21} > 0, \qquad b_{22} < 0, \\ &b_{11} + b_{22} < 0, \qquad b_{11}b_{22} - b_{12}b_{21} > 0, \end{aligned}$$

the effect of diffusion driven instability occurs: the constant solution $[\tilde{u}, \tilde{v}]$ is stable as a solution of ODE's

$$u_t = f(u, v), \quad v_t = g(u, v) \qquad \text{on } [0, +\infty)$$

but it is stable as a solution of (RD), (CC) only for some $d = [d_1, d_2] \in \mathbb{R}_+^2$ lying in the domain of stability D_S and unstable for the other ones (lying in the domain of instability D_U)—for the notation see Fig. 1, Notation 2.1 and Section 3. Further, spatial patterns of (RD), (CC) bifurcate from $[\tilde{u}, \tilde{v}]$ on the boundary C between D_S and D_U (see Fig. 1 and e.g. [20], [25]).

For the sake of simplicity we assume $\tilde{u} = \tilde{v} = 0$ in the sequel. We study only stationary solutions. Hence we solve the system

$$\text{(SRD)} \qquad \left. \begin{aligned} d_1 \Delta u + f(u, v) = 0 \\ d_2 \Delta v + g(u, v) = 0 \end{aligned} \right. \qquad \text{in } \Omega$$

with boundary conditions (MC) and (CC) in the form

$$\text{(1.2)} \qquad \begin{aligned} &u = v = 0 \quad \text{on } \Gamma_D, \\ &\frac{\partial u}{\partial n} = 0, \ \frac{\partial v}{\partial n} \in -\frac{m(v)}{d_2} \quad \text{on } \Gamma_U, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_N \end{aligned}$$

and

$$u = v = 0 \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N \cup \Gamma_U. \tag{1.3}$$

2. Weak formulation, general assumptions, model example

N o t a t i o n 2.1. $\mathbb{R}_+$—the set of all positive reals, $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$, $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R}$

$d^A \preceq d^B$ for any $d^A = [d^A_1, d^A_2], d^B = [d^B_1, d^B_2] \in \mathbb{R}^2_+$ if and only if $d^A_1 \leqslant d^B_1$ and $d^A_2 \leqslant d^B_2$

κ_j, e_j $(j = 1, 2, 3, \ldots)$—the eigenvalues and eigenvectors of $-\Delta$ with condition (1.3)

$C_j := \left\{d = [d_1, d_2] \in \mathbb{R}^2_+ ;\ d_2 = \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}\right\}$, $j = 1, 2, 3, \ldots$ (see Fig. 1)

C—the envelope of the hyperbolas C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

$D_U := \{d = [d_1, d_2] \in \mathbb{R}^2_+ ;\ d_2 > \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}$ for at least one $j = 1, 2, 3, \ldots\}$—the set of all $d \in \mathbb{R}^2_+$ lying to the left from C (domain of instability) (see Fig. 1)

$D_S := \mathbb{R}^2_+ \setminus (C \cup D_U)$—the set of all $d \in \mathbb{R}^2_+$ lying to the right from C (domain of stability) (see Fig. 1)

$\mathcal{T}$—the common tangent to all C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

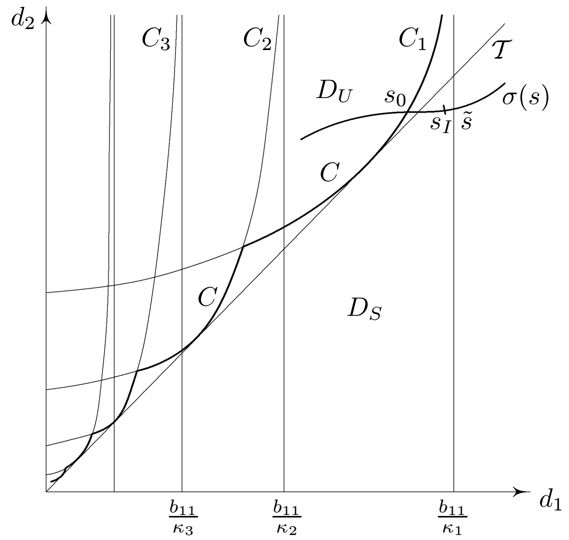

Fig. 1

$C^0(\mathrm{cl}\,\Omega)$—the space of continuous functions on $\mathrm{cl}\,\Omega$ equipped with the usual Chebyshev norm

$\mathbb{V}$ a real Hilbert space, $\mathbb{V}^2 = \mathbb{V} \times \mathbb{V}$ endowed with the inner product $\langle U, W\rangle = \langle u, w\rangle + \langle v, z\rangle$, $U = [u, v], W = [w, z] \in \mathbb{V}^2$

A, N_1, N_2—operators satisfying (2.4), (2.5)

$M = [\{0\}, M_2]$, $M_0 = [\{0\}, M_{02}]$—multivalued mappings of $\mathbb{V}^2$ into $2^{\mathbb{V}^2}$ defined in Model Example

$U = [u, v]$ elements of $\mathbb{V}^2$, $AU = [Au, Av]$, $N(U) = [N_1(U), N_2(U)]$ for $U = [u, v] \in \mathbb{V}^2$

$U^* = [\frac{b_{21}}{b_{12}} u, v]$ for $U = [u, v] \in \mathbb{V}^2$

$K := \{U \in \mathbb{V}^2;\ 0 \in M_0(U)\}$—closed convex cone with the vertex at the origin

We denote by $\to, \rightharpoonup$ the strong and weak convergence, respectively.

$$D(d) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},\ D^{-1}(d) = \begin{bmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{bmatrix},\ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ B^* = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$E_B(d) := \{U \in \mathbb{V}^2;\ D(d)U - BAU = 0\}$

$E_{B^*}(d) := \{U \in \mathbb{V}^2;\ D(d)U - B^*AU = 0\}$

$E_I(d) := \{U \in \mathbb{V}^2;\ D(d)U - BAU \in -M_0(U)\}$

critical point of a problem (P) (where (P) stands e.g. for (2.7) or (2.11))—a parameter $d \in \mathbb{R}^2_+$ for which (P) has a nontrivial solution

bifurcation point of a problem (P) (where (P) stands e.g. for (2.6) or (2.10))—a parameter $d^0 \in \mathbb{R}^2_+$ such that for any neighbourhood of $[d^0, 0, 0] \in \mathbb{R}^2_+ \times \mathbb{V}^2$ there exists $[d, U] = [d, u, v]$, $\|U\| \neq 0$ satisfying (P).

Notation 2.2. Set $\mathbb{V} := \{u \in W^{1,2}(\Omega);\ u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}$, $\mathbb{V}^2 := \mathbb{V} \times \mathbb{V}$,

$$\langle u, \varphi \rangle := \int_\Omega \sum_{j=1}^{n} u_{x_j} \varphi_{x_j} \,\mathrm{d}x \quad \text{for all } u, \varphi \in \mathbb{V}. \tag{2.1}$$

Then $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{V}$ and the corresponding norm $\| \cdot \|$ is equivalent to the usual Sobolev norm on the space $\mathbb{V}$ under the assumption (1.1) and the embeddings

$$\mathbb{V} \hookrightarrow L^2(\Omega), \mathbb{V} \hookrightarrow L^2(\partial\Omega) \tag{2.2}$$

are compact—see e.g. [10].

Set $n_1(u, v) = f(u, v) - b_{11}u - b_{12}v$, $n_2(u, v) = g(u, v) - b_{21}u - b_{22}v$ and define operators $A\colon \mathbb{V} \to \mathbb{V}$, $N_j\colon \mathbb{V}^2 \to \mathbb{V}$ $(j = 1, 2)$ by

$$\begin{aligned} \langle Au, \varphi \rangle &= \int_\Omega u\varphi \,\mathrm{d}x \quad \text{for all } u, \varphi \in \mathbb{V} \\ \langle N_j(U), \varphi \rangle &= \int_\Omega n_j(u, v)\varphi \,\mathrm{d}x \quad \text{for all } U = [u, v] \in \mathbb{V}^2,\ \varphi \in \mathbb{V}. \end{aligned} \tag{2.3}$$

It follows from embedding theorems (see e.g. [10]) that

(2.4) A is a linear, symmetric, positive and completely continuous operator.

Further, if $u, v \in W^{1,2}(\Omega)$ then it follows from the embedding theorem that $u, v \in L^q(\Omega)$ with any real $q \geqslant 1$ for $\mathfrak{n} \leqslant 2$ and $1 \leqslant q \leqslant \frac{2\mathfrak{n}}{\mathfrak{n}-2}$ for $\mathfrak{n} > 2$. If n_j satisfy a growth condition $n_j(\xi, \eta) \leqslant C(1 + |\xi|^{q-1} + |\eta|^{q-1})$ for any $\xi, \eta \in \mathbb{R}$ then $n_j(u, v) \in L^{q^*}(\Omega)$ with $q^* = \frac{q}{q-1}$ by the Nemytskii theorem (see e.g. [10]) and this together with the compactness of the embedding mentioned implies that

$$
\begin{aligned}
&N_1, N_2 \text{ are nonlinear, completely continuous operators from } \mathbb{V}^2 \text{ to } \mathbb{V} \\
&\lim_{\|U\| \to 0} \frac{\|N_j(U)\|}{\|U\|} = 0 \ (j = 1, 2)
\end{aligned}
\tag{2.5}
$$

(for the last condition, see [18], Lemma 1.A in Appendix).

Now, a weak solution of the problem (SRD), (1.3) is a solution of the operator equations

$$
\begin{aligned}
d_1 u - b_{11} A u - b_{12} A v - N_1(u, v) &= 0 \\
d_2 v - b_{21} A u - b_{22} A v - N_2(u, v) &= 0.
\end{aligned}
\tag{2.6}
$$

We also consider the linear problem corresponding to (2.6), i.e.

$$
\begin{aligned}
d_1 u - b_{11} A u - b_{12} A v &= 0 \\
d_2 v - b_{21} A u - b_{22} A v &= 0.
\end{aligned}
\tag{2.7}
$$

Model Example. (Cf. [16].) Let us consider a multivalued mapping $m\colon \mathbb{R} \to 2^{\mathbb{R}}$ which is a singlevalued real continuous function on $\mathbb{R} \setminus \{0\}$ and a multivalued one at $\xi = 0$ such that

$$
\begin{aligned}
&m(\xi) = 0 \text{ for } \xi > 0, \ m(\xi) \leqslant 0 \text{ for } \xi < 0, \\
&\lim_{\xi \to 0_-} m(\xi) = m^0 \ \text{ with some } m^0 \in (-\infty, 0), \quad m(0) = [m^0, 0].
\end{aligned}
$$

Set

$$
\begin{aligned}
&\underline{m}(\xi) = \overline{m}(\xi) = m(\xi) \qquad \text{for } \xi \neq 0, \\
&\underline{m}(0) = m^0, \quad \overline{m}(0) = 0
\end{aligned}
$$

and let us assume that

$$
|\underline{m}(\xi)|, |\overline{m}(\xi)| \leqslant k \cdot (1 + |\xi|) \text{ with some } k > 0.
\tag{2.8}
$$

Consider the situation from Notation 2.2 and define a multivalued mapping $M_2\colon \mathbb{V} \to 2^{\mathbb{V}}$ by

$$M_2(v) := \Big\{ z \in \mathbb{V};\ \int_{\Gamma_U} \underline{m}(v)\varphi \ \mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle \leqslant \int_{\Gamma_U} \overline{m}(v)\varphi \ \mathrm{d}\Gamma \qquad \text{for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ on } \Gamma_U \Big\}. \tag{2.9}$$

(The inequalities on Γ_U are understood in the sense of traces.) Then a solution of

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av - N_1(u,v) = 0\\ &d_2 v - b_{21}Au - b_{22}Av - N_2(u,v) \in -M_2(v) \end{aligned} \tag{2.10}$$

is a weak solution of the problem (SRD), (1.2)—see [9] for details. Further, introduce a positively homogeneous mapping $M_0\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ corresponding to $M(U) = [\{0\}, M_2(v)]$, $U = [u,v]$, which is defined by $M_0(U) = [\{0\}, M_{02}(v)]$ with

$$\begin{aligned} M_{02}(v) &:= \{z \in \mathbb{V};\ \langle z, v\rangle = 0,\ \langle z, \varphi\rangle \leqslant 0 \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ a.e. on } \Gamma_U\}\\ &\qquad \text{if } v \geqslant 0 \text{ a.e. on } \Gamma_U\\ M_{02}(v) &:= \emptyset \text{ if } v < 0 \text{ on a subset of } \Gamma_U \text{ of a positive measure.} \end{aligned}$$

Then a solution of

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av = 0\\ &d_2 v - b_{21}Au - b_{22}Av \in -M_{02}(v) \end{aligned} \tag{2.11}$$

is a weak solution of

$$\left.\begin{aligned} &d_1 \Delta u + b_{11}u + b_{12}v = 0\\ &d_2 \Delta v + b_{21}u + b_{22}v = 0 \end{aligned}\right. \qquad \text{in } \Omega$$

with the boundary conditions

$$\begin{aligned} &u = v = 0 \ \text{ on } \Gamma_D,\\ &\frac{\partial u}{\partial n} = 0,\ v \geqslant 0,\ \frac{\partial v}{\partial n} \geqslant 0,\ \frac{\partial v}{\partial n} \cdot v = 0 \text{ on } \Gamma_U,\\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{2.12}$$

Note that the problem (2.11) is still nonlinear because M_0 is cone-valued and nonlinear. Hence we cannot use the standard technique (as e.g. the degree theory for linear mappings) to obtain the bifurcation points.

Remark 2.1. We can also consider $m^0 = -\infty$ in Model Example. Then we define M in the same way as M_0 and do not assume (2.8).

R e m a r k 2.2. It is easy to see from the definition of M_0 that the inclusion problem (2.11) is equivalent to the variational inequality

$$\begin{aligned}&U \in K;\\&\langle D(d)U - BAU, V - U\rangle \geqslant 0 \text{ for any } V \in K\end{aligned} \tag{2.13}$$

with

$$K := \{U \in \mathbb{V}^2;\ 0 \in M_0(U)\} = \mathbb{V} \times \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ on } \Gamma_U\}. \tag{2.14}$$

Therefore, the inclusion (2.10) is a generalization of such problems (2.13) and also of variational inequalities

$$\langle D(d)U - BAU - N(U), V - U\rangle + \Psi(V) - \Psi(U) \geqslant 0 \text{ for any } V \in K$$

with a positive convex lower semicontinuous functional $\Psi\colon \mathbb{V}^2 \to (-\infty, +\infty]$, $\Psi \not\equiv +\infty$, where $M = \partial\Psi$—the subdifferential of Ψ (cf. e.g. [5]).

3. Properties of the linear equation

In the sequel, we consider a general real Hilbert space $\mathbb{V}$ and operators $A\colon \mathbb{V} \to \mathbb{V}$, $N\colon \mathbb{V}^2 \to \mathbb{V}$ satisfying (2.4), (2.5).

O b s e r v a t i o n 3.1. (Cf. [4], Section 2, [6], Section 4.) It follows from (2.4) that the characteristic values of A (i.e. the eigenvalues of the Laplacian with (1.3) for A from Notation 2.2) form a sequence $\{\kappa_i\}_{i=1}^{\infty}$, ($\kappa_i \to +\infty$ for $i \to +\infty$) of positive numbers. The set of all corresponding eigenvectors $\{e_i\}_{i=1}^{\infty}$ forms a complete orthonormal system in $\mathbb{V}$.

Proposition 3.1. *The eigenvalue problem*

$$D(d)U - BAU + \mu AU = 0 \tag{3.1}$$

has a system of eigenvalues

$$\mu_i^{(r)} = \tfrac{1}{2}[b_{11} + b_{22} - (d_1 + d_2)\kappa_i \pm\sqrt{\mathcal{D}}],\ r = 1, 2 \tag{3.2}$$

with $\mathcal{D} := [b_{11} + b_{22} - (d_1 + d_2)\kappa_i]^2 - 4\cdot[(d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21}]$, $i = 1, 2, \ldots$, *which are roots of*

$$\mu^2 - \mu[b_{11} + b_{22} - (d_1 + d_2)\kappa_i] + (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21} = 0. \tag{3.3}$$

In particular, $d = [d_1, d_2]$ is a critical point of (2.7) if and only if $\mu = 0$ is a solution of (3.3), i.e. if and only if

$$(d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21} = 0, \tag{3.4}$$

i.e. if d lies on a hyperbola $C_i = \left\{d = [d_1, d_2] \in \mathbb{R}^2_+ ;\ d_2 = \dfrac{b_{12}b_{21}/\kappa_i^2}{d_1 - b_{11}/\kappa_i} + \dfrac{b_{22}}{\kappa_i}\right\}$ *for some $i = 1, 2, \ldots$.*

For the proof see e.g. [4], Section 2.

Observation 3.2. (See [20] and [4] for the proof of the following statement.) Under the assumption (SIGN), for a given i there are two real roots $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ of (3.3) for any d lying to the left from C_i or in the right neighbourhood of C_i (including C_i). The smaller one (say $\mu_i^{(2)}(d)$) is always negative.

It follows from the definition of C_i that $\mu_i^{(1)}(d) < 0$ or $\mu_i^{(1)}(d) > 0$ for d to the right or to the left, respectively, from C_i and in a neighbourhood of C_i. For d lying to the right and sufficiently far from C_i, we have $\mu_i^{(r)}(d) \in \mathbb{C}\backslash\mathbb{R}$ with $\operatorname{Re}\mu_i^{(r)}(d) < 0$, $r = 1, 2$. Further, for any $d \in \mathbb{R}^2_+$, let us set $\mu(d) := \max\{\mu_i^{(1)}(d);\ \mu_i^{(1)}(d) \in \mathbb{R}\}$. Hence, $\mu(d)$ is the greatest eigenvalue of (3.1). Then the envelope C of all C_i, $i = 1, 2, \ldots$ is equal to $\{d \in \mathbb{R}^2_+ ;\ \mu(d) = 0\}$ and $\mu(d) < 0$ or $\mu(d) > 0$ for d from a neighbourhood of C and to the right or to the left from C, respectively.

Observation 3.3. It follows from Proposition 3.1 and Observation 3.2 that $E_B(d) \neq \{0\}$ if and only if $d \in \bigcup\limits_{j=1}^{\infty} C_j$. Moreover, let p be an index such that the characteristic value κ_p of A (i.e. the eigenvalue of the Laplacian with (1.3) for A from Notation 2.2) has a multiplicity k, $\kappa_p = \ldots = \kappa_{p+k-1}$. Then for any $d \in C_p = \ldots = C_{p+k-1}$, $d \notin C_q$ for $C_q \neq C_p$ we have

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p}^{p+k-1} \text{ with } U_i(d) = [\alpha_i(d)e_i, e_i], \tag{3.5}$$

where $\alpha_i(d) = \frac{d_2\kappa_p - b_{22}}{b_{21}} > 0$. Further, if $d \in C_p \cap C_q$ for some $C_q \neq C_p$, $\kappa_p \neq \kappa_q = \ldots = \kappa_{q+\ell-1}$ (κ_q has the multiplicity ℓ) then

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}. \tag{3.6}$$

For the proof see [4], Section 2.

We will show in Theorem 4.1 the existence of a bifurcation point to (2.10). The method of the proof of this fact will be the same as in [16]. One of the assumptions in [16] was $\operatorname{int} K \neq \emptyset$. Here, we have $\mathfrak{n} > 1$ and therefore $\operatorname{int} K = \emptyset$ in general. We consider an auxiliary problem with an additional parameter δ (see below) which has the property $\operatorname{int} K^{\delta} \neq \emptyset$ and which approximates our original problem for $\delta \to 0$.

Notation 4.1. Let $\delta > 0$ be fixed. Let G be a bounded domain in $\mathbb{R}^{\mathfrak{n}}$ with a Lipschitz boundary such that $\operatorname{cl}\Omega \subset G$. We define a mollification mapping $\Phi^{\delta}\colon \mathbb{V} \to W^{1,2}(G)$ in the following way: Let $\varphi^{\delta}\colon \mathbb{R}^{\mathfrak{n}} \to [0,+\infty)$ be a C^{∞}-smooth function such that $\varphi^{\delta}(0) > 0$, $\varphi^{\delta}(x) \leqslant \varphi^{\delta}(0)$ for any $x \in \mathbb{R}^{\mathfrak{n}}$, $\varphi^{\delta}(x) = 0$ for all $x \notin \mathcal{B}_{\delta}(0)$ (the ball with a radius δ centered at the origin) and $\int_{\mathbb{R}^{\mathfrak{n}}} \varphi^{\delta}(x)\,\mathrm{d}x = 1$. Then φ^{δ} is bounded on $\mathbb{R}^{\mathfrak{n}}$ and φ^{δ} converges in the sense of distributions to the Dirac measure centered at the origin for $\delta \to 0_+$. For an example of such a function see [23]. There exists a continuous "extension" mapping $E\colon W^{1,2}(\Omega) \to W_0^{1,2}(G)$ (see [23]). Let us define a mapping

$$\Phi^{\delta}(v,x) := \int_G \varphi^{\delta}(x-y)Ev(y)\ \mathrm{d}y \qquad \text{for any } v \in \mathbb{V},\ x \in G.$$

Hence, $\Phi^{\delta}(v,\cdot)$ is a continuous function on $\operatorname{cl}\Omega$ and it is easy to see that if $v_n, v \in \mathbb{V}$, $v_n \to v$ in $\mathbb{V}$ then $\Phi^{\delta}(v_n,\cdot) \to \Phi^{\delta}(v,\cdot)$ in $C^0(\operatorname{cl}\Omega)$. Further, define M^{δ}, M_0^{δ}, K^{δ} by $M^{\delta}(U) = [\{0\}, M_2^{\delta}(v)]$, $M_0^{\delta}(U) = [\{0\}, M_{02}^{\delta}(v)]$, $K^{\delta} = \mathbb{V} \times K_2^{\delta}$ with

$$\begin{aligned} M_2^{\delta}(v) := \Big\{ z \in \mathbb{V};\ &\int_{\Gamma_U} \underline{m}(\Phi^{\delta}(v,x))[\Phi^{\delta}(\varphi,x)]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}(\Phi^{\delta}(v,x))[\Phi^{\delta}(\varphi,x)]^- \,\mathrm{d}\Gamma \\ &\leqslant \langle z,\varphi\rangle \leqslant \int_{\Gamma_U} \overline{m}(\Phi^{\delta}(v,x))[\Phi^{\delta}(\varphi,x)]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \underline{m}(\Phi^{\delta}(v,x))[\Phi^{\delta}(\varphi,x)]^- \,\mathrm{d}\Gamma \\ &\hspace{20em} \text{for all } \varphi \in \mathbb{V} \Big\}; \end{aligned}$$

$$M_{02}^{\delta}(v) := \big\{ z \in \mathbb{V};\ \langle z,v\rangle = 0,\ \langle z,\varphi\rangle \leqslant 0 \text{ for all } \varphi \in \mathbb{V}, \Phi^{\delta}(\varphi,\cdot) \geqslant 0 \text{ on } \Gamma_U \big\} \quad \text{if } \Phi^{\delta}(v,\cdot) \geqslant 0 \text{ on } \Gamma_U;$$

$$M_{02}^{\delta}(v) := \emptyset \ \text{ if } \Phi^{\delta}(v,x_0) < 0 \text{ for some } x_0 \in \Gamma_U;$$

$$K_2^{\delta} := \{\varphi \in \mathbb{V};\ 0 \in M_{02}^{\delta}(\varphi)\}.$$

Here, φ^+, φ^- denote the positive and negative parts of φ, respectively, $\varphi = \varphi^+ - \varphi^-$. Note that we have $K_2^{\delta} = \{\varphi \in \mathbb{V};\ \Phi^{\delta}(\varphi,\cdot) \geqslant 0 \text{ on } \Gamma_U\}$ and $\operatorname{int} K_2^{\delta} \supseteq \{\varphi \in \mathbb{V};\ \Phi^{\delta}(\varphi,\cdot) > 0 \text{ on } \operatorname{cl}\Gamma_U\} \neq \emptyset$ because Φ^{δ} is $(\mathbb{V} \to C^0(\operatorname{cl}\Omega))$-continuous and the interior of K^{δ} is the preimage of an open set.

O b s e r v a t i o n 4.1. The mappings M^δ and M_0^δ obviously satisfy the following conditions:

(4.1) $0 \in M^\delta(0)$;

(4.2) K^δ is a closedconvex cone with the vertex at the origin, $\{0\} \neq K^\delta \neq \mathbb{V}^2$;

(4.3) if $U \in K^\delta$ then $U^* \in K^\delta$;

(4.4) $M_0^\delta(tV) = tM_0^\delta(V)$ for all $t > 0,\ V \in \mathbb{V}^2$;

(4.5) if $U \in \mathbb{V}^2$ then $\langle Z, U\rangle = 0$ for all $Z \in M_0^\delta(U)$;

(4.6) if $U \in \mathbb{V}^2$ then $\langle Z, \Psi\rangle \geqslant 0$ for all $\Psi \in K^\delta,\ Z \in -M_0^\delta(U)$.

Proposition 4.1. *Let $U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W,\ Z_n \to Z$ in $\mathbb{V}^2$ and $d_n \to d$ in $\mathbb{R}^2_+$ such that $D(d_n)W_n + Z_n \in -\frac{M^\delta(U_n)}{\|U_n\|}$. Then $W_n \to W,\ D(d)W + Z \in -M_0^\delta(W)$.*

The proof is given in [9].

There exists a system of continuous functions $p_\tau\colon \mathbb{R} \to \mathbb{R}$ with a real parameter $\tau \in [0, +\infty)$ such that

$$p_0 \equiv 0,\ p_\tau(\xi) = 0 \text{ for } \xi \geqslant 0,\ p_\tau(\xi) \in (m(\xi), 0] \text{ for } \xi < 0 \tag{4.7}$$

satisfying the following conditions:

$$\begin{aligned}
&\text{if } \tau_n \to \tau \in [0, +\infty),\ \xi_n \to \xi \ \text{ then } p_{\tau_n}(\xi_n) \to p_\tau(\xi);\\
&\text{if } \tau_n \to \tau \in (0, +\infty),\ \xi_n \to 0_- \ \text{ then } \tilde{p}_\tau := \liminf_{n\to+\infty} \frac{p_{\tau_n}(\xi_n)}{\xi_n} > 0;\\
&\text{if } \tau_n \to 0_+,\ \xi_n \to 0_- \ \text{ then } \frac{p_{\tau_n}(\xi_n)}{\xi_n} \to 0,\ \liminf_{n\to+\infty} \frac{p_{\tau_n}(\xi_n)}{\tau_n\xi_n} > 0;\\
&\text{if } \tau_n \to +\infty,\ \xi_n \to \xi,\ p_{\tau_n}(\xi_n) \to p\\
&\qquad\qquad \text{then } p \in m(\xi) \text{ or } p = m(\xi) \text{ for } \ \xi = 0 \text{ or } \xi \neq 0, \text{ respectively.}
\end{aligned} \tag{4.8}$$

Let us define for any $\tau \in [0, +\infty)$ a function $p_{0,\tau}\colon \mathbb{R} \to \mathbb{R}$ such that $p_{0,\tau}(\xi) = 0$ for all $\xi \geqslant 0$ and $p_{0,\tau}(\xi) = \tilde{p}_\tau \cdot \xi$ for all $\xi < 0$. Moreover, a system of operators $P_\tau^\delta, P_{0,\tau}^\delta\colon \mathbb{V}^2 \to \mathbb{V}^2$ with a parameter $\tau \in [0, +\infty)$ is defined by $P_\tau^\delta(U) = [0, P_{\tau,2}^\delta(v)]$, $P_{0,\tau}^\delta(U) = [0, P_{0,\tau,2}^\delta(v)]$ for $U = [u, v]$, where

$$\left.\begin{aligned}
\langle P_{\tau,2}^\delta(v), \psi\rangle &= \int_{\Gamma_U} p_\tau(\Phi^\delta(v, x))\Phi^\delta(\psi, x)\ \mathrm{d}\Gamma\\
\langle P_{0,\tau,2}^\delta(v), \psi\rangle &= \int_{\Gamma_U} p_{0,\tau}(\Phi^\delta(v, x))\Phi^\delta(\psi, x)\ \mathrm{d}\Gamma
\end{aligned}\right\} \text{ for all } v,\ \psi \in \mathbb{V}.$$

O b s e r v a t i o n 4.2. For such a system of operators and a fixed $\delta \in (0, \delta_0)$ the following conditions are clearly fulfilled:

$$\begin{cases} P^\delta_\tau(U) = 0 \quad \text{for all } U \in K^\delta, \\ \langle P^\delta_\tau(U), V\rangle \leqslant 0 \quad \text{for all } U \in \mathbb{V}\times\mathbb{V}, V \in K^\delta,\ \tau \in [0, +\infty); \end{cases} \tag{4.9}$$

$$\langle P^\delta_\tau(U), U\rangle \geqslant 0,\ \langle P^\delta_{0,\tau}(U), U\rangle \geqslant 0 \quad \text{for all } U \in \mathbb{V}\times\mathbb{V},\ \tau \in [0, +\infty). \tag{4.10}$$

The proofs of the following propositions for Model Example will be given in [9].

Proposition 4.2. *Let $U_n \rightharpoonup U$ in $\mathbb{V}^2$, $\tau_n \geqslant 0$, $d_n \to d \in \mathbb{R}^2_+$. Then*

$$\liminf_{n\to+\infty} \langle D^{-1}(d_n) P^\delta_{\tau_n}(U_n), U_n - U\rangle \geqslant 0.$$

If, moreover, $U = 0$, $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|}$ are bounded and $W_n = \frac{U_n}{\|U_n\|} \overset{\mathbb{V}^2}{\rightharpoonup} W$, then

$$\liminf_{n\to+\infty} \Big\langle \frac{D^{-1}(d_n) P^\delta_{\tau_n}(U_n)}{\|U_n\|}, W_n - W \Big\rangle \geqslant 0.$$

Proposition 4.3. *Let $U_n \overset{\mathbb{V}^2}{\rightharpoonup} U$, $\tau_n \to \tau \in [0, +\infty)$. Then $P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} P^\delta_\tau(U)$. For $\tau = +\infty$ and $P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} Z$ this Z belongs to $M^\delta(U)$. For $U = 0$ and $\tau = 0$ the convergence*

$$\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} 0$$

holds. Moreover, if $U = 0$, $W_n = \|U_n\|^{-1} U_n \overset{\mathbb{V}^2}{\rightharpoonup} W$ and $\tau_n \to \tau \in [0, +\infty)$, then $\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} P^\delta_{0,\tau}(W)$. For $\tau = +\infty$ and $\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} Z$ we have $Z \in M^\delta_0(W)$.

Proposition 4.4. *Let $U_n \overset{\mathbb{V}^2}{\to} 0$, $W_n = \|U_n\|^{-1} U_n \overset{\mathbb{V}^2}{\rightharpoonup} W \notin K^\delta, \tau_n \to \tau_0 > 0$ and $V \in \operatorname{int} K^\delta$. Then $\limsup\limits_{n\to+\infty} \|U_n\|^{-1} \big\langle P^\delta_{\tau_n}(U_n), V\big\rangle < 0$. For $\tau_0 = 0$, moreover,*

$$\limsup_{n\to+\infty} (\tau_n \|U_n\|)^{-1} \big\langle P^\delta_{\tau_n}(U_n), V\big\rangle < 0.$$

Proposition 4.5. *Let us assume that $U_n \overset{\mathbb{V}^2}{\rightharpoonup} U$, $Z_n \overset{\mathbb{V}^2}{\to} Z$, $d_n \to d \in \mathbb{R}^2_+$ and $\delta_n \to 0_+$. Then the following implications hold:*

$$D(d_n)U_n + Z_n \in -M^{\delta_n}(U_n) \Longrightarrow U_n \overset{\mathbb{V}^2}{\to} U,\ D(d)U + Z \in -M(U); \tag{4.11}$$

$$D(d_n)U_n + Z_n \in -M^{\delta_n}_0(U_n) \Longrightarrow U_n \overset{\mathbb{V}^2}{\to} U,\ D(d)U + Z \in -M_0(U). \tag{4.12}$$

Let us remark that (4.11) is essential for the proof of Theorem 4.1 and (4.12) is used for the proof of the destabilizing effect ($s_I > s_0$)—see Remark 4.2.

Let δ_0 be from (1.1) and let $d^0 \in C_p$ be a fixed point such that there is an eigenfunction e corresponding to the eigenvalue κ_p of the Laplacian with (1.3) such that

$$e \leqslant -\varepsilon \text{ on a } \delta_0\text{-neighbourhood of } \Gamma_U \text{ in} \operatorname{cl}\Omega \text{ for some } \varepsilon > 0. \tag{4.13}$$

Then the system $\{e_i\}_{i=1}^{\infty}$ can be chosen in such a way that $\kappa_p = \ldots = \kappa_{p+k-1}$, k is the multiplicity of κ_p and (4.13) holds with $e = e_p$. In particular, it follows from Observation 3.3 and the definition of K^δ that

$$-U_0 \in E_B(d^0) \cap \operatorname{int} K^\delta \text{ for any } \delta \in (0, \delta_0) \tag{4.14}$$

is fulfilled with $U_0 = U_p(d^0)\big(= [\alpha_p(d^0)e_p, e_p]$, see (3.5)$\big)$.

In the sequel we consider a curve σ given by a differentiable mapping $\sigma\colon \mathbb{R} \to \mathbb{R}^2_+$ satisfying

$$\begin{cases} \sigma(s) \in D_S \text{ for all } s \in (s_0, +\infty), \\ \sigma \text{ intersects the envelope } C \text{ at the point } \sigma(s_0) = d^0, \\ \sigma \text{ intersects the line } d_1 = \frac{b_{11}}{\kappa_1} \text{ at a point } \sigma(\tilde{s}),\ \tilde{s} > s_0, \\ \sigma_1(s) < \frac{b_{11}}{\kappa_1} \text{ for all } s \in (s_0, \tilde{s}), \\ \sigma_1(s) > \frac{b_{11}}{\kappa_1} \text{ for } s \in (\tilde{s}, \tilde{s} + \zeta_0) \text{ with some } \zeta_0 > 0. \end{cases} \tag{4.15}$$

It is essential that if $d^0 \in C \cap C_p$ and (4.14) holds with $U_0 = U_p(d^0)$ then

$$\text{the curve } \sigma \text{ is transversal to } C_p \text{ at } d^0. \tag{4.16}$$

Note that if, moreover, $d^0 \in C_p \cap C_q$, $C_p \neq C_q$ then σ has to be transversal to C_p but not necessarily to both C_p and C_q.

Remark 4.1. By introducing the curve $\sigma(s)$ we have changed the two-parametric system (2.10) with $[d_1, d_2] \in \mathbb{R}^2_+$ to the system

$$D(\sigma(s))U - BAU - N(U) \in -M(U) \tag{4.17}$$

with a single parameter $s \in \mathbb{R}$. Further, by a *critical point* of

$$D(\sigma(s))U - BAU = 0 \tag{4.18}$$

or (2.11) written with d_1, d_2 replaced by $\sigma_1(s), \sigma_2(s)$ we mean a parameter s_1 such that $E_B(\sigma(s_1)) \neq \{0\}$ or $E_I(\sigma(s_1)) \neq \{0\}$, respectively, and by a *bifurcation point* of (4.17) we mean a parameter $s_2 \in \mathbb{R}$ such that for any neighbourhood of $[s_2, 0, 0] \in \mathbb{R} \times \mathbb{V}^2$ there exists $[s, U] = [s, u, v]$, $\|U\| \neq 0$ satisfying (4.17). Therefore, by the assumption (4.15) s_0 is the largest critical point of (4.18), because a nontrivial solution of (4.18) exists only for $\sigma(s) \in C_j$ for some $j = 1, 2, \ldots$—see Observation 3.3.

Theorem 4.1. *Let* (SIGN), (1.1), (2.2), (2.4) *and* (2.5) *hold, let* $\sigma(s)$ *be a differentiable curve satisfying* (4.15), *let* $d^0 \in C_p$ *and* (4.16) *hold. Let* (4.14) *hold with* $U_0 = U_p(d^0)$ $\big(= [\alpha_p(d^0)e_p, e_p]$, *see* (3.5)$\big)$. *Consider a multivalued mapping* M *such that there exists a system of multivalued mappings* M^δ *and the corresponding homogeneous multivalued mappings* M_0 *and* M_0^δ, *the operators* P_τ^δ, $P_{0,\tau}^\delta$ $(\tau \in [0, +\infty)$, $\delta \in (0, \delta_0))$ *satisfying the assumptions* (4.1)–(4.6), (4.9), (4.10) *and for which Propositions* 4.1–4.4 *and* (4.11) *in Proposition* 4.5 *remain valid. Then there exists a bifurcation point* $s_I \in [s_0, \tilde{s}]$ *of the inclusion* (4.17). *Hence, there is* $\varrho_0 > 0$ *such that for any* $\varrho \in (0, \varrho_0)$ *there are* s_ϱ, U_ϱ *satisfying* (4.17), $\|U_\varrho\|^2 = \varrho$, $s_\varrho \in [s_0, \tilde{s}]$ *and such that if* $\varrho_n \to 0_+$, $s_{\varrho_n} \to s_I$ *then* $s_I \in [s_0, \tilde{s}]$.

Proof will be given in Section 7. For $\mathfrak{n} = 1$, cf. [16], Theorem 2.10.

R e m a r k 4.2. If, moreover, either $\operatorname{int} K \neq \emptyset$ or (2.11) is equivalent to (2.13) (this assumption is satisfied in many reasonable situations) and the conditions

$$\text{if } U \in K \text{ then } U^* \in K, \tag{4.19}$$

$$\text{if } U \in \mathbb{V}^2 \text{ then } \langle Z, \Psi \rangle \geqslant 0 \text{ for all } \Psi \in K,\ Z \in -M_0(U) \tag{4.20}$$

and (4.12) hold then we can prove $s_I > s_0$, which implies that s_ϱ, U_ϱ from Theorem 4.1 do not satisfy

$$D(\sigma(s))U - BAU - N(U) = 0 \tag{4.21}$$

—see the proof of destabilizing effect in Appendix.

R e m a r k 4.3. There are two main improvements in comparison to [16], Theorem 2.10. First, the localization of the bifurcation point is specified—we show that $s_I < \tilde{s}$, $\tilde{s}$ is from (4.15), i.e. $d_I = \sigma(s_I) \notin Z_0$ in the sense of [7], i.e. $d_1^I = \sigma_1(s_I) \leqslant \frac{b_{11}}{\kappa_1}$. Second, in [16] the case $\mathfrak{n} = 1$, $\dim E_B(d^0) = 1$ and $\operatorname{int} K \neq \emptyset$ was considered. Here, $\mathfrak{n} > 1$ is admitted and therefore the possible difficulties $\dim E_B(d^0) > 1$ and $\operatorname{int} K = \emptyset$ must be overcome. To get over the former one the operator L_δ is involved (see Notation 5.2), to get over the latter, the approximate problem (5.13)—see below—is considered. Notice that for δ fixed, the existence of a bifurcation point s_I^δ for this

δ-problem can be shown in the same way as in [16], cf. Remark 8.1 in Appendix for another technique overcoming the emptiness of $\operatorname{int} K$ by using the notion of pseudointerior.

Corollary 4.1. *Let* (SIGN) *and* (1.1) *hold, let* $\sigma(s)$ *be a differentiable curve satisfying* (4.15)*, let* $d^0 \in C_p$ *and* (4.16) *hold. Let* m *be the multivalued function from Model Example and let us assume that there exists an eigenfunction* e_p *corresponding to an eigenvalue* κ_p *of the Laplacian with* (1.3) *such that* (4.13) *is fulfilled with* $e = e_p$*. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (SRD), (1.2) *bifurcate at some* $s_I \in (s_0, \tilde{s}]$.

This follows from Theorem 4.1, Propositions 4.1–4.5, Remark 4.2 and the fact that no nontrivial constant functions can satisfy (1.3).

5. Reduction of dimension of the space $E_B(d^0)$

In this section we will keep the assumptions of Theorem 4.1. The following proposition holds (cf. [16], Remark 4.5):

Proposition 5.1. *Let* σ *satisfy* (4.15) *and* (4.16)*. Then*

$$\frac{(\kappa_p \sigma_2(s_0) - b_{22})^2}{b_{12} b_{21}} \sigma_1'(s_0) + \sigma_2'(s_0) < 0.$$

For the proof see Appendix.

Observation 5.1. Similarly as in [6], Section 4 we will consider an eigenvalue problem

$$(D^{-1}(d)BA - I)U = \mu U. \tag{5.1}$$

We will study the behaviour of eigenvalues of (5.1) with respect to the changing d along the curve $\sigma(s)$. The process will be the same as in [6]. Therefore, the detailed calculations are explained in Appendix and here only the main steps are sketched.

All eigenvalues of (5.1) are the roots of

$$\mu^2 d_1 d_2 \kappa_i^2 - \beta_i(d) \kappa_i \mu + \gamma_i(d) = 0, \tag{5.2}$$

i.e. the numbers

$$\mu_i^{(r)}(d) := \frac{\beta_i(d) \pm \sqrt{\omega(d)}}{2 d_1 d_2 \kappa_i}, \ r = 1, 2. \tag{5.3}$$

Here, $\beta_i(d) := d_1 b_{22} + d_2 b_{11} - 2d_1 d_2 \kappa_i$, $\gamma_i(d) := (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21}$, $\omega(d) := d_1^2 b_{22}^2 + d_2^2 b_{11}^2 - 2d_1 d_2 b_{11} b_{22} + 4d_1 d_2 b_{12} b_{21}$, $i = 1, 2, \ldots$. The set $\{d \in \mathbb{R}^2_+\,;\ \omega(d) = 0\}$ is a couple of half-lines, one of them is a common tangent $\mathcal{T}$ to all hyperbolas C_j, $j = 1, 2, \ldots$ (see also [20] and Figures 1 and 2). The calculations of the crucial signs of the eigenvalues $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ from (5.3) in the domains $\mathcal{D}_1, \ldots, \mathcal{D}_6$, are described in Appendix. They lead to the conclusion that for d lying to the left from C_i, there is one positive root of (5.2) and for d lying to the right from C_i, either none or both roots of (5.2) are positive.

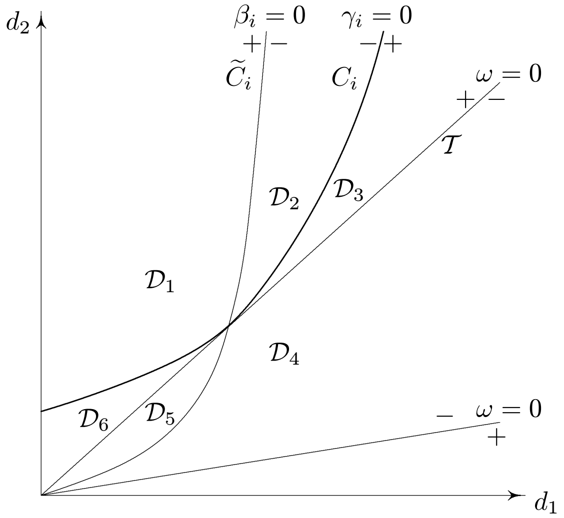

Fig. 2

N o t a t i o n 5.1. (Cf. [8], Notation 4.1.) The vectors

$$U_i^{(r)}(d) = \Big[\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d) d_2 \kappa_i}{b_{21}} e_i, e_i\Big], \ i \in \mathbb{N}, \ r = 1, 2 \tag{5.4}$$

are the eigenvectors of (5.1) corresponding to $\mu_i^{(r)}(d)$.

Let $\eta > 0$ be a small number. Let $d^0 \in C_p = \ldots = C_{p+k-1}$, $d^0 \notin \mathcal{T}$. Then the curve $\sigma(s)$ for $s \in (s_0 - \eta, s_0 + \eta)$ goes either from $\mathcal{D}_2$ into $\mathcal{D}_3$ or from $\mathcal{D}_1$ into $\mathcal{D}_6$ for η small—see Fig. 2. By $\mu_p(s)$ for $s \in (s_0 - \eta, s_0 + \eta)$ we denote the root of (5.2) changing the sign at d^0, i.e.

$$\begin{aligned} \mu_p(s) &= \mu_p^{(1)}(\sigma(s)) \quad \text{if } C_p \cap \mathcal{T} \preceq d^0 \\ &= \mu_p^{(2)}(\sigma(s)) \quad \text{if } d^0 \preceq C_p \cap \mathcal{T} \end{aligned} \tag{5.5}$$

(see Appendix for details).

Let $d^0 \in C_p = \ldots = C_{p+k-1}$, $d^0 \in \mathcal{T}$. Then the curve $\sigma(s)$ goes from the domains $(\mathcal{D}_1 \cup \mathcal{D}_2)$ into $(\mathcal{D}_4 \cup \mathcal{D}_5)$. By $\mu_p(s)$ we denote the positive root of (5.2) on $(s_0-\eta, s_0)$ (i.e. $\mu_p(s) = \mu_p^{(1)}(\sigma(s))$) and for $[s_0, s_0+\eta)$ we put $\mu_p(s) = \operatorname{Re}\mu_p^{(r)}(\sigma(s))$, $r = 1, 2$.

Let us denote by

$$U_i(s) = \Big[\frac{\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i}{b_{21}} e_i, e_i\Big],\ i = p, \ldots, p+k-1 \tag{5.6}$$

the corresponding eigenvectors if $d^0 \notin \mathcal{T}$ or $d^0 \in \mathcal{T}$ and $s \in (s_0-\eta, s_0]$, or their real parts in the case $d^0 \in \mathcal{T}$ and $s \in (s_0, s_0+\eta)$.

O b s e r v a t i o n 5.2. (Cf. [8], Observation 4.2.) Let $\mu_q(s) \neq \mu_p(s)$ for all q satisfying $\kappa_q \neq \kappa_p$. Then

$$\operatorname{Ker}(D^{-1}(\sigma(s))BA - (1+\mu_p(s))I) = \operatorname{Lin}\{U_i(s)\}_{i=p}^{p+k-1}$$

for all $s \in (s_0-\eta, s_0+\eta)$ in the case $d^0 \notin \mathcal{T}$ and for all $s \in (s_0-\eta, s_0]$ in the case $d^0 \in \mathcal{T}$. In particular, if $d^0 \in C_p$ and $d^0 \notin C_q$ for all $C_q \neq C_p$, then

$$E_B(d^0) = \operatorname{Lin}\{U_i(s_0)\}_{i=p}^{p+k-1}. \tag{5.7}$$

If $\mu_q(s) = \mu_p(s)$ for some q satisfying $\kappa_p \neq \kappa_q = \ldots = \kappa_{q+\ell-1}$, where κ_q has the multiplicity ℓ, then $\operatorname{Ker}(D^{-1}(\sigma(s))BA - (1+\mu_p(s))I) = \operatorname{Lin}\{U_i(s)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}$ for all $s \in (s_0-\eta, s_0+\eta)$. In particular, if $d \in C_p \cap C_q$ for some $C_q \neq C_p$, then

$$E_B(d^0) = \operatorname{Lin}\{U_i(s_0)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}. \tag{5.8}$$

N o t a t i o n 5.2. (Cf. [8], Notation 4.2.) Set $I(d^0) = \{i \in \mathbb{N} \setminus \{p\};\ d^0 \in C_i\}$. Set $I_p(d^0) = \{i \in I(d^0);\ C_i = C_p\}$ and $I_q(d^0) = I(d^0) \setminus I_p(d^0)$. Choose $\eta > 0$ such that $\mu_p(s)$ is well defined for any $s \in (s_0-\eta, s_0+\eta)$. Moreover, for $i \in I(d^0)$ set

$$\begin{aligned} &\nu_i(d^0) = 1 \quad \text{if } \mu_i(s_0) = \mu_i^{(1)}(d^0) \text{ or } \mu_i^{(1)}(d^0) = 0,\\ &\nu_i(d^0) = -1 \text{ if } \mu_i(s_0) = \mu_i^{(2)}(d^0) \text{ and } \mu_i^{(1)}(d^0) \neq 0, \end{aligned}$$

introduce a continuous cut-off function χ with a support in $(s_0-\eta, s_0+\eta)$ such that $\chi(s_0) = 1$, $\chi(\mathbb{R}) \subset [0,1]$ and for any $\delta > 0$, the linear completely continuous operator $L_\delta(s)$ in $\mathbb{V}^2$ (for any s fixed) by

$$L_\delta(s)\colon\ U \mapsto \delta\chi(s) \cdot \sum_{i \in I(d^0)} \nu_i(d^0) \frac{\langle U_i(s), U\rangle}{\|U_i(s)\|^2} \cdot U_i(s). \tag{5.9}$$

Let us notice that in [4] and [6] a simpler definition of L was taken without a sign term. Here we need also a proper sign in (5.11) below for the proof of the fact that $s_I > s_0$ in Theorem 4.1. Hence, the proof of Lemma 5.1 below is slightly more complicated but its assertion is the same as that in [6].

O b s e r v a t i o n 5.3. (Cf. [8], Remark 4.2.) (5.9) yields that $L_\delta(s) \equiv 0$ for $s \in \mathbb{R}$ if $I(d^0) = \emptyset$, i.e. if $\dim E_B(d^0) = 1$. From (5.4), (5.6), the form of U^* and the fact that $\langle e_i, e_j \rangle = 0$ for $i \neq j$ we deduce that for any $s \in (s_0 - \eta, s_0 + \eta)$ the identities $\langle U_i^{(r)}(\sigma(s)), U_j^{(r)}(\sigma(s)) \rangle = \langle U_i(s), U_j(s) \rangle = \langle U_i^*(s), U_j(s) \rangle = 0$ hold for all $j \neq i$, $r = 1, 2$ and

$$\begin{aligned}
&L_\delta(s)U_p(s) = L_\delta(s)U_p^*(s) = 0,\ L_\delta(s)U_i^{(r)}(\sigma(s)) = 0 \text{ for } i \notin I(d^0),\ r = 1, 2,\\
&L_\delta(s)U_i(s) = \delta\chi(s)U_i(s) \text{ for } i \in I(d^0), \quad \text{if } d^0 \text{ lies above or in } C_i \cap \mathcal{T},\\
&L_\delta(s)U_i(s) = -\delta\chi(s)U_i(s) \text{ for } i \in I(d^0), \quad \text{if } d^0 \text{ lies below } C_i \cap \mathcal{T}.
\end{aligned} \tag{5.10}$$

Moreover, we have for $\sigma(s)$ lying in the neighbourhood of C_p, $\sigma(s) \notin \mathcal{T}$ that

$$\begin{aligned}
&\langle D(\sigma(s))L_\delta(s)U, U_p(s) \rangle = \langle D(\sigma(s))L_\delta(s)U, U_p^*(s) \rangle = 0 \text{ for any } U \in \mathbb{V}^2,\\
&\langle D(\sigma(s))L_\delta(s)U_i(s), U_i^*(s) \rangle < 0 \quad \text{for } i \in I(d^0).
\end{aligned} \tag{5.11}$$

See Appendix for the proof of the last assertion. Note that if $d^0 \in C_p \cap C_q$, $C_p \neq C_q$ and $p < q$, then $d^0 \notin \mathcal{T}$ and d^0 lies below $C_p \cap \mathcal{T}$ and above $C_q \cap \mathcal{T}$.

Lemma 5.1. (Cf. [6], Lemma 4.1.) *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there is $\eta > 0$ such that the following assertions hold.*

(a) *Let $d^0 \in C_p \setminus \mathcal{T}$. Then for all $s \in (s_0 - \eta, s_0 + \eta)$, the eigenvalue $\mu_p(s)$ from (5.5) is simultaneously an algebraically simple eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the corresponding eigenvector $U_p(s)$. It changes the sign as s crosses s_0. The other eigenvalues have constant signs and constant multiplicities on $(s_0 - \eta, s_0 + \eta)$.*

(b) *Let $d^0 \in C_p \cap \mathcal{T}$. Then for $s \in (s_0 - \eta, s_0]$, $\mu_p(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the only normed eigenvectors $\pm \frac{U_p(s)}{\|U_p(s)\|}$. For $s \in (s_0 - \eta, s_0)$, $\mu_p(s)$ is positive and algebraically simple, $\mu_p(s_0) = 0$ is not algebraically simple. The sum of algebraic multiplicities of the other positive eigenvalues of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ is even for all $s \in (s_0 - \eta, s_0)$. For $s \in (s_0, s_0 + \eta)$, all eigenvalues of this operator are complex.*

In both cases (a), (b), $\operatorname{Ker}(D^{-1}(\sigma(s_0))BA - L_\delta(s_0) - I) = \operatorname{Lin}\{U_p(s_0)\}$ and the number $\Theta(s_0 - \varepsilon) - \Theta(s_0 + \varepsilon)$ is odd for all $\varepsilon \in (0, \eta)$ where $\Theta(s)$ is the sum of algebraic multiplicities of all positive eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$.

The proof, similar to that of [6], Lemma 4.1, is given in Appendix.

Let the parameter $\delta > 0$ be admissible for Lemma 5.1. For $\eta > 0$ as in Lemma 5.1 and such that

$$s_0 + \eta < \tilde{s} \tag{5.12}$$

where $\tilde{s}$ is from the assumption (4.15), we arrive at the following inclusions:

$$D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U \in -M^\delta(U), \tag{5.13}$$

$$D(\sigma(s))U - BAU + D(\sigma(s))L_\delta(s)U \in -M_0^\delta(U) \tag{5.14}$$

and the corresponding linear equation

$$D(\sigma(s))U - BAU + D(\sigma(s))L_\delta(s)U = 0, \tag{5.15}$$

which is the aim of this section.

6. Properties of solutions to the penalty equation

We will consider the system of penalty equations

$$D(\sigma(s))U - BAU - \frac{\tau}{1+\tau}N(U) + D(\sigma(s))L_\delta(s)U + P_\tau^\delta(U) = 0 \tag{6.1}$$

with the norm condition

$$\|U\|^2 = \frac{\varrho\tau}{1+\tau}. \tag{6.2}$$

Throughout this section $\delta > 0$ is a fixed parameter admissible for Lemma 5.1, hence we can use $\operatorname{int} K^\delta \neq \emptyset$. Moreover, $\varrho > 0$ is fixed and $\tau \in [0, +\infty)$ is a penalty parameter. The penalty equation (6.1) is a linear equation (5.15) for $\tau = 0$ while for $\tau \to +\infty$ we get the inclusion (5.13) (for the proof see Lemma 6.2).

Lemma 6.1. *If $[s_n, U_n, \tau_n] \in \mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}^+$, $s_n \to s$, $U_n \rightharpoonup U$, $\tau_n \to \tau \in [0, +\infty]$,*

$$D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n}N(U_n) + D(\sigma(s_n))L_\delta(s_n)U_n + P_{\tau_n}^\delta(U_n) = 0 \tag{6.3}$$

then $U_n \to U$. If, moreover, $\|U\| = 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$ then $W_n \to W$.

Since the operator $L_\delta(s)$ is completely continuous, the proof is identical to that of [16], Remark 3.1.

Lemma 6.2. *(Cf. [16], Lemma 3.2.) Let $[s_n, U_n, \tau_n] \in \mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}^+$, $s_n \to s$, $U_n \to U$, $\tau_n \to +\infty$ and let (6.3) hold. Then*

$$D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U \in -M^\delta(U).$$

Proof. From the continuity of L_δ, Proposition 4.3 and (6.3) it follows that

$$\begin{aligned} -Z_n := -P_{\tau_n}^\delta(U_n) &= D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n}N(U_n) + D(\sigma(s_n))L_\delta(s_n)U_n \\ &\to D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U = -Z \in -M^\delta(U). \end{aligned}$$

□

Lemma 6.3. *(Cf.* [17], *Lemma* 1.1.*) Any bifurcation point $s \in \mathbb{R}$ of* (5.13) *is simultaneously a critical point of* (5.14).

Proof. If s is a bifurcation point of (5.13) then there exist $s_n \to s$ and a sequence $\{U_n\}$ such that $\|U_n\| \to 0$, $\|U_n\| \neq 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$ and

$$D(\sigma(s_n))W_n - BAW_n - \frac{N(U_n)}{\|U_n\|} + D(\sigma(s_n))L_\delta(s_n)W_n \in -\frac{M^\delta(U_n)}{\|U_n\|}. \tag{6.4}$$

Using the compactness of A and L_δ, the assumption (2.5) and Proposition 4.1 we obtain $W_n \to W$ and

$$D(\sigma(s))W - BAW + D(\sigma(s))L_\delta(s)W \in -M_0^\delta(W). \tag{6.5}$$

□

Lemma 6.4. *If* $\operatorname{Ker}(D(\sigma(s_0)) - B^*A + L_\delta(s_0)) \cap \operatorname{int} K^\delta \neq \emptyset$ *then* $\{U \in \mathbb{V}^2;$ $D(\sigma(s_0))U - BAU + L_\delta(s_0)U \in -M_0^\delta(U)\} = \operatorname{Ker}(D(\sigma(s_0)) - BA + L_\delta(s_0)) \cap K^\delta$.

The proof is identical to the proof of [16], Lemma 3.3, if we put $U_0 = -U_p(s_0)$.

Lemma 6.5. *If* $\sigma_1(s) > \frac{b_{11}}{\kappa_1}$ *(i.e.* $\sigma(s) \in Z_0$ *in the notation of* [7]*) then the only solution of* (5.14) *is trivial. (The line* $d_1 = \frac{b_{11}}{\kappa_1}$ *is the asymptote to* C_1*—see Fig.* 1.*)*

Proof is done in a similar way as in [7], proof of Theorem 2.1. Note that the condition (M0) in the notation of [7] holds for any $\delta > 0$ small enough due to the assumption (4.5). Moreover, it follows from (5.12) that $L_\delta(s) \equiv 0$ for $s > \tilde{s}$.

Lemma 6.6. *If* $d = [d_1, d_2] \in \mathbb{R}^2_+$, $d_1 > \frac{b_{11}}{\kappa_1}$ *and* $\tau \in [0, +\infty)$ *then the equation*

$$D(d)U - BAU + P^\delta_{0,\tau}(U) = 0$$

has only the trivial solution.

The proof is identical to that of Lemma 3.4 in [7].

Lemma 6.7. *Let* ζ_0 *be from the assumption* (4.15). *For any* $\zeta \in (0, \zeta_0)$ *there exists* $\varrho_0 > 0$ *such that there is no nontrivial solution* U *of* (6.1) *with* $s = \tilde{s} + \zeta$, $\tau \in [0, +\infty)$ *and* $\|U\|^2 < \varrho_0$.

Proof follows from Lemmas 6.5 and 6.6 and can be done in the same way as that of [8], Lemma 3.9.

Lemma 6.8. *If* $[s_n, U_n, \tau_n] \to [s_0, 0, 0]$, $W_n = \frac{U_n}{\|U_n\|} \to \frac{U_p}{\|U_p\|}$ *and* (6.3) *holds then*

$$\liminf_{n\to+\infty} \frac{s_n - s_0}{\tau_n} > 0.$$

Proof is done in the same way as that of [16], Lemma 3.6 if we put $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$.

O b s e r v a t i o n 6.1. (Cf. [16], Remark 3.8.) The assumption (4.9) implies: If $[U_n, \tau_n] \in \mathbb{V}^2 \times \mathbb{R}^+$ and $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|} \rightharpoonup F$ then

$$\langle F, W\rangle = \lim_{n\to+\infty} \frac{\langle P^\delta_{\tau_n}(U_n), W\rangle}{\|U_n\|} \leqslant 0 \text{ for any } W \in K^\delta. \tag{6.6}$$

Moreover, let $F \neq 0$ and $V \in \mathbb{V}^2$, $W \in \operatorname{int} K^\delta$ be such that $\langle F, V\rangle > 0$, $\langle F, W\rangle = 0$. Then $\langle F, W + tV\rangle > 0$ for $t > 0$ and simultaneously $W + tV \in K^\delta$ for $t > 0$ small enough. Therefore $\langle F, W\rangle < 0$ for all $W \in \operatorname{int} K^\delta$ and any $F \neq 0$ satisfying (6.6).

Lemma 6.9. *There exists* $\varrho_0 > 0$ *such that if* $\varrho \in (0, \varrho_0)$, s_n, U_n, τ_n *satisfy* (6.1), (6.2), $U_n \notin K^\delta$, $[s_n, U_n, \tau_n] \to [s_0, U, \tau]$, $W_n = \frac{U_n}{\|U_n\|} \to W$, $s_n \geqslant s_0$ *and* $\tau \in [0, +\infty]$ *then* $W \notin K^\delta$.

Proof is similar to that of Lemma 3.7 in [16]. For the sake of completeness, it can be found in Appendix.

Lemma 6.10. *There exists* $\varrho_0 > 0$ *such that if* s, U, τ *satisfy* (6.1), $U \notin K^\delta$, $\|U\| < \varrho_0$ *then* $s \neq s_0$.

Proof can be done in the same way as the proof of [16], Lemma 3.9 where we take $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$ again, which is the only normed solution to (5.15) for $s = s_0$ belonging to K^δ.

Lemma 6.11. *There exists* $\varrho_0 > 0$ *such that if* s, U, τ *satisfy* (6.1), $s > s_0$, $0 \neq \|U\| < \varrho_0$ *then* $U \notin \partial K^\delta$.

The proof is identical to that of [16], Lemma 3.10. Again, we take $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$ and use the fact that it is the only normed solution to (5.15) for $s = s_0$ belonging to K^δ.

Let $\delta > 0$ be fixed and such that Lemma 5.1 is satisfied. We rewrite the system (6.1) into the form

$$U - T(s)U + H_\tau(s, U) = 0, \tag{7.1}$$

where

$$\begin{aligned} T(s)U &= D^{-1}(\sigma(s))BAU - \delta L_\delta(s)U, \\ H_\tau(s, U) &= D^{-1}(\sigma(s))\Big[-\frac{\tau}{1+\tau}N(U) + P_\tau^\delta(U)\Big]. \end{aligned} \tag{7.2}$$

If we define $P_\tau^\delta(U) = P_{-\tau}^\delta(U)$ for $\tau < 0$ then

$$\begin{cases} \text{for any } s \in \mathbb{R},\ T(s)\colon\ \mathbb{V}^2 \to \mathbb{V}^2 \text{ is linear completely continuous,} \\ \text{the mapping } s \mapsto T(s) \text{ of } \mathbb{R} \text{ into the space of linear continuous} \\ \text{mappings in } \mathbb{V}^2 \text{ (equipped with the operator norm) is continuous,} \\ \text{the mapping } Q\colon\ \mathbb{R} \times \mathbb{V}^2 \times \mathbb{R} \to \mathbb{V}^2 \text{ defined by} \\ Q(s, U, \tau) = T(s)U - H_\tau(s, U) \text{ is completely continuous;} \end{cases} \tag{7.3}$$

$$\begin{cases} \lim\limits_{\|U\|+|\tau|\to 0} \frac{\|H_\tau(s,U)\|}{\|U\|+|\tau|} = 0 \\ \text{uniformly with respect to } s \in [s_0 - \gamma, s_0 + \gamma],\ \gamma \in (0, +\infty) \end{cases} \tag{7.4}$$

are satisfied under the assumptions from Sections 1 and 4.

The proof of Theorem 4.1 is based on the following theorem (where by a *critical point of* T we mean the parameter $s \in \mathbb{R}$ such that there exists a nontrivial solution of $U - T(s)U = 0$ and by $\Theta_T(s)$ we denote the sum of algebraic multiplicities of all positive eigenvalues of the operator $T(s) - I$):

Theorem 7.1. *Let $\mathcal{K} \neq \mathbb{V}^2$ be a closed convex cone in $\mathbb{V}^2$ with its vertex at the origin and let the mappings T, H satisfy* (7.3) *and* (7.4)*. Assume that s_0 is the greatest critical point of T, s_0 is an isolated critical point of T,* $\operatorname{Ker}(I - T(s_0)) = \operatorname{Lin}\{U_0\}$, $-U_0 \in \operatorname{int}\mathcal{K}$ *and*

$$\Theta_T(s_0 + \xi) - \Theta_T(s_0 - \xi) \text{ is odd for any } \xi \in (0, \xi_0) \tag{7.5}$$

with some $\xi_0 > 0$. *Let the following assumptions hold for any* $\varrho \in (0, \varrho_0)$, $\varrho_0 > 0$ *small*, $[s, U, \tau]$ *and* $[s_n, U_n, \tau_n]$ *satisfying* (7.1), (6.2), $\tau \in [0, +\infty)$*:*

(7.6) *there exists* $C = C(\varrho_0) > 0$ *such that* $s \leqslant C$;

$$\text{(7.7)} \qquad \Big(U_n \notin \mathcal{K},\ \tau_n > 0,\ [s_n, U_n, \tau_n] \to [s_0, 0, 0], \frac{U_n}{\|U_n\|} \to U_0\Big) \Longrightarrow \underset{n_0}{\exists} \ \underset{n \geqslant n_0}{\forall} \ s_n > s_0;$$

$$\text{(7.8)} \qquad \Big(U_n \notin \mathcal{K},\ \tau_n > 0,\ [s_n, U_n, \tau_n] \to [s_0, U, \tau], \frac{U_n}{\|U_n\|} \to W \in \mathcal{K}\Big) \Longrightarrow \underset{n_0}{\exists} \ \underset{n \geqslant n_0}{\forall} \ s_n < s_0;$$

(7.9) *if* $U \notin \mathcal{K}$ *then* $s \neq s_0$;

(7.10) *if* $s > s_0, \|U\| \neq 0$ *then* $U \notin \partial\mathcal{K}$.

Then for any $\varrho \in (0, \varrho_0)$ *there exists a closed connected set* C_ϱ *in* $\mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}$ *containing* $[s_0, 0, 0]$ *such that*

(i) *if* $[s, U, \tau] \in C_\varrho$ *is such that* $[s_0, 0, 0] \neq [s, U, \tau]$ *then* (7.1), (6.2) *are fulfilled,* $s > s_0$, $U \notin \mathcal{K}$;

(ii) *for any* $\tau > 0$ *there exists at least one couple* $[s, U]$ *such that* $[s, U, \tau] \in C_\varrho$.

Proof of this theorem is based on Dancer's global bifurcation theorem ([1], Theorem 2) and on a general continuation theorem proved by Kučera in [14]. The main ideas of the proof are given in [16], proof of Theorem 4.2. Note that the role of the sets C_ϱ^+ and C_ϱ^- is reversed here in comparison with [16].

P r o o f o f T h e o r e m 4.1. We will prove Theorem 4.1 in several steps: In Step 1 we will show for fixed $\delta > 0$ and $\varrho > 0$ small the existence of a solution $\big[s_\varrho^\delta, U_\varrho^\delta\big]$ of (5.13). In Step 2 we obtain by a limiting process $\varrho \to 0_+$ (still with $\delta > 0$ fixed) a bifurcation point $s_I^\delta \in [s_0, \tilde{s} + \zeta_0]$ of (5.13). Finally, we will show in Step 3 the existence of a bifurcation point $s_I \in [s_0, \tilde{s}]$ of (4.17) by a limiting process $\delta \to 0_+$.

S t e p 1. For a fixed $\delta > 0$ we show that the assumptions of Theorem 7.1 are fulfilled with the operators from (7.2), $U_0 = \frac{U_p}{\|U_p\|}$ and with $\mathcal{K} = K^\delta$ from Notation 4.1: It follows from Remark 4.1 and the assumption (4.15) that s_0 from the assumptions of Theorem 4.1 is the greatest critical point of T and Lemma 5.1 gives $\operatorname{Ker}(I - T(s_0)) = \operatorname{Lin}\{U_p\}$, $-U_p \in \operatorname{int} K^\delta$. The assumption (7.5) follows from Lemma 5.1, the assumptions (7.6)–(7.10) follow from Lemmas 6.7–6.11. Hence it follows from Theorem 7.1 that for any $\varrho \in (0, \varrho_0)$ fixed there are $[s_n, U_n, \tau_n]$ satisfying (7.1) and (6.2) (i.e. (6.1) and (6.2)), $U_n \notin K^\delta$, $s_n \to s_\varrho^\delta \geqslant s_0$, $\tau_n \to +\infty$. We can assume $U_n \rightharpoonup U_\varrho^\delta$ and Lemmas 6.1, 6.2 imply that $U_n \to U_\varrho^\delta$ and U_ϱ^δ satisfies

$$\text{(7.11)} \qquad D(\sigma(s_\varrho^\delta))U - BAU - N(U) + D(\sigma(s_\varrho^\delta))L_\delta(s_\varrho^\delta)U \in -M^\delta(U).$$

Moreover, $U^\delta_\varrho \notin \operatorname{int} K^\delta$ and the limiting process in (6.2) implies $\|U^\delta_\varrho\|^2 = \varrho$. Further, Lemma 6.7 gives $s^\delta_\varrho \in [s_0, \tilde{s} + \zeta_0]$.

S t e p 2. We can construct s^δ_ϱ, U^δ_ϱ for any $\varrho \in (0, \varrho_0)$ and obtain by a limiting process $\varrho \to 0_+$ a bifurcation point $s^\delta_I \in [s_0, \tilde{s} + \zeta_0]$ of (5.13). Lemma 6.3 yields that s^δ_I is a critical point of (5.14). If $s^\delta_I = s_0$ for some $\delta > 0$ then Lemma 6.4 would imply $U^\delta_I \in K^\delta$ and

$$D(\sigma(s_0))U^\delta_I - BAU^\delta_I + D(\sigma(s_0))L_\delta(s_0)U^\delta_I = 0.$$

Therefore $U^\delta_I = -\frac{U_p}{\|U_p\|} \in \operatorname{int} K^\delta$ would hold. On the other hand we had $U^\delta_{\varrho_n} \notin K^\delta$ by Theorem 7.1 and the limiting process $\frac{U^\delta_{\varrho_n}}{\|U^\delta_{\varrho_n}\|} \to U^\delta_I$ gives a contradiction. This implies $s^\delta_I > s_0$ for any $\delta > 0$ small.

If $\operatorname{int} K \neq \emptyset$ and $\dim E_B(\sigma(s_0)) = 1$ then the assertion of Theorem 4.1 is proved, because we can take $\Phi^\delta(v) = v$ for any δ and we have $s_I = s^\delta_I \in (s_0, \tilde{s}]$.

S t e p 3. By the limiting process in (7.11) with $\delta_n \to 0_+$, $s^{\delta_n}_\varrho \to s_\varrho$, $U^{\delta_n}_\varrho \rightharpoonup U_\varrho$ (after choosing subsequences) we obtain by using (4.11) that $U^{\delta_n}_\varrho \to U_\varrho$, $\|U_\varrho\|^2 = \varrho$ and $[s_\varrho, U_\varrho]$ satisfies (4.17). This process can be done for any $\varrho \in (0, \varrho_0)$. Using the fact that ζ_0 can be chosen arbitrarily small we obtain by this procedure a bifurcation point $s_I \in [s_0, \tilde{s}]$ of (4.17).

R e m a r k 7.1. Let us notice that Steps 1 and 2 can be done in the same way as in [16]. Step 3, where δ is not fixed, is new in comparison to [16]. The fact $s_I > s_0$ can be proved under the additional assumptions from Remark 4.2—see the end of Appendix.

8. Appendix

P r o o f o f P r o p o s i t i o n 5.1. If $\sigma_1'(s_0) = 0$ then $\sigma_2'(s_0) < 0$ due to the orientation of the curve $\sigma(s)$ and there is nothing to prove. If $\sigma_1'(s_0) \neq 0$ then we can consider a curve $\sigma(s) = [\sigma_1(s), \sigma_2(s)]$ as $\sigma_2(s) = \tilde{\sigma}(\sigma_1(s))$ on $(s_0 - \eta, s_0 + \eta)$ with some $\eta > 0$ small and the hyperbola C_p as a curve

$$d_2 = h_p(d_1) = \frac{b_{12}b_{21}/\kappa_p^2}{d_1 - b_{11}/\kappa_p} + \frac{b_{22}}{\kappa_p}.$$

Differenting h_p with respect to d_1, we obtain at the point d_1^0 that

$$\frac{\mathrm{d}h_p(d_1^0)}{\mathrm{d}d_1} = -\frac{b_{12}b_{21}/\kappa_p^2}{(d_1^0 - b_{11}/\kappa_p)^2} = -\frac{b_{12}b_{21}}{(d_1^0\kappa_p - b_{11})^2}.$$

It follows that

$$\frac{\mathrm{d}h_p(d_1^0)}{\mathrm{d}d_1} = -\frac{(d_2^0\kappa_p - b_{22})^2}{b_{12}b_{21}}$$

by using (3.4) for d_1^0, d_2^0. Differenting $\tilde{\sigma}$ with respect to σ_1, we obtain $\frac{\mathrm{d}\tilde{\sigma}(\sigma_1(s))}{\mathrm{d}\sigma_1} = \frac{\sigma_2'(s)}{\sigma_1'(s)}$ for any $s \in (s_0 - \eta, s_0 + \eta)$. If the curve $\sigma(s)$ intersects C_p at the point $d^0 = \sigma(s_0)$ transversally then either $\frac{\sigma_2'(s_0)}{\sigma_1'(s_0)} = \frac{\mathrm{d}\tilde{\sigma}(d_1^0)}{\mathrm{d}d_1} < -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}$ in the case $\sigma_1'(s_0) > 0$ or $\frac{\sigma_2'(s_0)}{\sigma_1'(s_0)} = \frac{\mathrm{d}\tilde{\sigma}(d_1^0)}{\mathrm{d}d_1} > -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}$ in the case $\sigma_1'(s_0) < 0$. In both cases we obtain

$$\sigma_2'(s_0) < -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}\sigma_1'(s_0).$$

Our assertion follows.

Detailed version of Observation 5.1. (Cf. Section 2, [6], Section 4, [4], Section 2.) We can write (5.1) as a system

$$\begin{aligned} u - \frac{b_{11}}{d_1}Au - \frac{b_{12}}{d_1}Av &= -\mu u, \\ v - \frac{b_{21}}{d_2}Au - \frac{b_{22}}{d_2}Av &= -\mu v \end{aligned}$$

and the elements $U = [u, v] \in \mathbb{V}^2$ in the form

$$u = \sum_{j=1}^{\infty} \langle u, e_j \rangle e_j, \qquad v = \sum_{j=1}^{\infty} \langle v, e_j \rangle e_j. \tag{8.1}$$

Using these expansions and the fact that κ_i is a characteristic value of A, multiplying the first equation by $d_1\kappa_i e_i$ and the second by $d_2\kappa_i e_i$, we obtain

$$\begin{aligned} &\langle u, e_i \rangle (d_1\kappa_i - b_{11} + \mu d_1\kappa_i) - \langle v, e_i \rangle b_{12} = 0, \\ &\langle u, e_i \rangle b_{21} - \langle v, e_i \rangle (d_2\kappa_i - b_{22} + \mu d_2\kappa_i) = 0 \end{aligned}$$

for $i = 1, 2, \ldots$. A couple $\langle u, e_i \rangle$, $\langle v, e_i \rangle$ can be nontrivial for some i if and only if

$$(d_1\kappa_i - b_{11} + \mu d_1\kappa_i)(d_2\kappa_i - b_{22} + \mu d_2\kappa_i) - b_{12}b_{21} = 0. \tag{8.2}$$

Hence, μ is an eigenvalue of (5.1) if and only if μ is a root of

$$\begin{aligned} &\mu^2 d_1 d_2 \kappa_i^2 - \beta_i(d)\kappa_i\mu + \gamma_i(d) = 0 \\ &\text{with } \beta_i(d) = d_1 b_{22} + d_2 b_{11} - 2d_1 d_2 \kappa_i, \\ &\qquad \gamma_i(d) = (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21} \end{aligned} \tag{8.3}$$

for at least one i. Now, the coefficient $\beta_i(d)$ can be positive, negative or zero. (Note that the corresponding coefficient in (3.3) in Section 3 was negative by (SIGN) for any d in a neighbourhood of C_i.) The term $\gamma_i(d)$ is negative or positive for d lying to the left or to the right, respectively, from C_i. It is easy to simplify the term

$$\omega_i(d) := \beta_i^2(d) - 4d_1d_2\gamma_i(d) = d_1^2b_{22}^2 + d_2^2b_{11}^2 - 2d_1d_2b_{11}b_{22} + 4d_1d_2b_{12}b_{21}$$

and see that it does not depend on i. Therefore we will write only $\omega(d)$ instead of $\omega_i(d)$. The set $\{d \in \mathbb{R}^2_+ \,;\ \omega(d) = 0\}$ is the set of all d satisfying

$$d_1^2b_{22}^2 + d_2^2b_{11}^2 - 2d_1d_2b_{11}b_{22} + 4d_1d_2b_{12}b_{21} = 0.$$

Solving this equation for d_2 with d_1 as a parameter we obtain

$$d_2^{(r)} = \frac{d_1}{b_{11}^2}\left[-b_{12}b_{21} + \det B \pm 2\sqrt{-b_{12}b_{21}}\sqrt{\det B}\right],\ \ r = 1, 2.$$

Thus the set $\{d \in \mathbb{R}^2_+ \,;\ \omega(d) = 0\}$ is a couple of half-lines, one of them is a common tangent $\mathcal{T}$ to all hyperbolas C_j, $j = 1, 2, \ldots$ (see also [20] and Figures 1 and 2). Further, the set $\widetilde{C}_i = \{d \in \mathbb{R}^2_+ \,;\ \beta_i(d) = 0\}$ is a hyperbola with the property $\widetilde{C}_i \cap C_i = \mathcal{T} \cap C_i$.

The roots μ of (8.3) are

$$\mu_i^{(r)}(d) := \frac{\beta_i(d) \pm \sqrt{\omega(d)}}{2d_1d_2\kappa_i},\ \ r = 1, 2.$$

If $\gamma_i(d) < 0$ then $\omega(d) > 0$ and $|\beta_i(d)| < \sqrt{\omega(d)}$. Therefore there are two different real roots $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$, one is negative and the other one is positive. If $\gamma_i(d) > 0$ then $\omega(d)$ can be either negative (and we have a couple of complex roots) or nonnegative but $|\beta_i(d)| > \sqrt{\omega(d)}$ (and we have two real roots, both having the same sign). The possibilities for the signs of $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ are the following—see Fig. 2:

domain:	$\beta_i(d)$	$\gamma_i(d)$	$\omega(d)$		relation between eigenvalues:
$d \in \mathcal{D}_1$	$+$	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \widetilde{C}_i$	0	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} = -\mu_i^{(2)}$
$d \in \mathcal{D}_2$	$-$	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i$	$-$	0	$+$	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} = 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \mathcal{D}_3$	$-$	$+$	$+$	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} < 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \mathcal{T}$	$-$	$+$	0	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} < 0$
$d \in \mathcal{D}_4$	$-$	$+$	$-$		$\mu_i^{(1)} \neq \mu_i^{(2)},\ \mu_i^{(r)} \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{Re} \mu_i^{(r)} < 0$
$d \in \widetilde{C}_i$	0	$+$	$-$		$\mu_i^{(1)} = -\mu_i^{(2)} \in \mathrm{i}\mathbb{R},\ \operatorname{Re} \mu_i^{(r)} = 0$
$d \in \mathcal{D}_5$	$+$	$+$	$-$		$\mu_i^{(1)} \neq \mu_i^{(2)},\ \mu_i^{(r)} \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{Re} \mu_i^{(r)} > 0$
$d \in \mathcal{T}$	$+$	$+$	0	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} > 0$
$d \in \mathcal{D}_6$	$+$	$+$	$+$	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} > 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i$	$+$	0	$+$	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} = 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i \cap \widetilde{C}_i \cap \mathcal{T}$	0	0	0	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} = 0.$

These calculations lead to the conclusion that for d lying to the left there is one positive root of (8.3) and for d lying to the right from C_i, either none or both roots of (8.3) are positive.

P r o o f o f t h e s e c o n d p a r t o f (5.11). Using (5.9), (5.6), (8.2), (SIGN) and (5.3) we obtain

$$
\begin{aligned}
\langle D(\sigma(s))L_\delta(s)U_i(s), U_i^*(s)\rangle &= \delta\chi(s) \cdot \sum_{j \in I(d^0)} \nu_j(d^0) \cdot \langle D(\sigma(s))U_j(s), U_i^*(s)\rangle \\
&= \delta\chi(s)\nu_i(d^0)\Big[\frac{\sigma_1(s)(\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i)^2}{b_{12}b_{21}} + \sigma_2(s)\Big] \\
&= -\delta\chi(s)\nu_i(d^0)\frac{\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i}{b_{12}b_{21}} \\
&\qquad \times [\sigma_2(s)b_{11} + \sigma_1(s)b_{22} - 2\sigma_1(s)\sigma_2(s)\kappa_i - 2\sigma_1(s)\sigma_2(s)\kappa_i\mu_i(s)] \\
&= \delta\chi(s) \cdot \frac{[\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i]\sqrt{\omega(\sigma(s))}}{b_{12}b_{21}} < 0 \quad \text{for } i \in I(d^0).
\end{aligned}
$$

P r o o f o f L e m m a 5.1. Analogously as in Observation 5.1 we obtain that μ is an eigenvalue of the problem

$$D^{-1}(\sigma(s))BAU - L_\delta(s)U - U = \mu U$$

if and only if μ is a root of the quadratic equation

$$\mu^2 - \beta_i^\delta(s)\mu + \gamma_i^\delta(s) = 0 \tag{8.4}$$

with coefficients $\beta_i^\delta(s)$, $\gamma_i^\delta(s)$ depending continuously on s and δ. For the sake of efficiency, the structure of the proof differs from the structure of the lemma. We shall distinguish the following cases:

A1. Let $i \notin I(d^0)$. It follows from (5.10) that $\mu_i^{(r)}(\sigma(s))$ and $U_i^{(r)}(\sigma(s))$, $r = 1, 2$, from Observation 5.1 and Notation 5.1 are simultaneously eigenvalues and eigenvectors of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and (8.4) is equivalent to (5.2) for any $s \in \mathbb{R}$. In particular, this means by the definitions of $\mu_p(s)$, $U_p(s)$ that $\mu_p(s)$ and $U_p(s)$ is an eigenvalue and an eigenvector of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ for any $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0]$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively.

A2. If $i \notin I(d^0) \cup \{p\}$ then d^0 and also $\sigma(s)$ for any $s \in (s_0 - \eta, s_0 + \eta)$ lie to the right from C_i. (Recall that $d^0 \in C$.) It follows from Observation 5.1 that if $d^0 \in C_p \setminus \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then the sign of both $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ is constant on $(s_0 - \eta, s_0 + \eta)$ (more precisely, $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ are both negative or positive on $(s_0 - \eta, s_0 + \eta)$ for $C_i \cap \mathcal{T} \preceq d^0$ or $d^0 \preceq C_i \cap \mathcal{T}$, respectively). If $d^0 \in C_p \cap \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ are both negative or positive on $(s_0 - \eta, s_0)$ for $C_i \cap \mathcal{T} \preceq d^0$ or $d^0 \preceq C_i \cap \mathcal{T}$, respectively, and complex on $(s_0, s_0 + \eta)$.

A3. For $i = p$, $\mu_p(s)$ changes its sign at s_0 and the sign of the other root is constant on $(s_0 - \eta, s_0 + \eta)$ in the case $d^0 \in C_p \setminus \mathcal{T}$. More precisely, if $C_i \cap \mathcal{T} \preceq d^0$ then $\mu_p(s) = \mu_p^{(1)}(\sigma(s)) > 0$ on $(s_0 - \eta, s_0)$, $\mu_p(s) = \mu_p^{(1)}(\sigma(s)) < 0$ on $(s_0, s_0 + \eta)$, $\mu_p^{(2)}(\sigma(s)) < 0$ on $(s_0 - \eta, s_0 + \eta)$, and if $d^0 \preceq C_i \cap \mathcal{T}$ then $\mu_p(s) = \mu_p^{(2)}(\sigma(s)) < 0$ on $(s_0 - \eta, s_0)$, $\mu_p(s) = \mu_p^{(2)}(\sigma(s)) > 0$ on $(s_0, s_0 + \eta)$, $\mu_p^{(1)}(\sigma(s)) > 0$ on $(s_0 - \eta, s_0 + \eta)$. In the case $d^0 \in C_p \cap \mathcal{T}$ we have $\mu_p(s) > 0$ and the other root is negative on $(s_0 - \eta, s_0)$, both the roots being complex on $(s_0, s_0 + \eta)$.

B1. Let $i \in I(d^0)$. Let $d^0 \in C_p$, $d^0 \notin C_q$ for $C_q \neq C_p$. Then $i \in I_p(d^0)$, $\mu_i(s) = \mu_p(s)$ and $I_q(d^0) = \emptyset$. Let $C_i \cap \mathcal{T} \preceq d^0$. Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) - \delta\chi(s) < 0$ on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is negative on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$. Let $d^0 \preceq C_i \cap \mathcal{T}$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) + \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) + \delta\chi(s) > 0$ on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$ and that the other root is also positive on $(s_0 - \eta, s_0 + \eta)$. Let $d^0 \in C_i \cap \mathcal{T}$. Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4) again. It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that both $\mu_i(s) - \delta\chi(s)$ and the other root of (8.4) are negative on $(s_0 - \eta, s_0]$ and complex on $(s_0, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$. (See Observation 5.1.)

B2. Let $i \in I(d^0)$. Let $d^0 \in C_p \cap C_q$, $C_q \neq C_p$. Let $p > q$. Then $C_i \cap \mathcal{T} \preceq d^0 \preceq C_j \cap \mathcal{T}$ for $i \in I_p(d^0) \cup \{p\}$, $j \in I_q(d^0)$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s) = \mu_p(s) - \delta\chi(s)$ or $\mu_i(s) + \delta\chi(s) = \mu_q(s) + \delta\chi(s)$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). (Let us note that $\mu_i(s_0) = 0$ for any $i \in I(d^0)$.) It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) - \delta\chi(s) < 0$ or $\mu_i(s) + \delta\chi(s) > 0$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0 - \eta, s_0 + \eta)$, $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is negative or positive for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0 - \eta, s_0 + \eta)$, $\delta \in (0, \delta_0)$.

B3. Let $i \in I(d^0)$. Let $d^0 \in C_p \cap C_q$, $C_q \neq C_p$. Let $p < q$. Then $C_i \cap \mathcal{T} \preceq d^0 \preceq C_j \cap \mathcal{T}$ for $j \in I_p(d^0) \cup \{p\}$, $i \in I_q(d^0)$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) + \delta\chi(s) = \mu_p(s) + \delta\chi(s)$ or $\mu_i(s) - \delta\chi(s) = \mu_q(s) - \delta\chi(s)$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) + \delta\chi(s) > 0$ or $\mu_i(s) - \delta\chi(s) < 0$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0 - \eta, s_0 + \eta)$, $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is positive or negative for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0 - \eta, s_0 + \eta)$, $\delta \in (0, \delta_0)$.

Now, it follows from the relation of the eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and the roots of (8.4) mentioned above that there are no further eigenvalues and eigenvectors besides those discussed in A1–B3.

Let us show that for $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0)$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively, the algebraic and geometric multiplicities of any positive eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ coincide. First, we will show the coincidence of the algebraic and geometric multiplicities of any positive eigenvalue $\mu_i^{(r)}(d)$, $r = 1, 2$, of the operator $D^{-1}(\sigma(s))BA - I$.

The adjoint equation to (5.1) is

$$B^* D^{-1}(d) AU - U = \mu U$$

and similar considerations as in Observation 5.1 imply that the eigenvectors of this equation corresponding to $\mu_i^{(r)}(d)$ are

$$\widetilde{U}_i^{(r)}(d) = \Big[\frac{d_1}{d_2}\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i}{b_{12}}e_i, e_i\Big] = \Big[\frac{d_1}{d_2}\frac{b_{21}}{b_{12}}\alpha_i^{(r)}(d)e_i, e_i\Big],\ r = 1, 2.$$

(Recall that $U_i^{(r)}(d) = [\alpha_i^{(r)}(d)e_i, e_i],\ r = 1, 2$—see Observation 5.2.) An elementary calculation using (5.3) gives for $\mu_i^{(r)}(d) > 0$ that

$$\begin{aligned}
&|\langle U_i^{(r)}(d), \widetilde{U}_i^{(r)}(d)\rangle| = \Big|\frac{d_1}{d_2}\frac{(d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i)^2}{b_{12}b_{21}} + 1\Big| \\
&= -\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i}{d_2b_{12}b_{21}}\big[d_2b_{11} + d_1b_{22} - 2d_1d_2\kappa_i - 2d_1d_2\kappa_i\mu_i^{(r)}(d)\big] \\
&= -\frac{[d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i]\sqrt{\omega(d)}}{d_2b_{12}b_{21}} \neq 0 \text{ for } i = 1, 2, \dots,\ r = 1, 2,\ d \notin \mathcal{T}, \\
&\langle U_i^{(r)}(d), \widetilde{U}_j^{(r)}(d)\rangle = 0 \text{ for any } i \neq j,\ r = 1, 2,
\end{aligned} \tag{8.5}$$

cf. (5.11). Hence,

$$\det(\langle\widetilde{U}_i^{(r)}(d), U_j^{(r)}(d)\rangle)_{i,j\in J} \neq 0 \text{ for any } J \subset \mathbb{N},\ r = 1, 2,\ d \notin \mathcal{T}. \tag{8.6}$$

This yields that the algebraic and geometric multiplicities of $\mu_i^{(r)}(d)$ coincide for $i \in \mathbb{N},\ r = 1, 2,\ d \notin \mathcal{T}$ (see e.g. [24]). In particular, this holds for $d = \sigma(s)$ with $s \in \mathcal{U}_\eta(s_0)$, where $\mathcal{U}_\eta(s_0) := (s_0 - \eta, s_0 + \eta)$ for $d^0 \in C_p \setminus \mathcal{T}$, $\mathcal{U}_\eta(s_0) := (s_0 - \eta, s_0)$ for $d^0 \in C_p \cap \mathcal{T}$ (let us note that $\sigma(s) \notin \mathcal{T}$ for $s \in \mathcal{U}_\eta(s_0)$).

By a standard treatment of the adjoint operator we obtain

$$L_\delta^*(s)\widetilde{U}_i^{(r)}(\sigma(s)) = 0 \text{ for all } i \notin I(d^0),\ r = 1, 2,\ s \in \mathcal{U}_\eta(s_0).$$

This implies that $\widetilde{U}_i^{(r)}(\sigma(s))$ for $i \notin I(d^0)$, $r = 1, 2$, $s \in \mathcal{U}_\eta(s_0)$, is simultaneously an eigenvector of the adjoint operator $(D^{-1}(\sigma(s))BA)^* - L_\delta^*(s) - I$ corresponding to $\mu_i^{(r)}(\sigma(s))$. The above considerations imply the coincidence of the algebraic and geometric multiplicities of any $\mu_i^{(r)}(d) > 0$ with $i \notin I(d^0)$, $r = 1, 2$, $s \in \mathcal{U}_\eta(s_0)$, as the eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$.

If $d^0 \in C_p \cap \mathcal{T}$ then all eigenvalues of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ corresponding to $i \in I(d^0)$ are negative on $(s_0 - \eta, s_0)$ and complex on $(s_0, s_0 + \eta)$—see the first part of this proof.

If $d^0 \in C_p \setminus \mathcal{T}$ then, due to the continuous dependence on $s \in \mathbb{R}$, $\delta \geqslant 0$, we can choose $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there is $\eta > 0$ for which the determinant corresponding to (8.6) with the scalar products of the corresponding eigenvectors of the operators $D^{-1}(\sigma(s))BA-L_\delta(s)-I$ and $(D^{-1}(\sigma(s))BA)^*-L_\delta^*(s)-I$, respectively, with $i,j\in J\subset I(d^0)$, remains nonzero on $(s_0-\eta,s_0+\eta)$. Therefore, the algebraic and geometric multiplicities of any positive eigenvalue corresponding to $i\in I(d^0)$ coincide again.

Our considerations lead to the following conclusion. If $d^0\in C_p\setminus\mathcal{T}$ then $\mu_p(s)$ is the only eigenvalue of the operator $D^{-1}(\sigma(s))BA-L_\delta(s)-I$ changing its sign at s_0 and it is algebraically simple. The other eigenvalues have constant signs and multiplicities on $(s_0-\eta,s_0+\eta)$. If $d^0\in C_p\cap\mathcal{T}$ then $\mu_p(s)$ is a real positive algebraically simple eigenvalue on $(s_0-\eta,s_0)$. The other possible positive eigenvalues (which can correspond only to $i\notin I(d^0)$) form pairs $\mu_i^{(1)}(\sigma(s))\neq\mu_i^{(2)}(\sigma(s))$ where $\mu_i^{(1)}(\sigma(s))$, $\mu_i^{(2)}(\sigma(s))$ have the same algebraic multiplicity, i.e. the sum of algebraic multiplicities for any such pair is even. All eigenvalues are complex for $(s_0,s_0+\eta)$. The assertion of Lemma 5.1 follows.

P r o o f o f L e m m a 6.9. Assume that there are $\varrho^m\to 0_+$ such that for any m fixed there exist s_n^m, U_n^m, τ_n^m $(n=1,2,\dots)$ satisfying

$$\begin{aligned}(8.7)\quad & D(\sigma(s_n^m))U_n^m-BAU_n^m\\ &\qquad -\frac{\tau_n^m}{1+\tau_n^m}N(U_n^m)+D(\sigma(s_n^m))L_\delta(s_n^m)U_n^m+P_{\tau_n^m}(U_n^m)=0,\\ (8.8)\quad & \qquad \|U_n^m\|^2=\frac{\varrho^m\tau_n^m}{1+\tau_n^m}\end{aligned}$$

with $U_n^m\notin K^\delta$, $[s_n^m,U_n^m,\tau_n^m]\to[s_0,U^m,\tau^m]$, $\frac{U_n^m}{\|U_n^m\|}\to W^m\in K^\delta$ if $n\to+\infty$. We have $\|U_n^m\|^2=\frac{\varrho^m\tau_n^m}{1+\tau_n^m}\leqslant\varrho^m\to 0$ by (8.8). We can choose a subsequence $\{U_k\}_{k=1}^{+\infty}$ from $\{U_n^m\}_{n,m=1}^{+\infty}$ such that $Z_k=\frac{U_k}{\|U_k\|}\notin K^\delta$, $Z_k\rightharpoonup Z\in K^\delta$ and $\tau_k\to\tau\in[0,+\infty]$. Lemma 6.1 gives $Z_k\to Z$.

First let $\tau=0$. Dividing (8.7) by $\|U_k\|$, the limiting process gives

$$D(\sigma(s_0))Z-BAZ+D(\sigma(s_0))L_\delta(s_0)Z=0 \tag{8.9}$$

with help of (2.5) and Proposition 4.3. It means $Z=-\frac{U_p}{\|U_p\|}\in\operatorname{int}K^\delta$ because of $W^m\in K^\delta$ and the fact that $\pm\frac{U_p}{\|U_p\|}$ are the only normed solutions of (8.9). For $\tau\in(0,+\infty]$ the equation (8.7) gives that $\frac{P^\delta_{\tau_k}(U_k)}{\|U_k\|}$ are bounded and therefore we can assume $\frac{P^\delta_{\tau_k}(U_k)}{\|U_k\|}\rightharpoonup F$,

$$D(\sigma(s_0))Z-BAZ+D(\sigma(s_0))L_\delta(s_0)Z+F=0, \tag{8.10}$$

where we have employed (2.5) again. Multiplying (8.10) by $-U_p^*$, the equation

$$D(\sigma(s_0))U_p^* - B^*AU_p^* = 0$$

by Z and adding them we obtain $\langle F, -U_p^*\rangle = 0$ due to (5.10). Observation 6.1 implies $F = 0$, i.e. we have (8.9) and $Z = -\frac{U_p}{\|U_p\|}$ again. In both cases, this is a contradiction because $Z_k \notin K^\delta$ and $Z_k \to Z = -\frac{U_p}{\|U_p\|} \in \operatorname{int} K^\delta$ and our assertion is proved.

Remark 8.1. The other possibility to avoid the condition $\operatorname{int} K \neq \emptyset$ is to define a pseudointerior

$$K^- := \bigl\{U \in K;\ \underset{\substack{V\notin K\\ \tau>0}}{\forall} \langle P_\tau V, U\rangle < 0\ \&\ \underset{\substack{F\in\mathbb{V}^2\\ F\neq 0}}{\forall}\ \underset{W\in\mathbb{V}^2}{\exists} \langle F, W\rangle > 0,\ U \pm W \in K\bigr\}$$

(cf. [26], [6]) and assume $-U_p, -U_p^* \in K^-$ instead of the assumption $-U_p, -U_p^* \in \operatorname{int} K^\delta$ for any $\delta \in (0, \delta_0)$ in (4.14). In order to prove Lemmas in Section 5, one has to add a special assumption about the nonlinearity term N or about the sign of a scalar product of a certain type, respectively:

$$\begin{aligned} &\text{if } U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup \frac{U_p}{\|U_p\|} \\ &\text{then } \Bigl\langle \frac{N(U_n)}{\|U_n\|}, U_p^*\Bigr\rangle \geqslant 0 \text{ for } n \text{ large enough.} \end{aligned} \tag{8.11}$$

The meaning of this condition for (2.6) is the following: Let $s_n,\ U_n$ satisfy (2.6). Let $s_n \to s_0,\ U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup \frac{U_p}{\|U_p\|}$. After some calculation (similar to that in the proof of [16], Lemma 3.6), condition (8.11) leads us to the conclusion that $s_n \leqslant s_0$. This corresponds to the fact that a branch of bifurcating spatial patterns of (2.6) goes to the left from C, i.e. to the domain of instability of the trivial solution.

Proof of the destabilizing effect in Theorem 4.1 under the additional assumption from Remark 4.2.

We will show that there exists $\varepsilon > 0$ such that

$$s_I^\delta > s_0 + \varepsilon \text{ for all } \delta > 0 \text{ small enough.} \tag{8.12}$$

Assume that for $\delta_n \to 0$ we have $s_I^{\delta_n} \to s_0$, $U_n = U_I^{\delta_n} \rightharpoonup U$ satisfying

$$D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n \in -M_0^{\delta_n}(U_n), \tag{8.13}$$

where $U_I^{\delta_n}$ are from Step 2 of the proof of Theorem 4.1. With help of (4.12) the limiting process in (8.13) gives $U_n \to U$ and

$$D(\sigma(s_0))U - BAU \in -M_0(U),$$

i.e. $U \in E_I(d^0)$. Under the assumption of the equivalence of relations (2.11) and (2.13) we can use Lemma 7 together with Remark 5 from [26] to obtain $U \in K$ and $U \in E_B(d^0)$. Hence

$$U = \sum_{i \in I(d^0) \cup \{p\}} a_i U_i(s_0) \tag{8.14}$$

with some $a_i \in \mathbb{R}$ (see (5.7) or (5.8), respectively). Setting

$$F_n := D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n \tag{8.15}$$

we rewrite (8.13) into the form $F_n \in -M_0(U_n)$. Then the assumptions (4.19) and (4.20) imply

$$\langle F_n, U^* \rangle \geqslant 0. \tag{8.16}$$

To get a contradiction we prove that

$$\begin{aligned} \langle F_n, U^* \rangle &= \langle D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^* \rangle \\ &= \langle [D(\sigma(s_I^{\delta_n})) - D(\sigma(s_0))]U_n, U^* \rangle + \langle D(\sigma(s_0))U_n - BAU_n, U^* \rangle \\ &\quad + \langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^* \rangle < 0. \end{aligned} \tag{8.17}$$

Indeed, the first scalar product is negative for $s_I^{\delta_n} > s_0$ because we have

$$\langle [D(\sigma(s_I^{\delta_n})) - D(\sigma(s_0))]U_n, U^* \rangle = (s_I^{\delta_n} - s_0)R_n$$

where $R_n := \sigma_1'(\overline{s}_n)\langle u_n, u^* \rangle + \sigma_2'(\tilde{s}_n)\langle z_n, v^* \rangle$ with some $\overline{s}_n, \tilde{s}_n$ lying between $s_I^{\delta_n}$ and s_0. It follows from (8.14) and Proposition 5.1 that

$$\lim_{n \to +\infty} R_n = \sum_{i \in I(d^0) \cup \{p\}} a_i^2 \Big[\frac{(\kappa_i \sigma_2(s_0) - b_{22})^2}{b_{12} b_{21}} \sigma_1'(s_0) + \sigma_2'(s_0) \Big] < 0. \tag{8.18}$$

Note that (4.16) implies that the term in brackets in (8.18) is negative for all $i \in I_p(d^0) \cup \{p\}$ and nonpositive for $i \in I_q(d^0)$ in the case $d^0 \in C_p \cap C_q$, $C_p \neq C_q$—see Notation 5.2. The second scalar product in (8.17) vanishes because

$$\langle D(\sigma(s_0))U_n - BAU_n, U^* \rangle = \langle U_n, D(\sigma(s_0))U^* - B^*AU^* \rangle = 0.$$

If $U = \pm\frac{U_p}{\|U_p\|}$ then the last term in (8.17) is zero by (5.11). Further, we have

$$
\begin{aligned}
&\frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\quad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \nu_i(d^0)\frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}), U^*\rangle \\
&\quad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \nu_i(d^0)\frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}), a_iU_i^*(s_0)\rangle \\
&\quad \to \sum_{i\in I(d^0)} \frac{a_i^2[d_2^0\kappa_i - b_{22}]\sqrt{\omega(d^0)}}{b_{12}b_{21}} \text{ for } n\to+\infty
\end{aligned}
\tag{8.19}
$$

(see the proof of the second part of (5.11) with $s = s_0$). If $d^0 \notin \mathcal{T}$ then the limit in (8.19) is negative and therefore the last term in (8.17) is negative for large n.

If $d^0 \in \mathcal{T}$ then $\omega(d^0) = 0$ and therefore the limit in (8.19) is zero. But $I(d^0) = \{p+1, \dots, p+k-1\}$ (k is the multiplicity of κ_p), $\nu_i(d^0) = 1$ and $\langle D(\sigma(s_0))U_i(s_0), U_i^*(s_0)\rangle = \frac{a_i^2[d_2^0\kappa_i - b_{22}]\sqrt{\omega(d^0)}}{b_{12}b_{21}} = 0$ for any $i \in I(d^0)$. Therefore, by the definition of L_δ and by (5.11) we have

$$
\begin{aligned}
&\frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\quad = \frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\qquad - \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_0))U_i(s_0), a_iU_i^*(s_0)\rangle \\
&\quad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2} \\
&\qquad\quad \times \langle [D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}) - D(\sigma(s_0))U_i(s_0)], a_iU_i^*(s_0)\rangle \\
&\quad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}(s_I^{\delta_n} - s_0)a_iR_n^i
\end{aligned}
\tag{8.20}
$$

with $R_n^i := \sigma_1'(\overline{s}_n^i)\langle u_n, u_i^*\rangle + \sigma_2'(\tilde{s}_n^i)\langle z_n, v_i^*\rangle$ for suitable $\overline{s}_n^i, \tilde{s}_n^i \in (s_0, s_I^{\delta_n})$. Hence the last expression in (8.20) is negative for large n by the same argument used for the first term in the last part of (8.17) (cf. (8.18)). The assertion (8.17) follows and we have a contradiction with $s_I^{\delta_n} \to s_0$. Therefore, $s_I^{\delta_n} > s_0 + \varepsilon$ with some $\varepsilon > 0$ for any n and thus $s_I \geqslant s_0 + \varepsilon$.

It remains to show that $[s_\varrho, U_\varrho]$ do not satisfy (4.21): Assume by contradiction that there are $\varrho_n \to 0$, $s_{\varrho_n} \to s_I$, $U_{\varrho_n} \to 0$ satisfying

$$D(\sigma(s_{\varrho_n}))U_{\varrho_n} - BAU_{\varrho_n} - N(U_{\varrho_n}) = 0. \tag{8.21}$$

Dividing this equation by $\|U_{\varrho_n}\|$ we obtain after the limiting process

$$D(\sigma(s_I))U_I - BAU_I = 0$$

where U_I is an accumulating point of $\frac{U_{\varrho_n}}{\|U_{\varrho_n}\|}$. This is impossible because $s_0 < s_I$ is the greatest critical point of (4.18).

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