TRIVIAL GENERATORS FOR NONTRIVIAL FIBRES

Linus Carlsson, Umeå

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Abstract. Pseudoconvex domains are exhausted in such a way that we keep a part of the boundary fixed in all the domains of the exhaustion. This is used to solve a problem concerning whether the generators for the ideal of either the holomorphic functions continuous up to the boundary or the bounded holomorphic functions, vanishing at a point in $\mathbb{C}^n$ where the fibre is nontrivial, has to exceed n. This is shown not to be the case.

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1. Introduction

The boundary of an open set M will be denoted bM and the set of strictly pseudoconvex boundary points by $S(bM)$. By $\mathcal{B}(M)$ we denote one of the Banach algebras $H^\infty(M)$ (the bounded holomorphic functions on M) or $A(M)$ (holomorphic functions on M which can be continuously continued up to the boundary). A point in space will have the form $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and $B(p, r)$ is the ball with center at p and radius r. The set of strictly plurisubharmonic functions on a set M will be denoted $\mathcal{SPSH}(M)$. The projection, π from the spectrum $\mathcal{M}^{\mathcal{B}}$ into $\mathbb{C}^n$ is given by $\pi(\varphi) = (\varphi(z_1), \ldots, \varphi(z_n))$ and the inverse projection, $\pi^{-1}(p)$, is called the fibre over $p \in \mathbb{C}^n$.

In the article [6], Gleason asked whether the Banach algebra $A(B(0,1))$ was finitely generated, if this was the case then he proved that the maximal ideal consisting of functions vanishing at the origin, is generated by the coordinate functions. The question whether these ideals in the algebras of holomorphic functions are generated by the coordinate functions has been named the Gleason problem. The only known counterexamples to the Gleason problem are domains where the fibres are nontrivial.

In that case it easily follows that the coordinate functions cannot generate the ideal, see e.g. [2].

In this paper we go back to the original question of Gleason. There has for some years been a question around, concerning the number of generators in maximal ideals. The question was whether there is a relationship between the number of elements in the fibre and the number of generators. We show that there is no such relation.

Also we show that Proposition 1 in [1] is false.

2. Exhaustion of nonsmooth domains

In the rest of the article we use a smooth convex function $b_\varepsilon\colon\ \mathbb{R}\to\mathbb{R}_+$ equal to $|x|$ if $|x|>\varepsilon$. See e.g. [7]. If $r,t\in\mathcal{SPSH}(D)\cap C^k(D)$ then Guan also shows that

$$s(z):=r(z)+t(z)+b_\varepsilon(r(z)-t(z))$$

is a strictly plurisubharmonic, C^k smooth, function on D.

This is useful when creating a C^k smooth strictly pseudoconvex domain which is, except for a arbitrary small set, the intersection of two C^k smooth strictly pseudoconvex domains.

Proposition 1. *Let $D\subset\mathbb{C}^n$ be a bounded pseudoconvex domain. Assume that $M\subset\mathbb{C}^n$ is a nonempty, open set, such that $M\cap bD\subset\subset S(bD)\cap C^k$, $k\geqslant 2$. Then there is a family of domains $\{D_j\}_{j=1}^\infty$ which exhaust D with the following properties;*

(1) $D=\bigcup\limits_{j=1}^{\infty}D_j$

(2) *D_j is strictly pseudoconvex with C^k boundary,*

(3) $M\cap bD\subset\subset bD_j$,

(4) $\max\limits_{z\in D_j}\operatorname{dist}(bD,z)<1/j,\ j=1,2,\dots,n.$

Remark 2. We only demand that D has a C^k smooth boundary on a neighborhood of M.

Proof. Let M_1 and M_2 be such that $M\subset\subset M_1\subset\subset M_2$ and $M_2\cap bD\subset\subset S(bD)\cap C^2$.

Pick a $\chi\in C_0^\infty(M_2)$, $0\leqslant\chi\leqslant 1$, such that $\chi=1$ on M_1. Let $\varrho\in\mathcal{SPSH}(U_{M_2})$ be a defining function for D on M_2, where $U_{M_2}\subset\mathbb{C}^n$ is an open set with

$$M_2\cap bD\subset\subset U_{M_2}\cap bD\subset\subset S(bD)\cap C^2$$

such that

$$D\cap U_{M_2}=\{z\in U_{M_2}\colon\ \varrho(z)<0\}\,.$$

Fix $N_0 > 1$ so large that

$$\tilde{D} := D \cup \left\{ z \in U_{M_2} :\ \tilde{\varrho}(z) := \varrho(z) + \frac{\chi(z)}{N_0} < 0 \right\}$$

has a strictly pseudoconvex boundary at M_2, i.e. $b\tilde{D} \cap M_2 \subset\subset S(b\tilde{D})$. Since $\tilde{D}$ is pseudoconvex, we can find an exhaustion of strictly pseudoconvex domains $\tilde{D}_j$, with $b\tilde{D}_j \in C^\infty$. For some $N > 0$ we have that if $j > N$ then

$$M_1 \cap bD \subset\subset \tilde{D}_j.$$

For each j we fix a defining function $\varrho_j \in \mathcal{SPSH}(\tilde{D}_{j+1})$ such that

$$\tilde{D}_j = \{ z \in \mathbb{C}^n :\ \varrho_j(z) < 0 \}.$$

Fix a sequence $\varepsilon_j > 0$, such that $\varepsilon_j > \varepsilon_{j+1}$, with ε_1 small enough such that $r_j(z) := \varrho_j(z) + \tilde{\varrho}(z) + b_{\varepsilon_j}(\varrho_j(z) - \tilde{\varrho}(z))$ is a strictly plurisubharmonic defining function for $D_j := \{ z \in \mathbb{C}^n :\ r_j(z) < 0 \}$, and such that $bD_j \in C^k$ (Sard's theorem) and $M \cap bD \subset bD_j$. If necessary pick a subsequence j_k such that property (4) holds. The domains D_{j_k} satisfies properties (1) to (4). □

3. Nebenhülle and pseudoconvexity

The following definition is equivalent to the one given in [5].

Definition 3. The Nebenhülle of a domain $D \subset \mathbb{C}^n$ is defined as

$$N(D) = \text{interior}\Big(\bigcap D_\alpha \Big),$$

where the intersection is taken over all D_α which are smooth strictly pseudoconvex domains such that $D \subset\subset D_\alpha$.

Proposition 4. *Let D be a domain in $\mathbb{C}^n$. Let K be a nonempty compact subset of $S(bD)$. If there is a strictly pseudoconvex domain $\hat{D}$ and a neighborhood $V \subset S(bD) \cap C^2$ of K such that $D \subset \hat{D}$ and that $V \subset b\hat{D}$ then K is included in the boundary of the Nebenhülle of D, i.e. $K \subset bN(D)$.*

Proof. From the definition it is obvious that $N(D) \subset N(\hat{D})$ and since $\hat{D}$ is strictly pseudoconvex it follows that $N(\hat{D}) = \hat{D}$. Since $D \subset N(D)$, we have $K = K \cap bN(\hat{D})$ due to the hypothesis.

But since

$$K \subset \overline{D} \subset \overline{N(D)} \subset \overline{N(b\hat{D})}$$

we must in fact have that $K \subset bN(D)$. □

The following example is a counterexample to Proposition 1 in [1].

Example 5. The worm domain W, defined in [5] is pseudoconvex and has C^∞ smooth boundary. The worm domain satisfies the property that there exists a compact set K with nonempty interior in $S(bW)$ which is disjoint from the boundary of the Nebenhülle. By Proposition 4 there can not be a strictly pseudoconvex domain $\hat{D}$ and a neighborhood $V \subset S(bD)$ of K such that $D \subset \hat{D}$ and $V \subset b\hat{D}$.

3.1. Pseudoconvex domains in strictly pseudoconvex domains.

Lemma 6. *Let D be a pseudoconvex domain. Let $V \subset bD$ be an open set satisfying $V \subset S(bD) \cap b(N(D)) \cap C^1$.*

Assume $K \subset V$ is a nonempty compact set, then for every $\varepsilon > 0$ there exists a strictly pseudoconvex domain U with C^∞ boundary such that $D \subset\subset U$ and $d_n(K, bU) < \varepsilon$.

Here we use $d_n(K, bU) := \sup_{x\in K} d_n(x, bU)$ where $d_n(x, bU)$ is the Euclidian distance from x to bU in the direction of the normal vector pointing out of D.

Proof. Let $\varepsilon > 0$ and $x \in K$ be arbitrary. From Proposition 4 we choose a pseudoconvex domain $U_x \supset\supset D$ such that $d_n(x, bU_x) < \varepsilon/2$.

Let $V_x \subset\subset V$ be an open neighborhood of x such that $d_n(\overline{V_x}, bU_x) < \varepsilon$, this is possible since V is a C^1 surface and U_x can be chosen to be C^∞. Then $\{V_x\}$ is an open covering of K and since K is compact there is a finite number of V_x, call them $\{V_{x_j}\}_{j=1}^N$ such that

$$K \subset\subset \bigcup_{j=1}^N V_{x_j}.$$

Let $\tilde{U} = \bigcap_{j=1}^N U_{x_j}$ then $\tilde{U}$ is a pseudoconvex domain such that $D \subset\subset \tilde{U}$. Let U be a strictly pseudoconvex domain with C^∞ boundary so that $D \subset\subset U \subset\subset \tilde{U}$. □

Proposition 7. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $V \subset S(bD) \cap b(N(D))$ be an open set, which is C^k smooth, where $k \geqslant 2$.*

Assume that K is a nonempty, compact subset of V. Then there exists a bounded strictly pseudoconvex domain $\hat{D} \subset \mathbb{C}^n$ with C^k regular boundary such that

(1) $D \subset \hat{D}$,

(2) $K \subset b\hat{D}$.

Proof. Fix compacts K_1 and K_2 such that $K \subset K_1{}^\circ$, $K_1 \subset K_2{}^\circ$ and $K_2 \subset V$.

From Proposition 1 we get a strictly pseudoconvex domain $\tilde{D} \subset D$ such that $V \subset b\tilde{D} \in C^k$.

Let $\tilde{r} \in \mathcal{SPSH} \cap C^k(U_{\tilde{D}})$ defining function for $\tilde{D}$, where $U_{\tilde{D}} \subset \mathbb{C}^n$ is a domain such that $\tilde{D} \subset\subset U_{\tilde{D}}$. Let $\omega \subset U_{\tilde{D}}$ be a domain such that $\tilde{r} \in \mathcal{SPSH}(\omega)$ and $K_2 \subset \omega \cap bD \subset V$.

Let $\omega_1 \subset\subset \omega$ and $K_1 \subset \omega_1 \cap bD$. Let $\chi \in C^\infty(\mathbb{C}^n)$ be a cutoff function such that $\chi = 0$ on ω_1 and $\chi = 1$ outside ω.

Let $r(z) = \tilde{r}(z) - \chi(z)$ be locally defined on ω and $R = \{z \in \mathbb{C}^n : \ r(z) < 0\}$. Then $D \cap \omega \subset R \cap \omega$ and $K_1 \subset bR$.

Close enough to ω_1 the boundary bR will be strictly pseudoconvex since $r \in C^k(U_{\tilde{D}})$. Let ω_0 be an open set so that this property holds on a neighborhood and so that $\omega_1 \subset\subset \omega_0$.

From Lemma 6 we pick a strictly pseudoconvex domain Ω with C^∞ smooth boundary such that $D \subset\subset \Omega$ and

$$\Omega \cap \omega_0 \cap bR \subset\subset \omega_0 \cap bR.$$

Choose it close enough so that r is strictly plurisubharmonic in a neighborhood of $\Omega \cap \omega_0 \cap R$. Let t be a smooth strictly plurisubharmonic defining function in a neighborhood of Ω.

On ω_0 we define $s(z) = r(z) + t(z) + b_\varepsilon(r(z) - t(z))$. Choose $\varepsilon > 0$ so small that

$$D \subset S := \{z \in \mathbb{C}^n : \ s(z) < 0\}$$

and $s = r$ on ω_1 and $bS \cap b\omega = b\Omega \cap b\omega$.

Outside ω_0 we let $s = t$. From [7] we have that s is a strictly plurisubharmonic C^k function.

If the boundary of S is C^k we are done, but this needs not be the case, the derivatives of the defining function may vanish and thereby creating a cusp. Due to Sard's theorem, there exists a sequence $\lambda_j \searrow 0$ such that

$$S_{\lambda_j} := \{z \in \mathbb{C}^n : \ s_j(z) := s(z) - \lambda_j < 0\}$$

has C^k boundary for each λ_j and by Theorem 1.5.16 in [8] the domain is regular. Observe that $\Omega \subset\subset S_{\lambda_j}$.

We don't want any interference from t now, and therefore we choose two new domains $\omega_2, \omega_3 \in \mathbb{C}^n$, such that $\omega_2 \subset\subset \omega_3 \subset\subset \omega_1$, $K_1 \subset \omega_2 \cap D$ and $b\Omega \cap \omega_3 = \emptyset$.

Fix another cutoff function $\chi_2 \in C^\infty(\mathbb{C}^n)$ such that $\chi_2 = 0$ on ω_2 and $\chi_2 = 1$ outside ω_3. Define $r_j = (1 - \chi_2)r + \chi_2 s_j$ and

$$D_j = \{z \in \mathbb{C}^n : \ r_j(z) < 0\}.$$

Since D_j coincides with S_{λ_j} outside ω_3 and with D inside ω_2 the boundary bD_j is strictly pseudoconvex there. On $\omega_3 \setminus \omega_2$ we have $r_j = r - \lambda_j \chi_2$.

If we fix k big enough we have the same properties of r_k as r because of the continuity of the derivatives of r and because the C^2-norm of χ_2 on the closure of ω_3 is finite. We are therefore done with $\hat{D} = D_k$. □

3.2. An example.

In this section we show that for any given $m \geqslant 2$ and $n \geqslant 2$ there exists a bounded smooth domain $R_n^m \subset \mathbb{C}^n$ such that the envelope of holomorphy $\widetilde{R_n^m}$ is an m-sheeted Riemann domain spread over $\mathbb{C}^n$ with a point $p \in R_n^m$ such that $\#\pi^{-1}(p) \geqslant m$ and with the interesting property that the maximal ideal

$$J_p(R_n^m) = \{f \in \mathcal{B}(R_n^m)\colon\ f(p) = 0\}$$

is generated by n functions in $J_p(R_n^m)$. I am very pleased to announce that the idea to the case R_2^2 was shown to me by Nils Øvrelid.

Example 8. Let $m \geqslant 2$ and $n \geqslant 2$ be integers.

Let $\varrho_B(x) := |x|^2 - 1$ be a defining function for the unit ball in $\mathbb{R}^n$. Let

$$V = \{x = (x', x_n) \in \mathbb{R}^n\colon\ \varrho_V(x) := x_n - b_{0.01}(|x'|^2) < 0, |x|^2 - 4 < 0\}.$$

Define the function $Va\colon\ \mathbb{C}^n \to \mathbb{R}^n$ as $Va(z) = (|z_1|, |z_2|, \dots, |z_n|)$.
Set $\varrho_{B_n^1}(x) = \varrho_V(x) + \varrho_B(x) + b_{0.01}(\varrho_V(x) - \varrho_B(x))$ and define the domain

$$B_n^1 = \left\{z \in \mathbb{C}^n\colon\ \varrho_{B_n^1}(Va(z)) < 0\right\}.$$

The following is true for B_n^1

- $B_n^1 \subset B(0,1) \subset \mathbb{C}^n$.
- The boundary bB_n^1 is C^∞-smooth.
- B_n^1 is a Reinhardt domain.
- $\overline{B((0,0,\dots,0,\frac{1}{3}),0.11)} \cap \overline{B_n^1} = \emptyset$.
- $B(((0,0,\dots,0,\frac{1}{3}),0.11)$ is a subset of the envelope of holomorphy of B_n^1.

  In fact the envelope of holomorphy of B_n^1 is just the convex hull. This domain can be thought of as an hour-sand-glass.

Denote by B_n^2 the domain

$$B_n^2 = \left\{z \in \mathbb{C}^n\colon\ 10 \cdot \left(z - (0,0,\dots,0,3^{-1})\right) \in B_n^1\right\}$$

Fix a smooth curve $\gamma\colon [0,1] \to \mathbb{C}^n$ which only takes real values such that: $\gamma(0) = (1,0,0,\ldots,0) \in bB_n^1$, $\gamma(1) = (-0.1,0,\ldots,0,3^{-1}) \in bB_n^2$, $\gamma((0,1)) \cap (\overline{B_n^1 \cup B_n^2}) = \emptyset$, and that

$$\gamma([0,1]) \cap \overline{B(((0,0,\ldots,0,),0.11)}$$

is equal to the straight line segment starting at $\gamma(1)$ and ending at $(-0.11,0,\ldots,0,3^{-1})$; also we demand that $\gamma'(0)$ intersect bB_n^1 transversally, and finally that $\max(|\gamma(t)|, 0 \leqslant t \leqslant 1) < 1.05$.

For a set $M \subset \mathbb{C}^n$ let $M^{(\varepsilon)} = \{z \in \mathbb{C}^n : \ |z-\xi| < \varepsilon \text{ for some } \xi \in M\}$. Fix $\varepsilon_0 = 0.001$.

We construct a domain R_n^2 (see Theorem 4.1.43 with proof in [9] for the construction) with the following properties,

- $B_n^1 \cup B_n^2 \cup \gamma([0,1]) \subset R_n^2 \subset B_n^{1(\varepsilon_0)} \cup B_n^{2(\varepsilon_0)} \cup \gamma([0,1])^{(\varepsilon_0)}$,
- $(B_n^{1(\varepsilon_0)} \cap R_n^2) \setminus B(\gamma(0),\varepsilon_0) = B_n^{1(\varepsilon_0)} \setminus B(\gamma(0),\varepsilon_0)$,
- $(B_n^{2(\varepsilon_0)} \cap R_n^2) \setminus B(\gamma(0),\varepsilon_0) = B_n^{2(\varepsilon_0)} \setminus B(\gamma(0),\varepsilon_0)$.
- $bR_n^2 \in C^\infty$ (here we use Proposition 1).
- The envelope of holomorphy, $\widetilde{R_n^2}$, is a two sheeted Riemann domain spread over $\mathbb{C}^n$ where B_n^2 is lifted to the second sheet. The envelope of holomorphy $\widetilde{R_n^2}$ has a Stein neighborhood basis and $S(b(\widetilde{R_n^2})) \in C^\infty$.

Now assume that R_n^p has been created, let

$$R_n^{p+1} = R_n^p \cup \Big(\frac{1}{10^p} R_n^2 + \Big(0,0,\ldots,0,\frac{1}{3}\Big(1 + \frac{1}{10} + \ldots + \frac{1}{10^{p-1}}\Big)\Big)\Big),$$

so we retrieve our domain R_n^m as m copies of B_n^1 of different sizes nestled in such a way that we lift every B_n^j, $1 \leqslant j \leqslant m$ to a new sheet and thereby getting an m-sheeted Riemann domain spread over $\mathbb{C}^n$.

To prove the promised result in this section we will look at the zero set of an analytic function g, denoted $\mathcal{Z}_g$. The common zero set of a family of analytic functions $G = (g_1, g_2, \ldots, g_m)$ will be denoted $\mathcal{Z}_G$, that is

$$\mathcal{Z}_G = \{z \in \mathbb{C}^n : \ g_1(z) = g_2(z) = \ldots = g_m(z) = 0\}.$$

Lemma 9. *Let*

$$p = \Big(0,0,\ldots,0,\frac{1}{3}\Big(1 + \frac{1}{10} + \ldots + \frac{1}{10^{m-1}}\Big)\Big) \in R_n^m.$$

Let $\widetilde{R_n^m}$ denote the envelope of holomorphy of R_n^m. Then there exist a $V \subset\subset S(b(\widetilde{R_n^m})) \cap C^\infty$ which is open in $b(\widetilde{R_n^m})$ such that there exists functions $G := (g_1, g_2, \ldots, g_n) \in A^\infty(\widetilde{R_n^m})^n$ such that $\mathcal{Z}_G \cap \overline{\widetilde{R_n^m}} = \{p\}$ and

$$\mathcal{Z}_{g_j} \cap b(R_n^m) \subset V, 1 \leqslant j \leqslant n-1.$$

Proof. Let $\omega \subset\subset \mathbb{C}^n$ be the domain given by

$$\omega = R_n^m \cap B\Big(p, \frac{1}{10^m} + \frac{1}{10^{m+2}}\Big)$$

rounded off so we get a smooth strictly pseudoconvex domain. Let the distance from the origin to the boundary be denoted by $d = \operatorname{dist}(bR_n^m, 0)$. Define a neighborhood of the point p as

$$O_p = \Big\{z \in \omega\colon\ |z_n - p_n| < \frac{d}{2}10^{-m}\Big\}.$$

and let $V = bB_n^m \cap bO_p$.

Then V, lifted to the m-th sheet, satisfies the hypothesis since $\widetilde{R_n^m}$ has a Stein neighborhood basis. By Proposition 7 there exists a smooth strictly pseudoconvex domain Ω such that $R_n^m \subset \Omega$ and $V \subset b\Omega$ (the proposition is true in this case since the result is a local one).

Let φ be a cut off function on ω such that $\varphi(O_p) = 1$, $\varphi \in C^\infty(\omega)$, and $\varphi|_{b\omega\setminus V} = 0$. Observe that this is only done locally in the m-th sheet of the domain so we may continue φ to be identically 0 on $\Omega \setminus \omega$.

Let γ be the curve in the construction of R_n^m. Since γ is real, the argument of z_j, $j = 1, 2, \ldots, n-1$ on $\omega \setminus B_m^n$ stays away from $\frac{1}{2}\pi$ there exists an analytic branch of $\log(z_j)$ on $\operatorname{supp}(\bar\partial\varphi)$.

Let

$$\lambda_j = \begin{cases} \bar\partial\varphi \log(z_j), & \text{when } z \in \operatorname{supp}(\bar\partial\varphi), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lambda_j \in C^\infty_{(0,1)}(\bar\Omega)$ with $\bar\partial\lambda_j = 0$, so by Corollary 5.2.7 in [4] there is a solution $v_j \in C^\infty(\bar\Omega)$ such that $\bar\partial v_j = \lambda_j$.

Defining

$$g_j(z) = \exp(\varphi(z)\cdot\log(z_j) - v_j(z))$$

for $j = 1, 2, \ldots, n-1$ and $g_n(z) = z_n - p_n$ yields the desired functions. $\square$

Remark 10. The function $g_1 \in A^\infty(\widetilde{R_n^m})$ satisfy $g_1(p) = 0$ on the m-th sheet but $g_1(p) \neq 0$ on the first sheet, so g_1 separates the two sheets apart. Using the construction of g_1 above, we can construct a function that separates all the sheets at $\pi^{-1}(\pi(p))$ in $\widetilde{R_n^m}$, that is: The number of elements in the fibre over $p \in \mathbb{C}^n$ are at least m.

Claim 11. With R_n^m as in Example 8 we have that the maximal ideal $J_p = J_p(\widetilde{R_n^m})$ (where $\widetilde{R_n^m}$ is the envelope of holomorphy of R_n^m) is generated by n functions $g_i \in H^\infty(\widetilde{R_n^m})$, i.e. for any $f \in J_p$ there exist $f_i \in \mathcal{B}(D)$, $i = 1, 2, \ldots, n$, such that

$$f(z) = \sum_{i=1}^n g_i(z) f_i(z).$$

To prove this claim we will use the Koszul complex argument, following [10], we will use a trivial generalization of two of the lemmas there. We introduce the necessary notation for the classes of forms which we work in.

Notation 12. Let

$$K_r = \{u \in (C^\infty_{(0,r)} \cap L^\infty_{(0,r)})(D)\colon\ \bar\partial u \in L^\infty_{(0,r+1)}(D)\},$$

and $K_r^s = K_r \otimes_{\mathbb{C}} E_s$, where $E_s = \bigwedge E$ and E is just an n-dimensional space.

Notation 13. Let U_1 be a fixed open set in D, with $p \in U_1$. We denote by M_r^s the set $\{k \in K_r^s\colon\ k|_{U_1} = 0\}$.

Obviously $\bar\partial K_r^s \subset K_{r+1}^s$ and $\bar\partial M_r^s \subset M_{r+1}^s$

In his article [10], Øvrelid assumes that D is a domain in $\mathbb{C}^n$. Using the result of Theorem 4.10.4 in [8] one sees that the result of the following two Lemmas holds true when D is a smooth Riemann domain as well.

Lemma 14 (Lemma 1′. [10]). *If $k \in K_r^s$ and $\bar\partial k = 0$, $r \geqslant 1$, there exists a $k' \in K_{r-1}^s$, such that $\bar\partial k' = k$ and k' has a continuous extension to $\overline{D}$.*

With G as in Claim 11 and $\delta_G\colon K_r^s \to K_r^{s-1}$ as the interior product.

Lemma 15 (Lemma 3. [10]). *If $k \in M_r^s$ and $\delta_G k = \bar\partial k = 0$, there exists a $k' \in K_r^{s+1}$, with $\delta_G k' = k$ and $\bar\partial k' = 0$.*

Proof of Claim 11. Let V and G be as in the proof of Lemma 9. Given $f \in J_p$. Choose a smooth strictly convex set $\omega \subset R_n^m$ with $V \subset b\omega$, it follows that $p \in \omega$. By Proposition 7 (which works in this case since $\widetilde{R_n^m}$ has a Stein neighborhood basis) we get another bounded smooth strictly pseudoconvex Riemann domain Ω spread over $\mathbb{C}^n$ such that $\widetilde{R_n^m} \subset \Omega$ with $V \subset b\Omega$. Using the $\bar\partial$ result from Theorem 4.10.4 in [8] together with technique in the proof of proposition 2.2 in [3] we get a solution $f_i^0 \in H^\infty(\omega)$, $i = 1, 2, \dots, n$, such that

$$f(z) = \sum_{i=1}^{n} g_i(z) f_i^0(z), \quad z \in \omega.$$

Let φ_0 be a smooth cutoff function with $\operatorname{supp}(\varphi_0) \subset\subset \omega$, $\varphi_0(U_1) = 1$ where U_1 is a neighborhood of p in ω.

For $1 \leqslant j \leqslant n-1$ we fix open sets $\omega_j \subset \widetilde{R_n^m}$ such that $\mathcal{Z}_{g_j} \cap \widetilde{R_n^m} \subset \omega_j$ and $b\omega_j \cap b\widetilde{R_n^m} \subset\subset V$ so small that $(\overline{\omega_j} \setminus U_1) \cap \mathcal{Z}_{G^j} = \emptyset$, where $G^j = (g_1, g_2, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$ is the vector G with g_j omitted.

For each $1 \leqslant j \leqslant n-1$, let $\tilde{\varphi}_j$ be a smooth cutoff function, with $\operatorname{supp}(\tilde{\varphi}_j) \subset \omega_j$, $\tilde{\varphi}_j(O_{\mathcal{Z}_{g_j}}) = 1$ where $O_{\mathcal{Z}_{g_j}}$ is an open neighborhood of $\mathcal{Z}_{g_j}$ in $\widetilde{R^m_n}$ such that $O_{\mathcal{Z}_{g_j}} \subset \omega_j$ and $bO_{\mathcal{Z}_{g_j}} \cap b\omega_j \subset V$.

Define $\varphi_1 = (1-\varphi_0)(1-\tilde{\varphi}_1)$. Assuming that φ_{k-1} has been defined, let

$$\varphi_k = \varphi_{k-1}(1-\tilde{\varphi}_k), \quad k \leqslant n-1,$$

and $\varphi_n = 1 - \sum\limits_{j=0}^{n-1} \varphi_j$.

Letting

$$f_i^1(z) = f_i^0\varphi_0 + f\frac{\varphi_i}{g_i},$$

we get a smooth solution $f_i^1 \in C^\infty(\widetilde{R^m_n}) \cap L^\infty(\widetilde{R^m_n})$, that is

$$f = \sum_{i=1}^n g_i f_i^1.$$

Notice that $\operatorname{supp}(\bar\partial f_i^1) \cap b\widetilde{R^m_n} \subset V$ and can hence be extended trivially to Ω. Furthermore $\bar\partial f_i^1(z) = 0$ when $z \in U_1$.

Defining

$$F^1 = \sum_{j=1}^n f_i^1 \otimes e_i$$

we get (that the extension of) $\bar\partial F^1 \in M_1^1 = M_1^1(\Omega)$.

Applying Lemma 15 and then Lemma 14, we find a form $k \in K_0^2$ continuous on $\overline{\Omega}$, with $\bar\partial\delta_G k = \delta_G\bar\partial k = \bar\partial F^1$. Let F be the form defined by

$$F = F^1 - \delta_G k$$

then $\bar\partial F = 0$ on $\widetilde{R^m_n}$. Writing $F = \sum\limits_1^n f_i \otimes e_i$, it follows that $f_1, f_2, \ldots, f_n \in \mathcal{B}(\widetilde{R^m_n})$ and $f = \sum\limits_{i=1}^n g_i f_i$, which completes the proof. $\square$

Proposition 16. *The domain R^m_n, defined in Example 8, contains the point $p = \left(0,0,\ldots,0,\frac{1}{3}\left(1+\frac{1}{10}+\ldots+\frac{1}{10^{m-1}}\right)\right)$ with $\#\pi^{-1}(p) \geqslant m$ and the ideal $J_p(R^m_n)$ is generated by n functions in $\mathcal{B}(R^m_n)$.*

P r o o f. This is just a combination of Claim 11 and Lemma 9 together with Remark 10. $\square$

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Author's address: *Linus Carlsson*, Department of Mathematics and Mathematical Statistics, Umeå University, S-901 87 Umeå, Sweden, e-mail: `linus@math.umu.se`.