SYSTEMS OF REACTION-DIFFUSION EQUATIONS WITH SPATIALLY DISTRIBUTED HYSTERESIS

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Abstract. We study systems of reaction-diffusion equations with discontinuous spatially distributed hysteresis on the right-hand side. The input of the hysteresis is given by a vector-valued function of space and time. Such systems describe hysteretic interaction of non-diffusive (bacteria, cells, etc.) and diffusive (nutrient, proteins, etc.) substances leading to formation of spatial patterns. We provide sufficient conditions under which the problem is well posed in spite of the assumed discontinuity of hysteresis. These conditions are formulated in terms of geometry of the manifolds defining the hysteresis thresholds and the spatial profile of the initial data.

Keywords: spatially distributed hysteresis; reaction-diffusion equation; well-posedness

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1. Setting of the problem

1.1. Introduction and setting. Reaction-diffusion equations with spatially distributed hysteresis were first introduced in [6] to describe the growth of a colony of bacteria (Salmonella typhimurium) and explain emerging spatial patterns of the bacteria density. In [6], [7], numerical analysis of the problem was carried out, however without rigorous justification. First analytical results were obtained in [2], [17] (see also [1], [11]), where existence of solutions for multi-valued hysteresis was

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proved. Formal asymptotic expansions of solutions were recently obtained in a special case in [8]. Questions about the uniqueness of solutions and their continuous dependence on the initial data as well as a thorough analysis of pattern formation have remained open. In this paper, we formulate sufficient conditions that guarantee existence, uniqueness, and continuous dependence of solutions on initial data for systems of reaction-diffusion equations with discontinuous spatially distributed hysteresis. Analogous conditions for scalar equations have been considered by the authors in [4], [5].

Denote $Q_T = (0,1)\times(0,T)$, where $T>0$. Let $\mathcal{U}\subset\mathbb{R}^k$ and $\mathcal{V}\subset\mathbb{R}^l$ $(k,l\in\mathbb{N})$ be closed sets. We assume throughout that $(x,t)\in\overline{Q}_T$, $u(x,t)\in\mathcal{U}$, $v(x,t)\in\mathcal{V}$.

We consider the system of reaction-diffusion equations

$$\begin{cases} u_t = Du_{xx} + f(u,v,W(\xi_0,u)),\\ v_t = g(u,v,W(\xi_0,u)) \end{cases} \tag{1.1}$$

with the initial and boundary conditions

$$u|_{t=0}=\varphi(x),\quad v|_{t=0}=\psi(x),\quad u_x|_{x=0}=u_x|_{x=1}=0. \tag{1.2}$$

Here D is a positive-definite diagonal matrix; W is a hysteresis operator which maps an initial configuration function $\xi_0(x)$ $(\in\{1,-1\})$ and an input function $u(x,\cdot)$ to an output function $W(\xi_0(x),u(x,\cdot))(t)$. As a function of (x,t), $W(\xi_0,u)$ takes values in a set $\mathcal{W}\subset\mathbb{R}^m$ $(m\in\mathbb{N})$. Now we shall define this operator in detail.

Let $\Gamma_\alpha,\Gamma_\beta\subset\mathcal{U}$ be two disjoint smooth manifolds of codimension one without boundary (we call them *hysteresis thresholds*). For clarity of explanation, we assume that they are given by $\gamma_\alpha(u)=0$ and $\gamma_\beta(u)=0$ with $\nabla\gamma_\alpha(u)\neq 0$ and $\nabla\gamma_\beta(u)\neq 0$, respectively, where γ_α and γ_β are C^∞-smooth functions.

Denote $M_\alpha=\{u\in\mathcal{U}\colon \gamma_\alpha(u)\leqslant 0\}$, $M_\beta=\{u\in\mathcal{U}\colon \gamma_\beta(u)\leqslant 0\}$, $M_{\alpha\beta}=\{u\in\mathcal{U}\colon \gamma_\alpha(u)>0,\ \gamma_\beta(u)>0\}$. Assume that $M_\alpha\cap\Gamma_\beta=\emptyset$ and $M_\beta\cap\Gamma_\alpha=\emptyset$ (Figure 1).

Next, we introduce functions

$$W_1\colon D(W_1)=M_\alpha\cup\overline{M}_{\alpha\beta}\to\mathcal{W},\quad W_{-1}\colon D(W_{-1})=M_\beta\cup\overline{M}_{\alpha\beta}\to\mathcal{W}, \tag{1.3}$$

which we call *hysteresis branches.*

Condition 1.1. *The functions W_1 and W_{-1} are locally Hölder continuous.*

For any $\zeta_0\in\{1,-1\}$ (*initial configuration*) and $u_0\in C([0,T];\mathcal{U})$ (*$\mathcal{U}$-valued input*) we introduce the *configuration* function

$$\zeta\colon \{1,-1\}\times C([0,T];\mathcal{U})\to L_\infty(0,T),\quad \zeta(t)=\zeta(\zeta_0,u_0)(t)$$

as follows. Let $X_t = \{s \in (0,t]\colon\ u_0(s) \in \Gamma_\alpha \cup \Gamma_\beta\}$. Then $\zeta(0) = 1$ if $u_0(0) \in M_\alpha$, $\zeta(0) = -1$ if $u_0(0) \in M_\beta$, $\zeta(0) = \zeta_0$ if $u_0(0) \in M_{\alpha\beta}$; for $t \in (0,T]$, $\zeta(t) = \zeta(0)$ if $X_t = \emptyset$, $\zeta(t) = 1$ if $X_t \neq \emptyset$ and $u_0(\max X_t) \in \Gamma_\alpha$, $\zeta(t) = -1$ if $X_t \neq \emptyset$ and $u_0(\max X_t) \in \Gamma_\beta$ (Figure 1).

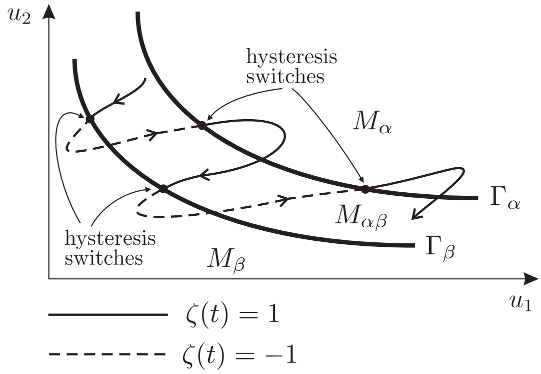

Figure 1. Regions of different behavior of the configuration ζ and the hysteresis W.

Now we introduce the *hysteresis operator* $W\colon\ \{1,-1\} \times C([0,T];\mathcal{U}) \to L_\infty(0,T)$ by the following rule (cf. [12], [18], [10]). For any initial configuration $\zeta_0 \in \{1,-1\}$ and input $u_0 \in C([0,T];\mathcal{U})$, the function $W(\zeta_0,u_0)\colon\ [0,T] \to \mathcal{W}$ (*output*) is given by

$$W(\zeta_0,u_0)(t) = W_{\zeta(t)}(u_0(t)), \tag{1.4}$$

where $\zeta(t)$ is the configuration function (taking values ± 1) and $W_{\pm 1}$ are the functions in (1.3) defining the hysteresis branches.

Assume that the initial configuration and the input function depend on spatial variable x. Denote them by $\xi_0(x)$ and $u(x,t)$, where $u(x,\cdot) \in C([0,T];\mathcal{U})$. Using (1.4) and treating x as a parameter, we define the *spatially distributed hysteresis*

$$W(\xi_0(x),u(x,\cdot))(t) = W_{\xi(x,t)}(u(x,t)), \tag{1.5}$$

where $\xi(x,t) = \zeta(\xi_0(x),u(x,\cdot))(t)$ is the *spatial configuration* (taking values ± 1) and $W_{\pm 1}$ are the functions in (1.3) defining the hysteresis branches.

1.2. Function spaces. Denote by $L_q(0,1)$, $q > 1$, the standard Lebesgue space and by $W_q^l(0,1)$ with natural l the standard Sobolev space. For a noninteger $l > 0$, denote by $W_q^l(0,1)$ the Sobolev space with the norm

$$\|\varphi\|_{W_q^{[l]}(0,1)} + \left(\int_0^1 \mathrm{d}x \int_0^1 \frac{|\varphi^{([l])}(x) - \varphi^{([l])}(y)|^q}{|x-y|^{1+q(l-[l])}}\,\mathrm{d}y\right)^{1/q},$$

where $[l]$ is the integer part of l. Introduce the anisotropic Sobolev spaces $W_q^{2,1}(Q_T)$ with the norm

$$\left(\int_0^T \|u(\cdot,t)\|^q_{W_q^2(0,1)}\,\mathrm{d}t + \int_0^T \|u_t(\cdot,t)\|^q_{L_q(0,1)}\,\mathrm{d}t \right)^{1/q}$$

and the space $W_q^{0,1}(Q_T)$ of $L_q(0,1)$-valued functions continuously differentiable on $[0,T]$ with the norm

$$\|u\|_{L_q(Q_T)} + \|u_t\|_{L_q(Q_T)}.$$

Denote by $C^\gamma(\overline{Q}_T)$, $\gamma \in (0,1)$, the Hölder space with the norm

$$\|u\|_{C^\gamma(\overline{Q}_T)} = \sup_{(x,t)\in\overline{Q}_T} |u(x,t)| + \sup_{(x,t),(y,s)\in\overline{Q}_T} \frac{|u(x,t)-u(y,s)|}{|x-y|^\gamma + |t-s|^\gamma}.$$

For vector-valued functions, we use the following notation. If, e.g., $u(x,t) \in \mathcal{U}$ and each component of u belongs to $W_q^{2,1}(Q_T)$, then we write $u \in W_q^{2,1}(Q_T;\mathcal{U})$.

Throughout, we fix q and γ such that $q \in (3,\infty)$ and $\gamma \in (0, 1-3/q)$. This implies that $u, u_x \in C^\gamma(\overline{Q}_T;\mathcal{U})$ for $u \in W_q^{2,1}(Q_T;\mathcal{U})$ (see Lemma 3.3 in [13], Chapter 2).

To define the space of initial data, we use the fact that if $u \in W_q^{2,1}(Q_T;\mathcal{U})$, then the trace $u|_{t=0}$ is well defined and belongs to $W_q^{2-2/q}((0,1);\mathcal{U})$ (see Lemma 2.4 in [13], Chapter 2). Moreover, one can define the space $W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ as the subspace of functions from $W_q^{2-2/q}((0,1);\mathcal{U})$ with the zero Neumann boundary conditions.

We assume that $\varphi \in W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ and $\psi \in L_\infty((0,1);\mathcal{V})$ in (1.2).

Definition 1.1. A pair $(u,v) \in W_q^{2,1}(Q_T;\mathcal{U}) \times W_\infty^{0,1}(Q_T;\mathcal{V})$ is a *solution of problem* (1.1), (1.2) if $W(\xi_0,u)$ is measurable with respect to (x,t) and (1.1), (1.2) hold.

Note that, due to hysteresis, the right-hand side in (1.1) is discontinuous in x and t, but it belongs to $L_q(Q_T;\mathcal{U})$. Hence, it is natural to use the Sobolev spaces $W_q^{2,1}(Q_T;\mathcal{U})$ and $W_{q,N}^{2-2/q}((0,1);\mathcal{U})$ for the component u and its initial data rather than Hölder spaces. By choosing $q > 3$, we ensure that the spatial derivative of the solution u is continuous. This allows us to define the notion of spatial transversality (see the next section), which is crucial for our proof of well-posedness of system (1.1). Since the v-component of system (1.1) has no diffusion, the function v can be discontinuous in x. However, we will see that it is bounded and has a generalized derivative with respect to t. This motivates the choice for the space $W_\infty^{0,1}(Q_T;\mathcal{V})$.

1.3. Spatial transversality. We will deal with the case where $\xi_0(x)$ has one discontinuity point. Generalization to finitely many discontinuity points is straightforward.

Condition 1.2.

(1) *For some* $\overline{b} \in (0,1)$, *we have*

$$\xi_0(x) = 1 \ (x \leqslant \overline{b}), \quad \xi_0(x) = -1 \ (x > \overline{b}). \tag{1.6}$$

(2) *For* $x \in [0, \overline{b}]$, *we have* $\varphi(x) \in M_\alpha \cup M_{\alpha\beta}$ *or, equivalently,* $\gamma_\beta(\varphi(x)) > 0$.
(3) *For* $x \in (\overline{b}, 1]$, *we have* $\varphi(x) \in M_\beta \cup M_{\alpha\beta}$ *or, equivalently,* $\gamma_\alpha(\varphi(x)) > 0$.
(4) *If* $\gamma_\alpha(\varphi(\overline{b})) = 0$, *then* $(\mathrm{d}/\mathrm{d}x)\gamma_\alpha(\varphi(x))\big|_{x=\overline{b}} > 0$.

It follows from Condition 1.2 that the hysteresis in (1.5) at the initial moment equals $W_1(\varphi(x))$ for $x \leqslant \overline{b}$ and $W_2(\varphi(x))$ for $x > \overline{b}$. Parts 2 and 3 in Condition 1.2 are necessary for the hysteresis to be well defined at the initial moment, while part 4 is an essential assumption. We refer to part 4 as the *spatial transversality* and say that $\varphi(x)$ *is transverse with respect to the spatial configuration* $\xi_0(x)$. This means that if $\varphi(\overline{b}) \in \Gamma_\alpha$, then the vector $\varphi'(\overline{b})$ is transverse to the manifold Γ_α at this point.

Consider time-dependent functions $u(x,t)$ such that $u, u_x \in C(\overline{Q}_T; \mathcal{U})$.

Definition 1.2. We say that a function u is *transverse on* $[0,T]$ (with respect to a spatial configuration $\xi(x,t)$) if, for every fixed $t \in [0,T]$, either $\xi(\cdot,t)$ has no discontinuity points for $x \in (0,1)$, or it has one discontinuity point and the function $u(\cdot,t)$ is transverse with respect to the spatial configuration $\xi(\cdot,t)$.

Definition 1.3. A function u *preserves spatial topology* (*of a spatial configuration* $\xi(x,t)$) *on* $[0,T]$ if, for $t \in [0,T]$, there is a continuous function $b(t) \in (0,1)$ such that $\xi(x,t) = 1$ for $x \leqslant b(t)$ and $\xi(x,t) = -1$ for $x > b(t)$.

We say that the solution from Definition 1.1 is *transverse* (*preserves spatial topology*) if the function $u(x,t)$ is transverse (preserves spatial topology).

Remark 1.1. The function $b(t)$ defining the discontinuity point of $\xi(x,t)$ plays the role of a free boundary. It resembles the free boundary arising in parabolic obstacle-type problems, where, loosely speaking, the hysteresis thresholds coincide (see, e.g., [3], [15] and the references therein).

1.4. Assumptions on the right-hand side. First, we assume the following.

Condition 1.3. *The functions* $f(u,v,w)$ *and* $g(u,v,w)$ *are locally Lipschitz continuous in* $\mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$.

Next, we formulate dissipativity conditions for f and g.

In the following condition, we denote by $\mathcal{U}_\mu$, $\mu \geqslant 0$, closed parallelepipeds in $\mathcal{U}$ with the edges parallel to coordinate axes such that $\varphi(x) \in \mathcal{U}_\mu$ for all $x \in [0,1]$.

Condition 1.4. *There is a parallelepiped $\mathcal{U}_0$ and, for each sufficiently small $\mu > 0$, there is a parallelepiped $\mathcal{U}_\mu$ and a locally Lipschitz continuous function $f_\mu(u,v)$ such that*

(1) *$|f_\mu(u,v)|$ converges to 0 uniformly on compact sets in $\mathcal{U} \times \mathcal{V}$ as $\mu \to 0$,*
(2) *for each $j = \pm 1$ and each point $u \in \partial\mathcal{U}_0 \cap D(W_j)$, $v \in \mathcal{V}$, the vector $f(u,v,W_j(u)) + f_\mu(u,v)$ points strictly inside $\mathcal{U}_0$,*
(3) *for each $j = \pm 1$ and each point $u \in \partial\mathcal{U}_\mu \cap D(W_j)$, $v \in \mathcal{V}$, the vector $f(u,v,W_j(u_\mu)) + f_\mu(u,v)$ points strictly inside $\mathcal{U}_\mu$ for all $u_\mu \in \mathcal{U}_\mu$.*

To formulate the assumption on g, we fix $\mathcal{U}_0$ satisfying Condition 1.4 and set

$$\mathcal{W}_0 = \bigcup_{j=\pm 1} \{W_j(u)\colon\ u \in \mathcal{U}_0\}. \tag{1.7}$$

Condition 1.5. *For any $T_0 > 0$, there is a compact $\mathcal{V}_0 = \mathcal{V}_0(T_0, \mathcal{U}_0) \subset \mathcal{V}$ such that $\psi(x) \in \mathcal{V}_0$ $(x \in (0,1))$ and the Cauchy problem*

$$v_t = g(u_0(x,t), v, w_0(x,t)), \quad v|_{t=0} = \psi(x) \tag{1.8}$$

has a solution $v \in W^{0,1}_\infty(Q_{T_0}; \mathbb{R}^l)$ satisfying $v(x,t) \in \mathcal{V}_0$ whenever

$$\begin{gathered} u_0 \in L_\infty(Q_{T_0}; \mathcal{U}), \quad w_0 \in L_\infty(Q_{T_0}; \mathcal{W}), \\ u_0(x,t) \in \mathcal{U}_0, \quad w_0(x,t) \in \mathcal{W}_0 \quad ((x,t) \in Q_{T_0}). \end{gathered}$$

R e m a r k 1.2. It follows from [14], Theorem 1, page 111, that system (1.8) has a unique solution $v \in W^{0,1}_\infty(Q_{T_0}; \mathbb{R}^l)$ for a sufficiently small $T_0 > 0$. Condition 1.5 additionally guarantees the absence of blow-up. In particular, the uniform boundedness of v holds if $|g(u,v,w)| \leqslant A(u,w)|v| + B(u,w)$, where $A(u,w)$ and $B(u,w)$ are bounded on compact sets (see Example 1.1). However, if $\mathcal{V} \neq \mathbb{R}^l$, one must additionally check that v does not leave $\mathcal{V}$, i.e., there exists a corresponding compact $\mathcal{V}_0$ lying *inside* $\mathcal{V}$. Alternatively, to fulfil Condition 1.5, one could assume the existence of invariant parallelepipeds for g, similarly to Condition 1.4.

E x a m p l e 1.1. The hysteresis operator and the right-hand side in the present paper apply to a model describing the growth of a colony of bacteria (Salmonella typhimurium) in a Petri plate (see, e.g., [7]). Let $u_1(x,t)$ and $u_2(x,t)$ denote the concentrations of diffusing buffer (pH level) and histidine (nutrient), respectively, while $v(x,t)$ denotes the density of nondiffusing bacteria. These three unknown functions satisfy the equations

$$
\begin{cases}
\dfrac{\partial u_1}{\partial t} = D_1 \Delta u_1 - a_1 W(\xi_0, u) v, \\
\dfrac{\partial u_2}{\partial t} = D_2 \Delta u_2 - a_2 W(\xi_0, u) v, \\
\dfrac{\partial v}{\partial t} = a W(\xi_0, u) v
\end{cases}
\tag{1.9}
$$

supplemented by the initial and no-flux (Neumann) boundary conditions. In (1.9), $D_1, D_2, a, a_1, a_2 > 0$ are given constants and $W(\xi_0, u)$ is the hysteresis operator. In this example, we have $\mathcal{U} = \{u \in \mathbb{R}^2 \colon u_1, u_2 \geqslant 0\}$, $\mathcal{V} = [0, \infty)$, $\mathcal{W} = [0, \infty)$. The hysteresis thresholds Γ_α and Γ_β are the curves in the plane given by $\gamma_\alpha(u) := -u_1 + a_\alpha/u_2 + b_\alpha = 0$ and $\gamma_\beta(u) := u_1 - a_\beta/u_2 - b_\beta = 0$, respectively, where $a_\alpha, a_\beta, b_\alpha, b_\beta > 0$ are some constants (Figure 1); the hysteresis branches are given by $W_1(u) = 1$ and $W_{-1}(u) = 0$.

2. Main results

In what follows, we assume that Conditions 1.1–1.5 hold.

Theorem 2.1 (local existence)**.** *There is a number $T > 0$ such that*

(1) *there is at least one solution of problem* (1.1), (1.2) *in Q_T;*
(2) *any solution in Q_T is transverse and preserves spatial topology.*

Theorem 2.2 (continuation of solutions)**.** *Let (u, v) be a transverse topology preserving solution of problem* (1.1), (1.2) *in Q_T for some $T > 0$. Then it can be continued to an interval $[0, T_{\max})$, where $T_{\max} > T$ has the following properties:*

1. *For any $t_0 < T_{\max}$, the pair (u, v) is a transverse solution of problem* (1.1), (1.2) *in Q_{t_0}.*
2. *Either $T_{\max} = \infty$, or $T_{\max} < \infty$ and (u, v) is a solution in $Q_{T_{\max}}$, but $u(\cdot, T_{\max})$ is not transverse with respect to $\xi(\cdot, T_{\max})$.*

Theorem 2.3 (continuous dependence on initial data)**.** *Assume the following.*

(1) *There is a number $T > 0$ such that problem* (1.1), (1.2) *with initial functions φ, ψ and initial configuration $\xi_0(x)$ defined by its discontinuity point $\overline{b}$ admits a unique transverse topology preserving solution (u, v) in Q_s for any $s \leqslant T$.*
(2) *Let $\varphi_n \in W_{q,N}^{2-2/q}((0,1); \mathcal{U})$, $\psi_n \in L_\infty((0,1); \mathcal{V})$, $n = 1, 2, \dots$, be a sequence of initial functions such that $\|\varphi - \varphi_n\|_{W_q^{2-2/q}((0,1);\mathcal{U})} \to 0$, $\|\psi - \psi_n\|_{L_q((0,1);\mathcal{V})} \to 0$ as $n \to \infty$.*

(3) *Let* $\xi_{0n}(x)$, $n=1,2,\ldots$, *be a sequence of initial configurations defined by their discontinuity points* $\overline{b}_n$ *similarly to* (1.6) *and* $\overline{b}_n \to \overline{b}$ *as* $n\to\infty$.

Then, for all sufficiently large n, *problem* (1.1), (1.2) *with the initial data* $(\varphi_n, \psi_n, \xi_{0n})$ *has at least one transverse topology preserving solution* (u_n, v_n). *Each sequence of such solutions satisfies*

$$\|u_n-u\|_{W_q^{2,1}(Q_T;\mathcal{U})}\to 0, \quad \|v_n-v\|_{W_q^{0,1}(Q_T;\mathcal{V})}\to 0, \quad \|b_n-b\|_{C[0,T]}\to 0$$

as $n\to\infty$, *where* $b(t)$ *and* $b_n(t)$ *are the respective discontinuity points of the configuration functions* $\xi(x,t)$ *and* $\xi_n(x,t)$.

Remark 2.1. If one knows a priori that all u_n are transverse on some interval $[0,T]\subset[0,T_{\max})$, then one can prove that u_n approximate u on $[0,T]$ even if u is not topology preserving on $[0,T]$.

Now we discuss the uniqueness of solutions. We will strengthen Condition 1.1 concerning the local Hölder continuity of $W_{\pm 1}$. Let $\mathcal{U}_0$ be the set from Condition 1.4.

Condition 2.1. *There are numbers* $K>0$ *and* $\sigma\in[0,1)$ *such that*

$$|W_1(u)-W_1(\hat{u})| \leqslant \frac{K}{(\gamma_\beta(u))^\sigma+(\gamma_\beta(\hat{u}))^\sigma}|u-\hat{u}|, \qquad \forall\, u,\hat{u}\in M_\alpha\cup\overline{M}_{\alpha\beta},$$

$$|W_{-1}(u)-W_{-1}(\hat{u})| \leqslant \frac{K}{(\gamma_\alpha(u))^\sigma+(\gamma_\alpha(\hat{u}))^\sigma}|u-\hat{u}|, \qquad \forall\, u,\hat{u}\in M_\beta\cup\overline{M}_{\alpha\beta}.$$

We refer the reader to [5] for the discussion about functions satisfying this condition.

Theorem 2.4 (uniqueness)**.** *Assume additionally that Condition* 2.1 *holds. Let* (u,v) *and* $(\hat{u},\hat{v})$ *be two transverse solutions of problem* (1.1), (1.2) *in* Q_T *for some* $T>0$. *Then* $(u,v)=(\hat{u},\hat{v})$.

3. Local existence, continuation and continuous dependence of solutions on the initial data

In this section we prove Theorems 2.1–2.3. Throughout the section, we fix $\mathcal{U}_0$ satisfying Condition 1.4 and $\mathcal{W}_0$ given by (1.7). Next, we fix some $T_0\in(0,1]$ and then $\mathcal{V}_0$ satisfying Condition 1.5.

The idea of the proof of Theorem 2.1 concerning the local existence is to show the existence of a fixed point of a nonlinear map $\mathcal{R}$ that is defined as follows. We take some function $u_1(x,t)$ and define the right-hand side of (1.1) via $W(\xi_0,u_1)$ instead of $W(\xi_0, u)$. After that, we solve the resulting system, which has no hysteresis depending on u any more, and obtain a solution (u, v). Then we set $\mathcal{R}\colon\ u_1 \mapsto u$. To make this scheme work, we have to choose a proper set for the functions u_1. It must be mapped into itself by $\mathcal{R}$, and the topology must be chosen in such a way that the operator $\mathcal{R}$ be continuous and compact. The required properties of the functions u_1 from the domain of $\mathcal{R}$ will be described in Lemma 3.4 and the properties of $u = \mathcal{R}(u_1)$ in Lemmas 3.5 and 3.6. Based on these lemmas, we shall implement the above scheme in Section 3.3.

3.1. Preliminaries. The following result is straightforward.

Lemma 3.1.

(1) *Let* $\lambda \in [0,1)$, $a \in C^\lambda[0,T]$, *and* $b(t) = \max\limits_{s\in[0,t]} a(s)$. *Then* $b \in C^\lambda[0,T]$ *and* $\|b\|_{C^\lambda[0,T]} \leqslant \|a\|_{C^\lambda[0,T]}$.

(2) *If* $a_j \in C[0,T]$ *and* $b_j(t) = \max\limits_{s\in[0,t]} a_j(s)$, $j = 1, 2$, *then* $\|b_1 - b_2\|_{C[0,T]} \leqslant \|a_1 - a_2\|_{C[0,T]}$.

Take some $T \leqslant T_0$. Consider functions $u_1, \hat{u}_1 \in L_\infty(Q_T; \mathcal{U})$ such that $u_1(x,t)$, $\hat{u}_1(x,t) \in \mathcal{U}_0$ $((x,t) \in Q_T)$. Next, consider functions $b_1, \hat{b}_1 \in C[0,T]$ such that $b_1(t), \hat{b}_1(t) \in [0,1)$ $(t \in [0,T])$. Define functions

$$w_1(x,t) = \begin{cases} W_1(u_1(x,t)), \ 0 \leqslant x \leqslant b_1(t), \\ W_{-1}(u_1(x,t)), \ b_1(t) < x \leqslant 1, \end{cases} \tag{3.1}$$

$$\hat{w}_1(x,t) = \begin{cases} W_1(\hat{u}_1(x,t)), \ 0 \leqslant x \leqslant \hat{b}_1(t), \\ W_{-1}(\hat{u}_1(x,t)), \ \hat{b}_1(t) < x \leqslant 1; \end{cases}$$

here we assume $W_{\pm 1}(u_1)$ and $W_{\pm 1}(\hat{u}_1)$ to be extended to $\mathcal{U}_0$ without loss of regularity.

Lemma 3.2. *For any* $p \in [1, \infty)$ *and* $t \in [0,T]$, *we have*

$$\|w_1(\cdot,t) - \hat{w}_1(\cdot,t)\|_{L_p((0,1);\mathcal{W})} \leqslant c_0(\|u_1(\cdot,t) - \hat{u}_1(\cdot,t)\|^{\sigma_0}_{L_\infty((0,1);\mathcal{U})} + |b_1(t) - \hat{b}_1(t)|^{1/p}),$$
$$\|w_1 - \hat{w}_1\|_{L_p(Q_T;\mathcal{W})} \leqslant c_0(T^{1/p}\|u_1 - \hat{u}_1\|^{\sigma_0}_{L_\infty(Q_T;\mathcal{U})} + \|b_1 - \hat{b}_1\|^{1/p}_{L_1(0,T)}),$$

where σ_0 *is a Hölder exponent for the functions* $W_{\pm 1}$ *and* $c_0 > 0$ *depends on* $\mathcal{U}_0$ *and* p, *but does not depend on* u_1, b_1, $\hat{u}_1$, $\hat{b}_1$, T.

P r o o f. We fix $t \in [0,T]$ and assume that $b_1(t) \leqslant \hat{b}_1(t)$ for this t. Then, using (3.1), we have

$$\int_0^1 |w_1 - \hat{w}_1|^p \,\mathrm{d}x = \int_0^{b_1(t)} |W_1(u_1) - W_1(\hat{u}_1)|^p \,\mathrm{d}x + \int_{\hat{b}_1(t)}^1 |W_{-1}(u_1) - W_{-1}(\hat{u}_1)|^p \,\mathrm{d}x$$
$$+ \int_{b_1(t)}^{\hat{b}_1(t)} |W_{-1}(u_1) - W_1(\hat{u}_1)|^p \,\mathrm{d}x.$$

Using the Hölder continuity and the boundedness of $W_{\pm 1}(u)$ for $u \in \mathcal{U}_0$, we obtain the first inequality in the lemma. Integrating it with respect to t from 0 to T, we obtain the second inequality. $\square$

Now we introduce sets that "measure" the spatial transversality. Denote by E_m, $m \in \mathbb{N}$, the set of triples (φ, ψ, ξ_0) such that $\varphi \in W^{2-2/q}_{q,N}((0,1);\mathcal{U})$, $\psi \in L_\infty((0,1);\mathcal{V})$, $\xi_0(x)$ is of the form (1.6), and the following hold:

(1) $\overline{b} \in [1/m, 1-1/m]$,

(2) $\gamma_\beta(\varphi(x)) \geqslant 1/m^2$ for $x \in [0, \overline{b}]$,

(3) $\gamma_\alpha(\varphi(x)) \geqslant 1/m^2$ for $x \in [\overline{b} + 1/m, 1]$,

(4) if $x \in [\overline{b}, \overline{b} + 1/m]$ and $\gamma_\alpha(\varphi(x)) \in [0, 1/m^2]$, then $(\mathrm{d}/\mathrm{d}x)\gamma_\alpha(\varphi(x)) \geqslant 1/m$,

(5) $\|\varphi\|_{W_q^{2-2/q}((0,1);\mathcal{U})} \leqslant m$ and $\|\psi\|_{L_\infty((0,1);\mathcal{V})} \leqslant m$.

It is easy to check that $E_m \subset E_{m+1}$. Furthermore, the sets E_m have the following properties (see Lemma 2.25 in [4]).

Lemma 3.3.

(1) *The union of all sets E_m, $m \geqslant 1$, coincides with the set of all data satisfying Condition* 1.2.

(2) *Assume*

(a) $(\varphi_m, \psi_m, \xi_m) \in E_m \setminus E_{m-1}$, $m = 2, 3, \ldots$;

(b) $\|\varphi_m - \varphi\|_{W_q^{2-2/q}((0,1);\mathcal{U})} \to 0$ *and* $\|\psi_m - \psi\|_{L_\infty((0,1);\mathcal{V})} \to 0$ *as* $m \to \infty$ *for some* $\varphi \in W_q^{2-2/q}((0,1);\mathcal{U})$ *and* $\psi \in L_\infty((0,1);\mathcal{V})$;

(c) $\overline{b}_m - \overline{b} \to 0$ *as* $m \to \infty$ *for some* $\overline{b} \in [0,1]$.

Then $\overline{b} \in \{0,1\}$ *or* $\varphi(x)$ *is not transverse with respect to* $\xi_0(x)$, *where* $\xi_0(x)$ *is given by* (1.6).

From now on, we fix $m \in \mathbb{N}$ such that $(\varphi, \psi, \xi_0) \in E_m$.

The next lemma follows from the implicit function theorem and Lemma 3.1. It describes the properties of the functions u_1 from the domain of the map $\mathcal{R}$ which we construct in Section 3.3.

Lemma 3.4. *Let $\lambda \in (0,1)$, $u_1, \partial u_1/\partial x \in C^\lambda(\overline{Q}_{T_0};\mathcal{U})$,*

$$\|u_1\|_{C^\lambda(\overline{Q}_{T_0};\mathcal{U})} + \left\|\frac{\partial u_1}{\partial x}\right\|_{C^\lambda(\overline{Q}_{T_0};\mathcal{U})} \leqslant c$$

for some $c > 0$, $u_1|_{t=0} = \varphi(x)$, and $(\varphi, \psi, \xi_0) \in E_m$. Then there are $T_1 = T_1(m, \lambda, c) \leqslant T_0$ and a natural number $N_1 = N_1(m, \lambda, c) \geqslant m$ which do not depend on u, φ, ξ_0 such that the following is true for any $t \in [0, T_1]$:

(1) *The equation $\gamma_\alpha(u_1(x,t)) = 0$ for $x \in [\overline{b}, 1]$ has at most one root. If this root exists, we denote it by $a_1(t)$; otherwise, we set $a_1(t) = \overline{b}$. One has $a_1(t) \in [\overline{b}, \overline{b} + 1/N_1]$, $a_1 \in C^\lambda[0, T_1]$ and $\|a_1\|_{C^\lambda[0,T_1]} \leqslant 1 + 2mc$.*
(2) *The hysteresis $W(\xi_0, u_1)$ and its configuration function $\xi_1(x,t)$ have exactly one discontinuity point $b_1(t)$; moreover, $b_1(t) = \max\limits_{s \in [0,t]} a_1(s)$, $b_1 \in C^\lambda[0, T_1]$.*

3.2. Auxiliary problem. Consider functions $u_1 \in L_\infty(Q_T;\mathcal{U})$ and functions $w_1 \in L_\infty(Q_T;\mathcal{W})$ such that

$$u_1(x,t) \in \mathcal{U}_0, \quad w_1(x,t) \in \mathcal{W}_0 \quad ((x,t) \in Q_T)$$

for some $T > 0$. Define functions

$$f_1(u, v, x, t) = f(u, v, w_1(x,t)), \quad g_1(v, x, t) = g(u_1(x,t), v, w_1(x,t)). \tag{3.2}$$

Consider the auxiliary problem

$$\begin{cases} u_t = Du_{xx} + f_1(u, v, x, t), \\ v_t = g_1(v, x, t), \\ u|_{t=0} = \varphi(x), \quad v|_{t=0} = \psi(x), \quad u_x|_{x=0} = u_x|_{x=1} = 0. \end{cases} \tag{3.3}$$

Set $f_U = \sup f(u, v, w)$ and $g_U = \sup g(u, v, w)$, where $(u, v, w) \in \mathcal{U}_0 \times \mathcal{V}_0 \times \mathcal{W}_0$.

The next result follows from the standard estimates for solutions of linear parabolic equations [13], from Conditions 1.3–1.5, and from the principle of invariant rectangles [16].

Lemma 3.5.

(1) *For any $T \leqslant T_0$, problem (3.3) has a unique solution $(u, v) \in W_q^{2,1}(Q_T;\mathcal{U}) \times W_\infty^{0,1}(Q_T;\mathcal{V})$ and*

$$u(x,t) \in \mathcal{U}_0, \quad v(x,t) \in \mathcal{V}_0 \quad ((x,t) \in Q_T),$$

$$\|u\|_{W_q^{2,1}(Q_T;\mathcal{U})} + \max_{t \in [0,T]} \|u(\cdot,t)\|_{W_q^{2-2/q}((0,1);\mathcal{U})} \leqslant c_1(\|\varphi\|_{W_q^{2-2/q}((0,1);\mathcal{U})} + f_U),$$

$$\|v\|_{W_\infty^{0,1}(Q_T;\mathcal{V})} \leqslant \|\psi\|_{L_\infty((0,1);\mathcal{V})} + 2g_U,$$

$$\|u\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} + \|u_x\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \leqslant c_2(\|\varphi\|_{W_q^{2-2/q}((0,1);\mathcal{U})} + f_U), \tag{3.4}$$

where $c_1, c_2 > 0$ depend only on T_0.

(2) *If* u_n, v_n, $n = 1, 2, \ldots$, *are solutions of problem* (3.2), (3.3) *with* u_1, w_1 *replaced by* u_{1n}, w_{1n} (*with the same properties*) *and*

$$\|u_{1n} - u_1\|_{L_\infty(Q_T;\mathcal{U})} + \|w_{1n} - w_1\|_{L_q(Q_T;\mathcal{W})} \to 0 \quad \text{as } n \to \infty,$$

then

$$\|u_n - u\|_{W_q^{2,1}(Q_T;\mathcal{U})} + \|v_n - v\|_{W_q^{0,1}(Q_T;\mathcal{V})} \to 0 \quad \text{as } n \to \infty.$$

The next lemma follows from Lemmas 3.4 and 3.5. Together with Lemma 3.5, it describes the image of the operator $\mathcal{R}$ which we construct in Section 3.3. In particular, it shows that the functions from the image remain spatially transverse on a sufficiently small time interval. Moreover, this time interval depends on m, but not on a particular initial data from E_m.

Lemma 3.6. *Let* (u, v) *be the solution of problem* (3.3) *defined in part* (1) *of Lemma* 3.5. *Then there are* $T_2 = T_2(m) \leqslant T_0$ *and a natural number* $N_2 = N_2(m) \geqslant m$ *such that, for any* $t \in [0, T_2]$, *conclusions* (1) *and* (2) *from Lemma* 3.4 *hold for* $u(x, t)$, *for the corresponding function* $a(t)$, *for the configuration function* $\xi(x, t)$ *of the hysteresis* $W(\xi_0, u)$, *for its discontinuity point* $b(t)$, *and for* T_2, N_2 *instead of* T_1, N_1. *Furthermore,* $(u(\cdot, t), v(\cdot, t), \xi(\cdot, t)) \in E_{N_2}$.

3.3. Proof of Theorem 2.1: local existence. Let us prove the first assertion.

Fix λ in Lemma 3.4 such that $\lambda \in (0, \gamma)$. Fix c_2 from Lemma 3.5. Set $c = c_{\lambda,\gamma} c_2(m + f_U)$, where $c_{\lambda,\gamma} > 0$ is the embedding constant such that $\|u\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} \leqslant c_{\lambda,\gamma}\|u\|_{C^\gamma(\overline{Q}_T;\mathcal{U})}$. Set $T = \min(T_1, T_2)$, where T_1, T_2 are defined in Lemmas 3.4 and 3.6.

Let $R^\lambda(\overline{Q}_T)$ be the set of functions $u(x, t)$ such that $u|_{t=0} = \varphi(x)$,

$$\begin{aligned} &u, u_x \in C^\lambda(\overline{Q}_T;\mathcal{U}), \quad u(x,t) \in \mathcal{U}_0 \quad ((x,t) \in Q_T), \\ &\|u\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} + \|u_x\|_{C^\lambda(\overline{Q}_T;\mathcal{U})} \leqslant c. \end{aligned} \tag{3.5}$$

The set $R^\lambda(\overline{Q}_T)$ is a closed convex subset of the Banach space endowed with the norm given by the left-hand side of the inequality in (3.5). Similarly, we define $R^\gamma(\overline{Q}_T)$.

Now we construct a map $\mathcal{R}\colon R^\lambda(\overline{Q}_T) \to R^\gamma(\overline{Q}_T)$ as follows. Take any $u_1 \in R^\lambda(\overline{Q}_T)$ and define $a_1(t)$ and $b_1(t)$ according to Lemma 3.4. Then define $w_1(x, t)$ by (3.1) and, using this w_1, define f_1, g_1 by (3.2). Finally, apply Lemma 3.5 and obtain a solution (u, v) of auxiliary problem (3.3). We now define $\mathcal{R}\colon u_1 \mapsto u$.

The operator $\mathcal{R}$ is continuous. Indeed, it is not difficult to check that the mapping $u_1 \mapsto a_1$ is continuous from $R^\lambda(\overline{Q}_T)$ to $C[0,T]$. Thus, the continuity of $\mathcal{R}$ follows by consecutively applying Lemmas 3.1 part (2), 3.2, 3.5 part (2), and the continuity of the embedding $W_q^{2,1}(Q_T;\mathcal{V}) \subset R^\gamma(\overline{Q}_T)$.

Furthermore, due to (3.4) and the choice of c, the operator $\mathcal{R}$ maps $R^\lambda(\overline{Q}_T)$ into itself. As an operator acting from $R^\lambda(\overline{Q}_T)$ into itself, it is compact due to its continuity from $R^\lambda(\overline{Q}_T)$ to $R^\gamma(\overline{Q}_T)$ and the compatness of the embedding $R^\gamma(\overline{Q}_T) \subset R^\lambda(\overline{Q}_T)$. Therefore, applying the Schauder fixed-point, we conclude the proof of the first assertion of the theorem.

The second assertion follows by applying the principle of invariant rectangles (see [16]) and Lemma 3.6. □

3.4. Proof of Theorem 2.2: continuation of solutions. Assume that there is $T_1 > 0$ such that the solution (u, v) cannot be continued to $[0, T_1]$ as a transverse solution of problem (1.1), (1.2).

Applying Theorem 2.1 and using Lemma 3.6, we obtain sequences $m_k \in \mathbb{N}$, $t_k > 0$ $(k = 1, 2, \dots)$ such that $m_{k+1} > m_k$, $t_{k+1} > t_k$, the solution (u, v) of problem (1.1), (1.2) can be continued as a transverse solution to the time interval $[0, t_k]$ and

$$(u(\cdot, t_k), v(\cdot, t_k), \xi(\cdot, t_k)) \in E_{m_k} \setminus E_{m_k - 1}, \tag{3.6}$$

where $\xi(x, t)$ is the spatial configuration of the hysteresis $W(\xi_0, u)$. Denote $T = \lim\limits_{k\to\infty} t_k$. By assumption, $T \leqslant T_1$. Since (u, v) is a solution of problem (1.1), (1.2) in Q_{t_k} for all k and

$$\begin{aligned}
\|u\|_{W_q^{2,1}(Q_{t_k};\mathcal{U})} &\leqslant c_1(\|\varphi\|_{W_q^{2-2/q}((0,1);\mathcal{U})} + f_U),\\
\|v\|_{W_\infty^{0,1}(Q_{t_k};\mathcal{V})} &\leqslant \|\psi\|_{L_\infty((0,1);\mathcal{V})} + 2g_U,
\end{aligned}$$

it follows that $u \in W_q^{2,1}(Q_T;\mathcal{U})$, $v \in W_\infty^{0,1}(Q_T;\mathcal{V})$, and (u, v) is a solution of problem (1.1), (1.2) in Q_T. Since $u(\cdot, t)$ is a continuous $W_q^{2-2/q}((0,1);\mathcal{U})$-valued function and $v(\cdot, t)$ is a continuous $L_\infty((0,1);\mathcal{V})$-valued function, we have

$$\begin{aligned}
&\|u(\cdot, t_k) - u(\cdot, T)\|_{W_q^{2-2/q}((0,1);\mathcal{U})} \to 0,\\
&\qquad \|v(\cdot, t_k) - v(\cdot, T)\|_{L_\infty((0,1);\mathcal{V})} \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{3.7}$$

Denote by $b(t)$ the discontinuity point of the configuration function $\xi(x, t)$. By construction, $b(t)$ is continuous and nondecreasing on $[0, t_k]$ for all k. Therefore, $b(t)$ is continuous on $[0, T]$. In particular, $b(t_k) \to b(T)$, $k \to \infty$. It follows from (3.6), (3.7), and part (2) of Lemma 3.3 that $u(x, T)$ is not transverse with respect to $\xi(x, T)$

(and Theorem 2.2 is proved in this case) or $b(T) = 1$. In the latter case, we can proceed similarly to the above, but effectively with hysteresis whose configuration function has no discontinuity points any more. $\square$

3.5. Proof of Theorem 2.3: continuous dependence on initial data. It suffices to prove the theorem for a sufficiently small time interval. Since $(\varphi, \psi, \xi_0) \in E_m$, it is easy to show that there is $n_1 = n_1(m) > 0$ such that $(\varphi, \psi, \xi_0), (\varphi_n, \psi_n, \xi_{0n}) \in E_{m+1}$ for all $n \geqslant n_1(m)$. Hence, by Theorem 2.1, there is $T \in (0, 1]$ for which problem (1.1), (1.2) has transverse topology preserving solutions (u, v) and (u_n, v_n) with the corresponding initial data. Moreover, any solution of problem (1.1), (1.2) in Q_T is transverse and preserves topology.

Let $a(t)$ and $a_n(t)$ be the functions corresponding to u and u_n as described in Lemma 3.6. Then the discontinuity points of the corresponding configuration functions $\xi(x,t)$, $\xi_n(x,t)$ are given by $b(t) = \max\limits_{s\in[0,t]} a(s)$ and $b_n(t) = \max\limits_{s\in[0,t]} a_n(s)$.

Assume that there is $\varepsilon > 0$ such that

$$(3.8)\quad \|u_n - u\|_{W_q^{2,1}(Q_T;\mathcal{U})} + \|v_n - v\|_{W_q^{0,1}(Q_T;\mathcal{V})} + \|b_n - b\|_{C[0,T]} \geqslant \varepsilon, \quad n = 1, 2, \dots,$$

for some subsequence of u_n, which we denote u_n again. Lemmas 3.4 and 3.5 imply that u_n and a_n are uniformly bounded in $W_q^{2,1}(Q_T;\mathcal{U})$ and $C^\gamma[0,T]$, respectively. Hence, we can choose subsequences of u_n and a_n (which we denote u_n and a_n again) such that

$$(3.9)\qquad \|u_n - \hat{u}\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \to 0, \quad \|(u_n)_x - \hat{u}_x\|_{C^\gamma(\overline{Q}_T;\mathcal{U})} \to 0, \quad n \to \infty,$$

$$(3.10)\qquad \|a_n - \hat{a}\|_{C[0,T]} \to 0, \quad n \to \infty,$$

for some function $\hat{u} \in C^\gamma(\overline{Q}_T;\mathcal{U})$ with $\hat{u}_x \in C^\gamma(\overline{Q}_T;\mathcal{U})$ and some $\hat{a} \in C[0,T]$.

Set $\hat{b}(t) = \max\limits_{s\in[0,t]} \hat{a}(s)$. Due to (3.10) and Lemma 3.1, we have

$$(3.11)\qquad \|b_n - \hat{b}\|_{C[0,T]} \to 0, \quad n \to \infty.$$

Using (3.9), (3.10), and Lemma 3.2, we conclude that

$$(3.12)\qquad W(\xi_0(x), \hat{u}(x,\cdot))(t) = \begin{cases} W_1(\hat{u}(x,t)), & 0 \leqslant x \leqslant \hat{b}(t), \\ W_2(\hat{u}(x,t)), & \hat{b}(t) < x \leqslant 1, \end{cases}$$

$$(3.13)\qquad \sup_{t\in[0,T]} (\|W(\xi_{0n}, u_n)(t) - W(\xi_0, u)(t)\|_{L_q((0,1);\mathcal{W})}) \to 0, \quad n \to \infty.$$

Now we show that

$$(3.14)\qquad \|v_n - \hat{v}\|_{W_q^{0,1}(Q_T;\mathcal{V})} \to 0, \quad n \to \infty,$$

for some $\hat{v}$. Take an arbitrary $\delta > 0$. It follows from the assumptions of the theorem and from (3.9) and (3.13) that

$$
\begin{aligned}
(3.15) \qquad \|\psi_n - \psi_k\|_{L_q((0,1);\mathcal{V})} \leqslant \delta, \quad \|u_n(\cdot,t) - u_k(\cdot,t)\|_{L_q((0,1);\mathcal{U})} \leqslant \delta,\\
\|W(\xi_{0n}, u_n)(t) - W(\xi_{0k}, u_k)(t)\|_{L_q((0,1);\mathcal{W})} \leqslant \delta,
\end{aligned}
$$

provided n, k are large enough. Estimates (3.15), the second equation in (1.1), and the local Lipschitz continuity of g yield

$$
\|v_n(\cdot,t) - v_k(\cdot,t)\|_{L_q((0,1);\mathcal{V})} \leqslant (1+2L)\delta + L\int_0^t \|v_n(\cdot,s) - v_k(\cdot,s)\|_{L_q((0,1);\mathcal{V})}\,\mathrm{d}s,
$$

where $L > 0$ does not depend on n, k. Hence, by Gronwall's inequality,

$$
(3.16) \qquad \|v_n(\cdot,t) - v_k(\cdot,t)\|_{L_q((0,1);\mathcal{V})} \leqslant k_1\delta,
$$

where $k_1 > 0$ does not depend on δ, n, k, and $t \in [0,T]$. A similar inequality for the time derivative of v_n follows from (3.15), (3.16), and from the second equation in (1.1). Since $\delta > 0$ is arbitrary, (3.14) holds.

Now consider (1.1), (1.2) with the subsequences φ_n, ψ_n, ξ_{0n}, u_n, v_n. Due to (3.9), (3.13), (3.14), and the local Lipschitz continuity of f, we have

$$
\|f(u_n, v_n, W(\xi_{0n}, u_n) - f(\hat{u}, \hat{v}, W(\xi_0, \hat{u})\|_{L_q(Q_T;\mathcal{U})} \to 0, \quad n \to \infty.
$$

Therefore, by the standard parabolic theory [13], we obtain from the first equation in (1.1) that

$$
\|u_n - \hat{u}\|_{W_q^{2,1}(Q_T;\mathcal{U})} \to 0, \quad n \to \infty.
$$

Combining the latter relation with (3.14) and the local Lipschitz continuity of f and g, we can pass to the limit, as $n \to \infty$, in (1.1), (1.2) with the subsequences $\varphi_n, \psi_n, \xi_{0n}, u_n, v_n$, and obtain that $(\hat{u}, \hat{v})$ is a solution of (1.1), (1.2) with the initial data (φ, ψ, ξ_0). Hence, due to the uniqueness assumption (see part 1 in the formulation of Theorem 2.3), $(u, v) = (\hat{u}, \hat{v})$ and $b(t) \equiv \hat{b}(t)$. Therefore, (3.8) is not true and we have the convergence for the whole sequence (u_n, v_n). $\square$

4. Uniqueness of solutions

In this section, we prove Theorem 2.4. For clarity of explanation, we restrict ourselves to the case where the initial data satisfy the equality $\gamma_\alpha(\varphi(\overline{b})) = 0$ in addition to Condition 1.2. (The case $\gamma_\alpha(\varphi(\overline{b})) \neq 0$ can be treated easily because then the hysteresis $W(\xi_0, u)$ remains constant on some time interval.)

Set

$$\overline{\varphi} := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\gamma_\alpha(\varphi(x))|_{x=\overline{b}} \ (> 0).$$

We fix T_1 such that the conclusions of Lemma 3.4 are true for u, $\hat{u}$ on $(0, T_1)$. Let $a(t)$, $b(t)$, $\hat{a}(t)$, $\hat{b}(t)$ be the functions defined in Lemma 3.4 for u and $\hat{u}$, respectively. We fix $T \in (0, T_1)$ and $\delta > 0$ such that the following inequalities hold for $t \in [0, T]$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\gamma_\alpha(u(x,t)) \geqslant \overline{\varphi}, \quad x \in [\overline{b} - \delta, \overline{b} + \delta], \tag{4.1}$$

$$\gamma_\beta(u(x,t)) < 0, \quad x \in [0, b(t)], \tag{4.2}$$

and the analogous inequalities hold for $\hat{u}$.

Due to (4.1) and (4.2), we have

$$W(\xi_0(x), u(x,\cdot)(t) = \begin{cases} W_1(u(x,t)), & 0 \leqslant x \leqslant b(t), \\ W_{-1}(u(x,t)), & b(t) < x \leqslant 1, \end{cases} \tag{4.3}$$

$$W(\xi_0(x), \hat{u}(x,\cdot)(t) = \begin{cases} W_1(\hat{u}(x,t)), & 0 \leqslant x \leqslant \hat{b}(t), \\ W_{-1}(\hat{u}(x,t)), & \hat{b}(t) < x \leqslant 1. \end{cases}$$

Let us now prove Theorem 2.4.

Step 1. Set $w = u - \hat{u}$ and $z = v - \hat{v}$. The functions w and z satisfy the equations

$$\begin{cases} w_t = w_{xx} + h_w(x,t), \\ z_t = h_z(x,t), \end{cases} \tag{4.4}$$

and zero boundary and initial conditions, where

$$h_w(x,t) = f(u, v, W(u)) - f(\hat{u}, \hat{v}, W(\hat{u})),$$
$$h_z(x,t) = g(u, v, W(u)) - g(\hat{u}, \hat{v}, W(\hat{u})).$$

Obviously, $h_w, h_z \in L_\infty(Q_T)$. The function w can be represented via the Green function $G(x, y, t, s)$ of the heat equation with the Neumann boundary conditions:

$$w(x,t) = \int_0^t \int_0^1 G(x, y, t, s) h_w(y, s) \,\mathrm{d}y \,\mathrm{d}s.$$

Therefore, using the estimate $|G(x, y, t, s)| \leqslant k_1/\sqrt{t-s}$, $0 < s < t$, with $k_1 > 0$ not depending on $(x,t) \in Q_T$ (see, e.g., [9]), we obtain

$$|w(x,t)| \leqslant k_1 \int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}} \int_0^1 |h_w(y,s)| \,\mathrm{d}y. \tag{4.5}$$

Set $Z(t) = \int_0^1 |h_z(y,t)|\,\mathrm{d}y$. Due to the second equation in (4.4),

$$Z(t) \leqslant \int_0^t \int_0^1 |h_z(y,s)|\,\mathrm{d}y\,\mathrm{d}s. \tag{4.6}$$

Step 2. Now we prove that, for some $k_2 > 0$ and all $s \in (0,T)$, we have

$$\begin{aligned} &\int_0^1 |h_w(y,s)|\,\mathrm{d}y \leqslant k_2(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}), \\ &\int_0^1 |h_z(y,s)|\,\mathrm{d}y \leqslant k_2(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}). \end{aligned} \tag{4.7}$$

Let us prove the first inequality (for the function h_w), assuming that $b(s) < \hat{b}(s)$. (The cases of h_z and $b(s) \geqslant \hat{b}(s)$ are treated analogously.) Since f is locally Lipschitz,

$$\begin{aligned} \int_0^1 |h_w(y,s)|\,\mathrm{d}y &\leqslant k_3 \int_0^1 (|w(y,s)| + |z(y,s)| + |W(u(y,s)) - W(\hat{u}(y,s))|)\,\mathrm{d}y \\ &\leqslant k_3\Big(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)} + \int_0^1 |W(u(y,s)) - W(\hat{u}(y,s))|\,\mathrm{d}y\Big), \end{aligned} \tag{4.8}$$

where $k_3 > 0$ and the constants $k_4, k_5, \ldots > 0$ below do not depend on $s \in [0,T]$.

Denote $\theta(y,s) = W(u(y,s)) - W(\hat{u}(y,s))$. Due to (4.3), we have

$$\theta(y,s) = \begin{cases} W_1(u) - W_1(\hat{u}), & 0 < y < b(s), \\ W_{-1}(u) - W_1(\hat{u}), & b(s) < y < \hat{b}(s), \\ W_{-1}(u) - W_{-1}(\hat{u}), & \hat{b}(s) < y < 1. \end{cases}$$

Below we separately consider the integrals of $\theta(y,s)$ over the intervals $(0, b(s))$, $(b(s), \hat{b}(s))$, $(\hat{b}(s), \overline{b} + \delta)$, and $(\overline{b} + \delta, 1)$, where δ was defined in (4.1).

Interval $(0, b(s))$**.** Inequality (4.2) implies that $\gamma_\beta(u(y,s)) < 0$, $\gamma_\beta(\hat{u}(y,s)) < 0$ on the closed set $\{(y,s)\colon\ y \in [0, b(s)],\ s \in [0,T]\}$. Hence, the values $\gamma_\beta(u(y,s))$ and $\gamma_\beta(\hat{u}(y,s))$ are separated from 0. Therefore, using Condition 2.1, we obtain

$$\int_0^{b(s)} |\theta(y,s)|\,\mathrm{d}y \leqslant k_4 \int_0^{b(s)} |u(y,s) - \hat{u}(y,s)|\,\mathrm{d}y \leqslant k_4 \|w\|_{C(\overline{Q}_T)}. \tag{4.9}$$

Interval $(b(s), \hat{b}(s))$**.** Boundedness of $W_1(\hat{u})$ and $W_{-1}(u)$ for $(y,s) \in \overline{Q}_T$ and Lemma 3.1 imply

$$\int_{b(s)}^{\hat{b}(s)} |\theta(y,s)|\,\mathrm{d}y \leqslant k_5 \int_{b(s)}^{\hat{b}(s)} \mathrm{d}y \leqslant k_5 \|b - \hat{b}\|_{C[0,T]} \leqslant k_5 \|a - \hat{a}\|_{C[0,T]}. \tag{4.10}$$

Using (4.1), we obtain for any $t \in [0,T]$ the inequalities

$$(4.11) \qquad \begin{aligned} |a(t) - \hat{a}(t)| &\leqslant \frac{1}{\overline{\varphi}} |\gamma_\alpha(u(a(t),t)) - \gamma_\alpha(\hat{u}(a(t),t))| \\ &\leqslant \frac{L_\alpha}{\overline{\varphi}} |u(a(t),t) - \hat{u}(a(t),t)| \leqslant \frac{L_\alpha}{\overline{\varphi}} \|u - \hat{u}\|_{C(\overline{Q}_T)}, \end{aligned}$$

where $L_\alpha > 0$ is a respective Lipschitz constant for $\gamma_\alpha(u)$ and hence does not depend on $T \in (0, T_1)$. Inequalities (4.10) and (4.11) yield

$$(4.12) \qquad \int_{b(s)}^{\hat{b}(s)} |\theta(y,s)| \, \mathrm{d}y \leqslant k_6 \|w\|_{C(\overline{Q}_T)}.$$

Interval $(\hat{b}(s), \overline{b} + \delta)$**.** Inequality (4.1) and the mean-value theorem imply that for $y \in [\hat{b}(s), \overline{b} + \delta]$ the following inequalities hold:

$$\gamma_\alpha(\hat{u}(y,s)) = \gamma_\alpha(\hat{u}(y,s)) - \gamma_\alpha(\hat{u}(\hat{a}(s),s)) \geqslant (y - \hat{a}(s))\overline{\varphi} \geqslant (y - \hat{b}(s))\overline{\varphi},$$
$$|\gamma_\alpha(u(y,s))| \geqslant (y - b(s))\overline{\varphi}.$$

Taking into account these two inequalities and using Condition 2.1, we obtain

$$(4.13) \qquad \int_{\hat{b}(s)}^{\overline{b}+\delta} |\theta(y,s)| \, \mathrm{d}y \leqslant k_7 \int_{\hat{b}(s)}^{\overline{b}+\delta} \frac{|u(y,s) - \hat{u}(y,s)|}{(y - \hat{b}(s))^\sigma} \, \mathrm{d}y \leqslant k_8 \|w\|_{C(\overline{Q}_T)}.$$

Interval $(\overline{b} + \delta, 1)$**.** Similarly to the interval $(0, b(s))$, we conclude that

$$(4.14) \qquad \int_{\overline{b}+\delta}^{1} |\theta(y,s)| \, \mathrm{d}y \leqslant k_9 \|w\|_{C(\overline{Q}_T)}.$$

Finally, (4.8)–(4.14) imply the first inequality in (4.7).

Step 3. Combining estimates (4.5)–(4.7), we obtain

$$\begin{aligned} |w(x,t)| &\leqslant k_{10}(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}) \int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}} \\ &= 2k_{10} T^{1/2} (\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}), \\ Z(t) &\leqslant k_2 T (\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}). \end{aligned}$$

Taking the supremum with respect to $t \in (0,T)$, we see that

$$\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)} \leqslant (2k_{10} T^{1/2} + k_2 T)(\|w\|_{C(\overline{Q}_T)} + \|Z\|_{L_\infty(0,T)}).$$

Thus, $w = 0$ and $z = 0$, provided that $T > 0$ is small enough. $\square$

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