SYMPLECTIC EMBEDDING OF THIN DISCS INTO A BALL

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Abstract. We perform symplectic embeddings of 'thin' discs into a small ball in arbitrary dimension, using the symplectic folding construction.

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1. Introduction

In this note, as a variant of the Gromov's non-squeezing theorem, we perform symplectic embeddings of 'thin' discs into a small ball in arbitrary dimension. More precisely, the 'thin disc' is defined by

$$D^{2n}_{\varepsilon,R} = [-\varepsilon,\varepsilon] \times B^{2n-1}_R$$

where B^k_R is a standard k-dimensional ball of radius R and we consider $D^{2n}_{\varepsilon,R}$ equipped with a standard symplectic form as a subset of $\mathbb{R}^{2n}$. Then our result is,

Theorem 1.1. *For any $R, r > 0$ and any ball B^{2n}_r of radius r with a standard symplectic structure, if we take ε small enough, then we can find symplectic embeddings of $D^{2n}_{\varepsilon,R}$ into B^{2n}_r.*

The proof is based on making use of the flexibility of symplectomorphisms of dimension 2, which coincide with the volume preserving maps. The crucial technique is the symplectic folding construction, which was initiated by Traynor [2] and developed by Lalonde and McDuff [1] to compare the displacement energy and the symplectic capacity. In the next section we recall it, and the proof of the theorem is given in the third section.

2. Symplectic folding construction

Let (M, ω) be a symplectic manifold, A be a compact subset, and $e(A)$ be its displacement energy, i.e.

$$e(A) = \inf\{\|\varphi\|\colon \varphi \in \mathrm{Ham}\,(M), \varphi(A) \cap A = \emptyset\}$$

here $\|\varphi\|$ is the Hofer's norm,

$$\|\varphi\| = \inf_H (\sup_{x,t} H(x,t) - \inf_{x,t} H(x,t)),$$

where $(x,t) \in M \times [0,1]$ and H ranges over the set of all compactly supported Hamiltonian functions $H\colon M \times [0,1] \to \mathbb{R}$ whose symplectic gradient vector fields generate a time 1 map equal to φ.

Let $H(x,t)$ be a Hamiltonian, φ_H^t its flow. Consider a hypersurface Q in $M \times [0,1] \times \mathbb{R}$ defined by

$$Q = (x, t, -H(x,t)).$$

The manifold $M \times [0,1] \times \mathbb{R}$ is equipped with a symplectic form $\omega + \mathrm{d}t \wedge \mathrm{d}z$ where z is the standard coordinate of $\mathbb{R}$. Then, Q has the characteristic foliation and its flow coincides with φ_H^t. For the set A above and an arbitrary positive number ε, there is a Hamiltonian H disjoining the set A, which is 0 near times $t = 0, 1$ and the associated hypersurface Q is contained in $M \times [0,1] \times [0, e(A) + \varepsilon]$. Let

$$R = [0, T] \times [0, e(A) + \varepsilon]$$

be a rectangle, with T any positive number. We define a positive number r by

$$r = 2Te(A).$$

Gluing the rectangles $[0,1] \times [0, e(A) + \varepsilon]$ and R along the pairs of edges

$$(\{1\} \times [0, e(A) + \varepsilon], \{0\} \times [0, e(A) + \varepsilon])$$

and

$$(\{0\} \times [0, e(A) + \varepsilon], \{T\} \times [0, e(A) + \varepsilon])$$

where the left components of the brackets are edges of R and the right components are those of the other rectangle, we get an annulus of area $(1+T)(e(A)+\varepsilon)$. Denote this annulus by X. Consider the symplectic manifold $M \times B$, where B is a symplectic 2 dimensional disc of area r. We can deform $M \times B$ into a neighbourhood (in $M \times \mathbb{R}^2$)

of the manifold $M \times Y$, where $Y = P_1 \cup L \cup P_2$ is the union of two rectangles P_1 and P_2 of the same shape $[0, T] \times [0, e(A)]$, joined by a line segment L. We define a map

$$i \colon A \times Y \to M \times X$$

as follows.

1. $A \times P_1$ maps to $M \times [0, T] \times [0, e(A) + \varepsilon]$ by the inclusion, slightly parallel translated to the second direction of the base to embed $[0, e(A)]$ into the interior of $[0, e(A) + \varepsilon]$.
2. $A \times L$ maps to the hypersurface Q, along the flow line of the characteristic foliation on Q.
3. $A \times P_2$ maps to $M \times [0, T] \times [0, e(A) + \varepsilon]$ by $\varphi_H^1 \times \mathrm{id}'$ where id' is the above mentioned slightly translated identity map of the base.

This map preserves the symplectic form, so by the symplectic neighbourhood theorem, we can extend this map and embed a small neighbourhood of $A \times B$ in $M \times B$ into $M \times X$.

3. The proof

The proof is by induction. We begin with the case of $n = 2$. Obviously, it is enough to embed $[-\varepsilon, \varepsilon] \times [-R, R]^3$ into B_r^{2n}. We view $[-\varepsilon, \varepsilon] \times [-R, R]^3$ as $[-\varepsilon, \varepsilon] \times [-R, R] \times [-R, R]^2$. Then, because symplectomorphisms equal volume preserving maps in dimension 2, we can easily symplectically isotope $[-\varepsilon, \varepsilon] \times [-R, R] \times [-R, R]^2$ to $B_{\varepsilon_0}^2 \times [-R, R]^2$, ε_0 sufficiently small. On the other hand, we can symplectically embed arbitrary large 2-disc D into B_r^4. For example, consider B_r^4 as an affine part of the complex projective plane and take a large degree smooth curve C. Then, again because symplectomorphisms equal with volume preserving maps in dimension 2, we can embed D into $C \cap B_r^4$ symplectically. By the symplectic neighbourhood theorem, we can embed $B_{\varepsilon_0}^2 \times [-R, R]^2$, and also $[-\varepsilon, \varepsilon] \times [-R, R] \times [-R, R]^2$ into B_r^4 symplectically, provided ε is sufficiently small. Next, suppose we have embeddings of $D_{\varepsilon,R}^{2n}$ into B_r^{2n} for all $n < k$, k some positive integer. We have to embed $D_{\varepsilon,R}^{2k}$ into B_r^{2k}. First, we view $D_{\varepsilon,R}^{2k}$ as $D_{\varepsilon,R}^{2(k-1)} \times [-R, R]^2$. By the induction hypothesis, we can embed symplectically $D_{\varepsilon,R}^{2(k-1)}$ into $B_{\varepsilon_0}^{2(k-1)}$, for ε_0 sufficiently small compared with r. So, it is enough to embed $B_{\varepsilon_0}^{2(k-1)} \times [-R, R]^2$ into B_r^{2k}. This can be done by the technique of symplectic folding. Namely, subdivide $[-R,\, R]^2$ into a number of small squares and deform it volume preservingly into a chain of those squares joined by narrow strips. Each (square, strip, square)-triplet looks like Y in the previous section. Then, we apply the folding construction to these triplets successively, and we will finally embed the whole space into B_r^{2k} spirally.

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References

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