UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we investigate the uniqueness problem of difference polynomials sharing a small function. With the notions of weakly weighted sharing and relaxed weighted sharing we prove the following: Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geqslant 7$ (or $n \geqslant 10$) is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z),2)$" (or $(\alpha(z),2)^*$), then $f(z) \equiv g(z)$. Our results extend and generalize some well known previous results.

Keywords: entire function; difference polynomial; uniqueness

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1. Introduction, definitions and results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let k be a positive integer or infinity and $a \in C \cup \{\infty\}$. Set $E(a,f) = \{z\colon f(z)-a=0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than k. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in [5], [13]. We denote by $N_{k)}(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity less or equel to k, and by

$\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \ldots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

Let $N_E(r, a; f, g)(\overline{N}_E(r, a; f, g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r, a; f, g)$ $(\overline{N}_0(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}\Big(r, \frac{1}{g-a}\Big) - 2\overline{N}_E(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}\Big(r, \frac{1}{g-a}\Big) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share a "IM".

We now explain in the following definition the notion of weakly weighted sharing which was introduced by Lin and Lin [8].

Definition 1 ([8]). Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}^E_{k)}(r, a; f, g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^O_{(k}(r, a; f, g)$ denotes the reduced counting function of those a-points of f which are a-points of g, and both of their multiplicities are not less than k.

Definition 2 ([8]). For $a \in C \cup \{\infty\}$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}\Big(r, \frac{1}{f-a}\Big) - \overline{N}^E_{k)}(r, a; f, g) = S(r, f),$$
$$\overline{N}_{k)}\Big(r, \frac{1}{g-a}\Big) - \overline{N}^E_{k)}(r, a; f, g) = S(r, g),$$
$$\overline{N}_{(k+1}\Big(r, \frac{1}{f-a}\Big) - \overline{N}^O_{(k+1}(r, a; f, g) = S(r, f),$$
$$\overline{N}_{(k+1}\Big(r, \frac{1}{g-a}\Big) - \overline{N}^O_{(k+1}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}\Big(r, \frac{1}{f-a}\Big) - \overline{N}_0(r, a; f, g) = S(r, f), \overline{N}\Big(r, \frac{1}{g-a}\Big) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f and g *weakly share a with weight k*. Here we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Now it is clear from Definition 2 that weakly weighted sharing is a scaling between IM and CM.

Recently, A. Banerjee and S. Mukherjee [1] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

Definition 3 ([1])**.** We denote by $\overline{N}(r,a;f|=p;g|=q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 4 ([1])**.** Let f, g share a "IM". Also let k be a positive integer or ∞ and $a \in C \cup \{\infty\}$. If

$$\sum_{p,q\leqslant k} \overline{N}(r,a;f|=p;g|=q) = S(r),$$

then we say f and g *share a with weight k in a relaxed manner*. Here we write f and g share $(a,k)^*$ to mean that f and g share a with weight k in a relaxed manner.

W. K. Hayman proposed the following well-known conjecture in [6].

Hayman's conjecture. If an entire function f satisfies $f^n f' \neq 1$ for all positive integers $n \in N$, then f is a constant.

It has been verified by Hayman himself in [7] for the case $n > 1$ and Clunie in [3] for the case $n \geqslant 1$, respectively.

It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. In 1997, corresponding to the famous conjecture of Hayman, Yang and Hua studied the unicity of differential monomials and obtained the following theorem.

Theorem A ([12])**.** *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geqslant 6$ a positive integer. If $f^n f'$ and $g^n g'$ share* 1 CM*, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

In 2001, Fang and Hong studied the unicity of differential polynomials of the form $f^n(f-1)f'$ and proved the following uniqueness theorem.

Theorem B ([4])**.** *Let f and g be two transcendental entire functions, $n \geqslant 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value* 1 CM*, then $f \equiv g$.*

In 2004, Lin and Yi extended the above theorem as to the fixed-point. They proved the following result.

Theorem C ([9]). *Let f and g be two transcendental entire functions, $n \geqslant 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.*

In 2010, Zhang [15] got an analogue result for translates.

Theorem D ([15]). *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geqslant 7$ is an integer. If $f^n(z)(f(z)-1) \times f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

Now one may ask the following question which is the motivation of the paper: Can the nature of small function $\alpha(z)$ be relaxed in the above theorem? Considering this question, we prove the following results.

Theorem 1. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geqslant 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z), 2)$", then $f(z) \equiv g(z)$.*

Theorem 2. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geqslant 10$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $(\alpha(z), 2)^*$, then $f(z) \equiv g(z)$.*

Without the notions of weakly weighted sharing and relaxed weighted sharing we prove the following theorem which also improves Theorem D.

Theorem 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geqslant 16$ is an integer. If $\overline{E}_{2)}(\alpha(z), f^n(z) \times (f(z)-1)f(z+c)) = \overline{E}_{2)}(\alpha(z), g^n(z)(g(z)-1)g(z+c))$, then $f(z) \equiv g(z)$.*

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 1 ([1]). *Let H be defined as above. If F and G share "$(1,2)$" and $H \not\equiv 0$, then*

$$T(r,F) \leqslant N_2\Big(r,\frac{1}{F}\Big) + N_2\Big(r,\frac{1}{G}\Big) + N_2(r,F) + N_2(r,G) - \sum_{p=3}^{\infty} \overline{N}_{(p}\Big(r,\frac{G}{G'}\Big) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 2 ([1]). *Let H be defined as above. If F and G share $(1,2)^*$ and $H \not\equiv 0$, then*

$$T(r,F) \leqslant N_2\Big(r,\frac{1}{F}\Big) + N_2\Big(r,\frac{1}{G}\Big) + N_2(r,F) + N_2(r,G) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}(r,F) - m\Big(r,\frac{1}{G-1}\Big) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 3 ([14]). *Let H be defined as above. If $H \equiv 0$ and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G)}{T(r)} < 1, \quad r \in I,$$

where $T(r) = \max\{T(r,F), T(r,G)\}$ and I is a set with infinite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 4 ([2]). *Let $f(z)$ be a meromorphic function in the complex plane of finite order $\sigma(f)$, and let η be a fixed non-zero complex number. Then for each $\varepsilon > 0$, one has*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r)$$

Lemma 5 ([11]). *Let $f(z)$ be an entire function of finite order $\sigma(f)$, c a fixed non-zero complex number, and*

$$P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0$$

where a_j $(j = 0, 1, \ldots, n)$ are constants. If $F(z) = P(z) f(z+c)$, then

$$T(r,F) = (n+1)T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r).$$

Lemma 6 ([10]). *Let F and G be two nonconstant entire functions, and $p \geqslant 2$ an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \leqslant N_2\Big(r,\frac{1}{F}\Big) + N_2\Big(r,\frac{1}{G}\Big) + 2\overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{G}\Big) + S(r,F) + S(r,G).$$

3. Proof of Theorem 1

Let

$$F(z) = \frac{f^n(z)(f(z)-1)f(z+c)}{\alpha(z)}, \qquad G(z) = \frac{g^n(z)(g(z)-1)g(z+c)}{\alpha(z)}.$$

Then $F(z)$ and $G(z)$ share "(1,2)" except the zeros or poles of $\alpha(z)$. By Lemma 5, we have

$$T(r,F(z)) = (n+2)T(r,f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f), \tag{3.1}$$

$$T(r,G(z)) = (n+2)T(r,g(z)) + O(r^{\sigma(g)-1+\varepsilon}) + S(r,g). \tag{3.2}$$

Suppose $H \not\equiv 0$, then by Lemma 1 and Lemma 4 we have

$$\begin{aligned} T(r,F) + T(r,G) &\leqslant 2N_2\Big(r,\frac{1}{F}\Big) + 2N_2\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g) \\ &\leqslant 4\overline{N}\Big(r,\frac{1}{f}\Big) + 4\overline{N}\Big(r,\frac{1}{g}\Big) + 2N\Big(r,\frac{1}{f(z)-1}\Big) + 2N\Big(r,\frac{1}{g(z)-1}\Big) \\ &\quad + 2N\Big(r,\frac{1}{f(z+c)}\Big) + 2N\Big(r,\frac{1}{g(z+c)}\Big) + S(r,f) + S(r,g) \\ &\leqslant 8T(r,f) + 8T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.3}$$

Substituting (3.1) and (3.2) into (3.3), we obtain

$$(n-6)[T(r,f)+T(r,g)] \leqslant O(r^{\sigma(f)-1+\varepsilon}) + O(r^{\sigma(g)-1+\varepsilon}) + S(r,f) + S(r,g)$$

which contradicts with $n \geqslant 7$. Thus we have $H \equiv 0$. Note that

$$\overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{G}\Big) \leqslant 3T(r,f) + 3T(r,g) + S(r,f) + S(r,g) \leqslant T(r)$$

where $T(r) = \max\{T(r,F), T(r,G)\}$. By Lemma 3, we deduce that either $F \equiv G$ or $FG \equiv 1$. Next we will consider the following two cases, respectively.

Case 1. $F \equiv G$, thus $f^n(z)(f(z)-1)f(z+c) \equiv g^n(z)(g(z)-1)g(z+c)$. Let $\varphi(z) = f(z)/g(z)$. If $\varphi^{n+1}(z)\varphi(z+c) \not\equiv 1$, we have

$$g(z) = \frac{\varphi^n(z)\varphi(z+c)-1}{\varphi^{n+1}(z)\varphi(z+c)-1}. \tag{3.4}$$

Then $\varphi(z)$ is a transcendental meromorphic function of finite order since $g(z)$ is transcendental. By Lemma 4, we have

$$T(r,\varphi(z+c)) = T(r,\varphi(z)) + S(r,\varphi). \tag{3.5}$$

If $\varphi^{n+1}(z)\varphi(z+c) = k(\neq 1)$, where k is a constant, then Lemma 4 and (3.5) imply that

$$(n+1)T(r,\varphi(z)) = T(r,\varphi(z+c)) + O(1) = T(r,\varphi(z)) + O(r^{\sigma(\varphi(z))-1+\varepsilon}) + O(\log r)$$

which contradicts with $n \geqslant 7$. Thus $\varphi^{n+1}(z)\varphi(z+c)$ is not a constant. Suppose that there exists a point z_0 such that $\varphi(z_0)^{n+1}\varphi(z_0+c) = 1$. Then $\varphi(z_0)^n\varphi(z_0+c) = 1$ since $g(z)$ is an entire function. Hence $\varphi(z_0) = 1$ and

$$\overline{N}\Big(r, \frac{1}{\varphi^{n+1}(z)\varphi(z+c)-1}\Big) \leqslant \overline{N}\Big(r, \frac{1}{\varphi(z)-1}\Big) \leqslant T(r,\varphi(z)) + O(1).$$

We apply the second Nevanlinna fundamental theorem to $\varphi(z)^{n+1}\varphi(z+c)$:

$$\begin{aligned} T(r,\varphi^{n+1}(z)\varphi(z+c)) &\leqslant \overline{N}(r,\varphi^{n+1}(z)\varphi(z+c)) + \overline{N}\Big(r, \frac{1}{\varphi^{n+1}(z)\varphi(z+c)}\Big) \\ &\quad + \overline{N}\Big(r, \frac{1}{\varphi^{n+1}(z)\varphi(z+c)-1}\Big) + S(r,\varphi) \leqslant 5T(r,\varphi(z)) + S(r,\varphi). \end{aligned}$$

By Lemma 5 we deduce

$$(n-3)T(r,\varphi(z)) \leqslant O(r^{\sigma(\varphi)-1+\varepsilon}) + S(r,\varphi), \tag{3.6}$$

which contradicts with $n \geqslant 7$. So $\varphi^{n+1}(z)\varphi(z+c) \equiv 1$. Thus $\varphi(z) \equiv 1$, that is $f(z) \equiv g(z)$.

Case 2. $F(z)G(z) \equiv 1$, that is

$$f^n(z)(f(z)-1)f(z+c)g^n(z)(g(z)-1)g(z+c) \equiv \alpha^2(z). \tag{3.7}$$

Since f and g are transcendental entire functions, we can deduce from (3.7) that $N(r,1/f) = S(r,f)$, $N(r,f) = S(r,f)$ and $N(r,1/(f-1)) = S(r,f)$. Then $\delta(0,f) + \delta(\infty,f) + \delta(1,f) = 3$, which contradicts the deficiency relation. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Let

$$F(z) = \frac{f^n(z)(f(z)-1)f(z+c)}{\alpha(z)}, \qquad G(z) = \frac{g^n(z)(g(z)-1)g(z+c)}{\alpha(z)}.$$

Then $F(z)$ and $G(z)$ share $(1,2)^*$ except the zeros or poles of $\alpha(z)$. Obviously

$$\begin{aligned} (4.1) \qquad 2N_2\Big(r,\frac{1}{F}\Big) + 2N_2\Big(r,\frac{1}{G}\Big) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{G}\Big) + S(r,F) + S(r,G) \\ \leqslant 11T(r,f) + 11T(r,g) + S(r,f) + S(r,g). \end{aligned}$$

According to (4.1) and Lemma 2, we can prove Theorem 2 in a similar way as in Section 3. □

5. Proof of Theorem 3

Let

$$F(z) = \frac{f^n(z)(f(z)-1)f(z+c)}{\alpha(z)}, \qquad G(z) = \frac{g^n(z)(g(z)-1)g(z+c)}{\alpha(z)}.$$

Then $\overline{E}_{2)}(1, f^n(z)(f(z)-1)f(z+c)) = \overline{E}_{2)}(1, g^n(z)(g(z)-1)g(z+c))$ except the zeros or poles of $\alpha(z)$. Obviously

$$\begin{aligned} (5.1) \qquad 2N_2\Big(r,\frac{1}{F}\Big) + 2N_2\Big(r,\frac{1}{G}\Big) + 3\overline{N}\Big(r,\frac{1}{F}\Big) + 3\overline{N}\Big(r,\frac{1}{G}\Big) + S(r,F) + S(r,G) \\ \leqslant 17T(r,f) + 17T(r,g) + S(r,f) + S(r,g). \end{aligned}$$

Using (5.1) and Lemma 6, we can prove Theorem 3 in a similar way as in Section 3. □

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