HAMILTONIAN COLORINGS OF GRAPHS WITH LONG CYCLES

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Abstract. By a hamiltonian coloring of a connected graph G of order $n \geqslant 1$ we mean a mapping c of $V(G)$ into the set of all positive integers such that $|c(x) - c(y)| \geqslant n - 1 - D_G(x, y)$ (where $D_G(x, y)$ denotes the length of a longest $x - y$ path in G) for all distinct $x, y \in G$. In this paper we study hamiltonian colorings of non-hamiltonian connected graphs with long cycles, mainly of connected graphs of order $n \geqslant 5$ with circumference $n - 2$.

Keywords: connected graphs, hamiltonian colorings, circumference

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The letters f–n (possibly with indices) will be reserved for denoting non-negative integers. The set of all positive integers will be denoted by $\mathbb{N}$. By a graph we mean a finite undirected graph with no loop or multiple edge, i.e. a graph in the sense of [1], for example.

0. Let G be a connected graph of order $n \geqslant 1$. If $u, v \in V(G)$, then we denote by $D_G(u, v)$ the length of a longest $u - v$ path in G. If $x, y \in G$, then we denote

$$D'_G(x, y) = n - 1 - D_G(x, y).$$

We say that a mapping c of $V(G)$ into $\mathbb{N}$ is a *hamiltonian coloring* of G if

$$|c(x) - c(y)| \geqslant D'_G(x, y)$$

for all distinct $x, y \in V(G)$. If c is a hamiltonian coloring of G, then we denote

$$\mathrm{hc}(c) = \max\{c(w); w \in V(G)\}.$$

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The *hamiltonian chromatic number* $\mathrm{hc}(G)$ of G is defined by

$$\mathrm{hc}(G) = \min\{\mathrm{hc}(c); c \text{ is a hamiltonian coloring of } G\}.$$

Fig. 1 shows four connected graphs of order six, each of them with a hamiltonian coloring.

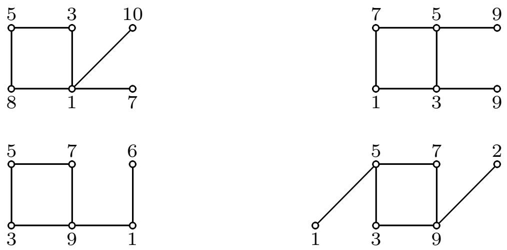

Fig. 1

The notions of a hamiltonian coloring and the hamiltonian chromatic number of a connected graph were introduced by G. Chartrand, L. Nebeský and P. Zhang in [2]. These concepts have a transparent motivation: a connected graph G is hamiltonian-connected if and only if $\mathrm{hc}(G) = 1$.

The following useful result on the hamiltonian chromatic number was proved in [2]; its proof is easy.

Proposition 1. *Let G_1 and G_2 be connected graphs. If G_1 is spanned by G_2, then* $\mathrm{hc}(G_1) \leqslant \mathrm{hc}(G_2)$.

It was proved in [2] that

$$\mathrm{hc}(G) \leqslant (n-2)^2 + 1$$

for every connected graph G of order $n \geqslant 2$ and that $\mathrm{hc}(S) = (n-2)^2 + 1$ for every star S of order $n \geqslant 2$. These results were extended in [3]: there exists no connected graph of order $n \geqslant 5$ with $\mathrm{hc}(G) = (n-2)^2$, and if T is a tree of order $n \geqslant 5$ obtained from a star of order $n-1$ by inserting a new vertex into an edge, then $\mathrm{hc}(T) = (n-2)^2 - 1$.

The following definition will be used in the next sections. Let G be a connected graph containing a cycle; by the circumference of G we mean the length of a longest cycle in G; similarly as in [2] and [3], the circumference of G will be denoted by $\mathrm{cir}(G)$. If G is a tree, then we put $\mathrm{cir}(G) = 0$.

1. It was proved in [2] that if G is a cycle of order $n \geqslant 3$, then $\mathrm{hc}(G) = n-2$. Proposition 1 implies that if G is a hamiltonian graph of order $n \geqslant 3$, then $\mathrm{hc}(G) \leqslant n-2$.

As was proved in [2], if G is a connected graph of order $n \geqslant 4$ such that $\operatorname{cir}(G) = n-1$ and G contains a vertex of degree 1, then $\operatorname{hc}(G) = n-1$. Thus, by Proposition 1, if G is a connected graph of order $n \geqslant 4$ such that $\operatorname{cir}(G) = n-1$, then $\operatorname{hc}(G) \leqslant n-1$.

Consider arbitrary j and n such that $j \geqslant 0$ and $n-j \geqslant 3$. We denote by $\operatorname{hc}_{\max}(n,j)$ the maximum integer $i \geqslant 1$ with the property that there exists a connected graph G of order n such that $\operatorname{cir}(G) = n-j$ and $\operatorname{hc}(G) = i$.

As follows from the results of [2] mentioned above,

$$\operatorname{hc}_{\max}(n,0) = n-2 \quad \text{for every } n \geqslant 3$$

and

$$\operatorname{hc}_{\max}(n,1) = n-1 \quad \text{for every } n \geqslant 4.$$

Using Proposition 1, it is not difficult to show that $\operatorname{hc}_{\max}(5,2) = 6$. Combining Proposition 1 with Fig. 1 we easily get $\operatorname{hc}_{\max}(6,2) \leqslant 10$. In this section, we will find an upper bound of $\operatorname{hc}_{\max}(n,2)$ for $n \geqslant 7$.

Let $n \geqslant 7$, let $0 \leqslant i \leqslant \lfloor \frac{1}{2}(n-2) \rfloor$, and let V be a set of n elements, say elements $u_0, u_1, \ldots, u_{n-4}, u_{n-3}, v, w$. We denote by $F(n,i)$ the graph defined as follows: $V(F(n,i)) = V$ and

$$E(F(n,i)) = \{u_0u_1, u_1u_2, \ldots, u_{n-4}u_{n-3}, u_{n-3}u_0\} \cup \{u_0v, u_iw\}.$$

Lemma 1. *Let $n \geqslant 7$. Then there exists a hamiltonian coloring c_i of $F(n,i)$ with*

$$\operatorname{hc}(c_i) = 3n - \lfloor \frac{1}{3}(n-2) \rfloor - 6 - i$$

for each i, $0 \leqslant i \leqslant \lfloor \frac{1}{3}(n-2) \rfloor$.

Proof. Put $j = \lfloor \frac{1}{3}(n-2) \rfloor$. Let $0 \leqslant i \leqslant j$. Consider a mapping c_i of $V(F(n,i))$ into $\mathbb{N}$ defined as follows:

$$c_i(u_0) = n-1,\ c_i(u_1) = n-3,\ \ldots,\ c_i(u_{j-1}) = n-2(j-1)-1,$$
$$c_i(u_j) = n-2j-1,\ c_i(u_{j+1}) = 3n-2j-7,\ c_i(u_{j+2}) = 3n-2j-9,\ \ldots,$$
$$c_i(u_{n-4}) = n+3,\ c_i(u_{n-3}) = n+1,\ c_i(v) = 1 \ \text{ and } \ c_i(w) = 3n-j-6-i.$$

(A diagram of $F(21,0)$ with c_0 can be found in Fig. 2.)

Consider arbitrary distinct vertices r and s of $F(n,i)$ such that $c_i(r) \geqslant c_i(s)$. Put $D_i'(r,s) = D_{F(n,i)}'(r,s)$. Obviously, $c_i(r) > c_i(s)$. If $(r,s) = (w, u_{j+1})$ or (u_{n-3}, u_0) or (u_{f+1}, u_f), where $0 \leqslant f \leqslant n-4$, then $c_i(r) - c_i(s) = D_i'(r,s)$. If $(r,s) = (u_j, v)$, then $D_i'(r,s) + 2 \geqslant c_i(r) - c_i(s) \geqslant D_i'(r,s)$. Otherwise, $c_i(r) - c_i(s) > D_i'(r,s)$. Thus c_i is a hamiltonian coloring of $F(n,i)$. We see that $\operatorname{hc}(c_i) = c_i(w)$. □

Let $n \geqslant 7$. We define $F'(n) = F(n,0) - u_0w + vw$.

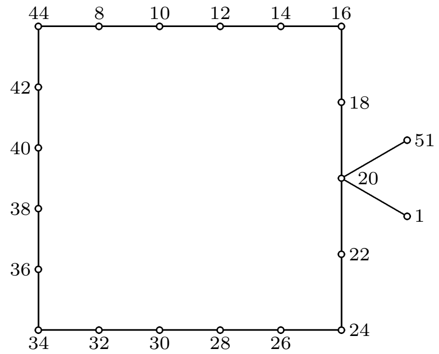

Fig. 2

Corollary 1. *Let $n \geqslant 7$. Then there exists a hamiltonian coloring c'_0 of $F'(n)$ with* $\mathrm{hc}(c'_0) = 3n - \lfloor \frac{1}{3}(n-2) \rfloor - 7$.

P r o o f. Put $c'_0 = c_1$, where c_1 is defined in the proof of Lemma 1. It is clear that c'_0 is a hamiltonian coloring of $F'(n)$. Applying Lemma 1, we get the desired result. □

Lemma 2. *Let $n \geqslant 7$. Then there exists a hamiltonian coloring c_i^+ of $F(n,i)$ with*

$$\mathrm{hc}(c_i^+) = 2n - 4 + 2\lfloor \tfrac{1}{2}(n-2) \rfloor - i$$

for each i, $\lfloor \frac{1}{3}(n-2) \rfloor + 1 \leqslant i \leqslant \lfloor \frac{1}{2}(n-2) \rfloor$.

P r o o f. Put $j = \lfloor \frac{1}{2}(n-2) \rfloor$ and $k = \lfloor \frac{1}{2}(n-2) \rfloor$. Let $j+1 \leqslant i \leqslant k$. Consider a mapping c_i^+ of $V(F(n,i))$ into $\mathbb{N}$ defined as follows:

$$c_i^+(u_0) = 3k+1,\ c_i^+(u_1) = 3k-1,\ \ldots,\ c_i^+(u_{k-1}) = k+3,\ c_i^+(u_k) = k+1,$$
$$c_i^+(u_{k+1}) = 2(n-3)+k+1,\ c_i^+(u_{k+2}) = 2(n-3)+k-1,\ \ldots,$$
$$c_i^+(u_{n-4}) = 3k+5,\ c_i^+(u_{n-3}) = 3k+3,\ c_i^+(v) = 1 \ \text{ and } \ c_i^+(w) = 2n-4+2k-i.$$

(A diagram of $F(21,7)$ with c_7^+ can be found in Fig. 3.)

Put $D'_i = D'_{F(n,i)}$. We see that $c_i^+(u_k) - c_i^+(v) = D'_i(u_k, v)$ and $c_i^+(w) - c_i^+(u_{k+1}) = D'_i(w, u_{k+1})$. It is easy to show that c_i^+ is a hamiltonian coloring of $F(n,i)$. We have $\mathrm{hc}(c_i^+) = c_i^+(w)$. □

Theorem 1. *Let $n \geqslant 7$. Then*

$$\mathrm{hc}_{\max}(n,2) \leqslant 3n - \lfloor \tfrac{1}{3}(n-2) \rfloor - 6.$$

P r o o f. Consider an arbitrary connected graph G of order n with $\mathrm{cir}(G) = n-2$. Obviously, G is spanned by a connected graph F such that $\mathrm{cir}(F) = n-2$ and F

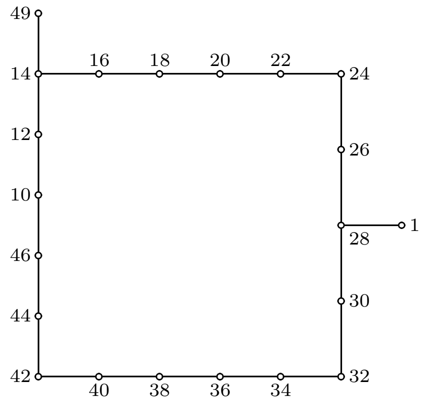

Fig. 3

has exactly one cycle. By Proposition 1, $\mathrm{hc}(G) \leqslant \mathrm{hc}(F)$. Thus we need to show that $\mathrm{hc}(F) \leqslant 3n - \lfloor \frac{1}{3}(n-2) \rfloor - 6$.

If F is isomorphic to $F'(n)$, then the result follows from Corollary 1. Let F be not isomorphic to $F'(n)$. Then there exists $i, 0 \leqslant i \leqslant \lfloor \frac{1}{2}(n-2) \rfloor$, such that F is isomorphic to $F(n,i)$. If $0 \leqslant i \leqslant \lfloor \frac{1}{3}(n-2) \rfloor$, then the result follows from Lemma 1. Let $\lfloor \frac{1}{3}(n-2) \rfloor \leqslant i \leqslant \lfloor \frac{1}{2}(n-2) \rfloor$. By Lemma 2, $\mathrm{hc}(F) \leqslant 2n - 4 + 2\lfloor \frac{1}{2}(n-2) \rfloor - i \leqslant 2n - 4 + 2\lfloor \frac{1}{2}(n-2) \rfloor - \lfloor \frac{1}{3}(n-2) \rfloor - 1 \leqslant 3n - \lfloor \frac{1}{3}(n-2) \rfloor - 7$, which completes the proof. □

Corollary 2. *Let $n \geqslant 7$. Then*

$$\mathrm{hc}_{\max}(n,2) \leqslant \tfrac{1}{3}(8n-14).$$

2. Consider arbitrary j and n such that $j \geqslant 0$ and $n - j \geqslant 3$. We denote by $\mathrm{hc}_{\min}(n,j)$ the minimum integer $i \geqslant 1$ with the property that there exists a connected graph G of order n such that $\mathrm{cir}(G) = n - j$ and $\mathrm{hc}(G) = i$. Since every hamiltonian-connected graph of order $\geqslant 3$ is hamiltonian, we get $\mathrm{hc}_{\min}(n,0) = 1$ for every $n \geqslant 3$. In this section we will find an upper bound of $\mathrm{hc}_{\min}(n,j)$ for $j \geqslant 1$ and $n \geqslant j(j+3)+1$.

We start with two auxiliary definitions. If U is a set, then we denote

$$E_{\mathrm{com}}(U) = \{A \subseteq U; |A| = 2\}.$$

If W_1 and W_2 are disjoint sets, then we denote

$$E_{\mathrm{combi}}(W_1, W_2) = \{A \in E_{\mathrm{com}}(W_1 \cup W_2); |A \cap W_1| = 1 = |A \cap W_2|\}.$$

Lemma 3. *Consider arbitrary j, k and n such that $j \geqslant 1, k \geqslant j+1$, and*

$$k + j(k+1) \leqslant n \leqslant k + (k-1)^2 + 2j.$$

Then there exists a k-connected graph G of order n such that $\mathrm{cir}(G) = n - j$ and $\mathrm{hc}(G) \leqslant 2j(k-1) + 1$.

Proof. Clearly, there exist $f_1, \ldots, f_{k-1}$ such that

$$j \leqslant f_g \leqslant k-1 \quad \text{for all } g,\ 0 \leqslant g \leqslant k-1$$

and

$$f_1 + \ldots + f_{k-1} = n - 2j - k.$$

Consider pairwise disjoint finite sets $U, W_1, \ldots, W_k$ and W_{k+1} such that $|U| = k$,

$$|W_g| = f_g \quad \text{for each } g, 0 \leqslant g \leqslant k-1$$

and $|W_k| = |W_{k+1}| = j$. We denote by G the graph with

$$V(G) = U \cup W_1 \cup \ldots W_k \cup W_{k+1}$$

and

$$E(G) = E_{\mathrm{com}}(V_1) \cup \ldots \cup E_{\mathrm{com}}(V_{k+1}) \cup E_{\mathrm{combi}}(U, V_1 \cup \ldots \cup V_{k+1}).$$

It is easy to see that G is a k-connected graph of order n and $\mathrm{cir}(G) = n - j$.

Put $D'(x,y) = D'_G(x,y)$ for $x, y \in U$. It is clear that

$$D'(u, u^*) = 2j$$

for all distinct $u, u^* \in U$,

$$D'(u, w) = j$$

for all $u \in U$ and $w \in W_1 \cup \ldots \cup W_{k+1}$,

$$D'(w, w^*) = 0$$

for all w and w^* such that there exist distinct $g, g^* \in \{1, \ldots, k+1\}$ such that $w \in W_g$ and $w^* \in W_{g^*}$, and

$$D'(w, w^*) = j$$

for all distinct w and w^* such that there exists $h \in \{1, \ldots, k+1\}$ such that $w, w^* \in W_h$.

Put $f_k = f_{k+1} = j$. Consider a mapping c of $V(G)$ into $\mathbb{N}$ with the properties that

$$c(U) = \{1, 2j+1, 4j+1, \dots, 2j(k-1)+1\}$$

and

$$c(W_g) = \{j+1, 3j+1, \dots, 2j(f_g-1)+j+1\}$$

for each $g, 1 \leqslant g \leqslant k+1$. It is easy to see that c is a hamiltonian coloring of G. Hence $\mathrm{hc}(G) \leqslant \mathrm{hc}(c) = 2j(k-1)+1$. □

Theorem 2. *Let n and j be integers such that $j \geqslant 1$ and $n \geqslant j(j+3)+1$, and let k be the smallest integer such that*

$$k \geqslant j+1 \quad \text{and} \quad (k-1)^2 + k \geqslant n - 2j.$$

Then

$$\mathrm{hc}_{\min}(n, j) \leqslant 2j(k-1)+1.$$

P r o o f. The theorem immediatelly follows from Lemma 3. □

Corollary 3. *Let $n \geqslant 5$ and let k be the smallest integer such that*

$$k \geqslant 2 \quad \text{and} \quad n \leqslant (k-1)^2 + k + 2.$$

Then

$$\mathrm{hc}_{\min}(n, 1) \leqslant 2k-1.$$

Corollary 4. *Let $n \geqslant 11$ and let k be the smallest integer such that*

$$k \geqslant 3 \quad \text{and} \quad n \leqslant (k-1)^2 + k + 4.$$

Then

$$\mathrm{hc}_{\min}(n, 2) \leqslant 4k-3.$$

3. As follows from results obtained in [2], if (a) $n \geqslant 3$, then for every $k \in \{1, 2, \dots, n-1\}$ there exists a connected graph G of order $n \geqslant 4$ such that $\mathrm{hc}(G) = k$, and if (b) G is a graph of order n such that $\mathrm{hc}(G) \geqslant n$, then $\mathrm{cir}(G) \neq n, n-1$.

For $n = 4$ or 5, it is easy to find a connected graph of order n with $\mathrm{hc}(G) = n$: $\mathrm{hc}(P_4) = 4$ and $\mathrm{hc}(2K_2 + K_1) = 5$. On the other hand, there exists no connected graph of order 6 with $\mathrm{hc}(G) = 6$. We can state the folowing question: Given $n \geqslant 7$, does there exist a connected graph G of order n with $\mathrm{hc}(G) = n$? Answering this question for $n \geqslant 8$ is the subject of the present section.

Let $1 \leqslant j \leqslant i$. Consider mutually distinct elements r, s, u, v, w and finite sets X and Y such that $|X| = i$, $|Y| = j$ and the sets X, Y and $\{r, s, u, v, w\}$ are pairwise disjoint. We define a graph $G(i, j)$ as follows:

$$\begin{aligned} V(G(i,j)) &= X \cup Y \cup \{r,s,u,v,w\} \quad \text{and } E(G(i,j)) \\ &= \{uv\} \cup E_{\mathrm{com}}(X) \cup \ E_{\mathrm{com}}(Y) \cup E_{\mathrm{combi}}(\{u,w\}, X \cup \{r\}) \\ &\quad \cup E_{\mathrm{combi}}(\{v,w\}, Y \cup \{s\}). \end{aligned}$$

Obviously, $\mathrm{cir}(G(i,j)) = i + j + 3 = |V(G(i,j)| - 2$.

Proposition 2. *Let $1 \leqslant j \leqslant i$. Put $D'(t_1,t_2) = D'_{G(i,j)}(t_1,t_2)$ for all $t_1,t_2 \in V(G(i,j))$. Then*

(1) $D'(x,y) = 0$ *for all* $x \in X$ *and all* $y \in Y$,

(2) $D'(x,s) = 0, D'(x,r) = D'(x,v) = 1$ *and* $D'(x,u) = D'(x,w) = 2$ *for all* $x \in X$,

(3) $D'(y,r) = 0, D'(y,s) = D'(y,u) = 1$ *and* $D'(y,v) = D'(y,w) = 2$ *for all* $y \in Y$,

(4) $D'(x_1,x_2) = 2$ *for all distinct* $x_1, x_2 \in X$,

(5) $D'(y_1,y_2) = 2$ *for all distinct* $y_1, y_2 \in Y$,

(6) $D'(r,s) = 0$,

(7) $D'(r,v) = D'(s,u) = 1$,

(8) $D'(u,v) = 2$,

(9) $D'(s,v) = D'(s,w) = j + 1$,

(10) $D'(v,w) = j + 2$,

(11) $D'(r,u) = D'(r,w) = \min(i + 1, j + 2)$,

and

(12) $$D'(u,w) = \min(i + 2, j + 3).$$

P r o o f is easy.

Lemma 4. *Let* $1 \leqslant j \leqslant i$. *Then* $\mathrm{hc}(G(i,j)) \geqslant i+j+5$.

P r o o f. Suppose, to the contrary, that there exists a hamiltonian coloring c of $G(i,j)$ such that $\mathrm{hc}(c) \leqslant i+j+4$. Thus $\mathrm{hc}(c) \leqslant 2i+4$. We may assume that there exists $t \in V(G(i,j))$ such that $c(t)=1$.

Put $X^+ = X \cup \{u,w\}$. By virtue of (2), (4) and (12),

$$|c(x_1^+) - c(x_2^+)| \geqslant 2 \quad \text{for all distinct } x_1^+, x_2^+ \in X^+. \tag{13}$$

By virtue of (2), (7) and (12),

$$c(r) \neq c(x^+) \neq c(v) \quad \text{for all } x^+ \in X^+, \tag{14}$$

$$c(r) \neq c(v), \ c(s) \neq c(u) \tag{15}$$

and

$$|c(u) - c(v)| \geqslant 2.$$

Obviously, $|X^+| = i+2$. As follows from (13),

$$\max c(X^+) \geqslant 2i+2+\min c(X^+). \tag{16}$$

Thus $\mathrm{hc}(c) \geqslant 2i+3$. Since $\mathrm{hc}(c) \leqslant i+j+4$, we get

$$i-1 \leqslant j \leqslant i. \tag{17}$$

If $\{c(r), c(v)\} = \{1,2\}$, then (14) implies that $\max c(X^+) \geqslant 2i+5$; a contradiction. If $\{c(r), c(v)\} = \{\mathrm{hc}(c), \mathrm{hc}(c)-1\}$, then $\max c(X^+) \leqslant 2i+2$; a contradiction. Thus

$$\{1,2\} \neq \{c(r), c(v)\} \neq \{\mathrm{hc}(c), \mathrm{hc}(c)-1\}. \tag{18}$$

Moreover, if

$$c(u) = \min c(X^+) \quad \text{and} \quad c(v) = c(u)+2$$

or

$$c(u) = \max c(X^+) \quad \text{and} \quad c(v) = c(u)-2,$$

then $\max c(X^+) \geqslant 2i+3+\min c(X^+)$.

Combining (11) and (12) with (17), we have

$$|c(r)-c(u)| \geqslant i+1, |c(r)-c(w)| \geqslant i+1 \quad \text{and} \quad |c(u)-c(w)| \geqslant i+2. \tag{19}$$

We denote by c' a mapping of $V(G(i,j))$ into $\mathbb{N}$ defined as follows:

$$c'(t) = \mathrm{hc}(c)+1-c(t) \quad \text{for each } t \in V(G(i,j)).$$

We see that c' is a hamiltonian coloring of $G(i,j)$ and that $\mathrm{hc}(c') = \mathrm{hc}(c)$. Obviously, $c(u) \leqslant c(v)$ or $c'(u) \leqslant c'(v)$. Without loss of generality we assume that $c(u) \leqslant c(v)$. Thus

$$c(v) \geqslant c(u) + 2$$

and if $c(u) = 1$ and $\mathrm{hc}(c) = 2i + 3$, then $c(v) \geqslant 4$.

We distinguish two cases.

Case 1. Assume that $j = i - 1$. Then $\mathrm{hc}(c) = 2i + 3$. By virtue of (9) and (10),

$$|c(s) - c(v)| \geqslant i, |c(s) - c(w)| \geqslant i \quad \text{and} \quad |c(v) - c(w)| \geqslant i + 1.$$

If $c(r) < c(u) < c(w)$ or $c(r) < c(w) < c(u)$ or $c(u) < c(w) < c(r)$ or $c(w) < c(u) < c(r)$, then (19) implies that $\mathrm{hc}(c) \geqslant 2i + 4$, which is a contradiction.

Let $c(w) < c(r) < c(u)$. As follows from (19), $c(u) = 2i + 3$ and therefore $c(v) \geqslant 2i + 5$; a contradiction.

Finally, let $c(u) < c(r) < c(w)$. Thus $c(w) = 2i + 3$ and therefore $c(u) = 1$ and $c(r) = i + 2$. Since $c(u) = 1$ and $\mathrm{hc}(c) = 2i + 3$, we get $c(v) \geqslant 4$. If $c(v) < c(s)$, then $c(s) \geqslant i+4$ and therefore $|c(s)-c(w)| \leqslant i-1$; a contradiction. Let $c(s) < c(v)$. Since $c(s) \neq c(u)$, we have $c(s) \geqslant 2$. This implies that $c(v) \geqslant i + 2$. Since $c(w) = 2i + 3$, we get $c(v) = i + 2$. Thus $c(v) = c(r)$, which contradicts (15).

Case 2. Assume that $i = j$. Recall that $\mathrm{hc}(c) \leqslant 2i + 4$. By virtue of (9) and (10),

$$|c(s) - c(v)| \geqslant i + 1, |c(s) - c(w)| \geqslant i + 1 \quad \text{and} \quad |c(v) - c(w)| \geqslant i + 2.$$

If $c(r) < c(w) < c(u)$ or $c(w) < c(r) < c(u)$, then (19) implies that $c(u) \geqslant 2i+3$ and therefore $c(v) \geqslant 2i + 5$, which is a contradiction.

Let $c(r) < c(u) < c(w)$. Then $c(w) = 2i+4$ and therefore $c(r) = 1$ and $c(u) = i+2$. This implies that $c(v) \geqslant i + 4$ and therefore $|c(v) - c(w)| \leqslant i$; a contradiction.

Let $c(u) < c(w) < c(r)$. Then $c(u) = 1$, $c(w) = i + 3$ and $c(r) = 2i + 4$. Since $3 \leqslant c(v) \neq c(r)$, we get $|c(v) - c(w)| \leqslant i$; a contradiction.

Let $c(w) < c(u) < c(r)$. Then $c(w) = 1$, $c(u) = i + 3$ and $c(r) = 2i + 4$. Assume that $c(s) < c(v)$; since $c(w) = 1$, we get $c(s) \geqslant i+2$ and therefore $c(v) \geqslant 2i+3$; since $c(r) = 2i + 4$ and $c(v) \neq c(r)$, we get $c(v) = 2i + 3$, which contradicts (18). Assume that $c(v) < c(s)$; since $c(u) = i + 3$, we get $c(v) \geqslant i + 5$ and therefore $c(s) \geqslant 2i + 6$; a contradiction.

Finally, let $c(u) < c(r) < c(w)$. Then $c(w) \geqslant 2i + 3$. If $c(v) < c(s)$, then $c(v) \geqslant 3$ and $c(s) \geqslant i+4$ and therefore $c(w) \geqslant 2i+5$; a contradiction. Assume that $c(s) < c(v)$.

If $c(s) \geqslant 2$, then $c(v) \geqslant i+3$ and therefore $c(w) \geqslant 2i+5$, which is a contradiction. Let $c(s) = 1$. Then $c(u) = 2$, $c(r) = i+3$ and $c(w) = 2i+4$. This implies that $c(v) = i+2$. Obviously, $\min c(X^+) = 2$. Since $c(v) = i+2$ and $c(r) = i+3$, we see that $c(x^+) \notin \{i+2, i+3\}$ for each $x^+ \in X^+$. Therefore $\max c(X^+) \geqslant 2i+3+\min c(X^+) = 2i+5$, which is a contradiction.

Thus the proof of the lemma is complete. □

Theorem 3. *For every $n \geqslant 8$, there exists a connected graph G of order n with $\operatorname{cir}(G) = n-2$ and $\operatorname{hc}(G) = n$.*

Proof. For every f and h such that $f \leqslant h$ we define

$$\mathrm{EVEN}[f,h] = \{g;\ f \leqslant g \leqslant h, g \text{ is even}\}$$

and

$$\mathrm{ODD}[f,h] = \{g;\ f \leqslant g \leqslant h, g \text{ is odd}\}.$$

We will use graphs $G(i,j)$ in the proof.

Consider an arbitrary $n \geqslant 8$. We distinguish four cases.

Case 1. Let $n = 4f+8$, where $f \geqslant 0$. Put

$$G_1 = G(2f+2, 2f+1).$$

Then the order of G_1 is n. Let c_1 be an injective mapping of $V(G_1)$ into $\mathbb{N}$ such that

$$c_1(r) = c_1(s) = 2f+5,\ c_1(u) = 1,\ c_1(v) = 3,\ c_1(w) = 4f+8,$$
$$c_1(X) = \mathrm{EVEN}[4, 4f+6] \ \text{ and } \ c_1(Y) = \mathrm{EVEN}[6, 4f+6].$$

(For $f = 0$, G_1 and c_1 are presented in Fig. 4.) Combining (1)–(12) with the definition of a hamiltonian coloring, we see that c_1 is a hamiltonian coloring of G_1. Clearly, $\operatorname{hc}(c_1) = 4f+8 = n$. Lemma 4 implies that $\operatorname{hc}(c_1) = \operatorname{hc}(G_1)$. Thus $\operatorname{hc}(G_1) = n$.

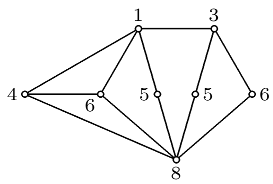

Fig. 4

C a s e 2. Let $n = 4f + 9$, where $f \geqslant 0$. Put

$$G_2 = G(2f + 2, 2f + 2).$$

Then the order of G_2 is n. Let c_2 be an injective mapping of $V(G_2)$ into $\mathbb{N}$ such that

$$c_2(r) = c_2(s) = 2f + 6,\ c_2(u) = 1,$$
$$c_2(v) = 3,\ c_2(w) = 4f + 9 \ \text{ and } \ c_2(X) = c_2(Y) = \mathrm{ODD}[5, 4f + 7].$$

(For $f = 0$, G_2 and c_2 are presented in Fig. 5.) By virtue of (1)–(12), c_2 is a hamiltonian coloring of G_2. Obviously, $\mathrm{hc}(c_2) = n$. As follows from Lemma 4, $\mathrm{hc}(G_2) = n$.

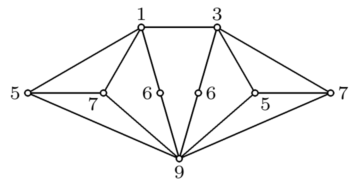

Fig. 5

C a s e 3. Let $n = 4f + 10$, where $f \geqslant 0$. Put

$$G_3 = G(2f + 3, 2f + 2).$$

The order of G_3 is n. Let c_3 be an injective mapping of $V(G_3)$ into $\mathbb{N}$ such that

$$c_3(r) = 2f + 5,\ c_3(s) = 2f + 6,\ c_3(u) = 1,\ c_3(v) = 3,\ c_3(w) = 4f + 10,$$
$$c_3(X) = \mathrm{EVEN}[4, 4f + 8] \ \text{ and } \ c_3(Y) = \mathrm{ODD}[5, 4f + 7].$$

(See Fig. 6 for $f = 0$.) By (1)–(12), c_3 is a hamiltonian coloring of G_3. By Lemma 4, $\mathrm{hc}(G_3) = \mathrm{hc}(c_3) = n$.

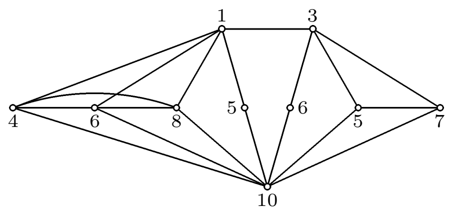

Fig. 6

C a s e 4. Let $n = 4f + 11$, where $n \geqslant 0$. Put

$$G_4 = G(2f + 4, 2f + 2).$$

The order of G_4 is n again. Let c_4 be an injective mapping of $V(G_4)$ into $\mathbb{N}$ such that

$$c_4(r) = 2f + 6,\ c_4(s) = 2f + 7,\ c_4(u) = 1,\ c_4(v) = 4,\ c_4(w) = 4f + 11,$$
$$c_4(X) = \mathrm{ODD}[3, 4f + 9] \ \text{ and } \ c_4(Y) = \mathrm{EVEN}[6, 4f + 8].$$

(See Fig. 7 for $f = 0$.) Combining (1)–(12) with Lemma 4, we see that $\mathrm{hc}(G_4) = \mathrm{hc}(c_4) = n$.

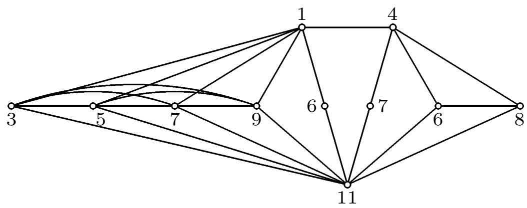

Fig. 7

Thus the proof is complete. □

The author conjectures that there exists no connected graph G of order 7 such that $\mathrm{hc}(G) = 7$.

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